HIRONORI SHIGA

One attempt to the $K3$ modular function I

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One Attempt to the $K3$ Modular Function $I$.

HIRONORI SHIGA (*)

0. – Introduction.

In this note the author reconstructs the Picard’s modular function as a modular function for a family of algebraic $K3$ surfaces with two complex parameters.

In 1883 Picard has constructed an analytic function of two variables analogous to the elliptic modular function (see [1]). He started from the following integral containing two complex parameters $x$ and $y$,

\[ I(x, y) = \int_{\infty}^{1} \frac{dt}{\sqrt{t(t-1)(t-x)(t-y)}}. \]

The function $I(x, y)$ of $x$ and $y$ is a multivalued analytic function on the domain $A = \{(x, y): xy(x-1)(y-1)(x-y) \neq 0\}$ in $\mathbb{C}^2$. This integral plays a similar role as the following integral,

\[ I'(x) = \int_{\infty}^{1} \frac{dt}{\sqrt{t(t-1)(t-x)}} \]

which induces the elliptic modular function $\lambda(\zeta)$. This integral is a multivalued analytic function of $x$ on $\mathbb{C} - \{0, 1\}$ and it is a solution of the following hypergeometric differential equation,

\[ x(x-1)z'' + (2x-1)z' + \frac{1}{4}z = 0. \]

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Let $w_1(x)$ and $w_2(x)$ be two independent solutions of (0.1). And let us consider the ratio $\zeta(x) = w_2(x)/w_1(x)$, this is the $s$-function of Schwartz.

The inverse function $x = \lambda(\zeta)$ of $\zeta(x)$ becomes a single valued automorphic function defined on the upper half plane. The fundamental region of $\lambda(\zeta)$ is defined by the inequalities $(\zeta_1 - \frac{1}{2})^2 + \zeta_2^2 \leq \frac{1}{4}$, $(\zeta_1 + \frac{1}{2})^2 + \zeta_2^2 \geq \frac{1}{4}$ and $-1 \leq \zeta_1 \leq 1$, where $\zeta = \zeta_1 + \sqrt{-1}\zeta_2$. We note that $\lambda(\zeta)$ realizes the universal covering space of the domain $C - \{0, 1\}$. Similarly the function $z = I(x, y)$ satisfies the following differential equation:

\[
\begin{align*}
9x(1-x)(x-y)r &= 3(5x^3-4xy-3x+2y)p + 3y(1-y)q + (x-y)z \\
3(x-y)s &= p - q \\
9y(1-y)(y-x)t &= 3x(1-x)p + 3(5y^3-4xy-3y+2x)q + (y-x)z,
\end{align*}
\]

where we use the conventional notations $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$. And easily we can rewrite (0.2) in the form of a total differential equation;

\[
d \begin{pmatrix} z \\ p \\ q \end{pmatrix} = \Omega \begin{pmatrix} z \\ p \\ q \end{pmatrix},
\]

where $\Omega$ is a matrix of rational 1-forms. The differential equation (0.2) is the equation for the Appell’s hypergeometric function $F_1(x, \beta, \beta', \gamma; x, y)$ (see [2]), in our case the parameters take the values $(x, \beta, \beta', \gamma) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, 1)$. And the equation (0.3) is completely integrable. Hence the dimension of the solution space of (0.3) (and also of (0.2)) is equal to three.

Let $\omega_1$, $\omega_2$ and $\omega_3$ be the three independent solutions of (0.3). And let us consider the ratios $\zeta_2(x, y) = \omega_2/\omega_1$ and $\zeta_3(x, y) = \omega_3/\omega_1$. Also these are multivalued analytic functions on $\Lambda$. And we obtain single valued holomorphic functions $x = q_1(\zeta_1, \zeta_3)$ and $y = q_3(\zeta_1, \zeta_3)$, as the inverse mapping of $(\zeta_1, \zeta_3) = (\zeta_2(x, y), \zeta_3(x, y))$. If we choose $\omega_i$ ($i = 1, 2, 3$) adequately, then $q_i(\zeta_1, \zeta_3)$ ($i = 1, 2$) is defined on the hyperball $\{(\zeta_1, \zeta_3): |\zeta_1|^2 + |\zeta_3|^2 < 1\}$. In such a way Picard constructed his modular function. But unfortunately the mapping $(q_1, q_2)$ does not realize the universal covering of the domain $\Lambda$.

We shall study this mapping. At first we shall translate the original integral (0.0) to a double integral on an algebraic surface $S(\lambda, \mu)$ containing two complex parameters ((1.4) in the section 1). The surface $S(\lambda, \mu)$ is defined by the equation

\[w^3 - uv^2(1-u-v)(1-\lambda u - \mu v) = 0,\]
where \((u, v, w)\) is an affine coordinate of \(P^3\) and the parameters \((\lambda, \mu)\) move on the domain \(A\). It will be shown that the minimal nonsingular model \(\tilde{S}(\lambda, \mu)\) of \(S(\lambda, \mu)\) is a K3 surface (in the section 2). Hence there is only one independent holomorphic 2-form \(\psi\) on \(\tilde{S}(\lambda, \mu)\) and it does not vanish. And the second homology group \(H_2(\tilde{S}(\lambda, \mu), \mathbb{Z})\) is a free Abelian group of rank 22 (see [4]). The surface \(\tilde{S}(\lambda, \mu)\) will be characterized as an elliptic surface \((X, p, A)\) (in the section 2) satisfying the following conditions:

i) the base space \(A\) is equal to \(P^1\),

ii) the general fibre \(p^{-1}(v)\) (\(v\) is a point on \(A = P^1\)) is an elliptic curve defined by the lattice \(\{m \cdot \exp(2\pi i/3) + n: m \text{ and } n \text{ are integers}\}\),

iii) there are five singular fibres \(p^{-1}(v_i)\) \((i = 1, 2, 3, 4)\) and \(p^{-1}(v_{\infty})\). The fibre \(p^{-1}(v_i)\) \((i = 1, 2, 3, 4)\) consists of three rational curves intersecting at one point. The singular fibre \(p^{-1}(v_{\infty})\) consists of seven nonsingular rational curves \(\theta_0, \ldots, \theta_6\). And these components have the following intersection multiplicities,

\[
\theta_0\theta_1 = \theta_0\theta_2 = \theta_0\theta_3 = \theta_1\theta_4 = \theta_2\theta_4 = \theta_3\theta_6 = 1
\]

and any other intersection multiplicity is equal to zero.

The former singular fibre is of type IV and the latter is of type IV* according to the study of Kodaira (see [3]),

iv) the total space \(X\) has a holomorphic section.

Next we shall study the surface \(\tilde{S}(\lambda, \mu)\) and we shall obtain the following properties.

a) We find a subgroup \(A(\tilde{S})\) of \(H_2(\tilde{S}, \mathbb{Z})\) which is composed of algebraic cycles with rank 16, and this subgroup coincides with the Neron-Severi group (that is the subgroup of all algebraic cycles) for almost all \((\lambda, \mu)\) on \(A\) (in the section 3).

b) We construct a basis system \(\Gamma_1, \ldots, \Gamma_{22}\) of \(H_2(\tilde{S}(\lambda, \mu), \mathbb{Z})\) such that \(\Gamma_1, \ldots, \Gamma_{22}\) induces a generator system of the quotient group

\[
H_2(\tilde{S}(\lambda, \mu) \otimes \mathbb{Q})/A(\tilde{S}(\lambda, \mu))
\]

(in the section 4).

c) If we set

\[
\tau_i(\lambda, \mu) = \frac{1}{\Gamma_1} \int_\Gamma \psi \quad (i = 1, \ldots, 6),
\]
there occurs a relation

\[ \eta_{i+1}(\lambda, \mu) = \exp \left( \frac{4\pi i}{3} \right) \eta_i(\lambda, \mu) \quad \text{for } i = 1, 3, 5 \]

4.1) in the section 4).

d) We construct the dual basis system \( G_1, \ldots, G_{22} \) of \( H_2(\mathcal{S}, \mathbb{Q}) \) such that \( G_i G_j = \delta_{ij} (1 \leq i, j \leq 22) \) (in the section 3 and 4), and we determine the matrix of the intersection multiplicities of this \( \text{system} \) (in the section 3).

e) We describe the generator system \( \delta_1, \ldots, \delta_4 \) of the monodromy transformation group of \( H_2(\mathcal{S}(\lambda, \mu), \mathbb{Z}) \) induced from the fundamental group \( \pi_1(\mathcal{A}) \) (in the section 5).

Finally we obtain the following results using the above consideration:

CONCLUSION:

1) The inverse mapping \( (\lambda, \mu) \Rightarrow (\varphi_1(\zeta_1, \zeta_2), \varphi_2(\zeta_1, \zeta_2)) \) of the period mapping \( (\zeta_1, \zeta_2) = (\eta_2(\lambda, \mu)/\eta_1(\lambda, \mu), \eta_2(\lambda, \mu)/\eta_1(\lambda, \mu)) \) for the family \( \mathcal{S}(\lambda, \mu) \) coincides with the Picard's modular function stated above.

2) The defining domain \( \varOmega \) for the function \( \varphi_i (i = 1, 2) \) is determined by the Riemann-Hodge relation \((4.8) \) in the section 4).

3) The generator system of the transformation group of \( \varOmega \) which corresponds to the automorphic functions \( \varphi_i (i = 1, 2) \) is given as the table in the part III of the section 5.

The author wishes to know whether it is possible to obtain other significant analytic functions of several variables in the same manner.

1. – Translation to a double integral.

Here we reduce the integral \( (0.0) \) to a double integral. There are two integral representation formulas for the Appell’s hypergeometric function \( F_1(x, \beta, \beta'; \gamma; x, y) \) (see [2]), \( F_1 \) is defined as the following

\[ F_1(x, \beta, \beta'; \gamma; x, y) = \sum_{m, n} \frac{(\alpha, m + n)(\beta, m)(\beta', n)}{(\gamma, m + n)m!n!} x^m y^n, \]

where \((\lambda, k)\) indicates the product \( \lambda(k + 1) \ldots (\lambda + k - 1) \).
If the parameters satisfy the condition \( \Re \alpha > 0 \) and \( \Re (\gamma - \alpha) > 0 \), we have

\[
F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 u^{\alpha-1} (1 - u)^{\beta - 1} (1 - xu)^{\beta' - 1} (1 - yu)^{-\gamma} \, du
\]

for any point \((x, y)\) on the polydisk \(|x| < 1, |y| < 1\), where \(\Gamma\) indicates the gamma function.

If the parameters satisfy the condition

\[
\Re \beta > 0, \quad \Re \beta' > 0 \quad \text{and} \quad \Re (\gamma - \beta - \beta') > 0,
\]

then we have

\[
F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\beta') \Gamma(\gamma - \beta - \beta')} \cdot \int_A u_1^{\beta - 1} v_1^{\beta' - 1} (1 - u_1 - v_1)^{\gamma - \beta - \beta' - 1} (1 - xu_1 - yv_1)^{-\gamma} \, du_1 \, dv_1
\]

for any point \((x, y)\) on the polydisk \(|x| < 1, |y| < 1\), where \(A\) is a triangle in the \((\Re u_1, \Re v_1)\)-space defined by inequalities \(u_1 \geq 0, v_1 \geq 0\) and \(1 - u_1 - v_1 \geq 0\).

If we use the variable \(t' = 1/t\), then it follows that

\[
I(x, y) = -\int_0^1 \frac{dt'}{\sqrt{t'^2(1 - t')(1 - t'x)(1 - t'y)}}.
\]

By the formula (1.1) the right hand side of the above equality is equal to

\[-\Gamma(1/3)\Gamma(2/3)F_1(1/3, 1/3, 1/3, 1; x, y).\]

Then by the formula (1.2) we have

\[
F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; x, y) = \frac{1}{\Gamma(1/3)^2} \int_A u_1^{1/3} v_1^{1/3} (1 - u_1 - v_1)^{-1} (1 - xu_1 - yv_1)^{-1} \, du_1 \, dv_1.
\]

And if we set \(w_1^3 = u_1 v_1 (1 - u_1 - v_1)(1 - xu_1 - yv_1)^2\), it follows that

\[
\int_A \frac{1 - xu_1 - yv_1}{w_1^3} \, du_1 \, dv_1 = -\frac{(\Gamma(1/3))^2}{\Gamma(2/3)} I(x, y).
\]
Here we consider the transformation:

\[
\begin{align*}
    u &= xu_1, \\
    v &= 1 - xu_1 - yv_1, \\
    w &= (xy^2(y - 1))^{-1/2}w_1, \\
    \lambda &= \frac{y - x}{x(y - 1)}, \\
    \mu &= 1/(1 - y).
\end{align*}
\]

After this transformation we obtain the following:

\[
(1.4) \quad \mu^{1/2}(\lambda - \mu)^{1/2}(\mu - 1)^{1/2} \int_D \int \frac{v}{w^2} du \, dv = (\Gamma(1/3)^{1/2}(\Gamma(2/3))^{-1}I(x, y),
\]

where \( D \) is a real 2-dimensional triangle defined by inequalities \( u/x \geq 0 \), \((1 - u - v)/y \geq 0 \) and \( 1 - \lambda u - \mu v \geq 0 \). And also we have the following:

\[
(1.5) \quad w^2 - yv^2(1 - u - v)(1 - \lambda u - \mu v) = 0.
\]

Hence we know that the Picard’s original integral is represented as the double integral of the left hand side of (1.4). Then in the following we study the property of this integral.

2. - Minimal nonsingular model of \( S(\lambda, \mu) \).

In this section we study the minimal nonsingular model of the algebraic surface (1.5). We define the compactification of this surface in \( P \times P^2 \) as follows:

\[
(2.1) \quad \begin{align*}
    \xi_0^2 - v^2\xi_1(\xi_0 - \xi_1 - v\xi_0)(\xi_0 - \lambda\xi_1 - \mu\xi_0) &= 0, \\
    v'\xi_2^2 - \xi_1(\xi_0 v' - \xi_1 v' - \xi_0)(\xi_0 v' - \lambda\xi_1 v' - \mu\xi_0) &= 0,
\end{align*}
\]

where \([\xi_0, \xi_1, \xi_2]\) is a homogeneous coordinate of \( P^2 \) and we set \( v' = 1/v \). In the following the parameters \((\lambda, \mu)\) move on \( A \). We denote the surface (2.1) by \( S(\lambda, \mu) \) or simply \( S \). We use the following notations,

- \( \tilde{S} \): the minimal nonsingular model of \( S \),
- \( \Lambda \): the compactified Riemann sphere of \( v \)-space,
- \( p' \): the projection mapping from \( S \) to \( \Lambda \),
- \( p \): the projection mapping from \( \tilde{S} \) to \( \Lambda \),

\[
\begin{align*}
    v_0 &= 0, \quad v_1 = 1, \quad v_2 = (\lambda - 1)/(\lambda - \mu), \quad v_3 = 1/\mu, \quad v_\infty = \infty.
\end{align*}
\]
The fibre $p^{-1}(v)$ is a nonsingular elliptic curve for every value $v$ except $v_i$ $(i = 0, 1, 2, 3, \infty)$. Hence $\tilde{S}$ is an elliptic surface. The fibre $p^{-1}(v_i)$ $(i = 1, 2, 3)$ is a rational curve with one isolated singularity which is locally isomorphic to the singularity $w^3 - uv = 0$. When we resolve this singularity there occur two rational curves with self intersection number $-2$, and the self intersection number of the proper image of $p^{-1}(v_i)$ is also equal to $-2$.

These three curves meet transversally at one point. Hence we get a singular fibre of type IV as $p^{-1}(v_i)$ $(i = 1, 2, 3)$ (see [3] section 6). Next we consider $p^{-1}(v_0)$. The surface $S$ has cusp singularity along this curve. When we proceed the $\sigma$-process along $p^{-1}(v_0)$ there occur three rational curves with self intersection number $-3$. Any curve of them intersects the proper image of $p^{-1}(v_0)$ transversally at one point, and these intersection points are different. The self intersection number of the proper image of $p^{-1}(v_0)$ is equal to $-1$, that is exceptional. After the blow down process of this curve we obtain a singular fibre of type IV as $p^{-1}(v_0)$.

The general fibre has the canonical form $y^2 = 4x^3 - Ox - g_3$, hence the invariant $j = g_1^3(g_3^3 - 27g_3)$ of this curve is equal to 0. Consequently the functional invariant $J$ of the elliptic surface $\tilde{S}$ is the constant function 0. For the elliptic surface with the functional invariant constant zero there are seven possibilities as its fibre:

- the regular fibre of the invariant 0,
- the singular fibre of type II, IV, $I^*_0$, $IV^*$, $II^*$ and
- the multiple singular fibre of type $\mu I_\alpha$ (see [3] section 9).

We note that the Euler number of these fibres are equal to 0, 2, 4, 6, 8, 10 and 0, respectively.

The surface $(\tilde{S}, p, \Lambda)$ has a holomorphic section $L$ given by $\{\xi_1 = \xi_2 = 0\}$ in (2.1). Namely $\tilde{S}$ is a basic member. According to the calculation we know that $L$ meets with every singular fibre on an irreducible component with multiplicity 1. We can describe the singular fibre as the decomposition with its irreducible components:

$$p^{-1}(v_i) = \Theta_{i_0} + \Theta_{i_1} + \Theta_i \quad (i = 0, 1, 2, 3),$$

$$p^{-1}(v_\infty) = 3\Theta_{\alpha_0} + 2\Theta_{\alpha_1} + 2\Theta_{\alpha_2} + 2\Theta_{\alpha_3} + \Theta_{\alpha_4} + \Theta_{\alpha_5} + \Theta_{\alpha_6},$$

where $\Theta_i$ $(i = 0, 1, 2, 3, \infty)$ is the component intersecting $L$.

According to Kodaira we have the following canonical bundle formula for a basic member (see [3] section 12).

**Theorem 2.1 (Kodaira).** Suppose an elliptic surface $(X, \Phi, \Lambda)$ is a basic
member. Then we have

\[ K_\Delta = \Phi^*(K_\Delta - F) \quad (K \text{ indicates the canonical bundle}), \]

where \( F \) is a certain line bundle on \( \Delta \) with \( c(F) = -p - 1 \).

According to this theorem we have \( c_2^1 = 0 \) for such a surface. Using the Noether's formula:

\[ p_g - q + 1 = \frac{1}{12} (c_1^2 + c_2), \]

we obtain \( c_2 \equiv 0 \pmod{12} \). For any elliptic surface \( X \) the Euler number \( \chi(X) \) is equal to the summation of the Euler numbers of all singular fibres. Hence we have

\[ c_2 = \chi(\tilde{S}) = \sum_{i=0}^3 \chi(p^{-1}(v_i)) + \chi(p^{-1}(v_\infty)). \]

Already we have \( (p^{-1}(v_i)) = 4 \) for \( i = 0, 1, 2, 3 \). Then \( p^{-1}(v_\infty) \) must be a singular fibre of type IV*, consequently we obtain

\[ c_2 = 24. \]

Now we show that \( \tilde{S} \) is a K3 surface, that is a minimal nonsingular compact complex surface with \( K = 0 \) and the irregularity \( q = 0 \) (it is equivalent with the condition \( b_1 = c_1 = 0 \)).

**Proposition 2.1.** Suppose an elliptic surface \( (X, \Phi, \Delta) \) is a basic member. Then \( X \) is a K3 surface if and only if \( c_2 = 24 \) and \( \Delta = P \).

**Proof.** (Necessity) Because \( K = 0 \) we know \( p_g = 1 \). According to the Noether's formula it follows \( c_2 = 24 \). From Theorem 2.1 we know \( c(F) = -2 \). And \( c(K_\Delta) \) is equal to \( 2g - 2 \), where \( g \) indicates the genus of \( \Delta \). Hence \( g \) must be 0.

(Sufficiency) By the assumption \( c_2 = 24 \) it follows \( p_g - q + 1 = 2 \). We have \( c(K_\Delta - F) = 0 \) because of Theorem 2.1. Then the assumption \( \Delta = P \) assures \( K_\Delta - F = 0 \). This implicates \( K_\Delta = 0 \), consequently we have \( q = 0 \).

According to this proposition we can conclude that \( \tilde{S} \) is a K3 surface.

**Conclusion 1.** The surface \( \tilde{S}(\lambda, \mu) \) is an elliptic surface satisfying the condition (i)-(iv) in the section 0 and is a K3 surface, where the parameters \( (\lambda, \mu) \) lie on \( \Lambda \).
Remark. The condition (i)-(iv) induces unique homological invariant which belongs to the functional invariant \( J = 0 \). Then it characterizes the surface \( \tilde{S} \) as an elliptic surface.

Let us consider the 2-form \( \psi \) on \( \tilde{S} \) which is canonically induced from the 2-form \( \psi = -du \wedge dv \), the integrand of (1.4), on \( S \). By the calculation we can obtain that \( \psi \) is the holomorphic 2-form on \( \tilde{S} \). In the rest of this paper we shall study the range of the integral (1.4).

3. - Basis of \( H_2(\tilde{S}, \mathbb{Q}) \).

In this section we construct a basis system of \( H_2(\tilde{S}, \mathbb{Q}) \). We consider the fixed surface \( \tilde{S}(\frac{-1}{3}, \frac{1}{2}) \) till the end of this section. Because the Euler number \( \chi(\tilde{S}) \) is equal to 24 and \( b_1 = 0 \), then \( H_2(\tilde{S}, \mathbb{Q}) \) is a 22-dimensional vector space over \( \mathbb{Q} \).

(I) Transcendental cycles. Let \( v \) be a point on \( A \) different from \( v_i \) \((i = 0, 1, 2, 3, \infty)\). And let us consider a closed arc \( a_i(v) \) which starts from \( v \) and goes around the critical point \( v_i \) in the positive sense \((i = 0, 1, 2, 3, \infty)\). Then \( a_i(v) \) induces a monodromy transformation \( (A_i) \) of the first homology group \( H_1(p^{-1}(v), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) of the general fibre \( p^{-1}(v) \). Let us choose a basis system \( (\gamma_1(v), \gamma_2(v)) \) of \( H_1(p^{-1}(v), \mathbb{Z}) \) so that the intersection multiplicity \( \gamma_1(v)\gamma_2(v) \) is equal to \(-1\) and so that we have

\[
\int \omega = \exp \left[ \frac{2\pi i}{3} \right] \int \omega,
\]

where \( \omega \) is the Abelian differential on \( p^{-1}(v) \). According to the study of Kodaira (see [3] section 9) we know that \( (A_i) \) is determined as the following:

\[
(A_i) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ for } i = 0, 1, 2, 3,
\]

\[
(A_\infty) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.
\]

We note that these transformations are of order 3. Let us consider a line segment \( l_i \) \((i = 0, 1, 2, 3)\) connecting \( v_i \) and \( v_\infty \) in the lower half \( v \)-plane.

We denote the open set \( A - \bigcup_{i=0}^{3} l_i \) by \( A_\emptyset \). We can determine the basis \( (\gamma_1(v), \gamma_2(v)) \) of \( H_1(p^{-1}(v), \mathbb{Z}) \) so as to vary continuously while \( v \) moves on \( A_\emptyset \).
If we determine the basis \((\gamma_1(v), \gamma_2(v))\) at one point, then for every value \(v\) on \(A\) the basis \((\gamma_1(v), \gamma_2(v))\) is uniquely determined up to the homotopic equivalence. We shall give a concrete construction of \(\gamma_1(v)\) and \(\gamma_2(v)\) in the next section, then we leave them indeterminate for the moment. Now we construct 2-cycles \(G_i\) \((i = 1, \ldots, 6)\) of \(\tilde{S}\) with the following procedure.

Let us make an oriented Jordan arc \(\alpha_i\) from \(v_0\) to \(v_i\). And let us make a closed oriented Jordan arc \(g_i\) which goes around the line segment \(l_i\) in the negative sense and intersects \(\alpha_i, l_2, l_3\) in this order. We denote the intersecting point of \(g_i\) and \(\alpha_i\) by \(r_i\). Let us take the 1-cycle \(\gamma_3(r_i)\) of the fibre \(p^{-1}(r_i)\). And let us make a continuation of \(\gamma_3(r_i)\) along the arc \(g_i\) till arriving at the intersecting point with \(l_i\). According to (3.1) we can proceed the continuation taking \(- (\gamma_1(v) + \gamma_2(v))\) from here. And we shall arrive at the intersecting point with \(l_i\). Similarly we can proceed the continuation along \(g_i\) changing the 1-cycle according to (3.1). The arc \(g_i\) intersects \(\bigcup_{i=0}^{3} l_i\) exactly three times, then this continuation determines a 2-cycle \(G_i\) in \(\tilde{S}\). And we make other five 2-cycles in the same manner.

Let us make oriented Jordan arcs \(\alpha_i\) \((i = 1, 2, 3)\) from \(v_0\) to \(v_i\) in the upper half plane, where we suppose that these arcs do not intersect each other. Next we make oriented closed Jordan arcs \(g_i\) \((i = 1, 2, 3)\) which goes around \(l_i\) in the negative sense, where we make \(g_i\) so that any one of them intersects \(\alpha_j\) \((j = 1, 2, 3)\) and \(l_k\) \((k = 0, 1, 2, 3)\) at most one time. We denote the intersecting point of \(g_i\) and \(\alpha_i\) by \(r_i\). We define six 2-cycles \(G_1, \ldots, G_6\) as in the diagram (3.1), where always we define the orientation of \(G_i\) as the ordered pair of the orientation of the base arc and the orientation of the 1-cycle of the fibre.

**Diagram 3.1**

<table>
<thead>
<tr>
<th>defined 2-cycle</th>
<th>base arc</th>
<th>starting 1-cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_1)</td>
<td>(g_1)</td>
<td>(\gamma_2(r_1))</td>
</tr>
<tr>
<td>(G_2)</td>
<td>(-g_1)</td>
<td>(\gamma_1(r_1))</td>
</tr>
<tr>
<td>(G_3)</td>
<td>(g_2)</td>
<td>(\gamma_2(r_2))</td>
</tr>
<tr>
<td>(G_4)</td>
<td>(-g_2)</td>
<td>(\gamma_1(r_2))</td>
</tr>
<tr>
<td>(G_5)</td>
<td>(g_3)</td>
<td>(\gamma_2(r_3))</td>
</tr>
<tr>
<td>(G_6)</td>
<td>(-g_3)</td>
<td>(\gamma_1(r_3))</td>
</tr>
</tbody>
</table>

(II) Intersection multiplicities of \(G_1, \ldots, G_6\). At first we define some notations. We denote the restriction of \(G_i\) to a fibre of one point \(\ast\) on a base arc by \(G_i(\ast)\). And we denote the intersection multiplicity of two cycles \(C\) and \(C'\) at a intersecting point \(\ast\) by \((CC')(\ast)\).
(a) We know that $G_i G_i = 0$ $(i = 1, \ldots, 6)$ by changing the base arc $g_j$ $(j = 1, 2, 3)$ to a homologous one in $\Delta' = \Delta - \bigcup_{i=0}^{3} \{v_i\} - \{v_\infty\}$. And we have $G_i G_{i+1} = 0$ $(i = 1, 3, 5)$ by the same reason.

(b) The base arc $g_1$ of $G_1$ and the base arc $g_3$ of $G_3$ meet each other at two points $a_1$ and $a_3$. We have

$$G_1 G_3 = (G_1 G_3)(a_1) + (G_1 G_3)(a_3) = \sum_{i=1,2} [(g_1 g_3)(a_i) \times G_1(a_i) G_3(a_i)].$$

On the other hand we have:

$$(g_1 g_3)(a_1) = 1, \quad (g_1 g_3)(a_3) = -1, \quad G_1(a_1) = \gamma_3(a_1), \quad G_3(a_3) = -(\gamma_1 + \gamma_2)(a_3), \quad G_1(a_2) = \gamma_3(a_2).$$

Hence we have $G_1 G_3 = 1$. And also we know the following by the same reason;

$$G_3 G_3 = G_2 G_4 = G_4 G_6 = 1.$$

(c) Also the base arc $g_1$ of $G_1$ and the base arc $-g_2$ of $G_4$ meet each other at two points $a_1$ and $a_2$. Then we have:

$$G_1 G_4 = \sum_{i=1,2} G_1 G_4(a_i) = \sum_{i=1,2} [(g_1 (-g_2)(a_i) \times G_1(a_i) G_4(a_i)] .$$

On the other hand we have:

$$(g_1 (-g_2)(a_1) = -1, \quad (g_1 (-g_2)(a_2) = 1, \quad G_4(a_1) = \gamma_4(a_1), \quad G_4(a_2) = \gamma_4(a_2).$$

Hence we have $G_1 G_4 = 2$. By the same argument we have $G_3 G_6 = 2$.

In the same manner we can calculate all the intersection multiplicities $a_{ij} = G_i G_j$ $(1 \leq i, j \leq 6)$. Consequently we obtain the intersection matrix $A = (a_{ij})$ as the following:

$$A = \begin{pmatrix}
0 & 0 & 1 & 2 & -2 & -1 \\
0 & 0 & -1 & 1 & -1 & -2 \\
1 & -1 & 0 & 0 & 1 & 2 \\
2 & 1 & 0 & 0 & -1 & 1 \\
-2 & -1 & 1 & -1 & 0 & 0 \\
-1 & -2 & 2 & 1 & 0 & 0
\end{pmatrix}.$$
(III) Algebraic cycles. We choose 16 divisors $G_7, ..., G_{22}$ on the fiber space $(S, p, P)$ as the following:
$G_7 = \text{one of the general fibres}, G_8 = \text{the global section } L, G_9 = \theta_{60}, G_{10} = \theta_{21},$
$G_{11} = \theta_{10}, G_{12} = \theta_{11}, G_{13} = \theta_{20}, G_{14} = \theta_{21}, G_{15} = \theta_{20}, G_{16} = \theta_{21}, G_{17} = \theta_{oo},$
$G_{18} = \theta_{oo}, G_{19} = \theta_{oo}, G_{20} = \theta_{oo}, G_{21} = \theta_{oo}, G_{22} = \theta_{oo}.$ We note that we eliminated every component of the singular fibres which intersects the global section $L$. And we have $G_i^2 = -2$ for every $i (i = 7, ..., 22)$, because they occur from a rational double singularities. We know the intersection multiplicities $G_i G_j (7 \leq i, j \leq 22)$ by observing their geometric situation. Here we give the configuration diagram (diag. 3.2).

![Diagram 3.2](image)

Consequently we obtain the matrix $B = (b_{ij})$ of the intersection multiplicities $b_{ij} = G_i \cdot G_j (1 \leq i, j \leq 16)$ as the following:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(3.4) $B = \ldots$

(IV) Intersection matrix. We can easily show that the determinants of $A$ and $B$ are not equal to zero. Next we consider the intersection of transcendental cycles $G_1, ..., G_6$ and algebraic cycles $G_7, ..., G_{22}$. It is apparent that the transcendental cycle does not intersect all algebraic cycles except $G_8$,
the global section. Then we examine the intersection of $G_i$ ($i = 1, \ldots, 6$) and $G_s$.

Let $\delta_i$ be a sufficiently small disk in the $v$-sphere which has the center $v_i$ ($j = 0, 1, 2, 3, \infty$). Let $v = c$ be a fixed point on $A'$, and let $\beta_i$ be a positively oriented circle in $\delta_i$ which goes around $v_i$. Let us consider a loop $\tilde{F}_i$ which starts from $c$ and goes to the initial point of $\beta_i$ and goes around $v_i$ along $\beta_i$ and finally returns to $c$ along the former path. Then the base arc $g_i$ of $G_i$ is represented as $-\tilde{F}_i = \beta_i + \tilde{\beta}_i + \beta_i$. And also the other $g_i$ can be represented as same. The restriction of the fibre space $(\tilde{S}, p, P)$ over $A_0$ is biholomorphically equivalent to the trivial one. Then it is sufficient to observe that the restriction of $G_i$ over $\beta_i$ does not intersect $L$. According to Kodaira (see [3]) all of $p^{-1}(\delta_j)$ ($j = 0, 1, 2, 3$) are biholomorphically equivalent, then we examine only the case when we have $j = 0$.

We have a following representation of $(\tilde{S}|\delta_0, p, \delta_0)$, that is the restriction of $\tilde{S}$ over $\delta_0$:

\begin{equation}
\eta_0^3 v - \eta_i(\eta_2^2 - \eta_1^2) = 0,
\end{equation}

where $[\eta_0, \eta_1, \eta_2]$ is the homogeneous coordinate of $P^2$. Set $s = \eta_i/\eta_0$ and $t = \eta_2/\eta_0$, then we obtain an affine representation

\begin{equation}
v - s(t^2 - s^2) = 0.
\end{equation}

And we know that the correspondence between (3.5) and (2.1) is given by setting $\eta_i/\eta_0 = \nu \xi_1/\xi_2$ and $\eta_2/\eta_0 = \nu \xi_2/\xi_2$. Hence $L$ is given by $\eta_0 = \eta_i = 0$ in (3.5). Let us regard the fibre $p^{-1}(v)$ in (3.5') as a two sheeted Riemann surface over $s$-sphere.

Then we obtain four ramified points $s = 0$, $\nu \xi_1 - v$, $\nu \xi_2 - v$, and $\nu^2 \xi_1 - v$, where $\nu = \exp (2\pi i/3)$. Here we define two circles $\gamma_1(v)$ and $\gamma_2(v)$ on the $s$-sphere as follows:

\begin{equation}
\begin{align*}
\gamma_1(v): s &= (\frac{1}{2})(1 + 2\nu\xi_1)\sqrt{-v}, \quad 0 \leq \theta \leq 2\pi, \\
\gamma_2(v): s &= (\frac{1}{2})(1 + 2\nu\xi_2)\nu\sqrt{-v}, \quad 0 \leq \theta \leq 2\pi.
\end{align*}
\end{equation}

We consider the closed Jordan arc on the Riemann surface $p^{-1}(v)$ which has the projection $\gamma_i(v)$ ($i = 1, 2$). We denote them by $\tilde{\gamma}_i(v)$. Then we obtain a canonical homology basis of $p^{-1}(v)$ because we have $\tilde{\gamma}_1 \tilde{\gamma}_2 = -1$.

Now let $u$ tend to zero by fixing the value $v$, then the value of $w$ tends to zero with the order $w = O(|w|^2)$. Remember that we determined $s = uv/w$ and $t = v/w$, then we know that the intersecting point of $p^{-1}(v)$ and the global section $L$ corresponds to a point at infinity $(s, t) = (0, \infty)$. 
Hence it is apparent that \( \tilde{y}_i(v) \) does not intersect \( L \) for every \( v \) on \( \partial_i - \{v_i\} \). Consequently we have

\[
G_i G_k = 0 \quad \text{for } i = 1, \ldots, 6 \text{ and } k = 7, \ldots, 22.
\]

Set \( X = \{\tilde{S}(\lambda, \mu) : (\lambda, \mu) \in A \} \). And let \( \tilde{A} \) be the universal covering space of \( A \). The fibre space \( X \) over \( \tilde{A} \) is topologically trivial. Because of this trivialization we can define the basis system \( G_1(\lambda, \mu), \ldots, G_{22}(\lambda, \mu) \) of \( H_4(S(\lambda, \mu), \mathbb{Q}) \) for every \( (\lambda, \mu) \) on \( A \).

Because of (3.3), (3.4) and (3.7) we obtain the following.

**Conclusion 2.** A basis system of \( H_4(S(\lambda, \mu), \mathbb{Q}) \) is given by \( \{G_1, \ldots, G_{22}\} \), and the intersection matrix \( M = (G_i G_j)_{1 \leq i, j \leq 22} \) is the direct sum \( M = A \oplus B \).

4. **Basis of \( H_4(\tilde{S}, \mathbb{Z}) \) and the Riemann-Hodge relation.**

(I) In this section we construct the basis of \( H_4(\tilde{S}, \mathbb{Z}) \) using the basis \( G_1, \ldots, G_{22} \) of \( H_4(\tilde{S}, \mathbb{Q}) \). And for the moment we consider the surface \( \tilde{S}(\lambda, \mu) \) with fixed parameters \( (\lambda, \mu) = (-\frac{1}{2}, \frac{1}{2}) \).

Let us consider the following automorphism \( \varrho_1 \) of the surface \( S \) defined by (1.5):

\[
\varrho_1 \left\{ \begin{array}{l}
w' = u, \\
v' = v, \\
w' = v \cdot \exp(2\pi i/3).
\end{array} \right.
\]

We can examine that \( \varrho_1 \) can be extended to an automorphism \( \varrho \) of \( \tilde{S} \). And \( \varrho \) is of order three on every simple component of the fibre and it is the identity on every multiple component of the fibre.

(II) Construction of transcendental 2-cycles. We consider a general fibre \( \varphi^{-1}(v) \) of the fibre space \( (\tilde{S}, \varphi, \mathbb{P}) \), where we suppose \( 0 < v < 1 \). We regard this general fibre as a three sheeted covering Riemann surface over \( u \)-sphere represented in (1.5). And we denote this Riemann surface by \( R(v) \).

We consider the following arcs \( d_i \) (\( i = 1, 2, 3 \)) on \( u \)-sphere:

- \( d_1 \): the line segment connecting two points \( u = 0 \) and \( u = \infty \),
- \( d_2 \): the line segment connecting two points \( u = 1 - v \) and \( u = \infty \),
- \( d_3 \): the line segment connecting two points \( u = -2 + v \) and \( u = \infty \).

When \( u \) satisfies the inequality \( 0 < u < 1 - v \) there are three different values of

\[
w = \{v^2 u(1 - u - v) (1 - \mu v - \lambda u)\}^{1/4},
\]
and their arguments are $0, 2\pi/3$ and $4\pi/3$. We can define single sheet of our covering Riemann surface by the continuation of one of these branches of $w$ over $u$-sphere $-\{d_1, d_2, d_3\}$. Then we define the first, second and third sheets of the cut Riemann surface $R(v) - \{u^{-1}(d_1), u^{-1}(d_2), u^{-1}(d_3)\}$ as the continuation of the value $w$ which satisfies $\arg w = 0$, $\arg w = 2\pi/3$ and $\arg w = 4\pi/3$ over the open arc $0 < u < 1 - v$, respectively.

Next we choose a real valued continuous function $\varepsilon(v)$ defined on the open arc $0 < v < 1$ which satisfies the inequality

$$\text{Min} \left( \frac{(1 - v)}{2}, \frac{(-2 + v)}{2} \right) > \varepsilon(v) > 0.$$ 

And we make following arcs $\beta_1$, $g_1$ and $\beta_2$ on $u$-sphere:

$\beta_1$: goes around the point $1 - v$ according to the parametrization

$$\varepsilon \theta(0 \leq \theta \leq 2\pi),$$

$\beta_2$: goes around the point $1 - v$ according to the parametrization

$$1 - v - \varepsilon \theta(0 \leq \varphi \leq 2\pi).$$

We denote the composite closed arc $g_1 \beta_2 g_1^{-1} \beta_1^{-1}$ by $c_1$. Let $\gamma_i(v)$ ($\gamma_1(v)$) be the lift of $c_1$ to the Riemann surface $R(v)$ which take the second (the first) sheet, respectively, along the arc $g_1$. According to the continuation along the base arc $\beta_i$ ($i = 1, 2$) it occurs a permutation $(1, 2, 3)$ of sheets of $R(v)$. Hence we know that $\gamma_1(v)$ and $\gamma_2(v)$ are closed arcs and that they have the intersection multiplicity $\gamma_1(v) \gamma_2(v) = -1$.

Here we consider the union

$$I'_i = \bigcup_{0 < r < 1} \gamma_i(r) \quad (i = 1, 2).$$

Let us examine that $I'_i$ tends to one point when $v$ tends to 0 or 1. In the section 3 (IV) we already obtained a homology basis $(\tilde{\gamma}_1(v), \tilde{\gamma}_2(v))$ of a general fibre $p^{-1}(v)$ of the fibre space $(\tilde{S}, p, P)$ with respect to the local representation (3.5). Then we consider again the representation (3.5). Let $p$ be a point of $\tilde{\gamma}_i(v)$ ($i = 1, 2$). The point $p$ is determined by $v$ and the parameter $\theta$. According to (3.5) and (3.6) we have

$$s = a(\theta) \sqrt{-v},$$

$$t = \sqrt{1 - (a(\theta))^2} a(\theta) \sqrt{-v},$$

where $a(\theta) = (1 + 2\varepsilon^2)/2.$
We know that $s$ and $t$ tends to zero as $v$ tends to zero, because $a(\theta)$ satisfies the condition $1 \leq |a(\theta)| \leq 2$. Hence $\gamma_i(v)$ $(i = 1, 2)$ tends to the origin of $(s, v, t)$-space when $v$ tends to 0. Consequently $\gamma_i(v)$ tends to one point when $v$ tends to 0. And this limit point is the intersection of three components of the singular fibre $p^{-1}(0)$, those are $\Theta_{00}$, $\Theta_{01}$ and $\Theta_0$.

By the same argument we know that $\gamma_i(v)$ tends to the intersecting point of three components of the singular fibre $p^{-1}(1)$ when $v$ tends to 1.

Hence if we attach these two limit points to $\Gamma_i$, we obtain a 2-cycle on $\tilde{S}$ which is homeomorphic to a sphere. We denote them by $\Gamma'_i$ $(i = 1, 2)$. By deforming the base arc $\{0 \leq v \leq 1\}$ of $\Gamma$, we may consider that $\Gamma'_i$ is situated over the arc $\alpha_i$ defined in the preceding section. So we define the orientation of $\Gamma'_i$ as the ordered pair of the orientation of $\alpha_i$ and the orientation of $\gamma_i(v)$.

Here we make following oriented arcs $g_i$, $g_5$, $\beta_i$ and $\beta_s$ on $u$-plane:

$g_i$: starts from $\varepsilon - 2 + v$ and goes to the end point $1 - v - \varepsilon$ in the upper half plane along a Jordan arc,

$g_5$: starts from $\varepsilon - 2 + v$ and goes to the end point $- \varepsilon$ along the real line,

$\beta_i$: goes around the origin according to a parametrization $- \varepsilon e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$,

$\beta_s$: goes around the point $- 2 + v$ according to a parametrization $- 2 + v + \varepsilon e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

The restriction of the fibre space $(\tilde{S}, p, \Lambda)$ over $\Lambda_0$ is biholomorphically equivalent to the direct product space. Then we can define $\gamma_i(v)$ uniquely (up to the homotopic equivalence) for every value $v$ on $\Lambda_0$. By the same procedure as the above we can define the following 2-cycles:

$$
\Gamma_s = \bigcup_{v \in \Lambda_0} \gamma_s(v) , \quad \Gamma_s = \bigcup_{v \in \Lambda_0} \gamma_s(v) ,
$$

$$
\Gamma_5 \bigcup_{v \in \Lambda_0} \gamma_5(v) , \quad \Gamma_5 = \bigcup_{v \in \Lambda_0} \gamma_5(v) ,
$$

where $\alpha_i$ is the arc defined in the section 3 (I). By the construction we have

$$
(4.1) \quad \varphi^{-1} \Gamma_i = \Gamma_{i+1} \quad \text{for} \quad i = 1, 3, 5 .
$$

Then it follows

$$
(4.1') \quad \int_{\Gamma_{i+1}} \varphi = \omega \int_{\Gamma_i} \varphi \quad (\omega = \exp(2\pi i/3)) \quad \text{for} \quad i = 1, 3, 5 .
$$
Let us examine the intersection multiplicity of $\Gamma_i$ and $G_j$ ($1 \leq i, j \leq 6$). The base arc $\alpha_i$ of $\Gamma_i$ and $\Gamma_i'$ intersects the base arc $g_i$ of $G_i$ at one point $r_i$. We know the following:

$$\alpha_i g_i = 1, \quad \Gamma_i(r_i) = \gamma_i(r_i), \quad \Gamma_i'(r_i) = \gamma_i'(r_i) \quad \text{and} \quad G_i(r_i) = \gamma_i(r_i).$$

Hence we obtain

$$\Gamma_i G_i = - (\alpha_i g_i) \times \gamma_i(r_i) \gamma_i'(r_i) = 1,$$

$$\Gamma_i G_i = - (\alpha_i g_i) \times \gamma_i'(r_i) \gamma_i(r_i) = 0.$$

By the same argument we obtain $\Gamma_i G_2 = 0$ and $\Gamma_i G_3 = 1$. And it is easily shown that $\Gamma_i G_j = \Gamma_i G_j = 0$ for $j = 3, 4, 5, 6$. We can discuss about $\Gamma_i$ ($i = 3, 4, 5, 6$) in the same manner. Consequently we have

$$(4.2) \quad \Gamma_i G_i = \delta_{ij} \quad (1 \leq i, j \leq 6),$$

where $\delta_{ij}$ indicates the Kronecker’s delta.

Let us consider the subgroup $A(\bar{S})$ of $\text{H}_2(\bar{S}, \mathbb{Z})$ which is generated by $G_7, \ldots, G_{22}$. The subgroup $A(\bar{S}) \otimes \mathbb{Q}$ of $\text{H}_2(\bar{S}, \mathbb{Z})$ is the one which is generated by $G_7, \ldots, G_{22}$. And already we obtained the direct sum decomposition

$$\text{H}_2(\bar{S}, \mathbb{Q}) = \{G_1, \ldots, G_6\} \oplus \{G_7, \ldots, G_{22}\}.$$

Let $C$ be an arbitrary element of $\text{H}_2(\bar{S}, \mathbb{Z})$ and put

$$C' = \sum_{j=1}^{6} a_j \Gamma_j, \quad \text{where} \quad a_j = CG_j.$$

According to (4.2) we obtain

$$(C - C')G_j = 0 \quad \text{for} \quad j = 1, \ldots, 6.$$

Then $C - C'$ belongs to the orthogonal complement of $G_1, \ldots, G_6$, namely it belongs to $A(\bar{S}) \otimes \mathbb{Q}$. Since $C - C'$ is an element of $\text{H}_2(\bar{S}, \mathbb{Z})$ it must belong to $A(\bar{S})$. Let $\Gamma_1, \ldots, \Gamma_{22}$ be a basis system of $A(\bar{S})$. By the above argument we know that $\{\Gamma_1, \ldots, \Gamma_6, \Gamma_7, \ldots, \Gamma_{22}\}$ is a basis system of $\text{H}_2(\bar{S}, \mathbb{Z})$.

(III) Now we consider the Riemann-Hodge relation. Set

$$\eta_k = \oint_{r_k} \psi \quad \text{for} \quad k = 1, \ldots, 22.$$
And we consider the cohomology group $H^2(\tilde{S}, \mathbb{R})$ and we regard this group as the cohomology group of real 2-forms by the de Rham correspondence. So we choose the basis $(\omega_1, ..., \omega_{22})$ of $H^2(\tilde{S}, \mathbb{R})$ so that

$$\int_{\gamma_j} \omega_k = \delta_{jk} \quad \text{for} \quad 1 \leq j, k \leq 22.$$  

Let $M'$ be a matrix of $a_{jk} = \int_\delta \omega_j \wedge \omega_k$ $(1 \leq j, k \leq 22)$. Then we can write the Riemann-Hodge relation as follows (see [4]):

\begin{equation}
\tilde{\eta} M' \tilde{\eta} = 0,
\end{equation}

\begin{equation}
\tilde{\eta} M' \tilde{\eta} > 0,
\end{equation}

where $\tilde{\eta} = (\eta_1, ..., \eta_{22})$.

Let us consider the dual basis $G_1^*, ..., G_{22}^*$ of $H_2(S, \mathbb{Q})$ such that $\Gamma_j G_i^* = \delta_{ij}$, where $G_i^* = G_i$ for $j = 1, ..., 6$. Then $\omega_i$ is cohomologous to $G_i^*$ as a current. Hence we have $M' = M$. Since the period $\int_D$ is equal to zero for a divisor $D$ on $\tilde{S}$, we have $\eta_i = 0$ for $i = 7, ..., 22$. Consequently we can write (4.5) using the matrix $A$ of the section 3:

\begin{equation}
\eta A^t \eta = 0,
\end{equation}

\begin{equation}
\eta A^t \tilde{\eta} > 0,
\end{equation}

where $\eta = (\eta_1, ..., \eta_6)$.

Here we consider the universal covering space $\tilde{A}$ of the domain $A$ of parameters. And let $(\lambda, \mu)$ be a point on $\tilde{A}$ which corresponds to the point $(\lambda, \mu)$ on $A$. Then the totality of the surfaces $\tilde{S}(\lambda, \mu)$ can be regarded as a fibre space over $\tilde{A}$ and it is topologically trivial. Then we can define the homology basis $\Gamma_1(\lambda, \mu), ..., \Gamma_{22}(\lambda, \mu)$ of $H_2(\tilde{S}(\lambda, \mu), \mathbb{Z})$. When we make the continuation of $\Gamma_1(\lambda, \mu)$ along a closed arc in $A$, it occurs a monodromy transformation. Any how we obtained a basis system of $H_2(\tilde{S}(\lambda, \mu), \mathbb{Z})$, where $(\lambda, \mu)$ varies on $A$. And the intersection matrix $M$ of their dual basis does not depend on the parameters $(\lambda, \mu)$. Then we obtain the relation (4.6) and (4.7) for every $(\lambda, \mu)$. From the relation (4.1') we can reduce the relation (4.7) as follows:

\begin{equation}
(\eta_1, \eta_2, \eta_3) \begin{pmatrix} 0 & \omega^2 & 1 \\ \omega & 0 & \omega^2 \\ 1 & \omega & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} < 0.
\end{equation}

Let $\sigma_1$, $\sigma_2$ and $\sigma_3$ be three eigen values of the matrix in this relation. Then
they are given as the solutions of \( t^3 - 3t + 1 = 0 \), and they satisfy the relation \( \sigma_1 < 0 < \sigma_2 < \sigma_3 \).

If we consider \([\eta_1, \eta_2, \eta_3]\) as a homogeneous coordinate in \( \mathbb{P}^2 \), according to (4.8) we obtain a domain \( \Omega \) in \( \mathbb{P}^2 \) which is biholomorphically equivalent to a hyperball.

**Remark.** Let \( NS(\tilde{\mathcal{S}}) \) be the Neron-Severi group of \( \tilde{\mathcal{S}} \) (that is the group of all divisors under the algebraic equivalence). The rank of \( NS(\tilde{\mathcal{S}}) \) is 16 for almost all \((\lambda, \mu)\) on \( \Lambda \). And it exceeds 16 if and only if \([\eta_1, \eta_2, \eta_3]\) is a rational point on \( \mathbb{P}^2 \).

From the argument of this section we obtained the following.

**Conclusion 3.** The period mapping \([\eta_1, \eta_2, \eta_3]\) for the family \( \tilde{\mathcal{S}}(\lambda, \mu) \) defines a multivalued analytic mapping from \( \Lambda \) to a hyperball \( \Omega \) in \( \mathbb{P}^2 \).

5. - Monodromy transformation and the reduction to the Picard’s mapping.

In this section we relate the monodromy transformation of \( H_2(\tilde{\mathcal{S}}(\lambda, \mu), \mathbb{Z}) \) which is induced from an element of \( \pi_1(\Lambda) \). And we show that the mapping \( \eta_1(\lambda, \mu), \eta_2(\lambda, \mu), \eta_3(\lambda, \mu) \) coincides with the mapping which is constructed by Picard. And we give the generators of the transformation group of \( \Omega \) which is induced from the monodromy transformation.

(I) Reduction to the original integral. From the construction of \( \Gamma_i \) \((i = 1, 2)\) these 2-cycles depend on the function \( \epsilon(v) \). But all of them are homotopic. Hence the period

\[
\eta_i = \int_{\Gamma_i} \psi \quad (i = 1, 2)
\]

does not depend on \( \epsilon(v) \). So we can consider the limit value of \( \eta_i \) as \( \epsilon(v) \) tends to zero, and that value is also equal to \( \eta_i \). Here we consider a real 2-dimensional triangle \( D_1 \) on \( \mathcal{S} \) as follows:

\[
D_1 = \{(u, v)|u \geq 0, v \geq 0, 1 - u - v \geq 0\}.
\]

This is the projection of the limit cycle of \( \Gamma_i \) \((i = 1, 2)\) to the \((u, v)\)-space. Then we obtain

\[
(5.1) \quad \eta_1 = (\omega^2 - \omega) \int_{D_1} \frac{v}{\omega^2} du \wedge dv,
\]

where we take the first sheet of \( \omega \), that takes a real value on \( D_1 \), and
\( \omega = \exp(2\pi i/3) \). Let us consider the other triangles as the followings:

\[
D_2 = \{(u, v)|\mu v/(\mu - 1) \geq 0, \mu(1-u-v)/(\mu - 1) \geq 0, \\
1 - \mu u - \mu v/(1-\mu) \geq 0\},
\]

\[
D_3 = \{(u, v)|\lambda u \geq 0, \mu v \geq 0, 1 - \lambda u - \mu v \geq 0\}.
\]

The triangle \(D_2(D_3)\) is the projection of the limit cycle of \(\Gamma_3(\Gamma_2\text{ and }\Gamma_1)\) to the \((u, v)\)-space, respectively. By the same argument we obtain

\[
\eta_3 = (1-\omega) \int_{D_3}^{v} \frac{v}{\omega^2} \, du \land dv,
\]

\[
\eta_5 = (1-\omega^2) \int_{D_3}^{v} \frac{v}{\omega^2} \, du \land dv.
\]

Because of the equality \(D = D_1 + D_2 - D_3\), \(D\) is the original triangle in (1.5), it holds that

\[
\int_{D}^{v} \frac{v}{\omega^2} \, du \land dv = \frac{1}{1-\omega^2} (\omega \eta_1 + \eta_3 + \omega^2 \eta_5).
\]

Hence we obtain the representation of the original Picard's integral (0.0) in terms of the period on \(\hat{S}\). Namely, according to (1.4) and (5.3) we have

\[
\int_{0}^{1} \frac{dt}{\sqrt{t(t-1)(t-x)(t-y)}} = c \mu^{s/3}(\mu - \lambda)^{s/3}(\mu - 1)^{s/3} \frac{\omega \eta_1 + \eta_3 + \omega^2 \eta_5}{1-\omega^2},
\]

where \(c\) is the gamma constant which appeared in (1.4).

(II) Monodromy transformation. Let \(p_0 = (\lambda_0, \mu_0)\) be the point \((-\frac{1}{2}, 0)\) on \(A\). We consider the following loops \(\delta_1, \ldots, \delta_5\) in \(A\), where we suppose that \(p_0\) is the initial point of every \(\delta_i\) (\(i = 1, \ldots, 5\)):

\[
\delta_1 \text{ goes round the point } \lambda = 0 \text{ in the positive sense on the hyperplane } \mu = \mu_0,
\]

\[
\delta_2 \text{ goes round the point } \lambda = 1 \text{ in the positive sense on the same plane},
\]

\[
\delta_3 \text{ goes round the point at infinity in the positive sense on the hyperplane } \lambda = 3\mu - 2.
\]
\[ \delta_4 \text{ goes round the point } \lambda = \mu_0 \text{ in the positive sense on the hyperplane } \mu = \mu_0, \]
\[ \delta_5 \text{ goes round the point } \mu = 0 \text{ in the positive sense on the hyperplane } \lambda = \lambda_0. \]

We regard \( A \) as \( P^2 \{ 6 \text{ complex lines} \} \). And let \( H \) be a general hyperplane in \( P^2 \). Then the generators of the fundamental group of \( H \cap A \) are also the generators of \( \pi_1(A) \), this is the theorem of Lefschetz. Hence \( (\delta_1, ..., \delta_6) \) constitutes a generator system of \( \pi_1(A) \). Every element \( \delta \) of \( \pi_1(A) \) induces a monodromy transformation \( \delta \) of \( H_2(\tilde{S}(\lambda_0, \mu_0), \mathbb{Z}) \). And every divisor of \( A(\tilde{S}) \) is invariant under this transformation. Then we consider the monodromy transformation of \( \Gamma_1, ..., \Gamma_6 \) in the following.

(i) Transformation \( \delta_1 \). When the point \( (\lambda, \mu) \) moves along the loop \( \delta_1 \), the critical points \( v_0, v_1, v_2 \) and \( v_\infty \) stay invariant and the point \( v_2 = (1 - \lambda)(\mu - \lambda) \) varies along a loop \( \omega_2 \) which goes round \( v_2 \) in the positive sense. This loop defines a Jordan region \( R(\omega_2) \) in the finite \( v \)-plane. Let \( \tilde{S}(\lambda, \mu; v) \) be a fibre over \( v \) of the elliptic fibre surface \( (\tilde{S}(\lambda, \mu, p, P) \). If \( v \) is a point on \( A' - \omega_1 \), where \( A' = P - \{ v_0, v_1, v_2, v_\infty \} \), then \( \delta_1 \) induces a monodromy transformation \( \delta_1(v) \) of \( H_1(\tilde{S}(\lambda_0, \mu_0; v), \mathbb{Z}) \). At first we study this transformation.

A general fibre \( \tilde{S}(\lambda, \mu; v) \) is realized as a covering Riemann surface by considering the representation (1.5). We denote this Riemann surface by \( R(\lambda, \mu; v) \). There are three triply ramified points over \( u = \infty, u = u_1 = (1 - \lambda)(\mu - \lambda) \) and \( u = u_2 = 1 - v \) on \( R(\lambda, \mu; v) \).

Let us make three cut arcs \( a_0, a_1 \) and \( a_2 \) on \( u \)-plane, they are line segments which connect \( u = 0 \) and \( \infty, u = u_1 \) and \( \infty, u = u_2 \) and \( \infty \), respectively. Then we can determine the \( i \)-th sheet \( w_i \) \((i = 1, 2, 3) \) so that we have \( w_1 = \exp(2\pi i/3)w_2 = \exp(4\pi i/3)w_3 \). Here we consider an oriented arc on \( u \)-plane which connects two points \( \alpha \) and \( \beta \). We denote the lift of this arc into the \( i \)-th sheet \( w_i \) by \( w_i(\alpha, \beta) \).

When a point \( (\lambda, \mu) \) moves on \( A \), the ramified points \( u = 0 \) and \( u = u_2 \) are invariant and only \( u = u_1(\lambda, \mu) \) varies. Let \( s_1 \) be a loop which is drawn by \( u_1 \) corresponding to \( \delta_1 \). This loop negatively goes round the Jordan region \( R(s_1) \) which is defined by \( s_1 \) in the finite \( u \)-plane. If \( u \) is an interior point of \( R(s_1) \), then \( \delta_1 \) does not induce a permutation of the branches of
\[ w = w(\lambda, \mu; u, v) = \sqrt[3]{v^2} \Lambda(u_1 - u)(u_2 - u). \]
And if \( u \) is an exterior point of \( R(s_1) \), then the branch \( w_i \) of \( w \) changes according to the permutation \((1, 2, 3)\).

If \( v \) is an exterior point of \( R(\omega_1) \), \( s_1 \) goes round \( 0 \) and \( u_1 \) in the negative
sense. And the canonical basis system of $H_1(S(\lambda_0, \mu; v), \mathbb{Z})$ are given as the following:

$$\begin{cases}
\gamma_1(v) = w_4(0, u_z) - w_3(0, u_z), \\
\gamma_2(v) = w_4(0, u_z) - w_2(0, u_z).
\end{cases}$$

Then the projection of $\gamma_i(v)$ is contained in $R(s_i)$. Hence in this moment $\delta_i'(v)$ is the identity.

And if $v$ is an interior point of $R(\omega_1)$, $s_i$ goes round only one ramified point $u = 0$ in the negative sense. We consider the oriented line segment connecting 0 and $u$ in this direction.

We denote this arc by $\epsilon$. Then $\epsilon$ can be deformed to an arc passing through the point $u = v - 2$. Then we obtain

$$\delta_i'(\gamma_2(v)) = \delta_i'(w_4(0, u_z) + w_2(u_z, 0))$$

$$= (w_4(0, u_z) + w_2(u_z, 0) + w_3(0, u_z)) + (w_4(u_z, 0) + w_3(0, u_z) + w_2(u_z, 0))$$

$$= 2w_2(u_z, 0) - w_2(u_z, 0) + w_3(0, u_z) - w_2(u_z, u_z).$$

By the direct observation we know

$$\delta_i'(\gamma_1)\gamma_1 = 1, \quad \delta_i'(\gamma_1)\gamma_2 = 0.$$ 

Hence we know $\delta_i'(\gamma_1) = \gamma_2$. And by the relation $\gamma_2 = \sigma^{-1}\gamma_1$, we obtain

$$\delta_i'(\gamma_2) = \sigma^{-1}\delta_i'(\gamma_1) = -(\gamma_1 + \gamma_2).$$

In the following we denote $-(\gamma_1 + \gamma_2)$ by $\gamma_3$, then we have

$$(5.5) \quad \delta_1' = \begin{pmatrix}
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_2 & \gamma_3 & \gamma_1
\end{pmatrix} \text{ if } v \text{ lies inside of } R(\omega_1),$$

$$\text{identity} \quad \text{ if } v \text{ lies outside of } R(\omega_1).$$

Now we study the transformation $\delta_1$. The base arc $\alpha_1$ of $\Gamma_1$, it is the line segment connecting $v = 0$ and $v = 1$, does not change as $(\lambda, \mu)$ moves along $\delta_1$. And $\alpha_1$ is contained in the exterior part of $\omega_1$. Then $\delta_1'(v)$ is identity for the value $v$ on $\alpha_1$. Then it follows that $\delta_1\Gamma_1 = \Gamma_1$. Similarly we have $\delta_1\Gamma_2 = \Gamma_2$. The critical point $v_3$ goes round the critical point $v_2$ in the positive sense drawing the loop $\omega_1$ as $(\lambda, \mu)$ moves along $\delta_1$. Then the base arc $\alpha_2$ of $\Gamma_2$ moves following the loop $\omega_1$. Let $v_j$ and $v_k$ ($0 \leq j, k \leq 3$) be two critical points, and we consider a Jordan arc $\alpha$ connecting $v_j$ and $v_k$ in $A_3 = A - \bigcup_{i=0}^{3} \alpha_i$. We denote the 2-cycle $\bigcup_{v \in \alpha} \gamma_i(v)$ by $\Gamma_i(j, k)$. 
Then the following holds:

\[ \Gamma_1 = \Gamma_1(0, 1), \quad \Gamma_2 = \Gamma_1(0, 2), \quad \Gamma_3 = \Gamma_1(0, 3). \]

And also we have

\[
\delta_1 \Gamma_3 = \Gamma_3(0, 2) + \Gamma_3(2, 3) + \Gamma_3(3, 2)
= \Gamma_3(0, 2) + (\Gamma_3(0, 3) - \Gamma_3(0, 2)) - (\Gamma_3(0, 3) - \Gamma_3(0, 2))
= -2\Gamma_4 + \Gamma_5 + 2\Gamma_6.
\]

If we regard the base arc \( \alpha_3 \) of \( \Gamma_3 \) as the composition of \( \alpha_3 \) and the arc \( \alpha_2 - \alpha_1 \), we know that the former is invariant as \( (\lambda, \mu) \) moves along \( \delta_1 \) and that the latter is also invariant. But we must notice that \( \alpha_3 - \alpha_2 \) is contained in the interior part of \( \omega_1 \). Hence we obtain the following, using the transformation (5.5):

\[
\delta_1 \Gamma_3 = \Gamma_3(0, 2) + \Gamma_3(2, 3) = \Gamma_1(0, 2) + \Gamma_3(0, 3) - \Gamma_3(0, 2) = \Gamma_3 - \Gamma_4 + \Gamma_6.
\]

Because of the relation (4.1) we obtain:

\[
\delta_1 \Gamma_3 = 2\Gamma_4 + 2\Gamma_4 - 2\Gamma_6 - \Gamma_6,
\delta_1 \Gamma_3 = 2\Gamma_4 - \Gamma_6 - \Gamma_6.
\]

Consequently we can describe the transformation \( \delta_1 \):

\[
(5.6) \quad \delta_1 \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 0 & 2 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix}.
\]

This transformation is of order three.

(ii) Transformation \( \delta_2 \). When the point \((\lambda, \mu)\) moves along \( \delta_2 \), the critical points \( v_a, v_1, v_3 \) and \( v_\infty \) are invariant. And \( v_2 \) moves along a loop \( \omega_2 \) which goes round \( v_a \) in the positive sense. This loop defines a Jordan region \( B(\omega_2) \) in the finite \( v \)-plane. By the same procedure as (i) we obtain the transformation \( \delta_2(v) \) of \( H_1(S(\omega_2, \mu; v), \mathbb{Z}) \) which is induced from \( \delta_2 \), where \( v \)
is fixed on $A' - \omega_2$:

\[
\delta'_2(v) = \begin{cases} 
\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix} & \text{if } v \text{ lies on } R(\omega_2), \\
\text{identity} & \text{if } v \text{ is outside of } R(\omega_2).
\end{cases}
\]

After the variation of the point $(\lambda, \mu)$ along $\delta_2$ the base arc $\alpha_1$ of $\Gamma_1$ is deformed to an arc which passes through $v_3$. This deformed arc is the composition of two subarcs: the one starts from $v_3$ and goes to $v_2$ and the other starts from $v_4$ and goes to $v_1$. The former lies in $R(\omega_2)$ and the latter lies outside of $R(\Gamma_2)$. Using (5.7) we have

\[\delta_2 \Gamma_1 = \Gamma_3(0, 2) - \Gamma_4(0, 2) + \Gamma_4(0, 1) = \Gamma_1 - \Gamma_3 + \Gamma_4.\]

After the variation of the point $(\lambda, \mu)$ along $\delta_3$ the base arc $\alpha_2$ of $\Gamma_2$ does not change geometrically, but the argument increases by $2\pi$ and this arc is contained in $R(\omega_3)$. And by the same variation the arc $\alpha_3$ is deformed to an arc which passes through the points $v_0, v_2, a_0$ in this order and ends at $v_3$, where $a_0$ is the line segment connecting $v_0$ and $\infty$.

We regard this deformed arc as a composition of two subarcs. The first starts from $v_0$ and goes to $v_2$, the second starts from $v_3$ and goes to $v_3$. The former lies in $R(\omega_3)$ and this part changes the argument by $2\pi$. The latter lies outside of $R(\omega_3)$. According to (5.7) we have

\[
\delta_3 \Gamma_2 = -\Gamma_2 - \Gamma_3,
\]
\[
\delta_3 \Gamma_3 = -\Gamma_3 - 2\Gamma_4 + \Gamma_4.
\]

Because of the relation (4.1) we obtain

\[
\delta_3 \Gamma_4 = \Gamma_3,
\]
\[
\delta_3 \Gamma_5 = 2\Gamma_3 + \Gamma_4 + \Gamma_5.
\]

Consequently we can describe the transformation $\delta_3$:

\[
\delta_3 = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \\ \Gamma_5 & \Gamma_6 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \\ \Gamma_5 & \Gamma_6 \end{pmatrix}.
\]

This transformation is of order three.
(iii) Transformation $\delta_a$. When the point $(\lambda, \mu)$ moves along $\delta_a$, the critical points $v_0$, $v_1$, $v_2$ and $\infty$ are invariant. The critical point $v_3$ varies along a loop $\omega_3$ which goes round $v_3$ in the positive sense. This loop defines a Jordan region $R(\omega_3)$ in the finite $v$-plane. By the usual method we obtain the transformation $\delta'_a(v)$ of $H_1(S(\lambda_0; \mu_0; v), Z)$ induced from $\delta_a$:

$$
\delta'_a = \begin{cases} 
\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_2 & \gamma_1 & \gamma_3 \\
\gamma_3 & \gamma_2 & \gamma_1 
\end{pmatrix} & \text{if } v \text{ lies on } R(\omega_3), \\
\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_2 & \gamma_1 & \gamma_3 \\
\gamma_3 & \gamma_2 & \gamma_1 
\end{pmatrix} & \text{if } v \text{ lies outside of } R(\omega_3).
\end{cases}
$$

If we use (5.9) we can describe the transformation $\delta_a$ by the same method as (i) and (ii):

$$
\delta_a = \begin{pmatrix} 
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6 
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & -2 \\
-1 & -1 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 1 & -1 & -2 \\
0 & 0 & -1 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix} 
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6 
\end{pmatrix}.
$$

This transformation is of order three.

(iv) Transformations $\delta_4$ and $\delta_5$. When the point $(\lambda, \mu)$ moves along $\delta_4$, the critical points $v_0$, $v_1$, $v_2$ and $\infty$ are invariant. And $v_3$ goes around $\infty$ in the positive sense. And when the point $(\lambda, \mu)$ moves along $\delta_5$, the critical points $v_0$, $v_2$ and $\infty$ are invariant. The critical point $v_3$ moves along a loop which is homotopic to zero in $A'$. The critical point $v_3$ goes around the point $\infty$ in the positive sense.

By the similar way we obtain the monodromy transformation $\delta_4$ and $\delta_5$ as follows:

$$
\delta_4 = \begin{pmatrix} 
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6 
\end{pmatrix} = \begin{pmatrix} 
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-2 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix} 
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6 
\end{pmatrix},
$$

$$
\delta_5 = \begin{pmatrix} 
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6 
\end{pmatrix} = \begin{pmatrix} 
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 2 & -2 & -1 & 1 & 0 \\
-2 & -1 & 1 & -1 & 0 & 1
\end{pmatrix} \begin{pmatrix} 
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6 
\end{pmatrix}.
$$
The transformation $\delta_1$ and $\delta_2$ are of order infinite.

(III) The monodromy transformation $\delta_i$ ($i = 1, 2, 3, 4, 5$) induces a linear transformation $\delta_i^*$ of the periods $\eta_i(\lambda, \mu)$ ($j = 1, 3, 5$). According to the results in (II) and (6.1') we can describe them as follows:

\[
\delta_1^* \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\omega^2 & 1 + 2\omega^2 \\ 0 & 1 - \omega^2 & \omega^2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},
\]
\[
\delta_2^* \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 + \omega^2 & 1 \\ 0 & -1 - \omega^2 & 0 \\ 0 & -1 - 2\omega^2 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},
\]
\[
\delta_3^* \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 & -1 - 2\omega^2 \\ 0 & \omega^2 & -1 - 2\omega^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},
\]
\[
\delta_4^* \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 - \omega^2 & 1 & 1 - \omega^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},
\]
\[
\delta_5^* \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 + 2\omega^2 & -2 - \omega^2 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}.
\]

The transformed value $\delta_i^*(\eta_i(\lambda, \mu))$ is nothing but the analytic continuation of $\eta_i(\lambda, \mu)$ along the arc $\delta_i$. According to the local Torelli type theorem of the period mapping for algebraic K3-surfaces (see [4]) $\eta_i(\lambda, \mu), \eta_3(\lambda, \mu)$ and $\eta_5(\lambda, \mu)$ are linearly independent. And if we observe the transformations $\delta_i^*$, it is easily shown that any $\eta_i(\lambda, \mu)$ ($j = 1, 3, 5$) is obtained as a linear combination of one period $\eta(\lambda, \mu) = \int \psi$ for some 2-cycle $\Gamma$ of $S$ and its analytic continuations. The function

\[
f(\lambda, \mu) = \frac{A(\lambda, \mu)}{1 - \omega^2} (\omega\eta_1 + \eta_3 + \omega^2\eta_5),
\]

where $A(\lambda, \mu) = \sigma_2(\mu - \lambda)^{5/3}(\mu - 1)^4$, of the right hand side in (5.4) is a solution for the differential equation (0.3) under the transformation of variables

\[
\lambda = (y - x)/(x(y - 1)), \quad \mu = 1/(1 - y).
\]

Then $A(\lambda, \mu)\eta_j(\lambda, \mu)$ ($j = 1, 3, 5$) are the three independent solutions for the
equation (0.3). Hence the ratios

\[ \zeta_1(\lambda, \mu) = \frac{\eta_6(\lambda, \mu)}{\eta_1(\lambda, \mu)}, \quad \zeta_2(\lambda, \mu) = \frac{\eta_8(\lambda, \mu)}{\eta_1(\lambda, \mu)} \]

coincide with the Picard’s original mapping. Consequently we obtain the following.

**CONCLUSION 4.** The period mapping \([\eta_1(\lambda, \mu), \eta_3(\lambda, \mu), \eta_6(\lambda, \mu)]\) for the family of surfaces \(S(\lambda, \mu)\) coincides with \((\zeta_1(x, y), \zeta_2(x, y))\) under the transformation (5.13). The inverse mapping \((\lambda, \mu) = (\varphi_1(\zeta_1, \zeta_2), \varphi_2(\zeta_1, \zeta_2))\) is equal to the Picard’s modular functions (up to a projective linear transformation). The functions \(\varphi_1\) and \(\varphi_2\) are holomorphic on \(\Omega\) which is determined by (4.8), and they are automorphic functions with respect to the transformation group generated by \(\delta_1^*, \ldots, \delta_8^*\).

**REFERENCES**


