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Nonlinear Perturbations of Linear Elliptic and Parabolic Problems at Resonance: Existence of Multiple Solutions.

PETER HESS (*)

1. – Introduction.

In this paper we are concerned with the existence of multiple solutions of the nonlinear equation

$$(1) \quad Lu + G(u) = f$$

in the real Hilbert space $H = L^2(\Omega)$, Ω a bounded domain in a finite-dimensional real Euclidean space. Here $L: H \supset D(L) \rightarrow H$ denotes a linear operator with dense domain $D(L)$ and compact resolvent; we assume that 0 is eigenvalue of L (and of the adjoint operator L^*), and that for the corresponding eigenspaces, $N(L) = N(L^*)$. Further G is the Nemytskii operator associated with the continuous function $g: \mathbf{R} \rightarrow \mathbf{R}$; we assume that the limits $g_{\pm} := \lim_{s \rightarrow \pm\infty} g(s)$ exist (in the proper sense), and that $g_- \leq 0 \leq g_+$. Then G maps H continuously into itself and has bounded range. Finally $f \in H$ is given.

By a well-known result which goes back to Landesman-Lazer [7], and for which various different proofs and extensions have been given (e.g. [4] and the comprehensive list of references therein), (1) is solvable at least for those $f \in H$ for which

$$(LL) \quad (f, w) < \int_{\Omega} (g_+ w^+ - g_- w^-) dx \quad \forall w \in N(L), w \neq 0.$$

Here w^+ (w^-) denotes the positive (negative) part of the function w , respectively, i.e. $w = w^+ - w^-$. We remark that if $g_- = g_+$, no $f \in H$ will satisfy (LL).

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Under some additional assumptions on $N(L)$ and g we show that equation (1) is solvable for certain $f \in H$ which do *not* satisfy (LL), and admits multiple solutions. We impose the following further conditions:

(I) The eigenfunctions of L enjoy the unique continuation property: if $w \in N(L)$ vanishes on a set of positive measure in Ω , then $w = 0$.

(II) There exists $\delta > 0$ such that

$$\begin{aligned} g(s) &\geq g_+ & \forall s \geq \delta, \\ g(s) &\leq g_- & \forall s \leq -\delta. \end{aligned}$$

Note that (II) is opposed to the original assumption

$$(2) \quad g_- < g(s) < g_+ \quad \forall s \in \mathbf{R}$$

made in the theorem of Landesman-Lazer. Set

$$\begin{aligned} \gamma_+ &:= \liminf_{s \rightarrow +\infty} (g(s) - g_+) s & (\geq 0), \\ \gamma_- &:= \liminf_{s \rightarrow -\infty} (g(s) - g_-) s & (\geq 0). \end{aligned}$$

The space H admits a decomposition $H = N(L) \oplus R(L)$. We set $H_1 := N(L)$, $H_2 := R(L)$ and denote by P_1 and P_2 the orthogonal projections on H_1 and H_2 , respectively. For $f \in H$ we write $f_1 := P_1 f$ and $f_2 := P_2 f$.

DEFINITION. Let \mathcal{S} be the nonempty, bounded, closed set in H_1 consisting of all functions f_1 for which

$$(f_1, w) \leq \int_{\Omega} (g_+ w^+ - g_- w^-) dx \quad \forall w \in N(L) = H_1.$$

We remark that the set \mathcal{S} is *independent* of $f_2 \in H_2$. Our main result is

THEOREM 1. *Let the mappings L and G be as described above, and suppose that either*

(α) *the functions in $N(L)$ have constant sign in Ω and both γ_+ , γ_- are positive, or*

(β) *the functions in $N(L)$ change sign in Ω and at least one of γ_+ , γ_- is positive.*

Then to each (fixed) $f_2 \in H_2$ there exists a bounded open set $\mathcal{S}_{f_2} \subset H_1$ containing \mathcal{S} , such that

(i) *equation (1) is solvable for all $f = f_1 + f_2$ with $f_1 \in \mathcal{S}_{f_2}$;*

(ii) equation (1) has at least two different solutions for $f = f_1 + f_2$ if $f_1 \in \mathcal{S}_{j_1} \setminus \mathcal{S}$.

As a consequence of Theorem 1 we further get

THEOREM 2. *Under the assumptions of Theorem 1, the mapping $L + G$ has closed range in H .*

Theorem 2 should be compared with the assertion that the range of $L + G$ is open under condition (2).

REMARK. If $N(L)$ is one-dimensional, it is readily seen that the results hold without hypothesis (I).

This research is related to two recent results concerning the particular situation where $g_- = 0 = g_+$. The first one is due to Fučík-Krbeč [5, Theorem 3] (cf. also [6] for some simplifications and improvements), the second one to Ambrosetti-Mancini [2, Theorem 3.1]. In [5, 6] attention is restricted to existence, while in [2] a multiplicity result is obtained by a global Lyapunow-Schmidt method. In order that the equation in $R(L)$ is uniquely solvable with continuous dependence on the given data, Ambrosetti-Mancini need some boundedness condition on the derivative g' .

If $g_- < g_+$, a multiplicity result is given in [1, Prop. 6.4] for perturbations in the *first* eigenvalue and functions $f \in L^\infty(\Omega)$.

Our approach to multiplicity results is similar to that in [2] in as much as degree theory is used. By employing the Leray-Schauder degree in suitable rectangles in H we are however able to avoid any local restriction on g .

The paper is organized as follows: Section 2 contains the proof of Theorem 1, Section 3 that of Theorem 2, while in Section 4 two examples are given of mappings L which satisfy the hypotheses of this paper: (a) an elliptic differential operator, (b) a parabolic differential operator with a periodicity condition in time.

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2. - Proof of Theorem 1.

(i) Let $f = f_1 + f_2$ with $f_1 \in \mathcal{S}$ and (fixed) $f_2 \in H_2$. Equation (1) is equivalent to the equation

$$(L + P_1)u + (G(u) - P_1u - f) = 0$$

which, since $L + P_1$ is invertible on H , is in turn equivalent to

$$(3) \quad u + (L + P_1)^{-1}(G(u) - P_1u - f) = 0.$$

Note that $(L + P_1)^{-1}: H \rightarrow H$ is a compact linear operator, and that G has bounded range in H . For $t \in [0, 1]$ and $u \in H$ we define the homotopy mapping

$$\mathcal{K}(t, u) = u + t(L + P_1)^{-1}(G(u) - P_1u - f).$$

Considering only the component in H_2 we see immediately that

$$(4) \quad \mathcal{K}(t, u) = 0 \quad \text{for } t \in [0, 1], u \in H \Rightarrow \|P_2u\| < b,$$

with some constant $b > 0$. For $n \in \mathbf{N}$ let

$$\mathfrak{B}_n = \{u \in H: \|P_1u\| < n, \|P_2u\| < b\}.$$

We claim that there exists $n_0 \in \mathbf{N}$ such that

$$(5) \quad \mathcal{K}(t, u) \neq 0 \quad \forall t \in [0, 1], \forall u \in \partial\mathfrak{B}_{n_0}.$$

Let us assume for the moment that (5) holds. By the homotopy invariance of the Leray-Schauder degree,

$$(6) \quad \begin{aligned} \deg(\mathcal{K}(1, \cdot), \mathfrak{B}_{n_0}, 0) &= \deg(\mathcal{K}(0, \cdot), \mathfrak{B}_{n_0}, 0) \\ &= \deg(I, \mathfrak{B}_{n_0}, 0) = 1. \end{aligned}$$

Since the degree is moreover invariant in components of $H \setminus \mathcal{K}(1, \partial\mathfrak{B}_{n_0})$, there exists an open neighborhood $\mathcal{U}(f_1)$ of f_1 in H_1 such that the degree = 1 also for $\tilde{f} \in H$ of the form $\tilde{f} = \tilde{f}_1 + f_2$ with $\tilde{f}_1 \in \mathcal{U}(f_1)$. For those \tilde{f} there exists a solution of (1) in \mathfrak{B}_{n_0} .

We set $\mathfrak{S}_{f_1} := \bigcup_{\tilde{f} \in \mathfrak{S}} \mathcal{U}(f_1)$. Then assertion (i) of Theorem 1 is proved.

It remains to establish (5). We argue by contradiction. Suppose for each $n \in \mathbf{N}$ we find $t_n \in [0, 1]$ and $u_n \in \partial\mathfrak{B}_n$ such that

$$(7) \quad \mathcal{K}(t_n, u_n) = 0.$$

By (4) it follows that

$$\|P_1u_n\| = n.$$

Applying the linear operator $L + P_1$ on both sides of (7) we get

$$(8) \quad (L + P_1)u_n + t_n(G(u_n) - P_1u_n - f) = 0 .$$

Hence $t_n \neq 0, \forall n \in \mathbf{N}$. We take the inner product of (8) with P_1u_n and obtain

$$(1 - t_n)\|P_1u_n\|^2 + t_n(G(u_n) - f, P_1u_n) = 0 .$$

We conclude that

$$(G(u_n) - f, P_1u_n) \leq 0 \quad \forall n ,$$

or, writing $P_1u_n = nw_n$ with $w_n \in H_1, \|w_n\| = 1$,

$$(9) \quad \int_{\Omega} (g(u_n) - f_1)nw_n \, dx \leq 0 .$$

Since $f_1 \in \mathcal{S}$, we know on the other hand that

$$(10) \quad \int_{\Omega} f_1nw_n \, dx \leq \int_{\Omega} (g_+(nw_n)^+ - g_-(nw_n)^-) \, dx .$$

Adding (9) and (10) we get

$$(11) \quad \int_{\Omega} (g(u_n) - g_+)(nw_n)^+ \, dx - \int_{\Omega} (g(u_n) - g_-)(nw_n)^- \, dx \leq 0 , \quad \forall n \in \mathbf{N} .$$

We investigate the first integral in (11); the second one is handled similarly. In the following limiting arguments we pass to subsequences repeatedly; in order not to complicate the notation we however do not change the indices thereby.

Considering the components of (8) in H_2 and recalling that $(L/D(L) \cap H_2)^{-1}: H_2 \rightarrow H_2$ is compact, we infer that the sequence $\{P_2u_n\}$ is relatively compact in H_2 . We may thus assume (for a subsequence)

$$P_2u_n \rightarrow z \quad (n \rightarrow \infty)$$

in H_2 and a.e. in Ω . Moreover there exists a function $y \in H$ such that, for some further subsequence,

$$|P_2u_n(x)| \leq y(x) \quad \forall n \in \mathbf{N}, \text{ a.e. } x \in \Omega .$$

(This useful fact occurs as an intermediate step in the standard proof of

completeness of L^p -spaces.) Since H_1 is finite-dimensional, we may also assume

$$w_n \rightarrow w \quad (n \rightarrow \infty)$$

in H_1 and a.e. in Ω , with $\|w\| = 1$. Hence $w(x) \neq 0$ for a.e. $x \in \Omega$, by hypothesis (I), and consequently

$$(12) \quad u_n \rightarrow \begin{cases} +\infty \\ -\infty \end{cases} \quad \text{a.e. on the sets} \quad \begin{cases} \{w > 0\}, \\ \{w < 0\}. \end{cases}$$

We now split up the first integral in (11):

$$\begin{aligned} \int_{\Omega} (g(u_n) - g_+)(nw_n)^+ dx &= \int_{\{u_n \geq \delta + \nu\}} \dots + \int_{\{u_n < \delta + \nu\}} \dots = \\ &= \int_{\{u_n \geq \delta + \nu\}} (g(u_n) - g_+) u_n dx - \int_{\{u_n < \delta + \nu\}} (g(u_n) - g_+) P_2 u_n dx + \\ &\quad + \int_{\{u_n < \delta + \nu\}} (g(u_n) - g_+)(nw_n)^+ dx = I_{1,n} - I_{2,n} + I_{3,n}, \quad \text{say.} \end{aligned}$$

The behaviour as $n \rightarrow \infty$ is now studied for each of the three integrals separately. In the following χ_{ω} denotes the characteristic function of the set $\omega \subset \Omega$.

$$a) \quad I_{1,n} = \int_{\Omega} \chi_{\{u_n \geq \delta + \nu\}} (g(u_n) - g_+) u_n dx.$$

Since the integrand is non-negative, we obtain by (12) and the Fatou lemma that

$$\liminf_{n \rightarrow \infty} I_{1,n} \geq \mu(\{w > 0\}) \cdot \gamma_+.$$

Here $\mu(\{w > 0\})$ is the Lebesgue measure of the subset $\{w > 0\}$ of Ω .

$$b) \quad I_{2,n} = \int_{\Omega} \chi_{\{u_n \geq \delta + \nu\}} (g(u_n) - g_+) P_2 u_n dx.$$

The integrand converges to 0 a.e. in $\{w > 0\}$ and $\{w < 0\}$ and is majorized by some multiple of the function $y \in H$; hence

$$I_{2,n} \rightarrow 0 \quad (n \rightarrow \infty)$$

by Lebesgue's theorem.

$$c) \quad I_{3,n} = \int_{\Omega} \chi_{\{u_n < \delta + \nu\}} (g(u_n) - g_+)(nw_n)^+ dx.$$

Since $\delta + y(x) > u_n(x) = nw_n(x) + P_2 u_n(x) \Rightarrow nw_n(x) < \delta + 2y(x)$, the func-

tion $|\chi_{\{u_n < \delta + v\}}(nw_n)^+|$ is bounded by $\delta + 2y \in H$. Moreover the integrand converges to 0 a.e. in $\{w > 0\}$ and $\{w < 0\}$. Again by Lebesgue's theorem,

$$I_{3,n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Similarly one treats the second integral in (11) and concludes that in both cases (α) and (β) ,

$$\begin{aligned} 0 &\geq \liminf \int_{\Omega} [(g(u_n) - g_+)(nw_n)^+ + (g(u_n) - g_-)(- (nw_n)^-)] dx \geq \\ &\geq \mu(\{w > 0\})\gamma_+ + \mu(\{w < 0\})\gamma_- > 0. \end{aligned}$$

This contradiction proves the existence of $n_0 \in \mathbf{N}$ such that (5) holds.

(ii) Let now $f \in H$ be such that $f_1 \in \mathcal{S}_{t_1} \setminus \mathcal{S}$. By the proof of Theorem 1(i) there exists a rectangle \mathfrak{B}_{n_0} in H such that $\deg(\mathcal{K}(1, \cdot), \mathfrak{B}_{n_0}, 0) = 1$. Further there is $\bar{w} \in N(L)$ such that

$$(13) \quad (f_1, \bar{w}) > \int_{\Omega} (g_+(\bar{w})^+ - g_-(\bar{w})^-) dx.$$

Since the integral on the right side in (13) is nonnegative, $f_1 \neq 0$ and thus $f \notin R(L)$. Let the constant $K \geq 0$ be such that the equation

$$Lu + G(u) = (1 + K)f$$

has no solution in H (note that G has bounded range in H). We consider the homotopy mapping

$$\mathcal{K}(t, u) = u + (L + P_1)^{-1}(G(u) - P_1u - (1 + t)f),$$

$t \in [0, K]$, $u \in H$. There exists a constant $c > b$ such that

$$(14) \quad \mathcal{K}(t, u) = 0 \quad \text{for } t \in [0, K], u \in H \Rightarrow \|P_2u\| < c.$$

For $n \in \mathbf{N}$ let

$$\mathcal{C}_n = \{u \in H: \|P_1u\| < n, \|P_2u\| < c\}.$$

We assert that for some $n_1 > n_0$,

$$(15) \quad \mathcal{K}(t, u) \neq 0 \quad \forall t \in [0, K], \forall u \in \partial\mathcal{C}_{n_1}.$$

For suppose, again to the contrary, that to each $n > n_0$ there exist $t_n \in [0, K]$ and $u_n \in \partial C_n$ such that

$$\mathcal{K}(t_n, u_n) = 0 .$$

Then

$$(16) \quad Lu_n + G(u_n) = (1 + t_n)f .$$

We may assume $t_n \rightarrow t$ ($n \rightarrow \infty$). By (14) we have $\|P_1 u_n\| = n$.

Arguing as in the proof of assertion (i), we infer that $u_n(x) \rightarrow \pm \infty$ a.e. on the sets $\{w \geq 0\}$ (cf. (12)).

Taking the inner product of (16) with the function \bar{w} of (13) we obtain

$$(G(u_n), \bar{w}) = (1 + t_n)(f_1, \bar{w}) ;$$

in the limit it follows

$$\int_{\{w>0\}} g_+ \bar{w} \, dx + \int_{\{w<0\}} g_- \bar{w} \, dx = (1 + t)(f_1, \bar{w}) \geq (f_1, \bar{w})$$

(the second inequality sign holding since $(f_1, \bar{w}) > 0$ by (13)). However

$$\int_{\Omega} (g_+(\bar{w})^+ - g_-(\bar{w})^-) \, dx \geq \int_{\{w>0\}} g_+(\bar{w})^+ \, dx - \int_{\{w>0\}} g_+(\bar{w})^- \, dx + \int_{\{w<0\}} g_-(\bar{w})^+ \, dx - \int_{\{w<0\}} g_-(\bar{w})^- \, dx .$$

We arrive at a contradiction to (13).

Thus by homotopy invariance of the degree,

$$0 = \deg(\mathcal{K}(K, \cdot), C_{n_1}, 0) = \deg(\mathcal{K}(0, \cdot), C_{n_1}, 0) = \deg(\mathcal{K}(1, \cdot), C_{n_1}, 0) .$$

We conclude by (6) and the additivity of the degree that

$$\deg(\mathcal{K}(1, \cdot), C_{n_1} \setminus \text{cl}(\mathfrak{B}_{n_0}), 0) = -1 ;$$

hence there exists a second solution of (1) in the set $C_{n_1} \setminus \text{cl}(\mathfrak{B}_{n_0})$. This proves Theorem 1.

3. – Proof of Theorem 2.

Let $\{f^n\}$ be a sequence of functions in H such that (1) admits solutions for each f^n , and suppose $f^n \rightarrow f$ in H . Writing $f = f_1 + f_2$, with $f_1 \in H_1$,

$f_2 \in H_2$, we distinguish between two cases:

- (a) $f_1 \in \mathcal{S}$. Then (1) is solvable for this f by Theorem 1(i).
- (b) $f_1 \notin \mathcal{S}$. There exists $\bar{w} \in N(L)$ such that

$$(17) \quad (f_1, \bar{w}) > \int_{\Omega} (g_+(\bar{w})^+ - g_-(\bar{w})^-) dx .$$

We claim that the solutions u_n of the equations

$$(18) \quad Lu_n + G(u_n) = f^n$$

remain bounded in H , as $n \rightarrow \infty$. Clearly $\|P_2 u_n\| \leq d, \forall n \in \mathbf{N}$, with some constant d . Assuming that $\|P_1 u_n\| \rightarrow \infty (n \rightarrow \infty)$, we derive as in the proof of Theorem 1(i) that

$$u_n(x) \rightarrow \pm \infty \quad \text{for a.e. } x \in \Omega .$$

Taking the inner product of (18) with \bar{w} and passing to the limit $n \rightarrow \infty$ we obtain as in the proof of Theorem 1(ii) that

$$(f_1, \bar{w}) \leq \int_{\Omega} (g_+(\bar{w})^+ - g_-(\bar{w})^-) dx ,$$

contradicting (17).

We thus may assume, by the compactness of $(L + P_1)^{-1}$, that $u_n \rightarrow u$ in H . The passage to the limit $n \rightarrow \infty$ in (18) is now immediate and proves the solvability of (1) for the function f also in this case.

4. - Let $\omega \subset \mathbf{R}^N (N \geq 1)$ be a bounded domain with smooth boundary, and let us denote by \mathcal{A} :

$$\mathcal{A}u = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - \lambda u \quad (\lambda \in \mathbf{R})$$

a formally selfadjoint, uniformly elliptic differential expression of second order, with real-valued coefficient functions $a_{ij} = a_{ji} \in C^1(\bar{\omega})$. Together with homogeneous Dirichlet boundary conditions, \mathcal{A} induces a selfadjoint differential operator A in $L^2(\omega)$ by

$$\begin{aligned} D(A) &= H_0^1(\omega) \cap H^2(\omega) , \\ Au &= \mathcal{A}u \quad (u \in D(A)) . \end{aligned}$$

It is known that the eigenfunctions of \mathcal{A} , i.e. the functions in $N(A)$, have the unique continuation property (e.g. [8]).

a) *The elliptic problem.* Here we set $\Omega := \omega$, $H := L^2(\omega)$, and $L := \pm A$. Then L satisfies all the assumptions made in the paper.

b) *The parabolic problem with periodicity condition in time.* Let $T > 0$ be given, and set $H := L^2(0, T; L^2(\omega)) = L^2(\Omega)$, where Ω denotes the cylinder $(0, T) \times \omega$ in \mathbf{R}^{1+N} .

Let \tilde{A} be the extension of the above introduced elliptic differential operator to H ; it is defined by

$$v = \tilde{A}u \Leftrightarrow u, v \in H, u(t) \in D(A) \text{ and } v(t) = Au(t) \text{ for a.a. } t \in (0, T).$$

\tilde{A} is a selfadjoint operator in H .

Let further $d/dt: H \supset D(d/dt) \rightarrow H$ be the linear operator given by

$$D\left(\frac{d}{dt}\right) = \{u \in H: u' \in H, u(0) = u(T)\}, \quad \frac{d}{dt}u = u' \quad \left(u \in D\left(\frac{d}{dt}\right)\right).$$

Here the time-derivative is meant in the distributive sense. Note that $u, u' \in H$ implies that u is (perhaps after modification on a nullset in $[0, T]$) a continuous and a.e. differentiable mapping of $[0, T]$ into $L^2(\omega)$. From the relation

$$(u', v) + (u, v') = (u(T), v(T))_{L^2(\omega)} - (u(0), v(0))_{L^2(\omega)},$$

which holds for all $u, v \in H$ with $u', v' \in H$, it follows that

$$\left(\frac{d}{dt}\right)^* = -\frac{d}{dt}$$

and thus

$$\left(\frac{d}{dt}u, u\right) = 0 \quad \forall u \in D\left(\frac{d}{dt}\right).$$

We claim that the mappings $L = \pm d/dt \pm \tilde{A}$ (where all 4 combinations are allowed) satisfy the conditions imposed on L , with

$$N(L) = N(L^*) = N\left(\frac{d}{dt}\right) \cap N(\tilde{A}) = N(A).$$

For the sake of definiteness suppose in the following that $L = d/dt + \tilde{A}$.

As in [3, Theorem 19] (where the initial-value problem is considered), one shows first that

$$(19) \quad R\left(\pm \frac{d}{dt} + \tilde{A} + (\lambda + 1)I\right) = H$$

(note that $\tilde{A} + (\lambda + 1)I$ is monotone and selfadjoint, hence a subdifferential). Further

$$(20) \quad \left(\frac{d}{dt} u, \tilde{A}u\right) = 0 \quad \forall u \in D\left(\frac{d}{dt}\right) \cap D(\tilde{A}).$$

From (19) it follows that

$$\left(\frac{d}{dt} + \tilde{A}\right)^* = -\frac{d}{dt} + \tilde{A};$$

by (20) we then conclude the above assertions on the nullspaces of L and L^* . Finally $(d/dt + \tilde{A} + (\lambda + 1)I)^{-1}: H \rightarrow H$ is compact by Aubin's lemma.

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