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Global existence for the Hamilton-Jacobi equations in Hilbert space


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Global Existence for the Hamilton-Jacobi Equations in Hilbert Space.

V. BARBU - G. DA PRATO

1. - Introduction.

In this paper we are concerned with the Hamilton-Jacobi equation

\[ \begin{align*}
q_t(t, x) + F(q_x(t, x)) + (Ax, q_x(t, x)) &= g(t, x) ; \quad x \in D(A), \; t \in [0, T] \\
q(0, x) &= q_0(x)
\end{align*} \]

in a Hilbert space \( H \). Here \( F \) is a convex Fréchet differentiable function on \( H \) and \( -A \) is the infinitesimal generator of a strongly continuous semigroup of linear continuous operators on \( H \). The subscripts \( t \) and \( x \) denote the partial differentiation with respect to \( t \) and \( x \) and \( g, q_0 \) are given real valued functions on \([0, T] \times H\) and \( H\), respectively.

The contents of this paper are outlined below.

In section 2 we shall exhibit several properties of the operator \( \varphi \rightarrow F(q_x) \). In particular it is shown that it arises as the generator of a semigroup of contractions on an appropriate subset \( E \) of the space \( C(H) \) defined below.

In section 3 it is studied equation (1.1) with \( A = 0 \). An explicit form of the solution in term of the semigroup \( S(t) \) is given for the homogeneous equation and it is proved the existence and uniqueness of a weak solution in the class of continuous convex functions \( \varphi \) satisfying the conditions

\[ \left( F'(\partial \varphi(x)), x \right) > 0, \quad x \in H; \quad 0 \in \partial \varphi(0). \]

Section 4 is concerned with equation

\[ \begin{align*}
q_t(t, x) + \frac{1}{2}|q_x(t, x)|^2 + (Ax, q_x(t, x)) &= g(t, x) ; \quad t \in [0, T]; \; x \in D(A) \\
q(0, x) &= q_0(x).
\end{align*} \]

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The main results of this section, Theorems 3 and 4 give existence and uniqueness of weak and classical solutions in the class of continuous convex functions on $H$. In particular, some existence results for the operator equation

\begin{equation}
\begin{aligned}
E_t + E_x A + A^* E + E_s E &= G, & t \in [0, T] \\
E(0) &= E_0
\end{aligned}
\end{equation}

are derived. Particular cases of equation (1.3) have been previously studied in [4], [5]. The situation in which $G$ is a linear continuous self-adjoint operator on $H$ (the Riccati equation) has been extensively studied in the past decade and we refer the reader to [12] and [15] for significant results and complete references.

In section 5 the relevance of eq. (1.1) in control theory and calculus of variations is explained. In section 6 we give some regularity properties for equation (1.3).

As far as we know, the present paper is the first attempt to study the Hamilton-Jacobi equations in Hilbert spaces. As regards the study of these equations in $\mathbb{R}^n$ the fundamental works of Kruzkov [16], Dougis [13], Fleming [14] must be cited. In [10] Crandall proposed a new method in the study of hyperbolic conservation laws equations based on the theory of nonlinear semigroup of contractions in Banach spaces (see also [6], [11]). The semigroup approach has been subsequently used in the study of Hamilton Jacobi equations in $\mathbb{R}^n$ by Aizawa [1], Burch [7], Burch and Goldstein [8] and other authors.

We conclude this section by listing briefly some definitions and notations that will be in effect throughout this paper. Let $H$ be a real Hilbert space with norm $| \cdot |$ and inner product $\langle \cdot, \cdot \rangle$.

Given a lower semicontinuous convex function $\varphi: H \to \mathbb{R} = \mathbb{R}^+ \cup \{ \infty \}$ we shall denote by $\partial \varphi: H \to H$ the subdifferential of $\varphi$, i.e.,

\begin{equation}
\partial \varphi(x) = \{ x^* \in H; \varphi(x) < \varphi(y) + \langle x^*, x - y \rangle, \text{ for all } y \in H \}
\end{equation}

and by $\varphi^*$ the conjugate of $\varphi$,

$$
\varphi^*(y) = \sup \{ (x, y) - \varphi(x); x \in H \}.
$$

If $\varphi$ is Fréchet (or more generally Gâteaux) differentiable at $x$ then $\partial \varphi(x)$ consists of a single element, namely the gradient of $\varphi$. In the sequel we shall use either the symbol $\varphi'$ or $\varphi_*$ for the gradient of $\varphi$ instead of the more conventional symbol $\nabla \varphi$. 

For each $R > 0$ we shall denote by $\Sigma_n$ the closed ball
$$\Sigma_n = \{ x \in H; \ |x| < R \}.$$ 

$C(\Sigma_n)$ will denote the Banach space of all continuous and bounded functions $\varphi: \Sigma_n \to \mathbb{R}^1$ which are bounded on bounded subsets, endowed with the norm $|\varphi|_n = \sup \{ |\varphi(x)|; x \in \Sigma_n \}.$

(1.5)

Let $C(H)$ be the space of all continuous functions $\varphi: H \to \mathbb{R}^1$ which are bounded on bounded subsets, topologized with the family of seminorms $\{ |\varphi|_R; R > 0 \}.$ By $C^1(H)$ we shall denote the space of all Fréchet differentiable functions $\varphi$ on $H$ with Fréchet differential $\varphi_*$ continuous, bounded on every bounded subset of $H$ and with $\varphi(0) = 0.$ $C^1(H)$ is a locally convex space endowed with the family of seminorms

(1.6)

$$|\varphi|_{1,n} = \sup \{ |\varphi_*(x)|; x \in \Sigma_n \}.$$

By Lip $(H)$ we shall denote the space of all functions $\varphi: H \to \mathbb{R}^1$ such that

(1.7) $$|\varphi|_{\text{Lip},R} = \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|}; x \neq y, x, y \in \Sigma_R \right\}, \quad \forall R > 0.$$ 

Further, we shall denote by $C^k(H, H)$, $k$ a natural number or zero, the space of all continuously $k$ times differentiable mappings $f: H \to H$ such that $f^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k - 1$ and

(1.8) $$\|f\|_{k,R} = \sup \left\{ \|f^{(k)}(x)\|_{L(H, H)}; x \in \Sigma_R \right\}, \quad \forall R > 0.$$ 

Here $f^{(j)}$ denotes the Fréchet differential of order $j$ of $f$ and $\|\cdot\|_{L(H, H)}$ the norm in the space $L(H, H)$ of linear continuous operators from $H$ into itself.

We shall denote by $C^2_{\text{Lip}}(H, H)$ the space of all continuous mappings $f: H \to H$ which are Lipschitzian on every bounded subset, endowed with the family of seminorms

(1.9) $$\|f\|_{2, R} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|}; x, y \in \Sigma_R, \ x \neq y \right\}.$$ 

By $C^k_{\text{Lip}}(H, H)$, where $k$ is a natural number, we shall denote the space of all $f \in C^k(H, H)$ such that

(1.10) $$\|f\|_{k,R} = \sup \left\{ \frac{\|f^{(k)}(x) - f^{(k)}(y)\|_{L(H, H)}}{|x - y|}; x, y \in \Sigma_R, \ x \neq y \right\}, \quad \forall R > 0.$$
Given a Banach space \( Z \) we shall denote by \( C([0, T]; Z) \) the space of all continuous functions from \([0, T]\) to \( Z \). If \( Z \) is one of the spaces \( C(H) \), \( C^t(H, H) \) or \( C^k(H, H) \) we set

\[
(1.11) \quad C([0, T]; Z) = \left\{ f \in C([0, T] \times H); f(t, \cdot) \in Z \text{ for } t \in [0, T] \right. \\
\quad \quad \quad \quad \left. \quad \text{and} \quad \| f \|_{L^R_T} = \sup_{t \in [0, T]} |f(t, \cdot)|_{L^R} < \infty \right\}
\]

(here \( \cdot \) is one of the seminorms (1.5), (1.6), (1.7), (1.8), (1.9) or (1.10)) and

\[
(1.12) \quad C^t([0, T]; Z) = \left\{ \varphi \in C([0, T]; Z); \varphi_t \in C([0, T]; Z) \right\}.
\]

By \( L^1(0, T; C(H)) \) we shall denote the space of functions \( f: [0, T] \to C(H) \) having the property that \( f(t, x) \in L^1(0, T) \) for every \( x \in H \) and

\[
(1.13) \quad \sup_{R > 0} \left\{ \int_0^T |f(t, x)| dt; x \in \Sigma_R \right\} < \infty
\]

for every \( R > 0 \).

2. — Assumptions and auxiliary results.

To begin with let us set forth the assumptions which will be in effect throughout this paper.

(a) \( H \) is a real Hilbert space with norm \( |\cdot| \) and inner product \((\cdot, \cdot)\).

(b) The function \( F \) is convex and belongs to \( C^t(H) \). \( F' \in C^0_{\text{Lip}}(H, H) \) and

\[
(2.1) \quad \lim_{|x| \to \infty} \frac{F(x)}{|x|} = \infty.
\]

(c) For every \( R > 0 \) there exists \( \omega_R > 0 \) such that

\[
(2.2) \quad (F'(x) - F'(y), x - y) \geq \omega_R |x - y|^2 \quad \forall x, y \in \Sigma_R.
\]

(d) The linear operator \(-A\) is the infinitesimal generator of a strongly continuous semigroup of contractions \( \exp(-At) \) on \( H \). By \( A^* \) we shall denote the dual operator and by \( D(A) \) the domain of \( A \) endowed with the graph norm.
We shall denote by \( K \) the set of all convex functions \( \varphi \in C(H) \) satisfying the following two conditions

\[
(F'(y), x) > 0 \quad \forall [x, y] \in \partial \varphi; \quad 0 \in \partial \varphi(0).
\]

**Lemma 1.** Let \( F \) satisfy assumptions (b) and (c) and let \( \varphi \in K \) be a given function. Then for every \( t > 0 \) the equation

\[
\partial \varphi(y) - (F')^{-1}(t^{-1}(x - y)) \ni 0
\]

has a unique solution \( y = y_t(x) \) satisfying

\[
|y_t(x)| < |x|.
\]

Moreover, for each \( t > 0 \) the mapping \( x \rightarrow y_t(x) \) belongs to \( C^{1}_{\text{Lip}}(H, H) \).

**Proof.** According to a well-known perturbation result due to Browder (see e.g. [2], p. 46) the operator \( \Gamma y = \partial \varphi(y) - (F')^{-1}(t^{-1}(x - y)) \) is maximal monotone on \( H \). Since \( F' \in C^{0}(H, H) \), \( (F')^{-1} \) is coercive and therefore \( \Gamma \) is onto \( H \). Hence eq. (2.4) has for each \( t > 0 \) at least one solution \( y = y_t(x) \). Writing (2.4) as

\[
y + tF'(\partial \varphi(y)) \ni x
\]

and using condition (2.3) it follows (2.5). The uniqueness of \( y \) as well as the Lipschitzian dependence of \( y(x) \) with respect to \( x \) follows by assumption (c).

In particular if \( \varphi \in C^{1}(H) \) then by (2.4) it follows that

\[
\varphi'(y_t(x)) = (F')^{-1}(t^{-1}(x - y_t(x))).
\]

For every \( \varphi \in K \) and \( t > 0 \), define

\[
(S(t)\varphi)(x) = (\varphi^* + tF^*)(x).
\]

Since \( \lim_{|x| \to \infty} (\varphi^*(x) + tF(x))/|x| = + \infty \), we may infer that for each \( \varphi \in K \) and \( t > 0 \), \( S(t)\varphi \) is a continuous convex function on \( H \) as well. Moreover, by Fenchel's duality theorem (see e.g. [3], p. 188), \( S(t) \) can be equivalently defined for \( t > 0 \), as

\[
(S(t)\varphi)(x) = \inf \{\varphi(y) + tF^*(t^{-1}(x - y)); \ y \in H\} = -\varphi(y_t(x)) + tF^*(t^{-1}(x - y_t(x))), \quad x \in H.
\]
It turns out that \( \{S(t); t \geq 0\} \) is a semigroup of contractions on \( C(H) \). More precisely, one has

**Lemma 2.** Let \( \{S(t); t \geq 0\} \) be the family of nonlinear operators on \( C(H) \) defined by formula (2.7). Then

\[
S(t)K \subset K \quad \text{for all } t > 0
\]
\[
S(t+s)\varphi = S(t)S(s)\varphi \quad \text{for all } t, s > 0, \varphi \in K
\]
\[
|S(t)\varphi - S(t)\psi|_R < |\varphi - \psi|_R \quad \text{for all } \varphi, \psi \in K, t > 0 \text{ and } R > 0.
\]

Moreover, for every \( t > 0 \), \( S(t) \) maps \( K \) into \( C^1(H) \) and

\[
\lim_{t \downarrow 0} t^{-1}(\varphi - S(t)\varphi) = F(\varphi') \quad \text{in } C(H)
\]

for each \( \varphi \in K \cap C^1(H) \) such that \( \varphi' \) is uniformly continuous on every bounded subset of \( H \).

**Proof.** Let \( \varphi \) be fixed in \( K \). As observed earlier, \( S(t)\varphi \) is a continuous real valued convex function on \( H \). By Lemma 1 and formula (2.8) it follows that \( S(t)\varphi \) is bounded on every bounded subset of \( H \) (because by assumption (2.1) \( F^* \) is bounded on bounded subsets). Moreover, by (2.7) it follows that (see e.g. [3], p. 100)

\[
\partial(S(t)\varphi)(x) = (\partial \varphi^* + tF')^{-1}(x), \quad \forall x \in H, \quad t > 0.
\]

Since \( F' \) is strictly monotone on \( H \), for each \( t > 0 \), the map \( (\partial \varphi^* + tF')^{-1} \) is single valued and Lipschitzian on every bounded subset as well. Hence \( S(t)\varphi \in C^1(H) \) for all \( t > 0 \) and by (2.6), (2.13) reduces to

\[
(S(t)\varphi)_x(x) = (\partial \varphi^* + tF')^{-1}(x) = (F')^{-1}(t^{-1}(x - y_t(x))) \quad \forall x \in H, \quad t > 0,
\]

Hence

\[
(F'(S(t)\varphi)_x(x), x) = t^{-1}(x - y_t(x), x) > 0 \quad \forall x \in H, \quad t > 0
\]

and again by Lemma 1 it follows that \( (S(t)\varphi)_x(0) = 0 \). Thus we have shown that \( S(t)\varphi \in K \) for every \( t > 0 \).

Let \( \varphi, \psi \) be two elements of \( K \). Since in virtue of Lemma 1, \( y_t(x) \in \Sigma_n \)
whenever $|x| < R$, it follows by (2.8),
\[(S(t)\psi)(x) - (S(t)\varphi)(x) < \psi(y_i(x)) - \varphi(y_i(x)) < |\psi - \varphi|_x \quad \text{for all } x \in \Sigma_R \text{ and } t > 0.
\]

(Here $y_i(x)$ is defined by Lemma 1). The latter implies (2.11).

Let us now prove the semigroup property (2.10). Using again the Fenchel theorem we have by (2.7) and (2.8)
\[
(S(t + s)\varphi)(x) = \inf \{ (\varphi^* + sF^*)(y) + tF^*(t^{-1}(x - y)); y \in H \} = \inf \{ (S(s)\varphi)(y) + tF^*(t^{-1}(x - y)); y \in H \} = (S(t)S(s)\varphi)(x).
\]

It remains to prove equality (2.12). To this end we fix $\varphi \in K \cap C^1(H)$ and observe that by (2.8) we have
\[
(2.15) \quad \varphi(x) - (S(t)\varphi)(x) = \\
= \varphi(x) - \varphi(y_i(x)) - tF^*(t^{-1}(x - y_i(x))) < (\varphi'(x), x - y_i(x)) - tF^*(t^{-1}(x - y_i(x))).
\]

Along with the well known conjugacy formula, (see e.g. [3], p. 91)
\[
(2.16) \quad F^*(F'(y)) + F(y) = (y, F'(y)), \quad \forall y \in H,
\]
relations (2.6) and (2.4) lead to
\[
(2.17) \quad t^{-1}\left(\varphi(x) - (S(t)\varphi)(x)\right) < \left(\varphi'(x) - \varphi'(y_i(x)), F'\left(\varphi'(y_i(x))\right) + F(\varphi'(y_i(x)))\right).
\]
On the other hand, since $F$ is convex, one has the inequality
\[
F'(\varphi'(y_i(x))) < F(\varphi'(x)) + \left(\varphi'(y_i(x)), \varphi'(y_i(x)) - \varphi'(x)\right)
\]
which along with (2.17) yields
\[
(2.18) \quad t^{-1}\left(\varphi(x) - (S(t)\varphi)(x)\right) - F(\varphi'(x)) < 0.
\]

Similarly, by (2.8) and (2.16) it follows that
\[
(S(t)\varphi)(x) - \varphi(x) < \left(\varphi'(y_i(x)), y_i(x) - x\right) + tF^*\left(t^{-1}(x - y_i(x))\right) = \\
= t\left(\varphi'(y_i(x)), F'(\varphi'(y_i(x)))\right) + tF^*\left(F'(\varphi'(y_i(x)))\right) = -tF(\varphi'(y_i(x))).
\]
Hence
\[ (2.19) \quad t^{-1}(\varphi(x) - (S(t)\varphi)(x)) - F(\varphi'(y_i(x))) > 0. \]

By (2.18) and (2.19) we see that
\[ (2.20) \quad 0 < F(\varphi'(x)) - t^{-1}(\varphi(x) - (S(t)\varphi)(x)) < F(\varphi'(x)) - F(\varphi'(y_i(x))) < \langle F'(\varphi'(x)), \varphi'(x) - \varphi'(y_i(x)) \rangle; \quad t > 0, \ x \in H. \]

Since \( \varphi' \) and \( F' \) are bounded on every \( \Sigma_\mathbb{R} \) it follows by (2.6) that
\[ |x - y_i(x)| < C_\mathbb{R} t \quad \forall x \in \Sigma_\mathbb{R}, \ t > 0 . \]

Inasmuch as \( \varphi' \) is uniformly continuous on every \( \Sigma_\mathbb{R} \), by (2.20) we deduce (2.12) as claimed.

**Remark.** Let \( L_0 \) be the operator defined in \( C(H) \) by
\[ (2.21) \quad L_0 \varphi = F(\varphi_\omega) \quad \forall \varphi \in D(L_0) \]
where \( D(L_0) \) consists of the set of all \( \varphi \in C^1(H) \cap K \) such that \( \varphi' \) is uniformly continuous on every bounded subset of \( H \).

By (2.11) and (2.12) it follows that \( L_0 \) is accretive in \( C(H) \) i.e.,
\[ (2.22) \quad |\varphi - \psi| + \lambda(L_0 \varphi - L_0 \psi)_\mathbb{R} > |\varphi - \psi|_\mathbb{R}, \quad \forall \mathbb{R} > 0; \ \lambda > 0 \]
for every pair \((\varphi, \psi) \in D(L_0) \times D(L_0)\).

On the other hand, (2.21) implies that \( L_0 \subset L_1 \) where \( L_1 \) is the infinitesimal generator of \( S(t) \).

3. **Equation (1.1) with \( A = 0. \)**

We begin with the homogeneous Cauchy problem
\[ (3.1) \begin{cases} \varphi_t(t, x) + F(\varphi_x(t, x)) = 0; & t > 0, \ x \in H \\ \varphi(0, x) = \varphi_0(x), & x \in H \end{cases} \]
where \( F \) satisfies assumptions (b), (c) and \( \varphi_0 \in K. \)
Define the function \( q : \mathbb{R}^+ \times H \rightarrow \mathbb{R}' \),

\[
q(t, x) = (S(t)q_0)(x), \quad t > 0, \ x \in H
\]

where \( S(t) \) is the semigroup defined by formula (2.7).

By Lemma 1 it follows that \( q(t, \cdot) \in C'(H) \) for every \( t > 0 \) and \( q(t, \cdot) \in K \) for every \( t > 0 \). Furthermore, it follows by (2.10) and (2.12) that for each \( q_0 \in K \), the right derivative \( (d^+/dt)S(t)q_0 \) exists at every \( t > 0 \) and

\[
\frac{d^+}{dt} S(t)q_0 + F((S(t)q_0)_x) = 0, \quad \forall t > 0.
\]

We have also used the fact that for each \( q_0 \in K \), \( (S(t)q_0)_x \) is uniformly continuous on bounded subsets (see (2.14)). In (3.3) \( d^+/dt \) is taken in the sense of topology of \( C(H) \). If \( q_0 \in C^4(H) \) then eq. (3.3) remains valid for \( t = 0 \). In particular, it follows by (3.3) that \( q \) is a strong solution to eq. (3.1) in the sense that

\[
\frac{d^+}{dt} q(t, x) + F(q(t, x)) = 0 \quad \text{for every } t > 0, \ x \in H.
\]

It is worth noting also that by (2.14) and Lemma 1 it follows that \( q(t, \cdot) \in C^4(H) \) for every \( t > 0 \) and

\[
\sup \left\{ |q(t, \cdot)|_{1,R}; \ t \in [\delta, T] \right\} < C_{\delta,R} \quad \text{for every } \delta \in ]0, T[ \text{ and } R > 0.
\]

If \( q_0 \in C^4(H) \) then by (2.6) and (2.14) it follows that \( q \in C([0, T]; C^4(H)) \), i.e.,

\[
\sup \left\{ |q(t, \cdot)|_{1,R}; \ t \in [0, T] \right\} < C_R \quad \forall R > 0.
\]

On the other hand the accretivity of the operator \( L_0(q) = F(q_0) \) on \( C(H) \) (see (2.22)) implies via a standard argument the uniqueness of the strong solution \( q \).

Summarising, we get

**Theorem 1.** Let \( F \) satisfy assumptions (b), (c). Then for each \( q_0 \in K \), the Cauchy problem (3.1) has a unique strong solution given by formula (3.2). More precisely, \( q(t, \cdot) \in C^4(H) \cap K \) for all \( t > 0 \), satisfies (3.5) and as a function of \( t \) from \( ]0, +\infty[ \), \( q(t) \) is continuous, everywhere differentiable from the right and satisfies eq. (3.4). Moreover the map \( q_0 \rightarrow q \) is a contraction from \( C(H) \) to \( C([0, T]; C^4(H)) \).

If in addition \( q_0 \in C^4(H) \) then \( q \in C([0, T]; C^4(H)) \) and equation (3.4) is satisfied for all \( t > 0 \).
REMARK. We have incidentally shown that the semigroup $S(t)$ has
smoothing effect on initial data (see [3], [9] for other classes of contraction
semigroups having this property).

We shall consider now the nonhomogeneous Cauchy problem

$$
\begin{align*}
\begin{cases}
\phi_i(t, x) + F(\phi_i(t, x)) = g(t, x); & t \in [0, T], \ x \in H \\
\phi(0, x) = \phi_0(x)
\end{cases}
\end{align*}
$$

where $F$ satisfies assumptions (b) and (c).

One assumes in addition that

(c) $K$ is a closed convex cone of $C(H)$.

Further we shall assume that

$$(3.8) \quad \phi_0 \in C^1(H) \cap K, \quad g \in C([0, T]; C^1(H)) \cap K$$

where $K$ is the closed convex cone of $C([0, T]; C(H))$ defined by

$$(3.9) \quad K = \{ \phi \in C([0, T]; C(H)); \ phi(t) \in K, \ \forall t \in [0, T] \}. $$

Consider the approximating equation

$$
\begin{align*}
\begin{cases}
\phi_i(t, x) + e^{-t}(\phi_i(t, x) - (S(\varepsilon)\phi)(t, x)) = g(t, x) \\
\phi(0, x) = \phi_0(x), \quad t \in [0, T], \ x \in H, \ \varepsilon > 0
\end{cases}
\end{align*}
$$

or equivalently

$$(3.11) \quad \phi(t, x) = \exp(\varepsilon^{-1}t)\phi_0(x) + \int_0^t \exp(\varepsilon^{-1}(t-s))g(s, x)ds +
\varepsilon^{-1}\int_0^t \exp(\varepsilon^{-1}(t-s))(S(\varepsilon)\phi)(s, x)ds, \quad t \in [0, T].$$

By assumptions (c), (3.8) and by Lemma 2 it follows that the operator defined
by the right hand side of eq. (3.11) maps every $C([0, T]; C(\Sigma))$
into itself and is contractant. Thus for every $\varepsilon > 0$, eq. (3.11) ((3.10)) has
a unique solution $\phi_\varepsilon \in K \cap C^1([0, T]; C(H))$. Since, as proved earlier, $S(\varepsilon)\psi \in C(H)$ for every $\phi \in K$ and $\varepsilon > 0$ it follows by (3.11) that $\phi_\varepsilon \in C([0, T];
C(\Sigma))$. Furthermore, recalling that (see (2.6) and (2.14)),

$$(3.12) \quad (S(\varepsilon)\psi)_\varepsilon(x) = \psi\left(\gamma_\varepsilon(x)\right), \quad x \in H, \ \varepsilon > 0$$
we see that \((q_{0})_{t} = q'_{0}\) is the solution to

\begin{equation}
\frac{d}{dt} q_{t}(t, x) + \varepsilon^{-1}(q_{t}(t, x) - q_{t}(t, y_{t}(t, x))) = g_{t}(t, x) \tag{3.13}
\end{equation}

where

\begin{equation}
y_{t}(t, x) + F'(q_{t}(t, y_{t}(t, x))) = x; \quad x \in \mathcal{H}, \quad t \in [0, T]. \tag{3.14}
\end{equation}

Then by an easy computation involving eq. (3.13) and the Gronwall lemma it follows that

\begin{equation}
|q_{t}(t)|_{1, n} \leq C_{n}(|q_{0}|_{1, n}, \sup_{0 \leq t \leq T} |g(t)|_{1, n}, \quad t \in [0, T]. \tag{3.15}
\end{equation}

(By \(C_{n}\) we shall denote several positive constants independent of \(\varepsilon\).) Parenthetically we notice that since by (3.14) and assumption (c) the mapping \(x \to q'_{t}(x, y_{t}(t, x))\) is Lipschitzian on every \(\Sigma_{n}\), it follows by (3.13) that if \(q'_{0} \in C_{\text{Lip}}^{0}(\mathcal{H}, \mathcal{H})\) and \(g_{t} \in C([0, T]; C_{\text{Lip}}^{0}(\mathcal{H}, \mathcal{H}))\) then

\begin{equation}
(q_{t})_{t} \in C([0, T]; C_{\text{Lip}}^{0}(\mathcal{H}, \mathcal{H})). \tag{3.16}
\end{equation}

Next by (2.20), (3.12) and (3.13) one has

\begin{align*}
|\varepsilon^{-1}(q_{t}(t, x) - (S(\varepsilon)q_{t})(t, x)) - F(q_{t}(t, x))| &< \\
&= -\varepsilon \left(F'(q_{t}(t, x)), \frac{d}{dt} q_{t}(t, x) + g_{t}(t, x)\right) \\
&= \varepsilon \left(F'(q_{t}(t, x)), g_{t}(t, x)\right) - \varepsilon \frac{d}{dt} F(q_{t}(t, x)).
\end{align*}

Integrating the latter over \(]0, T[\), we get by (3.15) the estimate

\begin{equation}
\int_{0}^{T} |\varepsilon^{-1}(q_{t}(t, x) - (S(\varepsilon)q_{t})(t, x)) - F(q_{t}(t, x))| dt < \\
< \varepsilon \int_{0}^{T} |F'(q_{t}(t, x))| g_{t}(t, x) dt + \varepsilon \left(F(q'_{0}(x)) - F(q'_{0}(T, x))\right) < C_{n}\varepsilon \quad \forall x \in \Sigma_{n}
\tag{3.17}
\end{equation}

and therefore by (3.10)

\begin{equation}
\begin{cases}
(q_{t})(t, x) + F((q_{t})_{t}(t, x)) = g(t, x) + \eta(t, x), \quad t \in [0, T] \\
(q_{0})(0, x) = q_{0}(x) \quad x \in \mathcal{H}
\end{cases} \tag{3.18}
\end{equation}
where
\[
\int_0^T |\eta_e(t, x)| dt < C_n \varepsilon \quad \forall x \in \Sigma_n, \ \varepsilon > 0.
\]

Now coming back to equation (3.11) it follows by Lemma 2 (part (2.11)) and the Gronwall lemma that the mapping \((\varphi, g, h) \mapsto g, h)\) is Lipschitzian from \(C(H) \times C([0, T]; C(H))\) to \(C([0, T]; C(H))\). More precisely, one has

\[
|G_e(\varphi, g, h)(t) - G_e(\varphi_s, g, h)| \leq \\
\leq |\varphi - \varphi_s| + \int_0^T |g(s) - h(s)| ds \quad \forall R > 0; \ t \in [0, T]
\]

for all \(\varphi, \varphi_s \in K\) and \(g, h \in K\). By (3.18) and (3.19) one concludes that

\[
|\varphi_e(t) - \varphi_s(t)| \leq C_n \varepsilon \sqrt{T} + \lambda \quad \forall t \in [0, T]; \ \varepsilon, \lambda > 0.
\]

Hence \(\lim_{\varepsilon \to 0} \varphi_{s} = \varphi\) exists in \(C([0, T]; C(H))\). Clearly \(\varphi \in K\) and by (3.15) we see that for each \(t \in [0, T]\), \(\varphi(t, \cdot)\) is Lipschitzian on every bounded subset of \(H\) and

\[
|\varphi(t)|_{L^\infty, R} \leq C_n \|\varphi_0\|_{L^\infty} + \sup_{t \in [0, T]} |g(t)|,
\]

for \(t \in [0, T]\).

Summarising, we have shown that there exists a sequence \(\{\varphi_{s}\} \subset C([0, T]; C(H))\) satisfying

(3.20) \(\varphi_{s} \in C([0, T]; C^1(H)) \cap K \quad \forall \varepsilon > 0\)

(3.21) \(\varphi_{s} \in C([0, T]; C(H)) \quad \forall \varepsilon > 0\)

(3.22) \(\varphi_{s} \to \varphi\) in \(C([0, T]; C(H))\) for \(\varepsilon \to 0\)

(3.23) \(\{\varphi_{s}\}\) is bounded in \(C([0, T]; C^1(H))\) and \((\varphi_{s})_{s} \in L^1(0, T; C(H))\)

(3.24) \(\varphi_{s}(0, x) = \varphi_{0}(x)\).

Here the convergence in the space \(L^1(0, T; C(H))\) is understood in the local convex topology given by the family of seminorms (1.13).

**Definition 1.** A function \(\varphi\) satisfying conditions (3.20) up to (3.24) is called weak solution to the Cauchy problem (3.7). We notice that by (3.19),
(3.22), (3.23) and (3.24) the weak solution \( \varphi = G(\varphi_0, g) \) is unique and

\[
G(\varphi_0, g) - G(\varphi_0, h) = \int_0^T g(\varphi(t)) \, dt
\]

for all \( \varphi_0, \varphi_0 \in K \) and \( g, h \in X \) satisfying condition (3.8). We have therefore proved the following theorem

**Theorem 2.** Assume that hypotheses (a), (b), (c) and (e) are satisfied. Then for any pair of functions \( (\varphi_0, g) \in K \times X \) satisfying condition (3.8), the Cauchy problem (3.7) has a unique weak solution \( \varphi \) which satisfies

\[
\sup \{ |\varphi(t)|_{L_0} : t \in [0, T] \} < \infty.
\]

Furthermore, the map \( (\varphi_0, g) \rightarrow \varphi \) is Lipschitzian from \( C(H) \times C([0, T]; C(H)) \) to \( C([0, T]; C(H)) \).

**Remarks.** 1° It is worth noting that another way to prove Theorem 2 is to apply the Bénilan existence result (see [2], [6]) to nonlinear evolution equation

\[
\frac{d\varphi}{dt} + L\varphi = g, \quad t \in [0, T]; \quad \varphi(0) = \varphi_0
\]

in the space \( C(H) \). Here \( L \) is the closure of \( L_0 \) (see (2.22)) in \( C(H) \times C(H) \).

2° Assumptions (b), (c) and (e) are verified by a large class of functions \( F \) which includes functions of the form

\[
F(x) = \zeta(|x|^2), \quad x \in H
\]

where \( \zeta \) is a real valued, convex and differentiable function on \([0, \infty[\) which satisfies the following conditions

\[
\zeta'(0) > 0; \quad \lim_{r \to +\infty} \zeta(r)/r = +\infty.
\]

4. – Existence and uniqueness for equation (1.2).

We shall study here the Cauchy problem

\[
\begin{cases}
\varphi_t(t, x) + \frac{1}{2} |\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x) \\
\varphi(0, x) = \varphi_0(x); \quad x \in D(A), \quad t \in [0, T]
\end{cases}
\]
where
\begin{equation}
q_\alpha \in C^1(H)
\end{equation}
and
\begin{equation}
g \in C([0, T]; C^1(H)).
\end{equation}

Further we shall assume that
\begin{equation}
q'_0 \in C^0_{\text{Lin}}(H, H); \quad g_\alpha \in C([0, T]; C^0_{\text{Lin}}(H, H)).
\end{equation}

In this case \( K \) is the set of all convex functions \( \varphi \in C(H) \) such that \( \partial \varphi(0) \in 0 \).

We start with the approximating equation
\begin{equation}
q(t, x) = \exp \left( -t/\epsilon \right) q_\alpha \left( \exp \left( -tA \right) x \right) + \\
\int_0^t \exp \left( -\left( t-s \right)/\epsilon \right) g(s, \exp \left( -\left( t-s \right)A \right) x) \, ds + \\
+ \epsilon^{-1} \int_0^t \exp \left( -\left( t-s \right)/\epsilon \right) \left( S(\epsilon) \varphi \right) \left( s, \exp \left( -\left( t-s \right)A \right) x \right) \, ds
\end{equation}
for \( t \in [0, T] \), \( x \in H \)

where by formula (2.8), \( S(\epsilon) \) is given by
\begin{equation}
S(\epsilon) \varphi \left( x \right) = \inf \left\{ \frac{|x - y|^2}{2\epsilon} + \varphi(y); \quad y \in H \right\} = \varphi(y_\epsilon) + \frac{|x - y_\epsilon|^2}{2\epsilon}
\end{equation}
and
\begin{equation}
y_\epsilon(x) = (1 + \epsilon \varphi')^{-1} x.
\end{equation}

Applying the contraction principle on the closed convex cone of \( C([0, T]; C(H)) \)
\begin{equation}
K = \{ \varphi \in C([0, T]; C(H)) : \varphi(t) \in K \ \forall t \in [0, T] \}
\end{equation}
we see that eq. (4.5) has a unique solution \( q_\epsilon \in K \). Moreover, in as much as \( (S(\epsilon) \varphi)'(x) = \varphi'(y_\epsilon(x)) \), we see that \( q_\epsilon \in C([0, T]; C^0(H)) \) and
\begin{equation}
q'_\epsilon(t, x) = \exp \left( -t/\epsilon \right) \exp \left( -tA^* \right) q'_0 \left( \exp \left( -tA \right) x \right) + \\
\int_0^t \exp \left( -\left( t-s \right)/\epsilon \right) \exp \left( -\left( t-s \right)A^* \right) g'(s, \exp \left( -\left( t-s \right)A \right) x) \, ds + \\
+ \epsilon^{-1} \int_0^t \exp \left( -\left( t-s \right)/\epsilon \right) \exp \left( -\left( t-s \right)A^* \right) q'_0 \left( s, y_\epsilon(s, \exp \left( -\left( t-s \right)A \right) x) \right) \, ds.
\end{equation}
By (4.5) it follows that for each \( x \in D(A) \), \( \varphi_\varepsilon(t, x) \) is differentiable on \( [0, T) \) and satisfies the equation

\[
(\varphi_\varepsilon)_t(t, x) + \varepsilon^{-1}(\varphi_\varepsilon(t, x) - S(\varepsilon)\varphi_\varepsilon(t, x)) + \langle Ax, (\varphi_\varepsilon)_x(t, x) \rangle = g(t, x); \quad t \in [0, T], \ x \in D(A).
\]

By an easy computation involving the Gronwall lemma and eq. (4.9) it follows

\[
|\varphi_\varepsilon(t)|_{1, R} \leq C_R \left( |\varphi_\varepsilon|_{1, R} + \sup_{0 \leq t \leq T} |g(t)|_{1, R} \right), \quad t \in [0, T].
\]

Next, since the mapping \( x \mapsto y_\varepsilon(x) \) is nonexpansive and in virtue of (2.6),

\[
\varphi_\varepsilon'(x, y_\varepsilon(t, x)) = \varepsilon^{-1}(x - y_\varepsilon(t, x))
\]

it follows by (4.4) and (4.9) that \( \varphi_\varepsilon(t) \in C^0_{\text{loc}}(H, H) \) for every \( t \in [0, T] \). Moreover, using once again the Gronwall lemma one finds the estimate

\[
\|\varphi_\varepsilon(t)\|_{0, R} \leq C_R \left( \|\varphi_\varepsilon \|_{0, R} + \sup_{0 \leq t \leq T} \|g'(t)\|_{0, R} \right)
\]

Next by inequality (2.20) and (4.11), (4.12)

\[
\|\varphi_\varepsilon(t)\|_{0, R} \leq C_R \left( \|\varphi_\varepsilon \|_{0, R} + \sup_{0 \leq t \leq T} \|g'(t)\|_{0, R} \right)
\]

for all \( x \in \Sigma_R \) and \( t \in [0, T] \).

Since the mapping \((\varphi_\varepsilon, g) \mapsto \varphi_\varepsilon \) is Lipschitzian from \( C(H) \times C([0, T]; C(H)) \) to \( C([0, T]; C(H)) \) (see (3.19)) we may infer by (4.13) that

\[
|\varphi_\varepsilon(t) - \varphi_\varepsilon(t)|_R \leq C_R(t + \lambda); \quad t \in [0, T]; \ \varepsilon, \lambda > 0
\]

and therefore \( \lim_{\varepsilon \to 0} \varphi_\varepsilon = \varphi \) exists in \( C([0, T]; C(H)) \). Clearly \( \varphi \in \mathcal{K} \) and

\[
\sup_{t \in [0, T]} |\varphi(t)|_{1d, R} < \infty, \quad \forall R > 0.
\]

By (4.10)-(4.12) and (4.13) we see that the function \( \varphi \) is a weak solution to problem (4.1) in the sense of Definition 1, i.e. there exists a sequence
Here the space $D(A)$ is endowed with the graph norm.

Summarising, we have proved the following theorem

THEOREM 3. Suppose that assumptions (a), (e) are satisfied and $q_0, g$ satisfy conditions (4.2), (4.3) and (4.4). Then the Cauchy problem (4.1) has a unique weak solution $q \in \mathcal{K}$ which satisfies (4.14). Moreover, the map $(q_0, g) \to q$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$ and for every $x \in D(A)$ the function $q(t, x)$ is absolutely continuous on $[0, T]$.

Our next concern is a regularity result for the solutions to equation (4.1). To this purpose we return to approximating sequence $\{q_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ and set

$$E^\varepsilon(t, x) = q_\varepsilon(t, x) \quad t \in [0, T], \ x \in H.$$ 

As seen above $E^\varepsilon \in C([0, T]; C^0(H, H))$ is the solution to (see (4.9))

$$E^\varepsilon(t, x) = \exp \left( -\frac{t}{\varepsilon} \right) \exp (-tA^*) q_0(\exp (-tA)x) +$$

$$+ \varepsilon^{-1} \int_0^t \exp \left( -\frac{(t-s)}{\varepsilon} \right) \exp (- (t-s) A^*) E^\varepsilon(s, \exp (- (t-s) A)x) ds +$$

$$+ \int_0^t \exp \left( -\frac{(t-s)}{\varepsilon} \right) \exp (- (t-s) A^*) g \left( s, \exp (- (t-s) A)x \right) ds$$

where

$$E^\varepsilon(t, x) = (S(\varepsilon) q_\varepsilon)'(t, x) = E^\varepsilon(1 + \varepsilon E^\varepsilon)^{-1}(t, x).$$

In addition to (4.2), (4.3) and (4.4) we shall assume that

$$q_0' \in C^1_{Lin}(H, H); \quad g' \in C([0, T]; C^1_{Lin}(H, H)).$$
We shall prove that under these conditions $E^\varepsilon \in C([0, T]; C^1_{\text{lip}}(H, H))$. To this end we introduce the following convex cone of $C^0(H, H)$:

\[(4.22) \quad \Pi = \{ E \in C^0(H, H); \ E \text{ monotone and } E(0) = 0 \}. \]

For every $E \in \Pi$ we set $E_\varepsilon = E(1 + \varepsilon E)^{-1}$ ($1$ is the identity operator in $H$). We notice that for every $\varepsilon > 0$ the operator $(1 + \varepsilon E)^{-1}$ is well defined and nonexpansive on $H$. In the next lemma we gather for later use some elementary properties of $E_\varepsilon$.

**Lemma 3.** For all $\varepsilon > 0$ and $R > 0$ one has

\[(4.23) \quad |E_\varepsilon|_{0,R} \leq |E|_{0,R} \quad \forall E \in \Pi \]

\[(4.24) \quad |E_\varepsilon|_{1,R} \leq |E|_{1,R} \quad \forall E \in \Pi \cap C^1(H, H) \]

\[(4.25) \quad \|E\|_{1,R} \leq \|E\|_{1,R} + \varepsilon \|E\|_{1,R} \|E\|_{0,R} + \|E\|_{1,R} \quad \forall E \in \Pi \cap C^1_{\text{lip}}(H, H). \]

Moreover, if $\|E\|_{1,R}$ and $\|\bar{E}\|_{1,R}$ are $< \alpha$ then there exists $\eta(\alpha) > 0$ such that

\[(4.26) \quad |E - \bar{E}|_{1,R} < (1 + \varepsilon \eta(\alpha)) |E - \bar{E}|_{1,R}. \]

The proof is standard and relies on the formula

\[E_\varepsilon(x) = E'((1 + \varepsilon E)^{-1}x)^{-1} \left(1 + \varepsilon E'((1 + \varepsilon E)^{-1}x)^{-1}\right). \]

(By $E'$ we shall denote the Fréchet derivative of the operator $E$.)

In the space $C([0, T]; C^1(H, H))$ consider the approximating equation

\[(4.27) \quad E(t, x) = \exp \left(-\frac{\varepsilon}{t}\right) \exp (-tA^*) E_0(\exp (-tA)x) +
\]

\[+ \varepsilon^{-1} \int_0^t \exp \left(-\frac{(t-s)}{\varepsilon}\right) \exp (-sA^*) G(s, \exp (-sA)x) ds +
\]

\[\int_0^t \exp \left(-\frac{(t-s)}{\varepsilon}\right) \exp (-sA^*) E(s, \exp (-sA)x) ds \quad x \in H; \ t \in [0, T] \]

where $E_0 = q_0'$ and $G = g'$. We consider the following closed convex cone of $C([0, T]; C^1(H, H))$

\[(4.28) \quad Q = \{ E \in C([0, T]; C^1_{\text{lip}}(H, H)); \ E(t) \in \Pi, \|E(t)\|_{1,R} < \alpha_R \}
\]

for every $t \in [0, T]$.

where $\alpha_R \to + \infty$ as $R \to + \infty$. 
Let $T$ be the operator defined by the right hand of equation (4.27). For the beginning we shall assume that $\|E_0\|_{1,R} < \alpha_R/2$ and $\|G(t)\|_{1,R} < \alpha_R/2$ for $t \in [0, T]$. Then for all sufficiently small $T$, $T$ maps $Q$ into itself and by (4.25) one has

$$||(TE)(t) - (T\tilde{E})(t)||_{1,R} < \rho_R \sup\{ |E(t) - \tilde{E}(t)|_{1,R} ; \ 0 < t < T\}$$

for all $E, \tilde{E} \in Q$. Here $0 < \rho_R < 1$ for every $R > 0$. Hence eq. (4.27) (equivalently (4.19)) has a unique solution $E = E^t \in C([0, T[ ; C^1_{Lip}(H, H))$ where $[0, T[ \]$ is some subinterval of $[0, T[$. Next after some calculations involving equation (4.19), estimates (4.23), (4.25) and the Gronwall lemma it follows that

$$(4.29) \quad \|E^t(t)\|_{1,R} < C_R \quad \forall t \in [0, T[ , \ e > 0$$

where $C_R$ is independent of $T'$. This implies by a standard procedure that $E^t \in C([0, T[ ; C^1_{Lip}(H, H))$ and inequality (4.29) extends on the whole interval $[0, T]$. On the other hand, using once again estimate (4.26) we see that

$$(4.30) \quad |\Phi_t(t, E_0, G) - \Phi_t(t, \tilde{E}_0, \tilde{G})|_{L^R} \leq C_R \left( |E_0 - \tilde{E}_0|_{L^R} + \int_0^t |G(t) - \tilde{G}(t)|_{L^R} \, dt \right)$$

for $t \in [0, T]$ and $k = 0, 1$, where $\Phi_t(t, E_0, G) = E^t$ is the solution to (4.27). On the other hand, we have

$$(4.31) \quad e^{-1}(E(x) - E_0(x)) - E^t(x)E(x) =$$

$$= \int_0^t \left( E'(sx + (1-s)y(x)) E(y(x)) - E'(x)E(x) \right) ds , \quad x \in H$$

where $y(x) = (1 + eE)^{-1} x$. Along with estimates (4.29) and (4.30) the latter yields

$$(4.32) \quad |e^{-1}(E^t(t) - E^t(0)) - (E^t)'(t)(E^t(t))_{L^R} < C_R e , \quad t \in [0, T].$$

Then by (4.32) it follows that

$$|E^t(t) - E^t(0)|_{L^R} < C_R(e + \lambda) , \quad t \in [0, T] ; \ e, \lambda > 0 .$$

Hence there exists $E \in C([0, T[ ; C^1(H, H))$ such that for $e \to 0$,

$$E_e \to E \quad \text{in} \ C([0, T] ; C^1(H, H)).$$
By the uniqueness of the limit we infer that \( E(t, x) = \varphi(x(t, x)) \) where \( \varphi \) is the weak solution to equation (4.1). We have therefore proved that

\[
(\varphi_x)_x \rightarrow \varphi_x \quad \text{in } C([0, T]; C_0(H, H)).
\]

Then by (4.17) we may infer that

\[
\varphi \in C^1([0, T]; D(A)) \cap C([0, T]; C'(H, H))
\]

and

\[
\begin{cases}
\varphi(t, x) + \frac{1}{2} |\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x) \\
\varphi(0, x) = \varphi_0(x) \quad \text{for } x \in D(A), \quad t \in [0, T].
\end{cases}
\]

This amounts to saying that \( \varphi \) is a classical solution to equation (4.1). We have therefore proved

**Theorem 4.** In Theorem 3 suppose in addition that \( \varphi_0 \) and \( g \) satisfy conditions (4.21). Then \( \varphi \) is a classical solution to equation (4.1).

Now we notice that by (4.10), \( E^\varepsilon = \varphi_\varepsilon \) satisfy

\[
\begin{align*}
E^\varepsilon(t, x) &= \exp(-tA^*)\varphi_0'(\exp(-tA)x) + \\
&\quad + \int_0^t \exp(-(t-s)A^*)E^\varepsilon_sE^\varepsilon\left(s, \exp(-(t-s)A)x\right) ds + \\
&\quad + \int_0^t \exp(-(t-s)A^*)g_\varepsilon\left(s, \exp(-(t-s)A)x\right) ds + \delta_\varepsilon(t, x)
\end{align*}
\]

where \( \delta_\varepsilon \to 0 \) in \( C([0, T]; C^0(H, H)) \) for \( \varepsilon \to 0 \), while by (4.33)

\[
E^\varepsilon \rightarrow E = \varphi_x \quad \text{in } C([0, T]; C^0(H, H)).
\]

Keeping in mind that the equation

\[
\begin{align*}
E(t, x) &= \exp(-tA^*)E_0(\exp(-tA)x) + \\
&\quad + \int_0^t \exp(-(t-s)A^*)E_sE\left(s, \exp(-(t-s)A)x\right) ds + \\
&\quad + \int_0^t \exp(-(t-s)A^*)G\left(s, \exp(-(t-s)A)x\right) ds \quad t \in [0, T], \quad x \in H
\end{align*}
\]
is the « mild » form of equation (1.3), we may say that \( E = qJz \) is a weak solution to this equation.

We have therefore the following existence result

**Theorem 5.** Under assumptions of Theorem 4, \( E(t, x) = qJ_{\alpha}(t, x) \) is a weak solution to operator equation (1.3) where \( E_0 = q_0' \) and \( G = g_\alpha \).

5. – An example in control theory.

The relevance of the Hamilton-Jacobi equations in control theory and the calculus of variations is well-known (see e.g. [3] and [12] for recent results concerning infinite-dimensional problems). Here we shall study the connection between equation (1.1) and the following optimal control problem:

Minimize

\[
(5.1) \quad \int_0^T \left( g(x(t)) + h(u(t)) \right) dt + q_0(x(T))
\]

over all \( u \in L^2(0, T; U) \) and \( x \in C([0, T]; H) \) subject to state equation

\[
(5.2) \quad \begin{cases}
  x' + Ax = Bu, & t \in [0, T] \\
  x(0) = x_0.
\end{cases}
\]

Here \( B \) is a linear continuous operator from \( U \) to \( H \), \( g: H \rightarrow \mathbb{R}^1 \), \( h: U \rightarrow \mathbb{R}^1 \), \( q_\alpha: H \rightarrow \mathbb{R}^1 \) are given lower semicontinuous convex functions and \( U \) is a real Hilbert space identified with its own dual and with inner product \( \langle \cdot, \cdot \rangle \).

We shall denote by \( W^{1,2}(0, T; H) \) the space

\[
\{ x \in L^2(0, T; H); x' \in L^2(0, T; H) \}
\]

where \( x' \) is the derivative in the sense of distributions. We shall assume that \( x_0 \in D(A) \) and \( -A \) is the infinitesimal generator of an analytic semigroup of contractions on \( H \). Then for each control \( u \in L^1(0, T; U) \), system (5.2) has a unique solution \( x_u \in W^{1,2}(0, T; H) \) with \( Ax_u \in L^2(0, T; H) \).

We associate with problem (5.1) the equation

\[
(5.3) \quad \begin{cases}
  \psi(t, x) - h^*\left(-B^*\psi(t, x)\right) - (Ax, \psi(t, x)) + g(x) = 0, & x \in D(A), \ t \in [0, T]. \\
  \psi(T, x) = q_0(x)
\end{cases}
\]

where \( h^* \) is the conjugate of \( h \) and \( B^* \) is the dual operator of \( B \).
We observe that by substitution $q(t, x) = \varphi(T - t, x)$, eq. (5.3) can be written in the form (1.1) where $F(y) = h^*(-B^*y)$ for all $y \in H$.

By analogy with Definition 1, we say that the function $\psi \in \mathcal{K}$ is a weak solution to equation (5.3) if there exists a sequence $\{\psi_n\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ such that for $\varepsilon \to 0$,

(5.4) $\psi_n \in C^1([0, T]; C(D(A)))$; $\psi_n(T) = \varphi_0$

(5.5) $\psi_n \to \psi$ in $C([0, T]; C(H))$

(5.6) $(\psi_n)_t - h^*(-B^*(\psi_n)_x) - (A\varphi, (\psi_n)_x) + g \to 0$ in $C([0, T]; C(H))$.

(5.7) The sequence $\{\psi_n\} \subset C^0([0, T]; C^0_{\text{LD}}(H, H))$ and it is bounded in $C([0, T]; C^1(H, H))$.

Here $\mathcal{K}$ is defined by (3.9) and $K$ is the set of all convex functions $\varphi \in C(H)$ such that $0 \leq \varphi(0)$ and

(5.8) $\langle (h^*)_x(-B^*y), B^*x \rangle < 0 \quad \forall [x, y] \in \partial \varphi$.

The results proved in sect. 3 and 4 give existence and uniqueness of the weak solution $\psi$ to equation (5.3) in several situations. For instance if $A = 0$ and $K$ is a closed convex cone of $C(H)$, Theorem 2 gives existence and uniqueness of a weak solution under the following assumptions:

(5.9) $\varphi_0$, $g \in K \cap C^1(H)$; $\varphi'_0$, $g' \in C^0_{\text{LD}}(H, H)$

(5.10) $h^* \in C^1(H)$ and $F(y) = h^*(-B^*y)$ satisfies (b) and (c).

If $h(u) = |u|^2/2$ and the range of $B$ is all of $H$ we may apply Theorem 3 to obtain existence and uniqueness under conditions (4.2), (4.3) and (4.4).

**Proposition 1.** Let $\psi \in \mathcal{K}$ be a weak solution to equation (5.3) where $(h^*)_x \in C^0_{\text{LD}}(H, H)$. Then for every $y \in D(A)$ and $t \in [0, T]$ one has

(5.11) $\psi(t, y) =$

$$= \inf \left\{ \int_0^T \left( \varphi(u(s)) + h(u(s)) \right) ds + \varphi_0(x(T)) ; \ u \in L^2(t, T; U), \ x(u(T)) = y \right\}.$$ 

Moreover, if $u^*$ is an optimal control in problem (5.1) then it is expressed as a
function of the optimal state $x^*$ by the feedback formula

\begin{equation}
(5.12) \quad u^*(t) = (h^*)_s(-B^*\partial \psi(t, x^*(t))) \quad \text{a.e. } t \in [0, T].
\end{equation}

Here $\partial \psi$ is the subdifferential of $\psi(t, \cdot)$.

Formula (5.12) gives the optimal feedback law of control problem (5.1). In particular if $\psi$ happens to be a classical solution to (5.3) (in particular this is the case if the conditions of Theorem 4 are satisfied) then $\partial \psi = \psi_\nu$ and it follows by (5.12) that $u^* \in C^1([0, T]; U)$.

**Proof of Proposition 1.** Let $t \in [0, T]$ and $u \in L^2(t, T; U)$ be fixed. Let $x_u$ the solution to (5.2) on $[t, T]$ such that $x_u(t) = y$. The obvious equality

\[
\frac{d}{ds} \psi_\nu(s, x_u(s)) = \langle \psi_\nu(s, x_u(s)), x_u'(s) \rangle + \langle \psi_\nu(s, x_u(s)), x_u'(s) \rangle, \quad \text{a.e. } s \in [t, T],
\]

along with (5.4) and (5.6) implies that $\psi_\nu(s, x_u(s))$ is absolutely continuous on $[t, T]$ and

\begin{equation}
(5.13) \quad \frac{d}{ds} \psi_\nu(s, x_u(s)) + g(x_u(s)) + h(u(s)) = h^*(-B^*(\psi_\nu)_s(s, x_u(s))) + h(u(s)) + \langle B^*(\psi_\nu)_s(s, x_u(s)), u(s) \rangle + \delta_u(s) \quad \text{a.e. } s \in [t, T],
\end{equation}

where $\delta_u \to 0$ uniformly on $[t, T]$.

Recalling that

\[ h(u) + h^*(\bar{u}) \geq \langle u, \bar{u} \rangle \quad \forall u, \bar{u} \in U \]

we deduce by (5.13) that

\[
\psi_\nu(t, y) \leq \int_t^T \left( g(x_u(s)) + h(u(s)) \right) ds + \psi_\nu(x_u(T)) - \int_t^T \delta_u(s) ds.
\]

Therefore

\begin{equation}
(5.14) \quad \psi(t, y) \leq \inf \left\{ \int_t^T \left( g(x_u(s)) + h(u(s)) \right) ds + \psi_\nu(x_u(T)); \ u \in L^2(t, T; U), \ x_u(t) = y \right\}.
\end{equation}

Now consider the Cauchy problem

\begin{equation}
(5.15) \quad \begin{cases}
\dot{x} + Ax = B(h^*)_s(-B^*(\psi_\nu)_s(s, x)), & s \in [t, T], \\
x(t) = y.
\end{cases}
\end{equation}
For each $\varepsilon > 0$ and $y \in D(A)$, problem (5.15) has a unique solution $x_\varepsilon \in W^{1,2}(t, T; H)$. Here is the argument.

Since $(\psi_\varepsilon)_s \in C([0, T]; C^0_{Lip}(H, H))$ and $(h^*)_s \in C^0_{Lip}(H, H)$, we deduce by a standard argument that (5.15) has a unique continuous local solution $x_\varepsilon$. In as much as $\psi_\varepsilon(s, -) \in K$ for all $s \in [0, T]$ it follows by (5.15) that

$$|x_\varepsilon(s)| < C \quad s \in [t, T']$$

where $[t, T']$ is the maximal interval of definition for $x_\varepsilon$.

Estimate (5.16) then implies by a standard device that $x_\varepsilon$ can be extended as a solution (in the « mild » sense) to (5.15) on the whole interval $[t, T]$. Clearly $x_\varepsilon \in W^{1,2}(t, T; H)$ and equation (5.15) is satisfied a.e. on $[t, T]$.

Now in (5.13) we take $u = u_\varepsilon = (h^*)_s(- B^*(\psi_\varepsilon)_s(s, x_\varepsilon))$ and obtain

$$\frac{d}{ds} \psi_\varepsilon(s, x_\varepsilon(s)) + g(x_\varepsilon(s)) + h(u_\varepsilon(s)) = \delta_\varepsilon(s) \quad \text{a.e. } s \in [t, T]$$

and therefore

$$\psi_\varepsilon(t, y) - \int_t^T (g(x_\varepsilon(s)) + h(u_\varepsilon(s))) ds + \varphi_\varepsilon(x_\varepsilon(T)) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0.$$

Along with (5.14) the latter implies (5.11) as claimed.

Let $u^* \in L^1(0, T; U)$ be any optimal control of the problem and let $x^* \in W^{1,2}(0, T; H)$ be the corresponding optimal state. By (5.11) it follows that

$$\psi(t, x^*(t)) = \int_t^T (g(x^*(s)) + h(u^*(s))) ds + \varphi_\varepsilon(x^*(T)) \quad t \in [0, T].$$

Next, in (5.13) we take $u = u^*$ and $x_\varepsilon = x^*$. We get

$$\psi_\varepsilon(t, x^*(t)) = \int_t^T \left( g(x^*(s)) + h(u^*(s)) \right) ds + \varphi_\varepsilon(x^*(T)) - \int_t^T \left( h^*(- B^*(\psi_\varepsilon)_s(s, x^*(s)) + h(u^*(s)) + \langle B^*(\psi_\varepsilon)_s(s, x^*(s)), u^*(s) \rangle + \delta_\varepsilon(s) \right) ds$$

and by (5.6), (5.17) we see that

$$\int_t^T \left( h(u^*(s)) + h^*(- B^*(\psi_\varepsilon)_s(s, x^*(s))) + \langle B^*(\psi_\varepsilon)_s(s, x^*(s)), u^*(s) \rangle \right) ds \rightarrow 0$$

uniformly on $[0, T]$.
which implies in particular that

\[ (5.18) \quad \int_0^T (h^*(-B^*(\psi_\varepsilon)_n(s, x^*(s)) - h^*(v)) \, ds < \]
\[ < \int_0^T \langle -B^*(\psi_\varepsilon)_n(s, x^*(s)) - v(s), u^*(s) \rangle \, ds + \eta_\varepsilon, \quad \forall v \in L^2(0, T; U) \]

where \( \eta_\varepsilon \to 0 \) for \( \varepsilon \to 0 \).

On the other hand, it follows by (5.7) that \( \{(\psi_\varepsilon)_n(t, x^*(t))\} \) is bounded in \( L^\infty(0, T; H) \). Thus we may assume that

\[ (5.19) \quad (\psi_\varepsilon)_n(t, x^*) \to q \quad \text{weak star in} \quad L^\infty(0, T; H) \]

and letting \( \varepsilon \) tend to zero in (5.18) we get (because the convex integrand is weakly lower semicontinuous),

\[ (5.20) \quad u^*(t) \in (h^*)_n(-B^*q(t)) \quad \text{a.e. } t \in ]0, T[ . \]

Similarly by (5.19) it follows that

\[ q(t) \in \partial \psi(t, x^*(t)) \quad \text{a.e. } t \in ]0, T[ . \]

which along with (5.20) implies (5.12) thereby completing the proof.

6. Additional regularity properties for the equation (1.3).

**Lemma 4.** For all \( \varepsilon > 0 \) and \( R > 0 \) one has

\[ (6.1) \quad \| E_\varepsilon \|_{l^2, R} < \left( 1 + \varepsilon \| E \|_{l^2, 0} \right) \left( \| E \|_{l^2, R} + 3 \| E \|_{l^1, R}^2 \right) \quad \forall E \in \Pi \cap C^2_{l^2}(H, H) \]

\[ (6.2) \quad \| E_\varepsilon - \bar{E} \|_{l^2, R} < \left( 1 + BR^2 \right) \left( 1 + 2\varepsilon BR(1 + R) + \varepsilon^2 BR^2 (1 + BR + 3R) \right) \| E - \bar{E} \|_{l^2, R} \quad \forall E, \bar{E} \in \Pi \cap C^2(H, H) \]

where \( B = \sup \{ \| E \|_{l^2, R}, \| \bar{E} \|_{l^2, R} \} \).
PROOF. The proof is standard and relies on the formula:

\begin{equation}
E^\varepsilon_\ast(x) = E^\varepsilon(J^\ast(x))(J^\ast(x), J^\ast(x)) + E(J^\ast(x)J^\ast(x))
\end{equation}

where

\begin{equation}
\begin{cases}
J^\varepsilon(x) = (1 + \varepsilon E)^{-1}(x) \\
J^\varepsilon(x) = \left(1 + \varepsilon E'\right)^{-1}(x) \\
J^\varepsilon(x) = -\varepsilon J^\varepsilon(x)E^\varepsilon(J^\ast(x))(J^\ast(x), J^\ast(x)).
\end{cases}
\end{equation}

**Lemma 5.** Assume that $E_0, E \in H \cap C^2_1(H, H)$ and $G, \tilde{G} \in C([0, T]; C^2_1(H, H))$ with $G(t) \in H, \forall t \in [0, T]$; then equation (4.27) has unique solutions $E, \hat{E} \in C([0, T]; C^2_1(H, H))$ and it is:

\begin{equation}
\|E(t, \cdot)\|_{2,R} \leq \exp \left(\alpha(t)\right)\|E_0\|_{2,\infty} + \int_0^t \exp \left(\alpha(t-s)\right)\|G(s, \cdot)\|_{2,\infty} ds
\end{equation}

\begin{equation}
|E(t, \cdot) - \hat{E}(t, \cdot)|_{1,R} \leq \exp \left(\alpha(t)\right)|E_0 - \hat{E}_0|_{1,\infty} + \int_0^t \exp \left(\alpha(t-s)\right)|G(s, \cdot) - \hat{G}(s, \cdot)|_{1,\infty} ds
\end{equation}

where $\alpha$ is independent from $\alpha$.

**Theorem 6.** Assume that $E_0 \in C^2_1(H, H) \cap H$ and $G \in C([0, T]; C^2_1(H, H))$ with $G(t) \in H, \forall t \in [0, T]$. Then there exists a unique solution $E \in C([0, T]; C^1(H, H))$ to equation (4.37).

**Proof.** Consider the approximating equation

\begin{equation}
E^\varepsilon(t, x) = \exp(-tA^\ast)E_0(\exp(-tA)x) + \int_0^t \exp(- (t-s)A^\ast) \cdot
\end{equation}

\begin{equation}
\left[\gamma^\varepsilon(E^\ast)(s, \exp(-(t-s)A)x) + G(s, \exp(-(t-s)A)x)\right] ds
\end{equation}

where

\[\gamma^\varepsilon(f) = (f - f_\ast)/\varepsilon \quad \forall f \in H.\]

We write $\gamma^\varepsilon$ in the following form:

\begin{equation}
\gamma^\varepsilon(f) = f_\ast f + R_\varepsilon(f)
\end{equation}

where

\begin{equation}
R_\varepsilon(f)(x) = \int f_\ast\left\{\tilde{x}(x) + (1 - \xi)(1 + J_\varepsilon(x)) f_\ast(x) - f_\ast(x) f(x)\right\} d\tilde{x}.
\end{equation}
Recalling (6.6), (6.9) and estimating $|E'(t, \cdot) - E_0'(t, \cdot)|_{1,H}$ via the Gronwall lemma we get:

\[
\lim_{\varepsilon \to 0} |R_{\varepsilon}(f)|_{1,H} = 0.
\]

Let now $\mu > 0$, it is:

\[
E^\mu(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot [\gamma_\mu(E^\mu) + R_\mu(E^\mu) - R_\mu(E^\mu)](s, \exp(-(t-s)A)x)ds.
\]

Recalling (6.6), (6.9) and estimating $|E'(t, \cdot) - E^\mu(t, \cdot)|_{1,H}$ via the Gronwall lemma we get:

\[
\lim_{\varepsilon \to 0} E^\varepsilon = E \quad \text{in } C([0, T]; C'(H, H))
\]

and from (6.5) it follows that $E$ is a solution to equation (4.37). To prove uniqueness consider two solutions $E_1$ and $E_2$, then for every $\beta > 0$ it is:

\[
E_i(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot [\gamma_\beta(E_i) + G - R_\beta(E_i)](s, \exp(-(t-s)A)x)ds \quad i = 1, 2.
\]

Using again (6.6), (6.9) and the Gronwall lemma we get $E_1 = E_2$.

Equation (4.37) is a «mild» form of equation (1.3). To find classical solution we consider the semigroup in $C_0(H, H)$ defined by

\[
G_i(f)(x) = \exp(-tA)f(\exp(-tA)x) \quad \forall f \in C'(H, H).
\]

We remark that $G_i$ applies II in itself; we put

\[
\begin{align*}
D(A) &= \left\{ f \in C'(H, H); \exists \lim_{h \to 0} \frac{1}{h} (G_h(f)(x) - f(x)) \in C'(H, H) \right\} \\
A(f)(x) &= \lim_{h \to 0} \frac{1}{h} (G_h(f)(x) - f(x))
\end{align*}
\]

**Lemma 6.** Assume that $f \in D(A) \cap C'(H, H)$ and $x \in D(A)$; then it is $f(x) \in D(A^*)$ and moreover:

\[
A(f)(x) = A^*f(x) + f(x)Ax.
\]
**PROOF.** Put \( g = \mathcal{A}(f) \) and take \( x, y \in D(A) \); it is:

\[
\langle g(x), y \rangle = \frac{d}{dh} \left\langle \exp(-hA^*) f(\exp(-hA)x), y \right\rangle_{h=0} = \\
= \frac{d}{dh} \left\langle f(\exp(-hA)x), \exp(-hA)y \right\rangle_{h=0} = -\langle f_x(x)Ax, y \rangle - \langle f(x), Ay \rangle,
\]

therefore the linear mapping

\[
y \mapsto \langle f(x), Ay \rangle = \langle g(x), y \rangle - \langle f_x(x)Ax, y \rangle
\]

is continuous; consequently it is \( f(x) \in D(A^*) \) and (6.11) is fulfilled.

We write now equation (1.3) in the following form

\[
(6.12) \quad \begin{cases} 
E_t + A(E) + E_x E = G \\
E(0) = E_0.
\end{cases}
\]

**THEOREM 7.** Assume that \( E_0 \in C^2_{Lip}(H, H) \cap \Pi \cap D(A) \), \( \mathcal{A}(E_0) \in C^1(H, H) \), \( G, G_t \in C([0, T]; C^2_{Lip}(H, H)) \) with \( G(t, \cdot) \in \Pi \).

Then the equation (1.3) has a unique classical solution.

**PROOF.** We can solve (in the «mild» form) the following linear problem:

\[
(6.13) \quad \begin{cases} 
V_t + A(V) + V_x E + E_x V = G_t \\
V(0) = G(0, \cdot) - A(E_0) - E_{x_2} E_0
\end{cases}
\]

where \( E \in C([0, T]; C^1(H, H)) \) is the solution to (4.37).

Let us consider the approximating problems:

\[
\begin{cases} 
E^n_t + A_n(E^n) + E^n_x E^n = G \\
E^n(0) = E_0 \\
V^n_t + A_n(V^n) + V^n_x V^n + E^n_x V^n = G_t \\
V^n(0) = G(0, \cdot) - A_n(E_0) - E_{x_2} E_0
\end{cases}
\]

where \( A_n(E) = A^*_n E + E_x A_n \) and \( A_n = n - n^2 (n + A)^{-1} \); it is clear that \( V^n = E^n_t \) and it is easy to show that

\[
V^n \to V, \quad E^n \to E \quad \text{in } C([0, T]; C^1(H, H))
\]

this implies \( V = E_t \) and the thesis follows.
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