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Pseudodifferential Operators and Hardy Kernels on $L^p(\mathbf{R}^+)$ (*).

JEFF E. LEWIS - CESARE PARENTI

Introduction.

We develop an algebra of pseudodifferential operators on $L^p(\mathbf{R}^+)$, 1 , which includes <math>H, the Hilbert transform restricted to \mathbf{R}^+ , and some classical Hardy kernels. Such operators arise in the study of singular integral equations in $L^p(\mathbf{R}^+)$ since $H^2 = -I + K$, where K is a Hardy kernel operator. The algebra of operators described here, called Mellin operators in $OP\Sigma_{1/p}$, are defined via the Mellin transform. As in the case of the Hilbert transform [Sh 1] or a Hardy kernel [FJL 1], the spectrum of the operator depends upon the L^p space on which it acts. As described in the remarks of Section 5, there are operators in $OP\Sigma_{1/p}$ for all p, $1 , which admit a parametrix in <math>OP\Sigma_{1/p}$ for some values of p, but do not have a parametrix in $OP\Sigma_{1/p}$ for other values of p; the parametrices in $OP\Sigma_{1/p}$, for different values of p, do not necessarily agree on $C_0^\infty(\mathbf{R}^+)$.

E. Shamir [Sh 1, Sh 2] studied the spectrum of the Hilbert transform on $L^p(\mathbf{R}^+)$. G. I. Eskin [E 1, E 2] has made an extensive study of operators defined via the Mellin transform and given applications to weighted L^2 spaces and boundary value problems. In [N] J. Nourrigat has defined a class of pseudodifferential operators on \mathbf{R}^+ defined by the Mellin transform and studied their properties on weighted L^2 spaces. B. A. Plamenevskii in [P] has studied an algebra of pseudodifferential operators in $\mathbf{R}^+ \times S^{n-1}$ defined using the Mellin transform. H. O. Cordes and E. A. Herman [CH] studied singular integrals on $L^2(\mathbf{R}^+)$.

In Section 1 we state the properties of the Mellin transform and Mellin multipliers to be used in the sequel. A representation for variable symbol Mellin operators is studied in Section 2. The space of symbols, $\Sigma_{1/p}$, and

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the space of Mellin operators, $OP\Sigma_{1/p}$, are defined in Section 3; the principal symbol σ_p is defined. The symbolic calculus is developed in Section 4. The operators in $OP\Sigma_{1/p}$ which admit a parametrix in $OP\Sigma_{1/p}$ are characterized by their symbols in Theorem 5; the remarks following Theorem 5 describe a typical situation. The index of an elliptic operator in $OP\Sigma_{1/p}$ is studied in Section 6. An application to an oblique derivative problem for Laplace's equation in a plane sector is given in Section 7.

1. - Preliminaries on the Mellin transform.

We shall deal with functions in $L^p = L^p(\mathbf{R}^+)$, 1 , with the norm

$$||f||_p = \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}.$$

It will be convenient to consider functions $g(x) \in L^p_*(\mathbf{R}^+)$ where

$$||g||_{p,*} = \left(\int_{0}^{\infty} |g(x)|^{p} \frac{dx}{x}\right)^{1/p}.$$

If $f(x) \in L^p$, we define $f_p(x) = x^{1/p} f(x) \in L^p_*(\mathbf{R}^+)$ and the functions $F(u) = f(\exp[-u])$, and $F_p(u) = f_p(\exp[-u])$, $u \in \mathbf{R}$. Note that

$$||f||_p = ||f_p||_{p,*} = ||F_p||_{L^p(\mathbf{R})} = \left(\int_{-\infty}^{+\infty} \exp\left[-u\right] |F(u)|^p du\right)^{1/p}.$$

If $f(x) \in C_0^{\infty}(\mathbf{R}^+)$ we define the Fourier transform of F_p as the function

(1.1)
$$\hat{F}_{p}(\xi) = \int_{-\infty}^{+\infty} \exp\left[-iu\xi\right] F_{p}(u) du = \int_{0}^{\infty} x^{1/p + i\xi - 1} f(x) dx , \quad \xi \in \mathbf{R} .$$

The Mellin transform of a function $f \in C_0^{\infty}(\mathbf{R}^+)$ is defined as

(1.2)
$$\tilde{f}(z) = \int_{0}^{\infty} x^{z-1} f(x) dx, \quad z \in \mathbf{C}.$$

It follows that for $f \in C_0^{\infty}(\mathbf{R}^+)$, $\tilde{f}(z)$ is an entire function and we have the inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \tilde{f}(z) dz,$$

where the notation $1/2\pi i \int_{a-i\infty}^{a+i\infty} ... dz$ denotes contour integration along the path $z = a + i\xi$, $-\infty < \xi < \infty$. Integration by parts shows that

$$\left(-x\frac{d}{dx}f\right)^{\sim}(z)=z\tilde{f}(z) \quad \text{ and } \quad \left((\log x)f\right)^{\sim}(z)=\frac{d}{dz}\tilde{f}(z).$$

For τ real, $\delta > 0$, we use the notation

$$S_{\tau,\delta} = \{z \in \mathbf{C} : \tau - \delta < \operatorname{Re} z < \tau + \delta\}.$$

If f is measurable on \mathbb{R}^+ and the integral (1.2) is absolutely convergent for all z in some strip $S_{\tau,\delta}$ we shall call the integral $\tilde{f}(z)$ the *Mellin transform* of f; under these conditions $\tilde{f}(z)$ is a holomorphic function in $S_{\tau,\delta}$. We make the following definition.

DEFINITION 1. Let b(z) be a bounded measurable function on the line Re z = 1/p. Then we say b is a Mellin multiplier on L^p iff the map

$$Bf(x) = rac{1}{2\pi i} \int_{1/y-i\infty}^{1/p+i\infty} x^{-z} b(z) \tilde{f}(z) dt, \ x > 0, \quad f \in C_0^{\infty}(\mathbf{R}^+),$$

is extendable as a bounded linear operator on L^p .

By (1.1), $\hat{F}_{p}(\xi) = \tilde{f}(1/p + i\xi)$, so that b is a Mellin multipler on L^{p} iff the function $\xi \to b(1/p + i\xi)$ is a Fourier multiplier on $L^{p}(\mathbf{R})$ [H].

We give the following examples which will be essential ingredients in the algebra of operators to be constructed in Section 3.

1) The Hilbert transform on $L^p(\mathbf{R}^+)$. The Hilbert transform of a function $f \in L^p$ is defined as

(1.3)
$$Hf(x) = p.v. \frac{1}{\pi} \int_{0}^{\infty} \frac{f(y)}{x - y} dy.$$

Following Shamir [Sh 1] and Eskin [E 1] we can represent H as

(1.4)
$$Hf(x) = \frac{1}{2\pi i} \int_{1/\pi - i\infty}^{1/\pi + i\infty} x^{-z} \left\{ i \frac{1 + \exp\left[2\pi i z\right]}{1 - \exp\left[2\pi i z\right]} \right\} \tilde{f}(z) dz , \quad f \in C_0^{\infty}(\mathbf{R}^+) .$$

It is well known that H is a bounded operator on L^p . Define the function

$$\theta(z) = \frac{1}{1 - \exp\left[2\pi i z\right]} = \frac{1}{2} \left(1 + \frac{1 + \exp\left[2\pi i z\right]}{1 - \exp\left[2\pi i z\right]}\right).$$

Then $\theta(z)$ is a Mellin multiplier on L^p , 1 .

2) Hardy kernels on L^p . Let k(x) be a measurable function on \mathbb{R}^+ such that for some a, b with 0 < a < b < 1,

$$\int_{0}^{\infty} x^{a-1} |k(x)| dx + \int_{0}^{\infty} x^{b-1} |k(x)| dx < \infty.$$

Then for all p, a < 1/p < b, the Hardy operator with kernel k is defined on L^p by

(1.5)
$$Kf(x) = \int_{0}^{\infty} k \left(\frac{x}{y}\right) f(y) \frac{dy}{y}.$$

Following [FJL 1], for $f \in C_0^{\infty}(\mathbb{R}^+)$,

(1.6)
$$Kf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \tilde{k}(z) \tilde{f}(z) dz,$$

where $\tilde{k}(z)$ is the Mellin transform of the kernel k which is defined and holomorphic for a < Re z < b.

3) The operator $T_{\zeta,p}$ (a particular Hardy kernel). Let $1 and <math>\zeta \in C$, Re $\zeta \neq 1/p$. For $f \in C_0^{\infty}(\mathbf{R}^+)$, define

(1.7)
$$T_{\zeta,p}f(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \frac{1}{\zeta - z} \tilde{f}(z) dz.$$

The function $b(z) = 1/(\zeta - z)$ is a Mellin multiplier on L^p , $1/p \neq \text{Re } \zeta$,

and the L^p norm of the operator $T_{\zeta,p}$ is bounded by $C|\text{Re }\zeta-1/p|^{-1}$ (C is independent of p). If $1/p < \text{Re }\zeta$, the kernel for $T_{\zeta,p}$ on L^p is given by

$$k_{\zeta,p}(x) = 0$$
, $0 < x < 1$, $k_{\zeta,p}(x) = x^{-\zeta}$, $x > 1$.

If Re $\zeta < 1/p$, the kernel for $T_{\zeta,p}$ on L^p is given by

$$k_{\zeta,p}(x) = -x^{-\zeta}, \quad 0 < x < 1, \quad k_{\zeta,p}(x) = 0, \quad x > 1.$$

The bound for the L^p norm of $T_{\zeta,p}$ is a consequence of Young's inequality.

2. - A class of bounded operators on $L^p(\mathbf{R}^+)$.

We now introduce a class of Mellin integral operators on $L^p(\mathbf{R}^+)$ with variable kernels.

THEOREM 1. Let a(x,z) be a function defined for x>0 and z in some strip $S_{1/p,\delta}$, $1< p<\infty$. Suppose that for all x, a(x,z) is holomorphic in $S_{1/p,\delta}$ and that there is an $\varepsilon>0$ and a constant C such that

$$\sup_{x>0} |a(x,z)| \leqslant C(1+|z|)^{-1-\epsilon}, \quad z \in S_{1/p,\delta}.$$

Then the operator defined by

(2.6)
$$Af(x) = \frac{1}{2\pi i} \int_{1/a - i\infty}^{1/a + i\infty} x^{-z} a(x, z) \tilde{f}(z) dz, \qquad f \in C_0^{\infty}(\mathbf{R}^+),$$

is extendable as a bounded operator on L^p .

Proof. Let $0 < \delta_1 < \delta$ and let $\Gamma_{1/p,\delta_1}$ denote the contour

$$\left\{z = \frac{1}{p} + \delta_1 + i\xi, \ -\infty < \xi < \infty\right\} \cup \left\{z = \frac{1}{p} - \delta_1 - i\xi, \ -\infty < \xi < \infty\right\}.$$

If Re z = 1/p, by the Cauchy integral formula

$$a(x,z) = \frac{1}{2\pi i} \int_{\Gamma_{1/p,\sigma}} \frac{a(x,\zeta)}{\zeta - z} d\zeta.$$

Using this representation for a(x, z) in (2.6) and applying Fubini's Theorem, we obtain

$$Af(x) = \frac{1}{2\pi i} \int\limits_{\Gamma_{1/p,\mathfrak{o}_{1}}} a(x,\,\zeta) \, \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-z} \, \frac{1}{\zeta-z} \, \tilde{f}(z) \, dz = \frac{1}{2\pi i} \int\limits_{\Gamma_{1/p,\mathfrak{o}_{1}}} a(x,\,\zeta) \, T_{\zeta,p} \, f(x) \, d\zeta \; .$$

An application of Minkowski's Integral Inequality gives

$$||Af||_{p} \leqslant \frac{1}{2\pi} \int_{\substack{x>0 \\ \Gamma_{1/p,o_{1}}}} \sup_{x>0} |a(x,\zeta)| ||T_{\zeta,p}f||_{p} d|\zeta| \leqslant C_{1} \delta_{1}^{-1} ||f||_{p}. \quad q.e.d.$$

THEOREM 2. Let a(x, z) be a function defined for x > 0 and for z in some strip $S_{1/p,\delta}$. Suppose that

- 1) a(x, z) is continuously differentiable in $\mathbb{R}^+ \times S_{1/p, \delta}$ and holomorphic in z,
- 2) For some $\varepsilon > 0$ there is a constant C such that for all x and z

$$|a(x,z)| \leqslant C \left(\frac{x}{1+x^2}\right)^{\epsilon} \left(\frac{1}{(1+|z|)^{2+\epsilon}}\right),$$

(2.8)
$$\left| x \frac{\partial}{\partial x} a(x, z) \right| \leq \frac{C}{(1 + |z|)^{1+\varepsilon}}.$$

Then the operator A defined by (2.6) is compact on L^p .

PROOF. By the proof of Theorem 1 and (2.7) it follows that the operators $f \to \chi_{(0,\lambda)}(x) \, Af(x)$ and $f \to \chi_{(\lambda^{-1},\infty)}(x) \, Af(x)$ have small L^p norm if λ is small. The map $f \to -x(d/dx) \, Af(x) = Tf(x)$ is represented by

$$Tf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \left\{ a(x,z)z - x \frac{\partial}{\partial x} a(x,z) \right\} \tilde{f}(z) dz.$$

By (2.8) and Theorem 1, T is bounded on L^p . From these observations it follows that the family $\{Af: ||f||_p \le 1\}$ is equicontinuous in $L^p(0, N)$ for every N and that

$$\lim_{N\to\infty} \sup_{\|f\|_p\leqslant 1} \int_{|x|>N} |Af(x)|^p dx = 0.$$

This establishes the compactness of A on L^p . q.e.d.

3. - Spaces of symbols and Mellin operators.

As a preliminary step we introduce some spaces of functions and their Mellin transforms.

DEFINITION 2. Let τ be real. If $\delta > 0$, by $\mathcal{F}_{\tau,\delta}$ we denote the class of functions $a(x) \in C^{\infty}(\mathbf{R}^+)$ such that the following property holds: for every δ_1 , $0 < \delta_1 < \delta$, and every j there is a constant $C = C(\delta_1, j, a)$ such that

$$\left| \left(x \frac{d}{dx} \right)^j a(x) \right| \leqslant C x^{-\tau} \left(\frac{x}{1+x^2} \right)^{\delta_1}.$$

By \mathcal{F}_{τ} we denote the space of functions a(x) such that $a \in \mathcal{F}_{\tau,\delta}$ for some δ .

DEFINITION 3. Let τ be real. If $\delta > 0$, by $\widetilde{\mathcal{F}}_{\tau,\delta}$ we denote the class of functions b(z) which are defined and holomorphic in the strip $S_{\tau,\delta}$ and such that the following property holds: for every δ_1 , $0 < \delta_1 < \delta$, and every j and k there is a constant $C = C(\delta_1, j, k, b)$ such that

$$\left| z^{j} \frac{d^{k}}{dz^{k}} b(z)
ight| \leqslant C$$

for all $z \in S_{\tau,\delta_1}$. By $\widetilde{\mathcal{F}}_{\tau}$ we denote the space of functions b(z) such that $b \in \widetilde{\mathcal{F}}_{\tau,\delta}$ for some δ .

The fact that the functions in $\widetilde{\mathcal{F}}_{\tau}$ are precisely the Mellin transforms of the functions in \mathcal{F}_{τ} is consequence of the following result whose proof is contained in the article of A. Avantaggiati [A, Sec. 2].

LEMMA 1. If $a \in \mathcal{F}_{\tau,\delta}$, then its Mellin transform $\tilde{a} \in \widetilde{\mathcal{F}}_{\tau,\delta}$. Conversely, given $b \in \widetilde{\mathcal{F}}_{\tau,\delta}$, define the function

$$a(x) = \frac{1}{2\pi i} \int_{z-i\infty}^{z+i\infty} x^{-z} b(z) dz, \qquad x > 0.$$

Then $a \in \mathcal{F}_{\tau,\delta}$ and $\tilde{a} = b$.

We define the symbols of the class of smoothing operators to be constructed.

DEFINITION 4. Let τ be real. If ε , $\delta > 0$, by $\Phi_{\tau,\delta,\varepsilon}$ we denote the class of functions such that:

1)
$$a(x,z) \in C^{\infty}(\mathbf{R}^+ \times S_{\tau,\delta}),$$

- 2) for all x, a(x, z) defines a holomorphic function on $S_{\tau, \delta}$,
- 3) for each ε_1 , $0 < \varepsilon_1 < \varepsilon$, and each δ_1 , $0 < \delta_1 < \delta$, and each M, j, k there is a constant $C = C(\varepsilon_1, \delta_1, M, j, k, a)$ such that

for $z \in S_{\tau,\delta}$.

By Φ_x we denote the class of functions a(x,z) which belong to $\Phi_{x,\varepsilon,\delta}$ for some ε,δ .

We recall that the function $\theta(z) = 1/(1 - \exp[2\pi i z])$ is holomorphic in the strip $0 < \operatorname{Re} z < 1$ and that for all M and j, uniformly in the strip $0 < \delta < \operatorname{Re} z < 1 - \delta < 1$,

(3.1)
$$\lim_{\mathbf{Im}\,z\to+\infty} \left| z^M \frac{d^j}{dz^j} (\theta(z) - 1) \right| = 0$$

and

(3.2)
$$\lim_{\mathrm{Im}\,z\to-\infty}\left|z^M\frac{d^j}{dz^j}\theta(z)\right|=0.$$

It follows that $\theta(z)(1-\theta(z)) \in \widetilde{\mathcal{F}}_{1/p}$ for 1 .

Finally we are ready for the definition of the space of symbols of an algebra of Mellin operators on $L^p(\mathbf{R}^+)$.

DEFINITION 5. Let $1 . Denote by <math>\Sigma_{1/p}$ the space of functions $a(x, z) \in C^{\infty}(\mathbf{R}^+ \times S_{1/p, \delta})$ for some $\delta = \delta(a) > 0$ and for which there is a representation of the following form in $\mathbf{R}^+ \times S_{1/p, \delta}$:

(3.3)
$$a(x,z) = a_{+}(x)\theta(z) + a(x)(1-\theta(z)) + a(z) + \alpha(x,z)$$

where

- 1) $a_{+}(x)$ and $a_{-}(x)$ are extendable as continuous functions on $\overline{R^{+}}$ in such a way that $a_{+}(x) a_{+}(0) \in \mathcal{F}_{0}$,
- $2) \ a(z) \in \widetilde{\mathcal{F}}_{1/p},$
- 3) $\alpha(x,z)\in \Phi_{1/p}$.

Definition 6. For each symbol $a \in \Sigma_{1/p}$ we define the Mellin operator

(3.10)
$$Af(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} a(x,z) \tilde{f}(z) dz , \quad f \in C_0^{\infty}(\mathbf{R}^+) .$$

The space of all such operators will be denoted by $OP\Sigma_{1/p}$. The function $a(x, z) \in \Sigma_{1/p}$ will be called the symbol of the Mellin operator A. If the symbol of the operator A is also in the class $\Phi_{1/p}$, we shall write $A \in OP\Phi_{1/p}$ and shall call A a smoothing operator.

DEFINITION 7. If A is a Mellin operator with symbol

$$a(x, z) = a_{+}(x)\theta(z) + a_{-}(x)(1 - \theta(z)) + a(z) + \alpha(x, z),$$

the function $a(x, z) - \alpha(x, z)$ will be called the principal symbol of A and be denoted by $\sigma_v(A)(x, z)$.

From Theorem 1 it follows that if $A \in OP\Sigma_{1/p}$, then A can be extended as a bounded operator on L^p ; moreover, if $\sigma_p(A)(x,z) \equiv 0$, A is a compact operator on L^p .

4. – The symbolic calculus for $OP\Sigma_{1/p}$.

We study the compositions and adjoints of operators in $OP\Sigma_{1/p}$.

THEOREM 3. Let $A, B \in OP\Sigma_{1/p}$. Then $AB \in OP\Sigma_{1/p}$. Moreover, if

$$\sigma_{\mathbf{r}}(A)(x,z) = a_{+}(x)\theta(z) + a_{-}(x)(1-\theta(z)) + a(z)$$

and

$$\sigma_p(B)(x,z) = b_+(x)\theta(z) + b_-(x)(1-\theta(z)) + b(z),$$

then

$$\sigma_{p}(AB)(x,z) = c_{+}(x)\theta(z) + c_{-}(x)(1-\theta(z)) + c(z),$$

where

$$(4.1) 1) c_{+}(x) = a_{+}(x)b_{+}(x)$$

$$(4.2) 2) c_{-}(x) = a_{-}(x)b_{-}(x)$$

$$\begin{array}{lll} (4.3) & 3) & c(z) & = \sigma_{\mathfrak{p}}(A)(0,z) \cdot \sigma_{\mathfrak{p}}(B)(0,z) - \\ & & - a_{+}(0)b_{+}(0)\theta(z) - a_{-}(0)b_{-}(0)\big(1 - \theta(z)\big) \,. \end{array}$$

PROOF. We shall first show that the composition of two smoothing operators is a smoothing operator. Let a(x, z), $b(x, z) \in \Phi_{1/p}$ and let A and B

be the corresponding Mellin operators. For $f \in C_0^{\infty}(\mathbf{R}^+)$,

$$\begin{split} Bf(z) &= \int\limits_0^\infty x^{z-1} \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-w} b(x,w) \hat{f}(w) \, dw \, dx \\ &= \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} \hat{b}(z-w,w) \hat{f}(w) \, dw \, , \end{split}$$

where $\tilde{b}(z, w)$ denotes the Mellin transform of b(x, w) in the x-variable. Thus

$$\begin{split} ABf(x) &= \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-z} a(x,z) \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} \tilde{b}(z-w,w) \tilde{f}(w) dw dz \\ &= \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-w} e(x,w) \tilde{f}(w) dw \,, \end{split}$$

where

$$egin{align} c(x,w) &= rac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-(z-w)} a(x,z) ilde{b}(z-w,w) dz \ &= rac{1}{2\pi i} \int\limits_{0-i\infty}^{0+i\infty} x^{-v} a(x,v+w) ilde{b}(v,w) dv \,. \end{split}$$

In the above calculations, the absolute convergence of the integrals justifies the use of Fubini's Theorem.

To verify that $c(x, w) \in \Phi_{1/p}$ we apply repeatedly the observation that if $a(x, z) \in \Phi_{1/p}$, then

$$z^{M}(\log x)^{N}\bigg(x\,\frac{\partial}{\partial x}\bigg)^{\!k}\bigg(\frac{\partial}{\partial z}\bigg)^{\!j}\,a(x,\,z)\in\varPhi_{1/p}\,.$$

We next show that if $A \in OP\Sigma_{1/p}$, $B \in OP\Phi_{1/p}$, then $AB \in OP\Phi_{1/p}$. Let the symbol of A be

$$a(x,z) = a_{+}(x)\theta(z) + a_{-}(x)(1-\theta(z)) + a(z)$$
 and let $b(x,z) \in \Phi_{1/p}$.

In this case we argue as above and again use Fubini's Theorem to obtain that

$$ABf(x) = \frac{1}{2\pi i} \int_{1/n - i\infty}^{1/n + i\infty} x^{-w} c(x, w) \tilde{f}(w) dw ,$$

where

$$c(x,w)=rac{1}{2\pi i}\int\limits_{0-i\infty}^{0+i\infty}x^{-v}a(x,v+w) ilde{b}(v,w)dv.$$

Consider, e.g., the contribution of

$$rac{a_+(0)}{2\pi i}\int\limits_{0-i\infty}^{0+i\infty}x^{-v} heta(v+w) ilde{b}(v,w)dv\,.$$

Modulo $\Phi_{1/p}$, this integral is $c_{+}(x, w)$ where

$$(4.4) c_{+}(x, w) = \frac{a_{+}(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} [\theta(v+w) - \theta(w)] \tilde{b}(v, w) dv,$$

Now

$$c_+(x,w) = rac{a_+(0)}{2\pi i} \int\limits_{0-i\infty}^{0+i\infty} x^{-v} \int\limits_0^1 rac{\partial heta}{\partial w} \left(w \,+\, tv
ight) v ilde{b}(v,w) \, dt \, dv \,.$$

To show that $c_+(x, w) \in \Phi_{1/p}$, we observe that for some $\delta > 0$, it is possible to shift the integral $(1/2\pi i) \int_{0-i\infty}^{0+i\infty} ... dv$ to either of the integrals

$$\frac{1}{2\pi i} \int_{-\delta - i\infty}^{-\delta + i\infty} \dots \, dv \qquad \text{or} \qquad \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \dots \, dv \, .$$

Hence $x^{-\delta}w^N(x(\partial/\partial x))^j(\partial/\partial w)^kc_+(x,w)$ is a linear combination of integrals of the form

$$(4.5) \quad I(x,w) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta-i\infty} x^{-(v+\delta)} (\log x)^s \int_0^1 t^m (w+tv)^M \theta^{(1+q)}(w+tv) \tilde{b}_r(v,w) dt dv,$$

where $b_r(x,w) \in \Phi_{1/p}$. The integrals I(x,w) are absolutely convergent and for w in some strip around Re w=1/p are bounded by $C|\log x|^s$. Repeating the argument with δ replaced by $-\delta$ shows that $c_+(x,w) \in \Phi_{1/p}$. In the same manner one shows that

$$\frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \{a_{-}(x)(1-\theta(v+w)) + a(v+w)\} \tilde{b}(v,w) dv$$

gives a function in $\Phi_{1/p}$.

The next step is to show that if $A \in OP\Phi_{1/p}$ and $B \in OP\Sigma_{1/p}$ then $AB \in OP\Phi_{1/p}$. Let a(x,z) be the symbol of A and suppose that $b(x,z) = b_+(x)\theta(z) + b_-(x)(1-\theta(z)) + b(z)$ is the symbol of B. Denote by B_+ the operator

$$B_+f(x) = \frac{1}{2\pi i} \int_{1/n-i\infty}^{1/n+i\infty} x^{-z} b_+(x) \theta(z) \tilde{f}(z) dz.$$

We have that

$$AB_{+}f(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/p+i\infty} x^{-w}c(x, w)\tilde{f}(w) dw + C_{1}f(x)$$

where

$$c(x, w) = \frac{\theta(w)}{2\pi i} \int_{0}^{0+i\infty} x^{-v} a(x, v + w) (b_{+}(x) - b_{+}(0))^{\sim}(v) dv$$

and C_1 is a smoothing operator. Since $(b_+(x) - b_+(0))^{\sim}(v) \in \widetilde{\mathcal{F}}_0$, it may be shown that $c(x, w) \in \Phi_{1/p}$. The other terms in the composition can be handled similarly.

As the next step we consider two operators A_+ , B_+ with symbols $a_+(x)\theta(z)$ and $b_+(x)\theta(z)$. Let Θ be the operator with symbol $\theta(z)$. Then

$$A_{+}B_{+} = a_{+}(x)b_{+}(0)\Theta^{2} + a_{+}(x)(\Theta b'_{+}(x)\Theta)$$

where

$$b_{\,+}^{\,\prime}(x) = b_{+}(x) - b_{+}(0) \in \mathcal{F}_{\mathbf{0}} \; .$$

The second part can be written as

$$Cf(x) = \frac{1}{2\pi i} \int_{1/n - i\infty}^{1/n + i\infty} x^{-w} c(x, w) \tilde{f}(w) dw,$$

where

$$c(x,w)=a_+(x)\theta(w)\,rac{1}{2\pi i}\int\limits_{0-i\infty}^{0+i\infty}x^{-v}\theta(v+w) ilde{b}'_+(v)dv\,.$$

Using the argument for the integral (4.4), we have that modulo a symbol in $\Phi_{1/p}$, $c(x, w) = a_{+}(x)b'_{+}(x)\theta^{2}(w)$.

At this point we observe that $\theta^2 = \theta(\theta - 1) + \theta$ so that the operator θ^2 contains the Hardy kernel operator with symbol $\theta(z)(\theta(z) - 1) \in \widetilde{\mathcal{F}}_{1/p}$. Thus

the principal symbol of A_+B_+ is given by

$$\sigma_{p}(A_{+}B_{+})(x,z) = a_{+}(x)b_{+}(x)\theta(z) + a_{+}(0)b_{+}(0)\theta(z)(\theta(z)-1)$$
.

Since the composition of two Hardy kernel operators with symbols in $\widetilde{\mathcal{F}}_{1/p}$ is a Hardy kernel operator with symbol in $\widetilde{\mathcal{F}}_{1/p}$, we leave to the reader the proof of the cases not considered explicitly and the calculation of the principal symbol. q.e.d.

COROLLARY. If $A, B \in OP\Sigma_{1/p}$ the commutator

$$[A, B] = AB - BA \in OP\Phi_{1/p}$$
.

We consider the adjoint of an operator $A \in OP\Sigma_{1/p}$. If 1/p + 1/q = 1 we define $A^*: L^q \to L^q$ to be the operator such that

$$\int\limits_0^\infty Af(x)\overline{g(x)}\,dx=\int\limits_0^\infty f(x)\overline{A^*g(x)}\,dx\,,\qquad f,\,g\in C_0^\infty(\pmb{R}^+)\,.$$

THEOREM 4. Let $A \in OP\Sigma_{1/p}$ and 1/p + 1/q = 1. Then $A^* \in OP\Sigma_{1/q}$; moreover, the principal symbol of A^* is

(4.6)
$$\sigma_{\mathfrak{g}}(A^*)(x,z) = \overline{\sigma_{\mathfrak{p}}(A)(x,1-\overline{z})}.$$

In particular if

(4.7)
$$\sigma_{n}(A)(x,z) = a_{+}(x)\theta(z) + a_{-}(x)(1-\theta(z)) + a(z),$$

then

(4.8)
$$\sigma_{\sigma}(A^*)(x,z) = \bar{a}_{+}(x)\theta(z) + \bar{a}_{-}(x)(1-\theta(z)) + \bar{a}(1-\bar{z}),$$

Re z near 1/q.

PROOF. We recall that the Hilbert transform H is representable as $H = i(2\Theta - 1) \in OP\Sigma_{1/p}$. Using the kernel representation (1.3) of H, we have that $H^* = -H \in OP\Sigma_{1/p}$. A calculation shows that

$$\sigma_q(H^*)(z) = \overline{\sigma_p(H)(1-\bar{z})}$$
.

Next consider the operator Af(x) = a(x)f(x) where $a(x) - a(0) \in \mathcal{F}_0$. Then $A*g(x) = \bar{a}(x)g(x)$. Representing Θ in terms of H and applying Theorem 3

proves the theorem for operators of the form

$$A = a_{+}(x)\Theta + a_{-}(x)(1-\Theta).$$

We now consider an operator A which is a Hardy operator with symbol $a(z) \in \widetilde{\mathcal{F}}_{1/p}$. There is a kernel $k(x) \in \mathcal{F}_{1/p}$ with $\tilde{k} = a$ such that $Af(x) = \int_0^\infty k(x/y) f(y) (dy/y)$. The adjoint A^* is representable with a kernel $k^*(x) = (1/x) \bar{k}(1/x)$. For Re w near 1/q we have that $\tilde{k}^*(w) = \overline{a(1-\overline{w})} \in \widetilde{\mathcal{F}}_{1/q}$. Finally we show that if $A \in OP\Phi_{1/p}$ then $A^* \in OP\Phi_{1/p}$. If the symbol of A is $a(x,z) \in \Phi_{1/p}$ and $f,g \in C_0^\infty(\mathbf{R}^+)$, represent

$$f(y) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} y^{-z} \tilde{f}(z) dz \quad \text{and} \quad g(y) = \frac{1}{2\pi i} \int_{1/q - i\infty}^{1/q + i\infty} y^{-w} \tilde{g}(w) dw.$$

Using Fubini's Theorem, we obtain that

$$\int\limits_0^\infty Af(y)\overline{g(y)}\,dy = \int\limits_0^\infty f(x) \overline{\frac{1}{2\pi i} \int\limits_{1/q-i\infty}^{1/q+i\infty} x^{-w} c(x,w) \widetilde{g}(w)\,dw}\,dx\,,$$

where

$$\overline{c(x,w)} = rac{1}{2\pi i} \int\limits_{1/y-i\infty}^{1/y+i\infty} x^{\overline{w}+z-1} \int\limits_{0}^{\infty} y^{-(z+\overline{w})} a(y,z) \, dy \, dz \; .$$

Performing the change of variables $z \to 1 - \overline{w} + v$, Re v = 0, we have that

$$\overline{c(x,w)} = rac{1}{2\pi i} \int\limits_{0-i\infty}^{0+i\infty} x^{-v} \widetilde{a}(v,(1-\overline{w})-v) \, dv \, .$$

Using arguments similar to those for (4.4) and (4.5), we can show that $c(x, w) \in \Phi_{1/q}$. q.e.d.

REMARK. If $A \in OP\Sigma_{1/p}$ the transposed operator tA is defined so that

$$\int\limits_0^\infty Af(x)\,g(x)\,dx = \int\limits_0^\infty f(x)\,{}^t\!Ag(x)\,dx,\,f,\,g \in C_0^\infty(\pmb{R}^+)\;.$$

Then if 1/p + 1/q = 1, ${}^{t}A \in OP\Sigma_{1/q}$. If $\sigma_{p}(A)$ is given by (4.6) then

$$\sigma_{q}({}^{t}A)(x,z) = a_{+}(x)(1-\theta(z)) + a_{-}(x)\theta(z) + a(1-z),$$

Re z near 1/q.

Remark. We observe that smoothing operators map L^p into $\mathcal{F}_{1/p}$.

Lemma 2. Let $A \in OP\Phi_{1/p}$. Then if $f \in L^p$, $Af \in \mathcal{F}_{1/p}$.

PROOF. Let the symbol of A be $a(x,z) \in \Phi_{1/p}$. Define the function k(x,t) by

$$k(x,t) = \frac{1}{2\pi i} \int_{1/\eta - i\infty}^{1/p + i\infty} t^{-z} a(x,z) dz.$$

Then for some $\delta > 0$ and each i, j there is a $C = C(\delta, i, j, k)$ such that

$$(4.9) \left| \left(x \frac{\partial}{\partial x} \right)^i \left(t \frac{\partial}{\partial t} \right)^j k(x, t) \right| \leqslant C t^{-1/p} \left(\frac{x}{1 + x^2} \right)^{\delta} \left(\frac{t}{1 + t^2} \right)^{\delta}.$$

Fix $\xi > 0$ and for $f \in C_0^{\infty}(\mathbf{R}^+)$ let

$$A_{\xi}f(x) = \int_{0}^{\infty} k\left(\xi, \frac{x}{y}\right) f(y) \frac{dy}{y}.$$

The Mellin transform of $A_{\xi}f$ is $a(\xi,z)\tilde{f}(z)\in\widetilde{\mathcal{F}}_{1/p}.$ Hence

$$A_{\xi}f(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} a(\xi, z) \tilde{f}(z) dz.$$

Putting $\xi = x$ we then have the representation

$$Af(x) = \int_0^\infty k\left(x, \frac{x}{y}\right) f(y) \frac{dy}{y}.$$

$$= \int_0^\infty y^{-1/p} k\left(x, \frac{x}{y}\right) [y^{1/p} f(y)] \frac{dy}{y}.$$

It follows that $x^{1/p}(x(d/dx))^j Af(x)$ is a linear combination of integrals of the form

$$I(x) = \int_{0}^{\infty} \left(\frac{x}{y}\right)^{1/p} k_r\left(x, \frac{x}{y}\right) \left[y^{1/p} f(y)\right] \frac{dy}{y},$$

where $k_*(x,t)$ satisfies estimates of the form (4.9). By Hölder's inequality,

$$|I(x)| \leqslant C \left(\frac{x}{1+x^2}\right)^{\delta} \left(\int\limits_0^{\infty} \left(\frac{t}{1+t^2}\right)^{\delta_a} \frac{dt}{t}\right)^{1/a} ||f||_p$$

where 1/p + 1/q = 1. q.e.d.

5. – Elliptic operators in $OP\Sigma_{1/p}$.

We characterize the operators $A \in OP\Sigma_{1/p}$ which are «elliptic», *i.e.*, for which there exists a parametrix $B \in OP\Sigma_{1/p}$ such that AB-I and BA-I are smoothing operators.

THEOREM 5. Let $A \in OP\Sigma_{1/p}$ with principal symbol

$$\sigma_n(A)(x,z) = a_{\perp}(x)\theta(z) + a_{-}(x)(1-\theta(z)) + a(z)$$
.

The following two conditions are equivalent:

- 1) There is an operator $B \in OP\Sigma_{1/p}$ such that $AB I \in OP\Phi_{1/p}$.
- 2) The following three conditions are satisfied by $\sigma_{v}(A)(x,z)$:

$$\left\{ \begin{array}{l} \inf\limits_{\xi\in R} \left| \,\sigma_p(A) \left(0,\frac{1}{p} \,+\, i\xi\right) \right| > 0 \;, \\ \inf\limits_{x>0} \left| a_+(x) \right| > 0 \;, \\ \inf\limits_{x>0} \left| a_-(x) \right| > 0 \;. \end{array} \right.$$

PROOF. Suppose that 1. is satisfied and let

$$\sigma_n(B)(x,z) = b_{\perp}(x)\theta(z) + b_{-}(x)(1-\theta(z)) + b(z)$$
.

By Theorem 3

$$\sigma_n(AB)(x,z) = 1 = 1 \cdot \theta(z) + 1(1 - \theta(z))$$
.

By (4.1) and (4.2) and the observation that

$$\sigma_{n}(AB)(0,z) = \sigma_{n}(A)(0,z) \cdot \sigma_{n}(B)(0,z),$$

we have the identities

$$egin{aligned} 1 &= a_+(x) b_+(x) \,, \ &1 &= a_-(x) b_-(x) \,, \ &1 &= \sigma_n(A)(0,z) \sigma_n(B)(0,z) \,. \end{aligned}$$

Condition 2 follows.

Conversely, suppose that 2. is satisfied. Note that for some $\delta > 0$,

$$\inf_{z\in S_{1/p,\,\delta}} \left|\sigma_p(A)(0,z)\right|>0$$
 .

We define an operator B with symbol

(5.2)
$$b(x,z) = \frac{1}{a_{+}(x)}\theta(z) + \frac{1}{a_{-}(x)}(1-\theta(z)) + b(z),$$

where

$$(5.3) b(z) = \frac{1}{\sigma_p(A)(0,z)} - \frac{1}{a_+(0)} \theta(z) - \frac{1}{a_-(0)} (1 - \theta(z)),$$

 $z \in S_{1/p,\delta}$. It may be shown, using the properties of $a_+(x)$, $a_-(x)$ and (3.1), (3.2), that $b(x,z) \in \Sigma_{1/p}$. A direct calculation using (4.1), (4.2), and (4.3) shows that B is a parametrix for A. q.e.d.

DEFINITION 8. If $A \in OP\Sigma_{1/p}$ and A satisfies condition 1. or 2. of Theorem 5, we shall say that A is an elliptic operator in $OP\Sigma_{1/p}$.

REMARK. We emphasize that the definition of ellipticity in $OP\Sigma_{1/p}$ depends on p. The following situation is typical.

Consider an operator A with principal symbol

$$\sigma_n(A)(x,z) = a(x,z) = a_{\perp}(x)\theta(z) + a_{-}(x)(1-\theta(z)) + a(z)$$
.

Suppose that $\inf |a_+(x)| > 0$, $\inf |a_-(x)| > 0$, and that $a(z) \in \widetilde{\mathcal{F}}_{1/p}$ for all p, $1 . Then the function <math>\psi(z) = (a(0,z))^{-1}$ is meromorphic in the strip $0 < \operatorname{Re} z < 1$; moreover, in any strip $0 < \delta < \operatorname{Re} z < 1 - \delta < 1$, $\psi(z)$ has only a finite number of poles. Hence for all p outside a discrete set, N(A), A is elliptic in $OP\Sigma_{1/p}$. Define b(x,z) by (5.2) where b(z) is defined

by (5.3). Then b(z) has the same poles and residues as $\psi(z)$. Then if $p \notin N(A)$, b(x,z) is the principal symbol of a parametrix for A in $OP\Sigma_{1/p_i}$. If p_1 and $p_2 \notin N(A)$, let $B_{p_i} \in OP\Sigma_{1/p_i}$ be parametrices for A in $OP\Sigma_{1/p_i}$, i=1,2. The calculation in [FJL 1] shows that for $f \in C_0^{\infty}(\mathbb{R}^+)$,

$$B_{p_1}f(x) - B_{p_2}f(x) = \int_0^\infty k\left(\frac{x}{y}\right)f(y)\,\frac{dy}{y}\,,$$

where the kernel k is given by

$$k(x) = \frac{1}{2\pi i} \int_{1/p_1 - i\infty}^{1/p_1 + i\infty} x^{-z} b(z) dz - \frac{1}{2\pi i} \int_{1/p_2 - i\infty}^{1/p_2 + i\infty} x^{-z} b(z) dz.$$

If $a(0, \zeta) = 0$ for some ζ , Re ζ between $1/p_1$ and $1/p_2$, then $k(x) \neq 0$ (see [FJL 1]).

REMARK. If A is elliptic in $OP\Sigma_{1/p}$ and Af = 0, $f \in L^p$, then $f \in \mathcal{F}_{1/p}$. This follows from Lemma 2.

6. – The index of an elliptic operator in $OP\Sigma_{1/p}$.

We will relate the index of an elliptic operator in $OP\Sigma_{1/p}$ to the winding numbers of the coefficients $a_{\pm}(x)$.

LEMMA 3. For v an integer define

$$\varphi_{\nu}(x) = \exp\left([2\pi i \nu] \frac{x}{x+1}\right).$$

Then

- 1) $\varphi_r \colon \overline{\mathbf{R}^+} \to S^1 = \{|z| = 1\}$ and the winding number of φ_r is r.
- 2) $\varphi_{\nu}(x) 1 \in \mathcal{F}_{0}$.
- 3) If a(x) is a function mapping $\overline{\mathbf{R}_+} \to \mathbf{C} \setminus \{0\}$ such that $a(x) a(0) \in \mathcal{F}_0$, and the winding number of a is v, then there is a continuous homotopy

$$F: [0, 1] \times \overline{R^+} \rightarrow C \setminus \{0\}$$
 such that

(i)
$$F(0, x) = a(0)\varphi_{\nu}(x)$$
,

(ii)
$$F(1, x) = a(x)$$
,

(iii)
$$F(t,0) = a(0), 0 \le t \le 1,$$

(iv)
$$F(t, x) - a(0) \in \mathcal{F}_0, \ 0 \le t \le 1.$$

PROOF. Parts 1 and 2 follow by a calculation. To prove 3, we remark that if we replace a(x) by $a(x)\varphi_{-\nu}(x)$ it is sufficient to construct the homotopy when $\nu=0$. In this case it is well known that there is a homotopy G(t,x) which satisfies (i)-(iii) and such that $G(t,\cdot) \in C^{\infty}(\mathbb{R}^+)$ for every t. Let $\varepsilon=\frac{1}{2}|a(0)|>0$ and choose $\delta>0$ such that

$$|G(t, x) - a(0)| < \varepsilon$$
 for $0 \le t \le 1$, $0 < x < 4\delta$ or $(4\delta)^{-1} < x < \infty$.

Construct a nonnegative partition of unity on $\overline{R^+}$ as $1 = \alpha_1(x) + \alpha_2(x) + \alpha_3(x)$ where $\alpha_1(x) = 1$ if $0 \le x \le \delta$, $\alpha_1(x) = 0$ if $x > 2\delta$, $\alpha_3(x) = 1$ for $x > \delta^{-1}$, $\alpha_3(x) = 0$ for $x < \frac{1}{2}\delta^{-1}$ and $\alpha_2(x) = 1 - \alpha_1(x) - \alpha_3(x)$. Define the homotopy F as

(6.1)
$$F(t,x) = (\alpha_1(x) + \alpha_3(x))(a(0) + t(a(x) - a(0))) + \alpha_2(x)G(t,x).$$

The verification of (i)-(iv) is left to the reader. q.e.d

As an application of the previous lemma we have the following theorem.

THEOREM 6. Let A be elliptic in $OP\Sigma_{1/p}$ and let

(6.2)
$$\sigma_n(A)(x,z) = a_{\perp}(x)\theta(z) + a_{\perp}(x)(1-\theta(z)) + a(z).$$

Suppose that v_+ and v_- are the winding numbers of a_+ and a_- . Then A as an operator on L^p has index $v = v_+ - v_-$.

The proof of Theorem 6 is accomplished by a sequence of lemmas.

LEMMA 4. With the notation of Theorem 6, A has the same index as the operator $A^{\#}$ with symbol

$$(6.3) a^{\#}(x,z) = a_{+}(0) \varphi_{\nu_{+}}(x) \theta(z) + a_{-}(0) \varphi_{\nu_{-}}(x) (1 - \theta(z)) + a(z) .$$

Proof. By Lemma 3 there are homotopies $F_{\pm}(t,x)$ which connect $a_{\pm}(x)$ to the functions $a_{\pm}(0)\varphi_{\nu_{\pm}}(x)$ in such a way that the operators A_t with symbols

$$a_t(x,z) = F_+(t,z)\theta(z) + F_-(t,x)(1-\theta(z)) + a(z)$$

are elliptic in $OP\Sigma_{1/p}$. Then $A_0=A$ and $A_1=A$ mod $OP\Phi_{1/p}$. Hence index $A^\#=\operatorname{index} A$ in L^p . q.e.d.

LEMMA 5. With the notation of Theorem 6 and Lemma 4, the operator A has the same index as the operator A, with symbol

$$(6.4) a_{\nu}(x,z) = \varphi_{\nu}(x)\theta(z) + (1-\theta(z)).$$

PROOF. Let A_0 be the operator with symbol $a_0^\#(z) \equiv a^\#(0,z)$ and B_0 be the operator with symbol $b_0(z) = (a_0^\#(z))^{-1}$. Then $B_0 A_0^\# = I = A_0^\# B_0$ so that $A^\#$ has the same index on L^p as the operator $B_0 A^\# = I + B_0 (A^\# - A_0^\#)$. Since $\sigma_p(A^\# - A_0^\#)(0,z) = 0$, Theorem 3 yields that

$$\sigma_p(B_0 A^{\#}) = \varphi_{r_+}(x)\theta(z) + \varphi_{r_-}(x)(1 - \theta(z)).$$

Then index $(A_r) = \text{index } (\varphi_{-r}, B_0 A) = \text{index } (A^{\#}).$ q.e.d.

PROOF OF THEOREM 6. It remains to calculate the index of A_r on L^p . Since $\sigma_p(A_r)(0,z) \equiv 1$, A_r is elliptic in $OP\Sigma_{1/r}$ for all r, $1 < r < \infty$. If $f \in L^p$, $A_r f = 0$, then $f \in \mathcal{F}_{1/r}$, $1 < r < \infty$. If $g \in L^q$, 1/p + 1/q = 1, $A_r^* g = 0$, then $g \in \mathcal{F}_{1/r}$, $1 < r < \infty$. Thus the index of A_r on L^p is the index of A_r on L^p . As an operator on L^p , L^p is in the algebra considered by Cordes and Herman [CH], and its symbol σA_r , as defined in [CH], has winding number r and hence index r. (The particular operator considered in [CH] was r and r are r and r and r and r and r are r and r and r and r are r and r and r and r are r and r are r and r and r and r are r and r and r are r and r and r are r are r and r are r are r and r are r and r are r and r are r are r and r are r and r are r and r are r and r are r are r and r are r and r are r and r are r are r and r are r and r are r are r and r are r and r are r and r are r and r are r a

7. - Application to an oblique derivative problem in a plane sector.

Operators in $OP\Sigma_{1/p}$ arise naturally in the oblique derivative problem in a plane sector.

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. We seek a solution of the following problem:

(7.1)
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \lim_{y \to 0^{+}} \left(\alpha_{1}(x) \frac{\partial u}{\partial y}(x, y) + \beta_{1}(x) \frac{\partial u}{\partial x}(x, y) \right) = \psi_{1}(x) \in L^{p}(\mathbf{R}^{+}), \\ \lim_{x \to 0^{+}} \left(\alpha_{2}(y) \frac{\partial u}{\partial x}(x, y) + \beta_{2}(y) \frac{\partial u}{\partial y}(x, y) \right) = \psi_{2}(y) \in L^{p}(\mathbf{R}^{+}), \end{cases}$$

where α_i and β_i are real functions such that $\alpha_i^2(t) + \beta_i^2(t) \equiv 1$.

If $\Phi_1(t)$, $\Phi_2(t) \in C_0^{\infty}(\mathbb{R}^+)$ we study the single layer potential with density Φ_1 along the x-axis and density Φ_2 along the positive y-axis, namely,

$$egin{align} u(x,\,y) &= rac{1}{2\pi} \int\limits_0^\infty \log \left((x-t)^2 \,+\, y^2
ight) arphi_1(t) dt \ &+ rac{1}{2\pi} \int\limits_0^\infty \log \left(x^2 \,+\, (y-t)^2
ight) arphi_2(t) dt \ &= u_1(x,\,y) \,+\, u_2(x,\,y) \;. \end{split}$$

Then it is known [St, FJL2] that in $L^p(\mathbf{R}^+)$,

$$\begin{split} &\lim_{y\to 0^+} \frac{\partial u_1}{\partial y}(x,y) = \varphi_1(x) \\ &\lim_{y\to 0^+} \frac{\partial u_1}{\partial x}(x,y) = p.v. \frac{1}{\pi} \int_0^\infty \frac{1}{x-t} \varphi_1(t) dt = H \varphi_1(x) , \\ &\lim_{x\to 0^+} \frac{\partial u_1}{\partial x}(x,y) = -\frac{1}{\pi} \int_0^\infty \frac{t}{y^2+t^2} \varphi_1(t) dt = -K_n \varphi_1(y) , \\ &\lim_{x\to 0^+} \frac{\partial u_1}{\partial y}(x,y) = \frac{1}{\pi} \int_0^\infty \frac{y}{y^2+t^2} \varphi_1(t) dt = K_\tau \varphi_1(y) . \end{split}$$

Note that K_n and K_τ are Hardy kernel operators with kernels

$$k_n(t) = rac{1}{\pi} rac{1}{1+t^2}$$
 and $k_{ au}(t) = rac{1}{\pi} rac{t}{1+t^2}$.

Since $k_n(t)$ and $k_{\tau}(t) \in \mathcal{F}_{1/p}$, $1 , we have that <math>K_n$ and K_{τ} are operators in $OP\Sigma_{1/p}$ and that

$$\sigma_{\scriptscriptstyle p}(K_{\scriptscriptstyle au})(z) = rac{-\cos\left((\pi/2)z
ight)}{\sin\left(\pi z
ight)} \quad ext{ and } \quad \sigma_{\scriptscriptstyle p}(K_{\scriptscriptstyle n})(z) = rac{\sin\left((\pi/2)z
ight)}{\sin\left(\pi z
ight)}.$$

Recall that the symbol of the operator H may be written as

$$\sigma_p(H)(z) = -rac{\cos{(\pi z)}}{\sin{(\pi z)}}.$$

Similar formulas hold for the boundary values of the gradient of u_2 .

We make the following assumptions on the coefficients of the boundary operators in (7.1):

$$\alpha_{j}(t) - \alpha_{j}(0), \quad \beta_{j}(t) - \beta_{j}(0) \in \mathcal{F}_{0}, \quad j = 1, 2.$$

Then the boundary operators applied to u give functions $\psi_1(t)$, $\psi_2(t) \in L^p(\mathbf{R}^+)$ where

(7.2)
$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \alpha_1(t)I + \beta_1(t)H & -\alpha_1(t)K_n + \beta_1(t)K_\tau \\ -\alpha_2(t)K_n + \beta_2(t)K_\tau & \alpha_2(t)I + \beta_2(t)H \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

We write the system (7.2) as $\vec{\psi} = A\vec{\varphi}$ where A is a matrix of operators in $OP\Sigma_{1/p}$. The matrix of principal symbols is given by

$$\sigma_p(A)(t,z) =$$

$$= \left(egin{array}{c} v_1(t) heta(z) + \overline{v_1(t)}(1- heta(z)) & rac{lpha_1(0)\cosig((\pi/2)zig)+eta_1(0)\sinig((\pi/2)zig)}{\sin(\pi z)} \ -rac{lpha_2(0)\cosig((\pi/2)zig)+eta_2(0)ig(\sin(\pi/2)zig)}{\sin(\pi z)} & v_2(t) heta(z) + \overline{v_2(t)}(1- heta(z)ig) \end{array}
ight)$$

where $v_i(t) = \alpha_i(t) + i\beta_i(t)$.

THEOREM 7. For j = 1, 2, let v_j be the winding numbers of $v_j(t)$. Suppose that

$$\inf_{\operatorname{Re} z = 1/p} \left| \det \sigma_p(A)(0,z) \right| > 0$$
.

Then

- 1) There is a matrix **B** of operators in $OP\Sigma_{1/p}$ such that AB I and BA I are matrices of smoothing operators.
- 2) As an operator on $L^p \times L^p$, the index of A is $2\nu_1 + 2\nu_2$.

PROOF. Since $\alpha_j^2 + \beta_j^2 = 1$, the function $\det \sigma_p(A)(t,z)$ is the symbol of an elliptic operator D in $OP\Sigma_{1/p}$. Denote by E a parametrix for D and let B be the matrix of operators

$$\boldsymbol{B} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \alpha_2(t)I + \beta_2(t)H & \alpha_1(0)K_n - \beta_1(0)K_\tau \\ \alpha_2(0)K_n - \beta_2(0)K_\tau & \alpha_1(t)I + \beta_1(t)H \end{pmatrix}.$$

The symbolic calculus establishes that, modulo a matrix of functions in $\Phi_{1/p}$, $\sigma_p(A)(t,z) \cdot \sigma_p(B)(t,z) = I$ and the first conclusion is established.

Let A_0 be the matrix of operators whose symbol is $\sigma_p(A)(0, z)$ and let B_0 be the matrix of operators whose symbol is $\sigma_p(B)(0, z) = [\sigma_p(A_0)(z)]^{-1}$. Then $B_0A_0 = A_1B_0 = I$ on $L^p \times L^p$ so that the index of A is the index of $B_0A = I + B_0(A - A_0)$. The matrix of principal symbols of B_0A is

(7.3)
$$\sigma_p(\boldsymbol{B_0}\boldsymbol{A})(t,z) =$$

$$= \begin{pmatrix} \frac{v_1(t)}{v_1(0)}\theta(z) + \frac{\overline{v_1(t)}}{\overline{v_1(0)}}(1-\theta(z)) & 0 \\ 0 & \frac{v_2(t)}{v_2(0)}\theta(z) + \frac{\overline{v_2(t)}}{\overline{v_2(0)}}(1-\theta(z)) \end{pmatrix}.$$

By Theorem 6, the index of B_0A is $2\nu_1 + 2\nu_2$.

REMARK. For the operator A_0 there are always values of p for which $\det \sigma_p(A_0)(z) = 0$ for some z, $\operatorname{Re} z = 1/p$.

Suppose that $\alpha_i(0) + i\beta_i(0) = \cos \gamma_i + i \sin \gamma_i$, j = 1, 2. Another representation of $\sigma_p(A_0)(z)$ is

$$\sigma_p(m{A_0})(z) = rac{1}{\sin{(\pi z)}} egin{pmatrix} \sin{(\pi z - \gamma_1)} & \sin{\left(rac{\pi}{2}z - \left(rac{\pi}{2} - \gamma_1
ight)
ight)} \ \sin{\left(rac{\pi}{2}z - \left(rac{\pi}{2} - \gamma_2
ight)
ight)} & \sin{(\pi z - \gamma_2)} \end{pmatrix}.$$

Then $\det \sigma_p(A_0)(z) = 0$ when z = (2k+1)/3 or $z = (2/\pi)(\gamma_1 + \gamma_2) + (2k+1)$. In particular $\det \sigma_p(A_0)(\frac{1}{3}) = 0$, and A_0 is not a Fredholm operator on $L^3 \times L^3$. This is in accordance with the results of [FJL 2] for double layer potentials for the Dirichlet problem for which the operators were not Fredholm for $p = \frac{3}{2}$.

REFERENCES

- [A] A. AVANTAGGIATI, Sistemi di equazioni a derivate parziali a coefficienti costanti in due variabili e problemi ellittici, Ricerche Mat., 24 (1975), pp. 203-279.
- [CH] H. O. CORDES E. A. HERMAN, Singular integral operators on a half-line, Proc. Nat. Acad. Sci. U.S.A., 56 (1966), pp. 1668-1673.
- [E 1] G. I. ŠSKIN, Boundary value problems for elliptic pseudodifferential operators (in Russian), Nauka, Moscow (1973).
- [E 2] G. I. Eskin, General mixed boundary problems for elliptic differential equations, C.I.M.E., Course on Pseudodifferential Operators with Applications, II Ciclo (1977), Liguori, Napoli.
- [FJL 1] E. B. FABES M. JODEIT, Jr. J. E. LEWIS, On the spectra of a Hardy kernel, J. Functional Analysis, 21 (1976), pp. 187-194.
- [FJL 2] E. B. Fabes M. Jodeit, Jr. J. E. Lewis, Double layer potentials for domains with corners and edges, Indiana U. Math. J., 26 (1977), pp. 95-114.
- [H] L. HÖRMANDER, Translation invariant operators on L^p spaces, Acta Math., 104 (1960), pp. 93-139.
- J. NOURRIGAT, Une classe d'opérateurs pseudo-différentiels à une variable,
 C. R. Acad. Sci. Sér. A, tome 283 (1976), pp. 457-460.
- [P] B. A. Plamenevskii, An algebra of pseudodifferential operators in spaces with weight (in Russian), Mat. Sb., **106** (1978), pp. 297-320.
- [Sh 1] E. Shamir, Reduced Hilbert transforms and singular integral equations, J. Analyse Math., 12 (1964), pp. 277-305.
- [Sh 2] E. Shamir, Elliptic systems of singular integral operators. I: The half-space case, Trans. Amer. Math. Soc., 127 (1967), pp. 107-123.
- [St] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

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