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The shape and smoothness of stable plasma configurations

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The Shape and Smoothness of Stable Plasma Configurations (*).

DAVID KINDERLEHRER (**) - JOEL SPRUCK (**)

dedicated to Hans Lewy

Consider an axially symmetric toroidal vessel which contains an ionized gas, for example hydrogen, held in equilibrium by an externally applied magnetic field. We are asked to find the magnetic vector of the vessel which in part is occupied by the gas and in part is vacuum. The magnetic vector admits a stream function [T1] which satisfies a different equation in each of the two regions, gas and vacuum. The boundary of the gas region, or plasma region, is a free boundary.

In this note we consider the simplest model of this phenomenon with the object of showing that the plasma set is bounded by an infinitely differentiable manifold. Our proof, which is very simple, is suggested by a method of H. Lewy [L1] (cf. also [L2]). We then illustrate how our result may be extended to higher dimensions (§ 5). In a subsequent paper, written jointly with L. Nirenberg, we show that the plasma boundary is an analytic manifold.

We wish to thank R. Temam for suggesting this problem to us.

1. – Variational formulations of the problem.

Let Ω be a bounded simply connected domain in the $z = x_1 + ix_2$ plane with smooth boundary $\partial\Omega$. Given $\varphi \in C^1(\partial\Omega)$ and $q(z)$ real analytic in $\bar{\Omega}$

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satisfying $q(z) \geqslant q_0 > 0$ in $\bar{\Omega}$, we set

$$(1.1) \qquad \begin{cases} V \ = \{v \in H^1(\Omega) \colon v = \varphi \text{ on } \partial\Omega\} \\ V_c = \left\{v \in V \colon \frac{1}{2}\int\limits_\Omega \min(v, 0)^2 q\, dx = c\right\}. \end{cases}$$

Problem 1. *To find* $u \in V_c \colon \frac{1}{2}\int\limits_\Omega u_x^2\, dx = \min\limits_{v \in V} \frac{1}{2}\int\limits_\Omega v_x^2\, dx.$

If $c > 0$ then $V_c \neq \emptyset$. Hence the number $\inf\left\{\frac{1}{2}\int\limits_\Omega v_x^2\, dx \colon v \in V_c\right\}$ exists. The connection between Problem 1 and the confinement of a gas is given in [T1].

Theorem 1. *Let* $c > 0$. (i) *Then there exists a solution* u *to Problem* 1 *which satisfies* $u \in C^{2,\alpha}(\Omega)$, $0 < \alpha < 1$ *and*

$$(1.2) \qquad \Delta u(z) + \lambda \min\big(u(z), 0\big)\, q(z) = 0 \quad \text{in } \Omega$$

for some number $\lambda > 0$.

(ii) *Let* w *be any solution to Problem* 1. *Then* $w \in C^{2,\alpha}(\Omega)$ *and satisfies* (1.2) *for some* λ *depending on* w.

The proof of Theorem 1 is quite standard and may be found for example in [T1]. The point of part ii) is that *any minimum* w must be a weak solution of equation (1.2), which is the Euler equation for problem 1. Hence by a theorem of Stampacchia [S1] w must be of class $C^{2,\alpha}$.

It is possible to consider a space V defined by other conditions, for example,

$$(1.3) \qquad V = \left\{v \in H^1(\Omega) \colon \frac{\partial v}{\partial \nu} = g \quad \text{on } \partial\Omega\right\}$$

or

$$V = \left\{ v \in H^1(\Omega) \colon v = \varphi \text{ on } \partial_1\Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} = g \text{ on } \partial_2\Omega\right\}$$

where $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$ and $\partial_i\Omega$ are open arcs and ν is the outward directed normal.

Now suppose that V is defined in such a manner that a solution u to *Problem* 1 satisfies

$$u > 0 \quad \text{on } \partial\Omega\,.$$

For example we may take $\varphi > 0$ in (1.1) or $g > 0$ in (1.3), recalling that $\partial/\partial\nu$ is the outward pointing normal derivative. We then define

$$
(1.4)\qquad \begin{cases} \Omega_+ = \Omega_+(u) = \{z \in \Omega : u(z) > 0\} \\ \Omega_- = \Omega_-(u) = \{z \in \Omega : u(z) < 0\} \\ \Gamma \ = \Gamma(u) = \partial\Omega_- \subset \Omega \, . \end{cases}
$$

From the maximum principle it is obvious that Ω_+ is connected.

A simple model of the problem of plasma containment by a magnetic field can be expressed in this framework [T1]. Let $I > 0$ be given. We seek a function $\psi \in C^1(\bar{\Omega})$ and a domain $\Omega_p \subset \Omega$, such that

$$
(1.5)\qquad \begin{cases} \Delta\psi = -\lambda q \psi \quad \text{in } \Omega_p , \quad \text{for some } \lambda > 0 \\ \Delta\psi = 0 \quad \text{in } \Omega_v = \Omega - \bar{\Omega}_p \\ \psi = 0 \quad \text{on } \Gamma = \partial\Omega_p \end{cases}
$$

$\psi = \gamma$, on $\partial\Omega$, γ a constant to be determined

$$
\int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\, ds = I \, .
$$

Choose $\varphi = 1$ and set

$$
\psi(z) = \frac{I}{\int(\partial u/\partial\nu)\, ds}\, u(z) \, , \qquad z \in \Omega
$$

where u is a solution to problem 1 with $\Omega_p = \Omega_-(u)$.

A similar problem occurs in the theory of the hydrodynamical vortex [Ber-F] [N].

Berestycki-Brezis [Be-Br], Puel [P] and Temam [T2] have also considered the plasma problem (1.5) where $\lambda > 0$ is prescribed a priori. It is easy to see that there is no solution for $\lambda < \lambda_1$, λ_1 the first eigenvalue of the Dirichlet problem in Ω with respect to the weight function q. Furthermore if $\lambda = \lambda_1$ then the first eigenfunction suitably normalized is the unique solution and hence $\Omega_p \equiv \Omega$, an uninteresting case. For $\lambda > \lambda_1$ the authors mentioned above have shown there is always at least one nontrivial solution. In [T2] the solution to this version of the plasma problem is obtained as a solution of the following interesting variational problem.

PROBLEM 2. *To find* $u \in V_\lambda$: $E(u) = \min_{v \in V_\lambda} E(q)$ *where*

$$(1.6)\quad \begin{cases} E(v) = \dfrac{1}{2}\displaystyle\int_\Omega v_x^2\,dx - \dfrac{\lambda}{2}\int_\Omega \min(v, 0)^2 q\,dx - Iv(\partial\Omega) \\ V_\lambda = \left\{v \in H^1(\Omega)\colon v|_{\partial\Omega} = \text{constant} \equiv v(\partial\Omega) \text{ and } \displaystyle\int_\Omega |\min(v, 0)|q\,dx = I/\lambda\right\}. \end{cases}$$

Once again we observe that *any minimum* w solving Problem 2 is of class $C^{2,\alpha}$.

2. – Topological properties of the plasma.

This section is devoted to the clarification of the topological nature of the set

$$\Omega_- = \{z \in \Omega\colon u(z) < 0\}$$

for a given solution u to *Problem* 1. Our object is to prove

THEOREM 2. *Let u be a solution to Problem* 1 *and let* $\Omega_- = \{z \in \Omega\colon u(z) < 0\}$. *Then Ω_- is a Jordan domain and $\Gamma = \partial\Omega_-$ is a Jordan curve of class* $C^{2,\alpha}$ $0 < \alpha < 1$.

In the next lemma we use that u minimizes the Dirichlet integral on a given set in an essential way. A similar idea has been used by Berger and Fraenkel [Ber-F] and Norbury [N].

LEMMA 2.1. *Let $u \in V_c$ be a solution to Problem* 1. *Then Ω_- is connected.*

PROOF. Suppose, for an argument by contradiction, that A_1 is a component of Ω_- and $A_2 = \Omega_- - A_1 \neq \emptyset$. Let us define

$$w(z) = \begin{cases} a_1 u(z) & z \in A_1 \\ a_2 u(z) & z \in A_2 \\ u(z) & z \in \Omega - \Omega_- \end{cases}.$$

Assume for the moment it is possible to find $a_i \geq 0$, $i = 1, 2$ so that $w \in V_c$, that is

$$(2.1)\qquad \tfrac{1}{2}\int_\Omega \min(w, 0)^2 q\,dx = \tfrac{1}{2}\int_\Omega \min(u, 0)^2 q\,dx = c$$

and

$$\tfrac{1}{2}\int_\Omega w_x^2\,dx = \tfrac{1}{2}\int_\Omega u_x^2\,dx = \inf_{v\in V_c} \tfrac{1}{2}\int_\Omega v_x^2\,dx\,. \tag{2.2}$$

Then w would also be a solution to *Problem* 1 and in particular, would be of class $C^{2,\alpha}(\Omega)$. However this can occur only if $a_1 = a_2 = 1$. For we can find points $z_i \in \partial A_i$ at which A_i satisfies an internal sphere condition. At such points $\operatorname{grad} u(z_i) \neq (0, 0)$ by the maximum principle. Thus $\operatorname{grad} w$ is continuous only if $a_1 = a_2 = 1$, i.e. $w \equiv u$.

To prove the lemma, we exhibit w. In view of (2.1), (2.2) our object is to solve the equations

$$a_1^2\int_{A_1}\min(u, 0)^2 q\,dx + a_2^2\int_{A_2}\min(u, 0)^2 q\,dx = 2c \tag{2.3}$$

$$a_1^2\int_{A_1} u_x^2\,dx + a_2^2\int_{A_2} u_x^2\,dx = \int_\Omega u_x^2\,dx - \int_{\Omega_+} u_x^2\,dx = \int_{\Omega_-} u_x^2\,dx\,. \tag{2.4}$$

Now we set

$$\varphi(z) = \begin{cases} u(z) & z \in A_1 \\ 0 & z \in \Omega - A_1 \end{cases}$$

and note that $\varphi \in H_0^{1,\infty}(\Omega)$ since $u = 0$ on ∂A_1. Now

$$\Delta u(z)\varphi(z) + \lambda q(z)\min(u(z), 0(\varphi(z) = 0$$

in Ω which implies that

$$-\int_\Omega u_{x_i}\varphi_{x_i}\,dx + \lambda\int_\Omega \min(u, 0)\varphi q\,dx = 0\,.$$

By the definition of φ this yields

$$\int_{A_1} u_x^2\,dx = \lambda\int_{A_1}\min(u, 0)^2 q\,dx\,.$$

A corresponding expression is valid for A_2. So let us set

$$\xi_i = \int_{A_i}\min(u, 0)^2 q\,dx$$

and write the equations (2.3), (2.4) as the system

$$a_1^2\xi_1 + a_2^2\xi_2 = 2c$$
$$a_1^2\lambda\xi_1 + a_2^2\lambda\xi_2 = \int_{\Omega_-} u_x^2 dx .$$

This is a system in the unknowns a_1^2, a_2^2 which admits the solution $a_1 = a_2 = 1$ corresponding to $w = u$. Hence the two equations are dependent and there is a whole line of solutions. Consequently the hypothesis that $A_2 \neq \emptyset$ is untenable. Q.E.D.

PROOF OF THEOREM 2. Recall that Ω_+ is connected. We will rely on a theorem of Hartman and Wintner [H-W]. Since $u \in C^{2,\alpha}(\Omega)$ and $u = 0$ on $\Gamma = \partial\Omega_-$, it suffices to show that $\operatorname{grad} u(z_0) \neq (0, 0)$ at each $z_0 \in \Gamma$ to conclude that Γ is a $C^{1,\alpha}$ curve. From (1.2)

$$|\Delta u(z)| \leqslant k|u(z)| \qquad z \in \Omega$$

for an appropriate constant $k > 0$. Assume that $u(z_0) = u_{x_i}(z_0) = 0$ for some $z_0 \in \Gamma$. Then by the theorem of Hartman and Wintner, there is an integer $\mu \geqq 2$ and a complex number $c = |c|e^{i\tau} \neq 0$ such that

$$\begin{aligned} u(z) &= \operatorname{Re}\{c(z - z_0)^\mu\} + O(|z - z_0|^\mu) \\ &= |c||z - z_0|^\mu \cos(\mu\theta + \tau) + O(|z - z_0|^\mu) , \end{aligned} \tag{2.5}$$

$\theta = \arg(z - z_0)$, and

$$\frac{\partial u}{\partial z}(z) = \mu c(z - z_0)^{\mu-1} + O(|z - z_0|^{\mu-1}) \tag{2.6}$$

in a neighborhood of z_0. From (2.6), the zeros of grad u are isolated. Given that $u_{x_i}(z_0) = 0$, $i = 1, 2$, there is a neighborhood $|z - z_0| < \varepsilon$ which is divided by 2μ smooth curves emanating from z_0 into $2\mu \geqslant 4$ sectors σ_j such that

$$u(z) > 0 \qquad \text{for } z \in \sigma_1 \cup \sigma_3 \cup \ldots \cup \sigma_{2\mu-1}$$

and

$$u(z) < 0 \qquad \text{for } z \in \sigma_2 \cup \sigma_4 \cup \ldots \cup \sigma_{2\mu} .$$

Choose $z_2 \in \sigma_2$ and $z_4 \in \sigma_4$. We may construct a simple arc β from z_2 to z_4 contained in Ω_- because Ω_- is open and connected. Further, we join z_2 and z_4 to z_0 in σ_2 and σ_4 respectively to obtain a Jordan curve $\gamma \subset \{z_0\} \cup \Omega_-$.

This γ must enclose σ_ν for some odd ν. Consequently the bounded component of $\boldsymbol{R}^2-\gamma$ contains σ_ν and the unbounded component of $\boldsymbol{R}^2-\gamma$ contains points of Ω_+ near $\partial\Omega$. Since this contradicts that Ω_+ is connected

$$\operatorname{grad} u(z_0) \neq (0, 0)\,. \qquad \text{Q.E.D.}$$

It is a simple matter to modify the proof of Theorem 2 to apply to solutions of *Problem* 2. We need only show that Ω_- is connected. Let w, A_1, A_2 be as in the proof of Lemma 2.1. We must now try to find a_1, a_2 so that $E(w)=E(u)$, and $\int_\Omega |\min(w,0)|q\,dx = \int_\Omega |\min(u,0)|q\,dx = I/\lambda$. As before

$$\int_{A_i} u_x^2 dx = \lambda \int_{A_i} \min(u,0)^2 q\,dx \qquad i=1,2\,. \tag{2.7}$$

Hence

$$E(u) = \tfrac{1}{2}\int_{\Omega_+} u_x^2 dx - Iu(\partial\Omega)$$

and

$$E(w) = E(u) + \frac{a_1^2}{2}\int_{A_1} u_x^2\,dx - \frac{a_1^2\lambda}{2}\int_{A_1}\min(w,0)^2 q\,dx + \frac{a_2}{2}\int_{A_2} u_x^2\,dx - \\ - \frac{a_2^2\,\lambda}{2}\int_{A_2}\min(w,0)^2 q\,dx = E(u)$$

by (2.7). Hence for the first equation any two numbers a_1, a_2 suffice.

The second equation

$$\frac{-I}{\lambda} = \int_\Omega \min(w,0)\,q\,dx = a_1\int_{A_1} uq\,dx + a_2\int_{A_2} uq\,dx$$

is linear. Hence there is a line of solutions. As before this violates the smoothness of the minimum. Hence Ω_- is connected. By the maximum principle, Ω_+ is also connected.

We may now proceed as before; namely, the existence of a point $z_0 \in \partial\Omega_-$ where $\operatorname{grad} u(z_0) = (0,0)$ is not consistent with the connectedness of Ω_+ and Ω_-. We restate this result as:

THEOREM 2′. *Let u be a solution to Problem 2 and let $\Omega_- = \{z \in \Omega: u(z) < 0\}$. Then Ω_- is a Jordan domain and $\Gamma = \partial\Omega_-$ is a Jordan curve of class $C^{2,\alpha}$, $0<\alpha<1$.*

3. – A local formulation.

The smoothness of the «free boundary» $\Gamma = \partial\Omega_-$ will be studied as a local problem. Let us suppose that $\lambda(z)$ is a real analytic function in a neighborhood of $z = 0$. Let $B = \{|z| < R\}$ be a small ball and Γ a simple arc of class $C^{2,\alpha}$, $0 < \alpha < 1$ in B passing through $z = 0$ and joining two points $a, b \in \partial B$. The ball B is thereby separated by Γ into two Jordan domains U_+ and U_-. We sssume Γ to be oriented positively, that is counterclockwise with respect to U_+.

Assume now that $u \in C^{2,\alpha}(\bar{B})$, $0 < \alpha < 1$ satisfies

$$\tag{3.1} \begin{cases} \Delta u = 0 \quad \text{in } U_+ \\ \Delta u + \lambda u = 0 \quad \text{in } U_- \\ u = 0\,, \quad |\nabla u| \neq 0 \text{ on } \Gamma\,. \end{cases}$$

The complex gradient of u,

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} - i\,\frac{\partial u}{\partial x_2}\right)$$

is holomorphic in U_+, where u is harmonic, and attains continuous values on Γ. Motivated by the ideas of Lewy [L1], we will derive a second function which is holomorphic in U_- and whose values on Γ are related to those of $\partial u/\partial z$ through an integral equation.

We introduce the Riemann function $R(z, \bar{z}, t, \bar{t})$ of the equation

$$\tag{3.2} \Delta v + \lambda v = 0 \quad \text{in } B\,.$$

It is a holomorphic function of the four independent complex variables $z, \bar{z}, t, \bar{t}$ for $z, t \in B$. Writing $R = R(z, z^*, t, t^*)$, for an appropriate range of the complex variables z, z^*, t, t^*, it satisfies the relations

$$\begin{aligned} &\tfrac{1}{4}R_{zz^*} + \lambda R = 0 \\ &R(t, t^*, z, z^*) = R(z, z^*, t, t^*) \\ &R(z, z^*, z, z^*) = 1 \\ &R_t(z, z^*, t, z^*) = 0 \\ &R_{t^*}(z, z^*, z, t^*) = 0\,. \end{aligned}$$

Fix a point z_0 on Γ. For $z \in U_-$ we define the function

$$\Phi(z) = \int_{z_0}^{z} R(z, \bar{z}_0, t, \bar{t}) \frac{\partial u}{\partial t} dt + R_{\bar{t}}(z, \bar{z}_0, t, \bar{t}) u \, d\bar{t} \tag{3.3}$$

along any path in U_- joining z_0 to z. $\Phi(z)$ is a well defined holomorphic function in U_- since the integrand in (3.3) is *exact*. This follows since R and u as functions of t, $\bar{t}$ satisfy (3.2) in U_-.

More useful than Φ is its derivative Φ'.

LEMMA 3.1. *Let $z_0 \in \Gamma$ and define $\Phi(z)$ by (3.3) for $z \in U_-$. Then*

$$\Phi'(z) = \frac{\partial u}{\partial z}(z) + \int_{\widehat{z_0 z}} R_z(z, \bar{z}_0, t, \bar{t}) \frac{\partial u}{\partial t} dt \qquad z \in \Gamma \tag{3.4}$$

and $\Phi' \in C^{1,\alpha}(\Gamma)$.

This follows immediately from (3.3) and the relations

$$R(z, \bar{z}_0, z, \bar{z}) = 1, \quad u|\Gamma = 0, \ u \in C^{2,\alpha}(B).$$

To conclude this section we interpret the information derived so far in terms of the Plemelj formulae.

THEOREM 3. *Let $u(z) \in C^{2,\alpha}(B)$ satisfy (3.1) and set*

$$f(z) = \frac{\partial u}{\partial z}(z), \qquad z \in B.$$

Let $R(z, \bar{z}, t, \bar{t})$ denote the Riemann function of (3.2) and for $z_0 \in \Gamma$ fixed, set

$$Af(z) = -\int_{\widehat{z_0 z}} R_z(z, \bar{z}_0, t, \bar{t}) f(t) \, dt \qquad z \in \Gamma.$$

Let $|z_0| < r < R$ and let $\Gamma_0 = \widehat{a_0 b_0} \subset \Gamma \cap B_r$ satisfy $0 < |a_0 - z_0| = |b_0 - z_0| = \delta \leq r - |z_0|$. Then there is a function $h(z)$ holomorphic in $B_\delta(z_0)$ such that

$$f(z) = \frac{1}{2} Af(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{t - z} Af(t) \, dt + h(z) \qquad \text{for } z \in \Gamma_0.$$

PROOF. Let z_0 and Γ_0 be as described in the hypotheses and set

$$\Gamma_0 \cup \alpha_+ = \partial(U_+ \cap B_\delta(z_0)) \quad \text{and} \quad -\Gamma_0 \cup \alpha_- = \partial(U_- \cap B_\delta(z_0)) \, .$$

Then by Cauchy's Theorem

$$f(z) = +\frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{t-z}f(t)\,dt + \frac{1}{2\pi i}\int_{\alpha_+}\frac{1}{t-z}f(t)\,dt\,, \qquad z \in U_+,\ |z-z_0| < \delta$$

$$0 = \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{t-z}\Phi'(t)\,dt + \frac{1}{2\pi i}\int_{\alpha_-}\frac{1}{t-z}\Phi'(t)\,dt\,, \qquad z \in U_+,\ |z-z_0| < \delta\,.$$

Hence for $z \in \Gamma_0$, by the Plemelj formulae,

$$f(z) = \frac{1}{2}f(z) + \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{t-z}f(t)\,dt + \frac{1}{2\pi i}\int_{\alpha_+}\frac{1}{t-z}f(t)\,dt\,, \qquad z \in \Gamma_0$$

$$0 = -\frac{1}{2}\Phi'(z) - \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{t-z}\Phi'(t)\,dt + \frac{1}{2\pi i}\int_{\alpha_-}\frac{1}{t-z}\Phi'(t)\,dt\,, \qquad z \in \Gamma_0\,.$$

Adding these two equations we obtain by *lemma* 3.1

$$f(z) = \frac{1}{2}Af(z) + \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{t-z}Af(t)\,dt + h(z)\,, \qquad z \in \Gamma_0$$

where

$$h(z) = \frac{1}{2\pi i}\int_{\alpha_+}\frac{1}{t-z}f(t)\,dt + \frac{1}{2\pi i}\int_{\alpha_-}\frac{1}{t-z}\Phi'(t)\,dt\,, \qquad |z-z_0| < \delta\,. \qquad \text{Q.E.D.}$$

4. – Regularity of the free boundary.

In this paragraph we show that when the conditions (3.1) are fulfilled, then Γ is an infinitely differentiable curve. We assume, as in § 3, that $B = \{|z| < R\}$ and that Γ is a simple arc of class $C^{2,\alpha}$ joining two points $a, b \in \partial B$ and passing through $z = 0$.

LEMMA 4.1. *Let* $\varphi(z) \in C^{1,\alpha}(\bar{\Gamma})$ *and define*

$$\Phi(z) = \frac{1}{\pi i}\int_{\Gamma}\frac{1}{t-z}\varphi(t)\,dt \qquad z \in \Gamma\,.$$

Then $\Phi(z) \in C^{1,\alpha}(\Gamma)$ *and*

$$\frac{\partial \Phi}{\partial z}(z) = \frac{1}{\pi i}\int_\Gamma \frac{1}{t-z}\frac{\partial \varphi}{\partial t}(t)\,dt + \frac{1}{\pi i}\left(\frac{\varphi(b)}{z-b} - \frac{\varphi(a)}{z-a}\right), \qquad z \in \Gamma. \tag{4.1}$$

PROOF. We include the proof of this elementary lemma for the reader's convenience. First we observe that since $\varphi \in C^{1,\alpha}(\bar{\Gamma})$, it may be considered the restriction of a function $\varphi^* \in C^{1,\alpha}(\bar{B})$ in such a way that

$$\frac{\partial \varphi}{ds} = \frac{\partial \varphi^*}{\partial z}\frac{dz}{ds} \quad \text{for } z \in \Gamma,$$

where s denotes arc length on Γ. The integral in (4.1) is thus well defined if we agree not to distinguish between φ and its extension φ^*.

The function Φ is understood to be a principal value integral on Γ so

$$\Phi(z) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)-\varphi(z)}{t-z}\,dt + \frac{\varphi(z)}{\pi i}\log\frac{b-z}{z-a} \quad \text{for } z \in \Gamma,$$

where the integral is absolutely convergent because φ is smooth. Now observe that

$$\frac{\partial}{\partial z}\frac{\varphi(t)-\varphi(z)}{t-z} = \frac{\varphi(t)-\varphi(z)-\varphi_z(z)(t-z)}{(t-z)^2}$$

a quantity easily seen to be absolutely integrable on Γ, again because φ is smooth. Hence

$$\begin{aligned}
\frac{\partial}{\partial z}\Phi(z) &= \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)-\varphi(z)-\varphi_z(z)(t-z)}{(t-z)^2}\,dt + \frac{1}{\pi i}\varphi_z(z)\log\frac{b-z}{z-a} \\
&\quad + \frac{\varphi(z)}{\pi i}\left(\frac{1}{a-z} - \frac{1}{b-z}\right) \qquad z \in \Gamma \\
&= \frac{1}{\pi i}\int_\Gamma \frac{\varphi_t(t)-\varphi_z(z)}{t-z}\,dt - \frac{1}{\pi i}\left.\frac{\varphi(t)-\varphi(z)-\varphi_z(z)(t-z)}{t-z}\right|_a^b \qquad (4.2)\\
&\quad + \frac{1}{\pi i}\varphi_z(z)\log\frac{b-z}{z-a} + \frac{\varphi(z)}{\pi i}\left(\frac{1}{a-z} - \frac{1}{b-z}\right) \\
&= \frac{1}{\pi i}\int_\Gamma \frac{\varphi_t(t)-\varphi_z(z)}{t-z}\,dt + \frac{1}{\pi i}\varphi_z(z)\log\frac{b-z}{z-a} + \frac{1}{\pi i}\left(\frac{\varphi(b)}{z-b} - \frac{\varphi(a)}{z-a}\right) \\
&= \frac{1}{\pi i}\int_\Gamma \frac{1}{t-z}\frac{\partial \varphi}{\partial t}(t)\,dt + \frac{1}{\pi i}\left(\frac{\varphi(b)}{z-b} - \frac{\varphi(a)}{z-a}\right)
\end{aligned}$$

where (4.2) is the result of an integration by parts. Q.E.D.

LEMMA 4.2. *Suppose that* $\Gamma \in C^{\mu,\alpha}$ *and* $\varphi \in C^{\mu,\alpha}(\Gamma)$. *Let*

$$\Phi(z) = \frac{1}{\pi i} \int_\Gamma \frac{1}{t-z} \varphi(t)\, dt \qquad z \in \Gamma .$$

Then

$$\Phi(z) \in C^{\mu,\alpha}(\Gamma) .$$

PROOF. Let us first suppose that $\varphi = 0$ in a neighborhood of the end-points a, b of Γ. For functions $\zeta(s)$ defined on Γ we define the operator

$$P(\zeta) = q(s) \frac{d\zeta}{ds}(s) , \qquad q(s) = \left(\frac{dz}{ds}\right)^{-1} .$$

It follows from Lemma (4.1) that, assuming $\Phi \in C^{\mu-1,\alpha}(\Gamma)$ and $\varphi \in C^{\mu,\alpha}(\Gamma)$,

$$P^\mu(\Phi) = \frac{1}{\pi i} \int_\Gamma \frac{1}{t-z} \frac{\partial^\mu \varphi}{\partial t^\mu}(t)\, dt \qquad z \in \Gamma .$$

The right hand side is Hölder continuous since $\Gamma \in C^1$ and $\partial^\mu\varphi/\partial t^\mu \in C^{0,\alpha}(\Gamma)$, by well known estimates. Since

$$P^\mu(\Phi) = q(s)^\mu \frac{d^\mu}{ds^\mu} \Phi(s) + \text{terms of lower order in } \Phi ,$$

the conclusion follows under our assumption about φ.

For the general case, given z, let $\eta(t)$ be a C^∞ function satisfying

$$\eta(t) = \begin{cases} 1 & \text{near } z \\ 0 & \text{near } a, b \end{cases}$$

and write

$$\Phi(z) = \frac{1}{\pi i} \int_\Gamma \frac{1}{t-z} \eta(t)\varphi(t)\, dt + \frac{1}{\pi i} \int_\Gamma \frac{1}{t-z} (1 - \eta(t)(\varphi(t)\, dt .$$

The case just considered applies to the first integral. The second integral is holomorphic near z. Q.E.D.

To prove that Γ is a C^∞ curve, we employ an inductive procedure based on the representation in *Theorem* 3.

THEOREM 4. *Let $\lambda(z)$ be a positive real analytic function in a neighborhood of B. Let $u \in C^{2,\alpha}(B)$ satisfy*

$$\begin{aligned} &\Delta u = 0 \quad \text{in } U_+ \\ &\Delta u + \lambda u = 0 \quad \text{in } U_- \\ &u = 0\,, \quad |\nabla u| \neq 0 \quad \text{on } \Gamma\,. \end{aligned}$$

Then Γ is a C^∞ curve.

PROOF. Let $f(z) = \partial u/\partial z(z)$, holomorphic in U_+ and continuous on Γ. The assertion that Γ is a curve of class, say, $C^{k,\alpha}$, is equivalent to claiming the existence of a function $\zeta(z)$, defined for z in a neighborhood of Γ and of class $C^{k,\alpha}$ there, such that

$$\bar z = \zeta(z) \quad \text{for } z \in \Gamma\,.$$

To show that Γ is C^∞, it suffices to show that ζ is in C^∞, indeed, it suffices to show that $\zeta(z)$ is a C^∞ function of z for $z \in \Gamma$. In the case at hand, $f dz + \bar f d\bar z = 0$ on Γ, since $u = 0$ on Γ, so

$$\frac{d\zeta}{dz}(z) = -\frac{fz}{\overline{f(z)}} \quad z \in \Gamma\,.$$

Therefore, if $f \in C^{\mu,\alpha}(\Gamma)$ and $\zeta \in C^{\mu,\alpha}(\Gamma)$, then $\zeta \in C^{\mu+1,\alpha}(\Gamma)$, or Γ is of class $C^{\mu+1,\alpha}$.

So suppose now that $\Gamma_0 = \widehat{a_0 b_0} \subset \Gamma$, $|a_0| = |b_0| = r$, and $f \in C^{\mu,\alpha}(\bar\Gamma_0)$, $\bar\Gamma_0$ is of class $C^{\mu+1,\alpha}$, and there exists a function $h_\mu(z)$ holomorphic in B_r such that

$$(4.3) \qquad \begin{cases} f^{(\mu)}(z) = \dfrac{1}{2}\varphi^{(\mu)}(z) + \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_0}\frac{1}{t-z}\varphi^{(\mu)}(t)\,dt + h_\mu(z) \qquad z \in \Gamma_0 \\ \varphi(z) = -\displaystyle\int_{\widehat{0z}} R_z(z, 0, t, \bar t) f(t)\,dt\,. \end{cases}$$

Here $g^{(\mu)}(z) = (d^\mu/dz^\mu) g(z)$. The above holds for $\mu = 0$ and $\Gamma_0 = \Gamma$ by Theorem 3.

With these hypotheses about the smoothness of f and Γ we see that

$$(4.4) \qquad -\varphi^{(\mu)}(z) = \frac{d^{\mu-1}}{dz^{\mu-1}}\left(R_z(z, 0, z, \bar z) f(z)\right) + \sum_{j=0}^{\mu-2}\frac{d^j}{dz^j}\left(\frac{\partial^{\mu-j}}{\partial z^{\mu-j}} R(z, 0, z, \bar z) f(z)\right) + \int_{\widehat{0z}}\frac{\partial^{\mu+1}}{\partial z^{\mu+1}} R(z, 0, t, \bar t) f(t)\,dt \qquad z \in \Gamma_0\,.$$

In (4.4), $\partial/\partial z$ denotes differentiation with respect to the first place in $R(z, z^*, t, t^*)$ and, suppressing the notation $\bar{z} = \zeta(z)$, $\bar{z}$ is regarded as a function of z. Hence (4.4) contains derivatives of f and $\bar{z}$ up to order $\mu - 1$. Therefore $\varphi^{(\mu)} \in C^{1,\alpha}(\bar{\Gamma}_0)$. By *Lemma* 4.1,

$$\frac{1}{\pi i}\int_{\Gamma_0}\frac{1}{t-z}\,\varphi^{(\mu)}(t)\,dt \in C^{1,\alpha}(\Gamma_0)\,.$$

Using Lemma 4.1 once again, we differentiate (4.3) to see that

$$f^{(\mu+1)}(z) = \frac{1}{2}\,\varphi^{(\mu+1)}(z) + \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{t-z}\,\varphi^{(\mu+1)}(t)\,dt + h_{\mu+1}(z) \qquad z \in \Gamma_0$$

$$h_{\mu+1}(z) = \frac{1}{\pi i}\left(\frac{\varphi^{(\mu)}(b_0)}{b_0 - z} - \frac{\varphi^{(\mu)}(a_0)}{a_0 - z}\right) + \frac{d}{dz}\,h_\mu(z) \qquad z \in \Gamma_0\,.$$

From this and Lemma 4.2 we conclude that (4.3) holds for $\mu+1$ and $f \in C^{\mu+1,\alpha}(\Gamma')$ and Γ' is in $C^{\mu+2,\alpha}$ for any subarc $\Gamma' \subset \bar{\Gamma}' \subset \Gamma$. Q.E.D.

5. – Generalizations of the problem.

Let Ω be a simply connected domain in R^n. It is natural to ask whether our results extend to solutions of the minimum problem corresponding to the divergence form equation

$$\frac{\partial}{\partial x_j}\left(a_{ij}(x)\,u_{x_i}(x)\right) + \lambda q(x)\min\left(u(x), 0\right) = 0 \qquad \text{in } \Omega\,.$$

It is easy to see that *Lemma* 2.1 continues to hold, that is $\Omega_- = \{x \in \Omega\colon u(x) < 0\}$ is connected. When $n = 2$ we may still deduce that $|\nabla u| \neq 0$ on $\Gamma = \partial\Omega_-$ for the more general equation. But since this conclusion was based on a two dimensional level line argument, when $n \geqslant 3$ we may only assert that the points of Γ where $|\nabla u| \neq 0$ are open and dense in Γ.

We now proceed to generalize the local regularity theorem proved in § 4. For $n = 2$ and a_{ij}, q real analytic the methods employed in § 3 and § 4 via the Riemann function yield the desired result. Rather than working out the details in this particular case, we will prove an extension of this result valid in n dimensions by using methods of classical potential theory.

Let $B = \{|z| < R\} \subset R^n$ be a small ball and Γ a simply connected hypersurface of class $C^{2,\alpha}$, $0 < \alpha < 1$ passing through $z = 0$ and separating B into two components U_+ and U_-. For simplicity we assume $n \geqslant 3$.

Assume now that $u \in C^{2,\alpha}(\bar{B})$, $0 < \alpha < 1$ satisfies

$$
(5.1) \qquad \begin{cases} L^+u \equiv \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)u_{x_j}\right) + b_i^+(x)u_{x_i} + c^+(x)u = 0 & \text{in } U_+ \\ L^-u \equiv \dfrac{\partial}{\partial x_i}\left(a_{ij}u_{x_j}\right) + b_i^-(x)u_{x_i} + c^-(x)u + 0 & \text{in } U^- \\ \qquad u = 0\,, \quad |\nabla u| \neq 0 & \text{on } \Gamma \end{cases}
$$

where the a_{ij} are symmetric and for simplicity we assume all coefficients are of class $C^\infty(B)$.

We introduce $G^\pm(x, y)$, fundamental solutions of the equations $L^\pm v = 0$ in B. In terms of G^+ we represent u in U_+ recalling that $u = 0$ on Γ as follows:

$$
u(x) = \int_\Gamma G^+(x, y)\,a_{ij}u_{x_i}\nu_j(y)\,dS_y + h^+(x) \qquad x \in U_+
$$

where ν is the exterior normal to Γ with respect to U_+ and h^+ is of class $C^\infty(B)$. Let l be any direction such that $l\cdot\nu > 0$.

Then

$$
\frac{\partial u}{\partial l}(x) = \int_\Gamma \frac{\partial G^+}{\partial l}(x, y)\,a_{ij}u_{x_i}\nu_j(y)\,dS_y + \frac{\partial h^+}{\partial l}, \qquad x \in U_+
$$

where $\partial/\partial l$ refers to differentiation in the x variables. Now letting x tend to Γ we obtain via well known properties of the single layer potential [M]

$$
(5.2) \qquad \frac{\partial u}{\partial l}(x) = \frac{a_{ij}(x)u_{x_i}(x)\,\nu_j(x)\;l\cdot\nu}{2a_{ij}(x)\nu_i(x)\nu_j(x)} + \int_\Gamma \frac{\partial G^+}{\partial l}(x, y)\,a_{ij}u_{x_i}\nu_j(y)\,dS_y + \frac{\partial h^+}{\partial l}(x) \qquad x \in \Gamma\,.
$$

Since u vanishes on Γ we see that

$$
(5.3) \qquad a_{ij}(x)\,u_{x_i}(x)\,\nu_j(x) = a_{ij}(x)\,\nu_i(x)\,\nu_j(x)\,\frac{\partial u}{\partial \nu}(x)
$$

and

$$
\frac{\partial u}{\partial l}(x) = l\cdot\nu\,\frac{\partial u}{\partial \nu}(x)\,.
$$

Using (5.3) to simplify (5.2) we arrive at the formula

$$
(5.4) \qquad \frac{1}{2}\frac{\partial u}{\partial l}(x) = \int_\Gamma \frac{\partial G^+}{\partial l}(x, y)\,\frac{a_{ij}\nu_i\nu_j}{l\cdot\nu}(y)\,\frac{\partial u}{\partial l}(y)\,dS_y + \frac{\partial h^+}{\partial l}(x)\,, \qquad x \in \Gamma\,.
$$

In a similar way working with G^- in U_- we obtain

$$(5.5)\qquad \frac{1}{2}\frac{\partial u}{\partial l}(x) = -\int_{\Gamma}\frac{\partial G^-}{\partial l}(x, y)\,\frac{a_{ij}\nu_i\nu_j}{l\cdot\nu}(y)\,\frac{\partial u}{\partial l}(y)\,dS_\nu + \frac{\partial h^-}{\partial l}(x)\,, \qquad x\in\Gamma$$

where $h^-\in C^\infty(B)$ and ν, as before, is the exterior normal with respect to U_+.

Adding (5.4) and (5.5) we arrive at the following representation theorem for solutions of (5.1).

THEOREM 5. *Let $u(x)\in C^{2,\alpha}(B)$ satisfy* (5.1). *Then*

$$(5.6)\qquad \frac{\partial u}{\partial l}(x) = \int_{\Gamma}\frac{\partial}{\partial l}K(x, y)\,\frac{a_{ij}\nu_i\nu_j}{l\cdot\nu}(y)\,\frac{\partial u}{\partial l}(y)\,dS_\nu + \frac{\partial h}{\partial l}, \qquad x\in\Gamma$$

where ν is a normal field to Γ, $l\cdot\nu>0$, $h\in C^\infty(B)$, and the kernel $K(x, y)$ is C^∞ for $x\neq y$ and satisfies

$$K(x, y) = O(r^{3-n})\,, \qquad K_{x_i} = O(r^{2-n})\,, \qquad K_{x_ix_j} = O(r^{1-n})\,, \qquad r = |x-y|\,.$$

PROOF. Formula (5.6) follows from adding equations (5.4) and (5.5) where we have set $K(x, y) = (G^+ - G^-)(x, y)$ and $h = h^+ + h^-$. The point of the formula is that the kernel K satisfies the nice estimates stated above. This holds since the operators L^+ and L^- have the same principal part. Q.E.D.

Using *Theorem* 5 we can essentially copy our old proof of the C^∞ nature of Γ.

THEOREM 6. *Let $u\in C^{2,\alpha}(B)$ satisfy*

$$\begin{aligned}
&\frac{\partial}{\partial x_i}(a_{ij}u_{x_j}) + b_i^+u_{x_i} + c^+u = 0 &&\text{in } U_+\\
&\frac{\partial}{\partial x_i}(a_{ij}u_{x_j}) + b_i^-u_{x_i} + c^-u = 0 &&\text{in } U_-\\
&u = 0, \quad |\nabla u|\neq 0 &&\text{on } \Gamma
\end{aligned}$$

where $a_{ij}(x)$, $b_i^\pm(x)$, $c^\pm(x)$ are C^∞ in a neighborhood of B.

Then Γ is a C^∞ hypersurface.

PROOF. We sketch the essential ideas.

Let us represent Γ as the graph of a function

$$x_n = \varphi(x_1, \ldots, x_{n-1})$$

in a neighborhood of the origin containing B, so that the x_n direction is normal to Γ at the origin. To show that Γ is C_∞ it suffices to show that $\varphi \in C_\infty$. We compute

$$\frac{\partial u}{\partial l} = l \cdot \nu \frac{\partial u}{\partial l} = \Big(l_n - \sum_{i<n} l_i \varphi_{x_i}\Big) u_{x_n}$$

since $\nu = (1 + \varphi_x^2)^{-1}(-\varphi_{x_1}, \dots, -\varphi_{x_{n-1}}, 1)$. Hence

$$l_n - \sum_{i\leqslant n} l_i \varphi_{x_i} = \frac{\partial u}{\partial l} \Big/ \frac{\partial u}{\partial x_n}.$$

It follows that if $\partial u/\partial l$ is a $C^{\mu,\alpha}$ function of $x_1, \dots, x_{n-1}$ for all vectors l with $l \cdot \nu > 0$ then $\varphi \in C^{\mu+1,\alpha}$ and

$$\frac{a_{ij}\nu_i \nu_j}{l \cdot \nu} \frac{\partial u}{\partial l} \in C^{\mu,\alpha}.$$

Because of the good estimates for the kernel K in *Theorem* 5 $(\partial/\partial l)K(x, y)$ is a « smoothing kernel » and formula (5.6) implies that $\partial u/\partial l \in C^{\mu+1,\alpha}$. Proceeding by induction we arrive at the desired conclusion. Q.E.D.

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