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Internal waves in fluids with rapidly varying density


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Internal Waves in Fluids with Rapidly Varying Density (*).

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Introduction.

We consider an incompressible, inviscid fluid completely filling the region between two horizontal planes at \( y = 0 \) and \( y = 1 \). An acceleration due to gravity of magnitude \( g \) is assumed to act in the negative \( y \) direction. We choose orthogonal \((x, z)\) axes in the plane at \( y = 0 \) and assume henceforth that all quantities are independent of \( z \); that is, we examine two dimensional flows. Our aim will be to show that, with density variation present, the fluid will support traveling waves of permanent form; i.e. flow patterns which appear steady to an observer moving with a fixed velocity in the \( x \) direction. If the moving observer sees a steady flow field at \( x = \pm \infty \) which is horizontal with constant velocity \( c \), then an observer for whom the fluid at \( x = \pm \infty \) is at rest will see a wave traveling to the left, with the crest having velocity \( c \). In this paper we deal with steady flows and merely mention at one point the substitution of variables needed to produce a traveling wave.

One possible steady flow is that which is everywhere horizontal with velocity \( c \). We call this a trivial flow. Naturally, such a flow observed in a moving frame so that it is at rest at \( x = \pm \infty \), will be at rest everywhere. We shall see that to have a nontrivial wave we will have to have a nonconstant density. It is the interaction of gravity and the variable density that makes wave propagation possible. We set down one further assumption before proceeding to a discussion of a model. We will assume that the density is nondiffusive; i.e., that the gradient of the density \( \rho = \rho(x, y) \) at each point is orthogonal to the velocity vector at that point. Density stratification due to temperature variation, salinity, or dissimilar

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layers of fluid persists for time scales which are long compared to time scales for wave propagation so nondiffusivity is often assumed. A consequence of this last assumption is that \( \rho \) is constant along streamlines. We’ll see in § 1 that we can thus assume that \( \rho = \rho(\psi) \) where \( \psi \) is a « stream function ». Our analysis will depend on the explicit form of \( \rho(\psi) \). A method for choosing \( \rho \) will be explained in § 1, but can be roughly described as follows: consider a flow pattern for which the deviation from a trivial flow is essentially confined to a bounded region (one expects this behavior for a « solitary » wave). At \( x = \pm \infty \) the flow is horizontal and the variable \( \psi \) measuring the « flux » between 0 and \( y \) is an increasing function \( \psi = \psi(y) \). Thus, giving a density \( \rho = \rho_0(y) \) produces a function \( \rho(\psi) = \rho_0(\psi(\psi)) \). Taking \( \rho(\psi) \) as given one can ask for divergence free solutions of the Euler equation consistent with having the given \( \rho = \rho(\psi) \) along streamlines. This is the point of departure for an analytical treatment.

Before describing our results we give some background to the problem, but only in briefest terms, for the article of Benjamin [1] includes a thorough discussion of the mechanics underlying internal waves and describes the principal results. The earliest work on permanent waves in stratified flow was that of Keulegan [2] and Long [3] who treated a two fluid system with fixed boundaries and exhibited the characteristics of solitary waves through perturbation analysis. Peters and Stoker [4] treated a similar problem but with a free upper surface and showed that such a system supported surface as well as internal solitary waves. Benjamin’s article [1] unifies the earlier results and exhibits both cnoidal (periodic) and solitary waves in a variety of flow situations. Benjamin’s point of departure is Long’s equation [5] (cf. also Dubreil-Jacotin [6]), but incorporating a simplification due to Yih [7]. The equation is (1.5) in the present paper.

The first results in the exact theory, that is, finding solutions of the Euler equations or equivalently, of the Long-Yih equation, are due to Ter-Krikorov [8] who considers a smoothly varying density with a free boundary. He shows the existence of small amplitude periodic waves which, in the limit of increasing period, become solitary waves. His work employs methods analogous to those of Friedrichs and Hyers [9] who treated the problem of the surface solitary wave for a fluid of constant density. In a paper with J. Bona and D. Bose [10] we analyzed the Long-Yih equation in the strip \( 0 < y < 1 \), for a substantial class of densities \( \rho_0(y) \) and showed the existence of periodic and solitary waves of finite amplitude. This we did for both fixed « energy » and fixed velocity. The allowable class of densities included, for example, \( \rho_0(y) = \alpha(1 - \beta y)^n \) with \( \alpha, \beta, \) and \( n \) positive, though as \( n \) approaches the value 4 the waves are restricted in amplitude. The method used was basically variational, but involved the use of sym-
metric rearrangements of functions as well. The method appears to break down if the density $\rho_0$ is not convex or concave, in a rough sense. In the present paper we treat a density which undergoes a rapid transition between two constant values and which in a suitable limit approaches a density which is piecewise constant. As noted, the formulation in [10] seems ill-suited to the problem at hand in which we consider a density which rapidly changes from concave to convex. Instead, as is done in [1] and [8] we formulate a problem for $y$ with $x$ and $\psi$ as independent variables. Ter-Krikorov analyzed an equation for $y(x, \psi)$ directly, not using a variational formulation. Benjamin’s perturbation scheme is based on a certain order of vanishing of a flow force integral and elucidates the behavior of the lowest order terms in a perturbation expansion of solutions of the Euler equations. Here we use a variational method to arrive at solutions of the Euler equation. There are many parallels with the work in [10], but as we must deal with a singular quasilinear elliptic expression rather than the Laplacian, the technical problems are completely different.

To describe the results we obtain we must introduce some further notation. We assume that the density $\rho_\omega(y)$ is a constant $\rho_0$ for $0 < y < 1 - \eta_1$ ($0 < \eta_1 < 1$), decreases smoothly to a constant $\rho_1$ ($0 < \rho_1 < \rho_0$) over a transition region $1 - \eta_1 < y < 1 - \eta_1 + \delta$ and remains $\rho_1$ for larger $y$. The methods would apply to more general densities, but to simplify an already lengthy analysis, we restrict the density as indicated. In all of the results the transition width $\delta$ and the "wave energy" $R$ (roughly an $H^1$ norm measuring the deviation from a trivial flow) must be suitably small. We first show the existence of waves which are periodic in $x$ with period $2k > 0$ and have energy $R > 0$. There can, of course, be flows with no vertical component of velocity—"conjugate" flows in the sense of Benjamin [11]. In fact the period must be sufficiently large to insure a vertical component (cf. Corollary 7.1). A basic feature of the flow is determined by the sign of $\epsilon = \rho_0(1 - \eta_1)^2 - \rho_1 \eta_1^{-2}$. For $\epsilon > 0$ and $k$ large we find waves of elevation while for $\epsilon < 0$, they are waves of depression. These results follow from Corollary 7.1 as well. The distinction, based on the sign of $\epsilon$, is consistent with earlier perturbation analyses. The wave amplitude is even in $x$, about the crest, while the amplitude and velocity decay exponentially over the interval $0 < x < k$ independently of $k$ (cf. § 4 and § 7). The estimates obtained allow us to let $\delta$ approach zero and obtain periodic waves for a discontinuous density. The existence of small amplitude periodic waves for two fluids can be reduced to a problem of bifurcation from a simple eigenvalue (cf. Zeidler [12] and references there). However, the result obtained that way appears not to allow a nontrivial limit as the period increases to infinity. We can let $k \to \infty$ with $\delta > 0$ or with $\delta = 0$ and obtain solitary
waves with exponential decay characteristics over the entire interval $0 < x < \infty$. These results are proven in the last section. By proving results which are independent of the transition width $\delta$ we show that the models with $\delta > 0$ (which one might believe are more physically reasonable) and the model with $\delta = 0$ are essentially interchangeable.

The outline of the paper is the following: In § 1 we give a formulation of the problem. In § 2 we exhibit the variational nature of the problem and state the main results of the paper in Theorem 2.1. The original variational problem is singular, but can be approached by means of a regularized problem. The nonsingular problem is solved in § 2 and some preliminary estimates done. To show that the solution obtained in § 2 is a solution of the original problem we require the regularity estimates of § 3. In § 4 we show that the solution can be assumed to have certain symmetry properties. The bounds in § 5 show that there are waves with speeds above the speed of “infinitesimal long waves”; i.e., that we have super-critical wave speeds. In § 6 we obtain lower bounds on the amplitude of waves, independently of the period $2k$. This is a prerequisite to obtaining non-trivial waves in the limit of increasing periods. The exponential decay characteristics are shown in § 7. In § 8 we recapitulate the various results obtained and state Theorem 8.1 which covers the assertions of Theorem 2.1 when $\delta > 0$ and $k < \infty$. The remainder of § 8 is occupied with the limiting forms of solutions corresponding to having a transition width $\delta = 0$ or a limit $k = \infty$ of increasing periods. Theorems 8.2-8.4 detail the limiting behavior and include the remaining cases of Theorem 2.1.

1. Formulation of the problem.

We begin with a description of a problem in spatial independent coordinates, i.e. in $S = \{(x, y) | -\infty < x < \infty, 0 < y < 1\}$ and later transform it so that $x$ and a stream variable $\psi$ are independent coordinates. As we will ultimately state results in the latter formulation we will proceed formally, postponing regularity requirements for the second formulation. We let $U(x, y)$ and $V(x, y)$ be the horizontal and vertical components, respectively, of a vector field $\mathbf{q} = (U, V)$ defined in $S$. Let $p = p(x, y)$ denote the pressure in a fluid occupying the region $S$; $\rho(x, y)$, its density; and $g$, the acceleration of gravity in the negative $y$ direction. We ask that the steady state Euler equations

\begin{align}
q(UU_x + VU_y) &= -p_x \\
q(UV_x + VV_y) &= -p_y + \rho g
\end{align}
be satisfied and that the steady continuity equation

\begin{equation}
\rho \, \text{div} \, \mathbf{q} + \mathbf{q} \cdot \text{grad} \, \rho = 0
\end{equation}

be satisfied. Here we use subscripts for partial derivatives; we will also use other standard notation for derivatives in what follows. If we have a solution of (1.1), (1.2), then \( \tilde{U}(x, y, t) = U(x + ct, y) - c \), \( \tilde{V} = V(x + ct, y) \), \( \tilde{\rho} = \rho(x + ct, y) \), and \( \tilde{\sigma} = \sigma(x + ct, y) \) satisfy the time dependent Euler equation and continuity equation (cf. [13], Chapter 1). The \((\tilde{U}, \tilde{V})\) flow field represents the \((U, V)\) field as seen by an observer moving in the positive \( x \) direction with velocity \( c \). In particular, if \((U, V)\) approaches \((c, 0)\) as \( x \to \pm \infty \) then to an observer at \( x = 0 \) \((\tilde{U}, \tilde{V})\) represents a wave of permanent shape traveling in the negative \( x \) direction with velocity \( c \), in fluid which is \(*\) at rest \(*\) at \( x = \pm \infty \).

As noted in the introduction, we will use a formulation of the steady flow problem introduced by Long [5] (cf. also [6]) and simplified by Yih [7]. It involves a further supposition that the density \( \rho \) is nondiffusive; i.e. \( \mathbf{q} \cdot \text{grad} \, \rho = 0 \). As a consequence \( \rho \) will be constant on each streamline. Further, if \( l(\rho) \) is any differentiable function of \( \rho \), then

\[ \text{div} \, (l(\rho) \mathbf{q}) = l(\rho) \, \text{div} \, \mathbf{q} + l'(\rho) \mathbf{q} \cdot \text{grad} \, \rho = 0 \]

and thus \( l(\rho) \mathbf{q} \) has an associated pseudo-stream-function. In particular, if \( l(\rho) = \rho^{1/4} \) then there is a pseudo-stream-function \( \psi = \psi(x, y) \) such that

\begin{equation}
\frac{\partial \psi}{\partial y} = \rho^{1/4} \, U, \quad \frac{\partial \psi}{\partial x} = -\rho^{1/4} \, V
\end{equation}

and since \( \mathbf{q} \cdot \text{grad} \, \rho = 0 \), \( \rho \) is a function of \( \psi \), \( \rho(\psi) \). Further, from Bernoulli's theorem, along each streamline the stagnation pressure or \(*\) total head \(*\) \( H \) is constant. Thus

\begin{equation}
H = p + \tfrac{1}{2} \rho(U^2 + V^2) + \rho y = H(\psi).
\end{equation}

If one eliminates \( p \) from (1.1) using (1.2)-(1.4) one finds that

\begin{equation}
\Delta \psi(x, y) + g y \psi'(\psi) = H'(\psi)
\end{equation}

where \( \Delta \psi = \psi_{xx} + \psi_{yy} \) and primes denote derivatives. We call (1.5) the Long-Yih equation. Conversely, if one chooses arbitrary functions \( \rho(\psi) \) and \( H(\psi) \) and can solve (1.5) for \( \psi = \psi(x, y) \), then with \( (U, V) \) defined by (1.3) and \( p \) defined using (1.4), the equations (1.1) and (1.2) will be satisfied.
Our ultimate aim is to obtain a solution of (1.1), (1.2) which will correspond to a solitary wave; that is, for which the flow at \( x = \pm \infty \) approaches a horizontal flow of velocity \( (c, 0) \) in a fluid with a given density stratification \( \varrho \) at \( \infty \), depending only on the vertical coordinate, say \( \varrho = \varrho_\infty(y) \). In choosing these conditions at \( x = \pm \infty \) we are effectively choosing the functions \( \varrho(\psi) \) and \( H(\psi) \). To see this we examine what \( \varrho \) and \( H \) would be in a totally horizontal flow with \( (U, V) = (c, 0) \). Since \( V = 0 \), \( \psi \) in (1.3) is a function of \( y \) alone; we’ll denote it by \( \hat{\psi}(y) \). Then (1.3) becomes

\[
\frac{\partial \hat{\psi}}{\partial y} = \varrho^{1/2}_\infty(y) c .
\]

We initially assume that \( \varrho_\infty \) is positive and infinitely differentiable. If we assume \( \hat{\psi}(0) = \psi_0 \), then

\[
\hat{\psi}(y) = \psi_0 + \int_0^y \varrho^{1/2}_\infty(s) c \, ds ,
\]

a strictly increasing function of \( y \) with a strictly increasing inverse function \( \hat{\psi}(\psi) \). Then \( \varrho \) as a function of the stream parameter \( \psi \) is

\[
\varrho(\psi) = \varrho_\infty(\hat{\psi}(\psi)) .
\]

From the second Euler equation (cf. (1.1)) we see that \( p \) must be hydrostatic:

\[
p(y) = -\int_0^y \varrho_\infty(s) g \, ds
\]

and hence

\[
H(\psi) = \frac{1}{2} \varrho(\psi) c^2 + \varrho(\psi) g \hat{\psi}(\psi) .
\]

Naturally, the function \( \psi(x, y) = \hat{\psi}(y) \) is a solution of (1.5) for any choice of \( c \) and thus we have a one-parameter family of solutions of (1.5). We call a flow corresponding to \( \hat{\psi}(y) \) and a given \( c \) a trivial flow. In studying the Long-Yih equation (1.5) we will use the functions \( \varrho \) and \( H \) described in (1.8) and (1.10), and eventually ask that \( \psi(x, y) - \hat{\psi}(y) \to 0 \) as \( |x| \to \infty \), the idea being that for a solitary wave the flow for large \( x \) should look like a trivial flow. As a step toward obtaining solitary waves we also treat waves which are periodic in \( x \). For these the trough between waves will approximate a trivial flow. To obtain nontrivial flows we’ll have to impose conditions on the density \( \varrho_\infty \). We can see that if \( \varrho_\infty \) is a constant, \( \varrho(\psi) \) and
$H(\psi)$ are both constants and (1.5) reduces to Laplace’s equation. If we
specify $\psi(x, 0) = \psi_0$ and $\psi(x, 1) = \psi_1 = \psi_0 + \int_0^1 \mathcal{L}_{-\frac{3}{2}}(s) \, ds$, then the only har-
monic function which is bounded in the strip $S$ and satisfies the boundary
conditions is a linear function of $y$ corresponding to a trivial flow. Thus a
necessary condition for nontrivial flows is a variable density $\varrho_{\alpha}(y)$.

It should be remarked that the functions $\chi(\psi)$ and $H(\psi)$ are defined
by (1.8) and (1.10), respectively, only in the range $\psi_0 < \psi < \psi_1$. In a treat-
ment of the problem using $\psi$ as a dependent variable (as in [10]) it is common
to extend the functions outside $[\psi_0, \psi_1]$ and eventually show that any solu-
tion obtained has range in $[\psi_0, \psi_1]$. An extension will not be necessary here as we will use $\psi$ as an independent variable.

We’ll assume that no reversal of flow occurs in any vertical section;
i.e., that $U > 0$ or, equivalently, that $\psi$ is strictly increasing in $y$ for each $x$.

For such a $\psi$ the inverse function $y = y(x, \psi)$ exists and it is this function
we’ll work with. If we assume that $\psi$ has two continuous derivatives and $\psi_\psi > 0$, then from (1.5), and the relations $\psi_x + \psi_y \psi_x = 0$ and $\psi_x \psi_y = 1$
we see that $y(x, \psi)$ satisfies

$$
-\frac{\partial}{\partial x} \left[ \frac{y_x}{y_y} \right] + \frac{1}{2} \frac{\partial}{\partial \psi} \left[ \frac{1 + y_x^2}{y_y^2} \right] + \varrho' y = H'.
$$

Our basic problem is finding solutions of (1.11) satisfying

$$
y(x, \psi_0) = 0, \quad y(x, \psi_1) = 1.
$$

To describe the regularity required of solutions and to have notation
for use in the sequel we introduce some function spaces. Let $D_k = \{(x, y) | |x| < k, \psi_0 < \psi < \psi_1\}$ and denote $D_\infty$ by $D$. The space $C^k(D)$ consists of
terms which are $2k$ periodic in $x$ with the norm

$$
\|y\|_{C^k(D)} = \sup_{D_\infty} |y(x, \psi)|;
$$

$C^l(D)$ consists of those $y \in C^k(D)$ having continuous derivatives through order $l$
with

$$
\|y\|_{C^l(D)} = \sup_{D_\infty} |D^\beta y|
$$

where $\beta = (\beta_1, \beta_2)$ is a pair of nonnegative integers, $|\beta| = \beta_1 + \beta_2$, and $D^\beta$ represents the partial derivative $\frac{\partial^{|\beta|}}{\partial x^{\beta_1} \partial y^{\beta_2}}$. For $0 < \alpha < 1$ we let $|y|_\alpha$
be the smallest constant $C$ for which

$$|y(x, \psi) - y(\tilde{x}, \tilde{\psi})| < C(\|x - \tilde{x}\| + \|\psi - \tilde{\psi}\|)^{\alpha/2}$$

and let $C^{\alpha+\beta}(\mathcal{D})$ consist of the $y \in C^{\alpha}_{k}$ for which $|\partial^\beta y|_a < \infty$ for $|\beta| < l$ with the norm

$$\|y\|_{C^{\alpha}_{k}} = \|y\|_{C^{\alpha}_{k}} + \sup_{|\beta| = l} |\partial^\beta y|_a .$$

Functions in $C^{\alpha+\beta}(\mathcal{D})$ have natural extensions to the closure $\mathcal{D}$, preserving the norm and we write $C^{\alpha+\beta}(\overline{\mathcal{D}})$ for the space of extended functions. The space $C^{\alpha}_{k}(\mathcal{D})$ denotes that subset of $C^{\alpha+\beta}_{k}(\mathcal{D})$ having continuous extensions to $\mathcal{D}$. We use $L^{2}_{k}(\mathcal{D})$ to denote the space of measurable functions $y$, $2k$ periodic in $x$, which are square integrable on $\mathcal{D}$, and define a norm

$$\|y\|_{L_{k}^{2}} = \left( \int_{\mathcal{D}} |y(x, \psi)|^2 \, dx \, d\psi \right)^{1/2} .$$

We let $H^{1}_{k}(\mathcal{D}) = \{ y \in C^{1}_{k}(\mathcal{D}) | y(x, \psi_{0}) = y(x, \psi_{1}) = 0 \}$ and let $H^{1}_{k}$ be the completion of $C^{1}_{k}(\mathcal{D})$ in the norm

$$\|y\|_{H^{1}_{k}} = \|y\|_{L_{k}^{2}} + \left( \int_{\mathcal{D}} |\nabla y|^2 \, dx \right)^{1/2} .$$

For an interval $I$, $H^{1}(I)$ denotes the standard space of functions which vanish at the endpoints and have one derivative in $L^{2}$.

A space without a subscript $k$ will denote the corresponding function space on the whole strip $\mathcal{D}$. If we are considering functions on another 2 dimensional strip, such as $S$ above, we use $C^{\alpha}_{k}(S)$, etc. to denote the analogous spaces.

For any arbitrary open set $\mathcal{D}' \subset \mathcal{D}$ we use $L^{\infty}(\mathcal{D}')$ to denote the space of measurable, essentially bounded functions, with the standard norm denoted $\|y\|_{L^{\infty}(\mathcal{D}')}$. We let $W^{1,\infty}(\mathcal{D}')$ denote the collection of continuous functions on $\mathcal{D}'$ with $L^{\infty}$ distribution derivatives and let

$$\|y\|_{W^{1,\infty}(\mathcal{D}')} = \|y\|_{L^{\infty}} + \|y_{x}\|_{L^{\infty}} + \|y_{\psi}\|_{L^{\infty}} .$$

For a density $\varrho_{0}(y)$ which is continuously differentiable we'll find a solution $y$ of (1.11), (1.12) in $C^{2+\alpha}_{k}(\overline{\mathcal{D}})$ or in $C^{2+\alpha}_{k}(\overline{\mathcal{D}}) \cap L^{2}(\mathcal{D})$. In § 8 we'll consider a limiting case of a discontinuous $\varrho_{0}$ and in that case we need a weaker formulation. If $\overline{\mathcal{D}}$ is a subset of $\mathcal{D}$ on which $\varrho'$ and $\mathcal{H}'$ are con-
continuous we call $y(x, \varphi) \in C^1(\bar{D})$ (with $y_\varphi > 0$) a weak solution of (1.11) on $\bar{D}$ if for all $C^\infty$ test functions $\varphi$ with support in $\bar{D}$,

\begin{equation}
\int_{\bar{D}} \left[ \frac{y_\varphi}{y_\varphi} \varphi_x - \frac{1}{2} \left( 1 + \frac{y_\varphi^2}{y_\varphi^2} \right) \varphi_y + g \varphi y_\varphi - H'(\varphi) \right] dx \, d\varphi = 0.
\end{equation}

Let $D^+ = \{(x, \varphi) \in D|0 < \varphi < \varphi_1\}$ and $\bar{D} = D - \bar{D}^-$, the bar denoting the closure. Suppose $g'(\varphi)$ and $H'(\varphi)$ are continuous on $D^+\cup D^-$, with a possible discontinuity at $\varphi = 0$.

**Definition 1.1.** If $y \in C^0(\bar{D}) \cap C^{1+\alpha}(\bar{D}^+) \cap C^{1+\beta}(\bar{D}^-)$ for some $\alpha > 0$, $\beta > 0$; $y_\varphi \in C^0(D)$; and $y_\varphi > 0$; we call $y$ a solution of (1.11) if the following two conditions are satisfied:

i) $y$ is a weak solution of (1.11) in $D^+$ and $D^-$;

ii) the pressure $p(x, \varphi)$ computed from (1.4) in each of $D^+$, $D^-$ has a continuous extension to $D$.

The notion of solution incorporates what is physically expected at a streamline where the density is discontinuous; that is, continuity of the slope $y_\varphi$ on a streamline and of the pressure. One could equally well introduce the notion of a weak solution on all of $D$ from which the continuity of pressure would follow.

2. - A variational problem.

At the outset we focus attention on solutions of (1.11) which are $2k$ periodic in $x$. In section 8 we will allow $k$ to approach $\infty$ to obtain solitary waves. Let $S_k = \{(x, y)||x| < k, 0 < y < 1\}$ and $S_\infty = S$. The Long-Yih equation (1.5) is formally the Euler equation for the functional

\begin{equation}
M(\varphi) = \int_{S_k} \left[ \frac{1}{2} |\nabla \varphi|^2 - g \varphi \varphi_\varphi + H(\varphi) \right] dx \, dy
\end{equation}

defined for $\varphi \in H^1_k$. In this section we assume $\varphi$ is smooth. Then $M$ is Frechet differentiable and

\begin{equation}
\langle M'(\varphi), \chi \rangle = \int_{S_k} \left[ \nabla \varphi \nabla \chi - g \varphi \varphi_\varphi \chi + H'(\varphi) \chi \right] dx \, dy
\end{equation}
where $\langle \cdot, \cdot \rangle$ represents the pairing between $H_x^1$ and its dual. At a critical point of $M$, i.e. where $M'(\psi) = 0$, we see that (2.2) yields the weak form of the Long-Yih equation. A functional which formally has equation (1.11) as its Euler equation is

$$\Phi(y) = \int_{D_x} \left[ \frac{1}{2} \frac{1 + y_x^2}{y_y} + gq'(\psi) \frac{y^2}{2} - H'(\psi)y \right] dx \, dy $$

in which the terms correspond (up to an added constant in the case of the second and third terms) to those in $M$, respectively, under the change of variables. Choosing a domain for $\Phi$ presents difficulties due to the presence of $y_y$ in a denominator in the integrand. We will accommodate this by altering the functional, but only after some further reformulation. The set $\mathcal{A} = \{ y \in C^1(D) | y_y > 0, y(x, \psi_1) = 0, y(x, \psi_2) = 1 \}$ is a domain in which $\Phi'$ exists, though not one suited to invoking variational principles. The function $\hat{\psi}(y)$ (cf. (1.7)) is the pseudo-stream-function for a trivial flow and is a solution of (1.5) as one easily verifies. Its inverse function $\hat{\psi}(\psi) \in A$ is a solution of (1.11) and thus a critical point of $\Phi$; i.e. $\langle \Phi'(\hat{\psi}), u \rangle = 0$ where $\langle \cdot, \cdot \rangle$ denotes the pairing of $C^1$ with its dual; here the pairing takes the form $\int \hat{\psi}'(\psi) \psi_y = 0$ for any $\psi \in C^1$ which vanishes on $\psi = \psi_1, \psi_2$. Consider $y \in \mathcal{A}$ and let $u = y - \hat{\psi}$. Defining

$$N(y) = \int_{D_x} \left[ \frac{1}{2} \frac{1 + y_x^2}{y_y} \right] dx \, dy $$

we find

$$N(\hat{\psi} + u) = \int_{D_x} \left[ \frac{1}{2} \frac{1}{y_y} - \frac{1}{2} u_y+ \frac{1}{2} \left( \frac{u_x^2 + \hat{\psi}'(\hat{\psi} + u_y) \right) \right] =
$$

$$= N(\hat{\psi}) - \langle N'(\hat{\psi}), u \rangle + \int_{D_x} \left[ \frac{1}{2} \frac{u_x^2 + \hat{\psi}'(\hat{\psi} + u_y) \right] .$$

It is then easy to see that

$$\Phi(y) = \Phi(\hat{\psi} + u) = \Phi(\hat{\psi}) + \langle \Phi'(\hat{\psi}), u \rangle + E(u)$$

where

$$E(u) = \int_{D_x} \left[ \frac{1}{2} \frac{\hat{\psi}'+u^2}{y_y} + gq'(\psi) \frac{u^2}{2} \right] dx \, dy .$$

Since $\Phi'(\hat{\psi}) = 0$, $y$ is a critical point of $\Phi$ if and only if $u$ is a critical point of $E$, the domain of $E$ being naturally inherited from that of $\Phi$. A further
change of variables will reduce $E$ to a more convenient form. As is done in [8] we let
\begin{equation}
\eta(\psi) = \tilde{\psi}(\psi) - \tilde{\psi}(0) = \int_0^\psi \frac{ds}{\rho^{-1}(\tilde{\psi}(s))})
\end{equation}
and
\begin{equation}
u(x, \eta) = u(x, \psi(\eta)) = y(x, \psi(\eta)) - \tilde{\psi}(\psi(\eta)); \hspace{1cm} \theta(\eta) = \rho(\psi(\eta))
\end{equation}
where $\psi(\eta)$ is inverse to $\eta(\psi)$. Then $E$ corresponds to
\begin{equation}
G(w) = \int_{\Delta w} \left[ \frac{1}{2} \rho^2 \rho(\eta) \frac{v_1^2}{1 + w_n} + g(\rho(\eta) \frac{w_n^2}{2} \right] dx d\eta
\end{equation}
where $\Delta_w = \{(x, \eta) | \eta < k, \eta(\eta_0) = \eta < \eta_1 = \eta(\eta_1)\}$ and $\Delta_\infty = \Omega$. Since the derivative $\eta_\eta$ is bounded above and below, one shows easily that $u$ is a critical point of $E$ if and only if $w$ is a critical point of $G$, the domain of $G$ now consisting of $w \in C^1(\Omega)$ for which $w_n > -1$.

From (2.8), $\eta$ is related to the original $y$ scale by a shift of size $-\tilde{\psi}(0) = -\eta_0 < 0$. However, $\eta$ is a stream coordinate. Note that from (1.8) and (2.9)
\begin{equation}
\tilde{\psi}(\eta) = \rho(\tilde{\psi}(\psi(\eta))) = \rho(\eta - \eta_0)
\end{equation}
so $\tilde{\psi}$ has no dependence on $c$. In our subsequent analysis of the problem we will use $x$ and $\eta$ as independent coordinates and will omit the tilde on $\eta$.

When we have occasion to refer back to $E$ we will make clear the distinction between the variables $\eta$ and $\psi$. For notational convenience we will sometimes replace the pair $(x, \eta)$ by $(x_1, x_2)$.

We have still not specified the function $\rho$ and do it at this point. From the last paragraph we see that it suffices to describe its dependence on $\eta$. For given constants $\rho_0$ and $0 < \rho_1 < \rho_0$ we let $\rho = \rho(\eta)$ be a non-increasing $C^\infty$ function such that
\begin{equation}
\begin{cases}
\rho(\eta) = \rho_0; & \eta_0 < \eta < 0, \\
\rho_0 > \rho(\eta) > \rho_1; & 0 < \eta < \delta, \\
\rho(\eta) = \rho_1; & \delta < \eta < \eta_1.
\end{cases}
\end{equation}
The particular form of $\rho$ on $0 < \eta < \delta$ is not important. In fact the methods we use would allow some smooth variation in density outside of the interval $[0, \delta]$. However, for simplicity we treat the case described here, our main
objective being the description of internal waves with a rapid variation in
density, the extreme case being a discontinuity. Note that the description
(2.11) corresponds to a rapid density change at height \( y = |\eta_o| = 1 - \eta_1 \)
in the original spatial variable.

The functional \( G \) is defined for any \( w \in C_1^k \) satisfying \( w_\eta > -1 \). If we
let \( \lambda = g/\sigma^2 \) and define

\[
\begin{align*}
F(w) &= \int_{\partial \Omega} \frac{1}{2} \theta(\eta) \frac{w_x^2 + w_\eta^2}{1 + w_\eta} \, dx \, d\eta \\
B(w) &= -\int_{\partial \Omega} \theta'(\eta) \frac{w_\eta^2}{2} \, dx \, d\eta
\end{align*}
\]

(2.12)

then a critical point of \( G \) corresponds to a solution of the nonlinear eigen-
value problem

\[
F'(w) = \lambda B'(w)
\]

(2.13)

which one can view as arising from a constrained variational problem. Of
course the integrand in \( F \) inherits the defect of having a denominator
\( 1 + w_\eta \) which can be zero. On the other hand we’ll see that it is the source
of solitary waves; in its absence, (2.13) would be linear and, as some
elementary Fourier analysis will show (cf. § 7), cannot have a nontrivial
solution \( w \) with \( \nabla w \) in \( L^2(\Omega) \).

We’ll see (cf. (2.21)) that the linearization of (2.13) at \( w = 0 \):

\[
\begin{aligned}
\frac{\partial}{\partial x} \theta \frac{\partial w}{\partial x} + \frac{\partial}{\partial \eta} \theta \frac{\partial w}{\partial \eta} &= \lambda \theta w ,
\end{aligned}
\]

\( w = 0 \) at \( \eta = \eta_0, \eta_1 \), has a lowest eigenvalue \( \mu = \mu_0 \) (\( \delta > 0 \)) corresponding
to a velocity \( c_0 = (g/\mu)^{1/2} \), a so-called « critical » velocity. With this notation
and noting that \( w(x, \eta) \) represents the deviation from horizontal of the stream-
line having the label \( \eta \) we now state the main results of the paper.

**Theorem 2.1.** Let \( \epsilon = \epsilon_0 \eta_0^{-2} - \epsilon_1 \eta_1^{-2} \) and suppose \( \epsilon \neq 0 \). Then there are
positive numbers (all constants appearing depend on \( c_0, 0 < \epsilon_1 < c_0, \) and \( \eta_0 \)
in addition to indicated parameters) \( \delta, \bar{R}, \) and \( \bar{k} = \bar{k}(R) \) such that for \( 0 < \delta < \delta, \)
\( 0 < R < \bar{R} \) and \( \bar{k}(R) < \bar{k} < +\infty \) the problem (1.11)-(1.12) has a nontrivial
solution \( (\lambda, y) \) (\( \lambda \) is implicit) with the following properties

1) \( 0 < \lambda < \mu_0(1 - C_R^{4/3}) \); i.e., the associated speed \( c = (g/\lambda)^{1/2} \) is
« supercritical ». 
2) $y$ has period $2k$ in $x$ (for $k < \infty$).

The streamline displacement

$$w(x, \eta) = y(x, \psi(\eta)) - \dot{\psi}(\psi(\eta)) \quad (\text{cf. } (1.7), (2.8), (2.9))$$

satisfies.

3) $F(w) = R^2$.

4) The associated flow is a wave of elevation ($w > 0$) if $e > 0$ and depression ($w < 0$) if $e < 0$.

5) $w(x, \eta) = w(-x, \eta)$ and for $0 < x_1 < x_2 < k$, $w(x_1, \eta) - w(x_2, \eta) > 0$ $(< 0)$ if $e > 0$ $(< 0)$.

6) $|w| < C_1 \exp[-px]$ and $|\nabla w| < C_2 \exp[-px]$, on $0 < x < k$ for a $p > 0$, $C_i = C_i(R, p)$, $i = 1, 2$.

The vertical component of velocity $V$ is $\neq 0$ and satisfies $|V| < C \exp[-px]$ on $0 < x < k$.

Theorem 2.1 is included in a series of results stated in § 8 which also detail the regularity properties of solutions and the convergence properties as $\delta \to 0$ or $k \to \infty$. The intervening sections contain the substance of the proofs.

All the results in Theorem 2.1 will follow from a study of equation (2.13) to which we now turn. To circumvent the unboundedness in $F$, we alter its integrand to remove the singularity, solve a nonsingular problem, and show that by restricting the size of the solution we obtain a solution of the original problem. Let $\xi = \xi(s)$ be a $C^\infty$ function, defined for $s > 0$, which is 1 for $0 < s < 1$, decreases to zero at $s = 2$, and is zero for $s > 2$. Let

$$f(p_1, p_2) = \frac{1}{2} \frac{p_1^2 + p_2^2}{1 + p_2}$$

so that $g(\nabla w)$ is the integrand in $F$. We define

$$a(p_1, p_2) = \xi f(p_1, p_2) + (1 - \xi_r) \frac{p_1^2 + p_2^2}{2}$$

(2.14)

where $\xi_r = \xi((p_2^2 + p_2^2)/r^2)$ and $0 < r^2 < \frac{1}{4}$. We will see shortly that $a(p_1, p_2)$ has a positive Hessian for suitably restricted $r$. Suppose we replace $F$ by

$$A(w) = \int_{\mathcal{D}_0} g(\eta) a(w_x, w_\eta) \, dx \, d\eta$$

(2.15)
for \( w \in \mathcal{H}_k^1 \). One shows easily that \( A \) is Frechet differentiable. If we find a solution of the quasilinear elliptic problem

\[
A'(w) = \lambda B'(w)
\]

with \(|\nabla w| < r\), then \( w \) will be a solution of \( F'(w) = \lambda B'(w) \) since \( A' \) and \( F' \) coincide for such a \( w \). We'll see that the existence problem for (2.16) is not difficult. The main work in the paper will be to obtain estimates on \( w \) and its derivatives. A different convex extension of \( f \) might allow for sharper estimates and a larger value of \( \tilde{R} \) in Theorem 2.1. We have, however, chosen not to carry explicit constants through all the elliptic estimates.

We begin with technical lemmas regarding \( a(p_1, p_2) \). Denote \( \partial a / \partial p_i \) by \( a_i \), \( \partial^2 a / \partial p_i \partial p_j \) by \( a_{ij} \), and third derivatives by \( a_{ijk} \) in obvious notation. Similarly, denote derivatives of \( f \) by \( f_i \) and \( f_{ij} \). We use the convention that repeated indices are summed from 1 to 2 and let \( O(r) \) be a term bounded by a constant times \( r \), the constant being a computable numerical constant.

**Lemma 2.1.** There exists an \( r_0 \), \( 0 < r_0 < 1/\sqrt{2} \) such that for \( 0 < r < r_0 \), there are positive constants \( \sigma_i, i = 1, 2, 3, 4, 5 \) and \( \nu > 0 \) such that \( a = a(p_1, p_2) \) defined by (2.14) satisfies

1) \( \frac{1}{2} \sigma_1 (p_1^2 + p_2^2) < a(p_1, p_2) < \frac{1}{2} \sigma_5 (p_1^2 + p_2^2) \);
2) \( \sigma_2 (p_1^2 + p_2^2) < a_1 p_1 + a_2 p_2 < \sigma_4 (p_1^2 + p_2^2) \);
3) \( a_1^2 + a_2^2 < \sigma_5 (a_1 p_1 + a_2 p_2) \);
4) \( a_{ij} \zeta_i \zeta_j > \nu (\zeta_i^2 + \zeta_j^2) \) for all \( p_i, p_j \);
5) \( \text{grad } a = (0, 1) + O(r) \);
6) \( a_{ijk} (p_1, p_2) = a_{ijk} (0, 0) + O(r) \); at \( (0, 0) \), \( a_{111} = 0 \), \( a_{112} = 2 \), \( a_{122} = 0 \), and \( a_{222} = 6 \), the other derivatives being equal to one of these by symmetry.

**Proof.** The function \( a \) can be written

\[
a(p_1, p_2) = \frac{p_1^2 + p_2^2}{2} \left( 1 - \frac{\xi \cdot p_2}{1 + p_2} \right)
\]

and so part 1) follows provided \(|\xi \cdot p_2/(1 + p_2)| < 1\). Since \( \xi = 0 \) for \( p_1^2 + p_2^2 > 2r^2 \) we can assume \( |p_2| < r \sqrt{2} \) and then any choice of \( r < 1/2 \sqrt{2} \) will suffice. Of course the \( \sigma_i \) depend on the choice of \( r \). For part 2) we write \( a \) as

\[
a(p_1, p_2) = \frac{p_1^2 + p_2^2}{2} - \frac{1}{2} \frac{(p_1^2 + p_2^2) p_2}{1 + p_2} \xi \cdot
\]
from which we calculate

\[
\begin{align*}
   a_1 &= p_1 - p_1 \frac{p_2 \frac{\xi_r}{1 + p_2 - \frac{(p_1^2 + p_2^2) p_1 p_2 \frac{\xi_r}{r^2}}{1 + p_2}}}{2 (1 + p_2)^2} \\
   a_2 &= p_2 - \frac{p_2^2 \frac{\xi_r}{1 + p_2} - \frac{1}{2} (p_1^2 + p_2^2) \frac{\xi_r}{1 + p_2}}{1 + p_2} - \frac{(p_1^2 + p_2^2) p_2 \frac{\xi_r}{r^2}}{1 + p_2 r^2}.
\end{align*}
\]

Thus

\[
a_1 p_1 + a_2 p_2 = (p_1^2 + p_2^2) \left[ 1 - \frac{p_2 \frac{\xi_r}{1 + p_2}}{2 (1 + p_2)^2} - \frac{(p_1^2 + p_2^2) p_2 \frac{\xi_r}{r^2}}{(1 + p_2 r^2)} \right].
\]

From this last expression it is an easy matter to see that the inequalities in 2) hold for a sufficiently small \( r \). Item 3) follows similarly.

As regards the second derivatives, they deviate from the Kronecker \( \delta_{ij} \) only for \( p_1^2 + p_2^2 < 2r^2 \) and then by at most a finite sum of terms, each a constant times an expression \( p_1^{m_1} p_2^{m_2} (1 + p_2)^{-k} \) with \( k = 1 \) or 2; \( l = 0, 1 \) or 2; \( m_1 > 0, m_2 > 0 \), and \( m_1 + m_2 - 2l > 1 \). That is, \( a_{ii} = \delta_{ii} + O(r) \) and hence 4) and 5) hold. Part 6) is shown similarly.

**Remark 1.** Since \( a = f \) for \( p_1^2 + p_2^2 < r^2 \) properties 1)-4) of the lemma hold for \( f \) in that range. In fact,

\[
f_{ij} = \begin{pmatrix}
   \frac{1}{1 + p_2} & -p_1 \\
   -p_1 & \frac{1 + p_1^2}{(1 + p_2)^2}
\end{pmatrix}
\]

which has a determinant \( (1 + p_2)^{-4} \) and a positive trace for \( p_2 > -1 \).

**Remark 2.** We assume henceforth that the cutoff value \( r \) satisfies \( 0 < r < r_0 \). Note that the constants \( \sigma_1, \ldots, \sigma_5 \), and \( \nu \) in the previous lemma can all be made arbitrarily close to 1 by choosing \( r_0 \) sufficiently small. For this reason we will ignore the dependence of estimates on the constants \( \sigma_1, \ldots, \sigma_5 \) and will eventually omit the \( \nu \) dependence as well. For now we prefer to exhibit the ellipticity constant \( \nu \) in the proofs.

**Corollary 2.1.** With \( A \) as defined by (2.15) we have

\[
1) \quad \frac{\sigma_1}{2} \int \partial_t |\nabla w|^2 < A(\nu) < \frac{\sigma_5}{2} \int \partial_t |\nabla w|^2
\]
PROOF. The inequalities 1) and 2) follow immediately from parts 1) and 2) of the preceding lemma since \( \rho > 0 \). Part 3) is a consequence of 1) and 2).

For use in the next lemma we introduce \( w_+(x, \eta) = \max(0, w(x, \eta)) \) and \( w^- = w - w^+ \). The map \( w \to w^+ (\text{or } w^-) \) is continuous from \( L^2_k \) into itself or into \( L^2(\Omega, \omega) \) with the weight \( \omega' \) as can be seen by a simple computation (cf. [14], p. 41).

**Theorem 2.2.** For any \( B > 0 \) the problem (2.16)

\[
A'(w) = \lambda B'(w)
\]

has solutions \((\lambda_i, w_i), i = 1,2\) with \( \lambda_i > 0, w_i \in H^1_k \cap C^\infty(\overline{\Omega}) \) and \( A(w_i) = R^2 \). The solution \( w_1 \) satisfies \( w_1 > 0 \) in \( \Omega \) and is characterized by

\[
B(w_1) = \sup_{w \in H^1_k \setminus \{0\}} \frac{B(w)}{A(w)} = \frac{B(w)}{A(w)}.
\]

(2.18)

The solution \( w_2 \) satisfies \( w_2 < 0 \) and is similarly characterized with \( B(w^-) \) replacing \( B(w^+) \).

PROOF. From part 1) of Corollary 2.1 we see that the set \( \{w | A(w) = R^2\} \)

is bounded in \( H^1_k \) and thus is compactly embedded in \( L^2_k(\Omega) \). Let \( \sigma^+ \) be the supremum in (2.18) and let \( v_n, n = 1,2,3, \ldots \), be a maximizing sequence. There is a subsequence of \( v_n \) converging weakly in \( H^1_k \) and strongly in \( L^2_k(\Omega) \) to a \( v_1 \in H^1_k \). Assume we've renamed the functions so that \( v_n \) is the convergent sequence. Denote \( B(w^+) \) by \( B_+(w) \). The map \( w \to B_+(w) \) is continuous on \( H^1_k \) and differentiable (cf. [14], p. 41) with

\[
\langle B_+(w), \chi \rangle = -\int \sigma'(w) \chi = \langle B'(w^+), \chi \rangle.
\]

Since \( v_n \) converges to \( w_1 \) in \( L^2_k \), \( v^+_n \) converges to \( w^+_1 \) and by continuity, \( B_+(w_1) = \sigma^+ \). Clearly \( \sigma^+ > 0 \) so \( w_1 \neq 0 \). Since \( A \) has a convex integrand and \( v_n \) converges weakly in \( H^1_k \), \( A(v_n) \to R^2 \). We claim that \( A(w_1) = R^2 \). For if not, then for some \( t > 1 \), \( A(tw_1) = R^2 \), since as \( t \to \infty \), \( A(tw_1) \to R^2 \).
\[ B_{+}(t \sigma_{1}) = t \sigma_{1} > 0 \]

But for \( t > 1 \), \( B_{+}(t \sigma_{1}) = t \sigma_{1} > \sigma_{1} \) contradicting the characterization of \( \sigma_{1} \).

Since \( A(w_{1}) \) and \( B_{+}(w_{1}) \) are Frechet differentiable and \( w_{1} \) is a constrained maximum it follows that \( A'(w_{1}) = \lambda_{1} B'_{+}(w_{1}) \) for a real \( \lambda_{1} \). That is

\[
\int_{\Omega_{s}} \left( \rho a_{1}(\nabla w_{1}) \varphi_{s} + \rho a_{2}(\nabla w_{1}) \varphi_{s} \right) = - \lambda_{1} \int_{\Omega_{s}} \varphi \left( \partial_{t} w_{1}^{+} \right)
\]

for all \( \varphi \in H^1_{0} \). Choosing \( \varphi = w_{1} \) and using part 2) of Corollary 2.1 we see that \( \lambda_{1} > 0 \). The function \( w_{1} \) is a weak solution of the equation

\[
\frac{\partial}{\partial t} \partial_{t} \sigma_{1} \frac{\partial w_{1}}{\partial \sigma_{1}} = \lambda_{1} \varphi \left( \partial_{t} w_{1}^{+} \right)
\]

where \( \partial_{t} \sigma_{1} = \int_{\sigma_{1}} a(t \nabla w_{1}) \, dt \) and thus we can apply a weak maximum principle ([15], p. 168) to show \( w_{1} > 0 \) (cf. [16] for a similar application). In this case, since \( w_{1} \) is periodic, the boundary consists of points where \( \eta = x_{3} = 0 \) or 1 and on that set \( w_{1} = 0 \) in the \( H^1 \) sense (cf. the definition of \( H^1_{0} \)). We conclude that \( w_{1} > 0 \) and thus \( w_{1}^{+} = w_{1} \). Then \( w_{1} \) is a weak solution of the quasilinear elliptic equation

\[
\frac{\partial}{\partial t} \partial_{t} \sigma_{1} \frac{\partial w_{1}}{\partial \sigma_{1}} = \lambda_{1} \varphi \left( \partial_{t} w_{1}^{+} \right)
\]

in which \( a(p_{1}, p_{2}) \) and \( \varphi(x_{3}) \) are \( C^{\infty} \) functions. As a consequence of results in Chapter 4 of [17] the function \( \varphi(x_{1}, x_{2}) \) is a \( C^{\infty} \) function on \( \Omega \). The strong maximum principle ([15], p. 33) applied to (2.19) shows that \( w_{1} > 0 \) in \( \Omega_{s} \). The case of \( (A_{2}, w_{2}) \) is treated analogously, completing the proof of the theorem.

Before pursuing estimates on \( w \) independent of \( \sigma \) (cf. § 3) we compile some additional technical lemmas regarding the problems (2.13) and (2.16). Note that the formal linearization of (2.13) about \( w = 0 \) is

\[
\frac{\partial}{\partial \sigma_{1}} \varphi \frac{\partial w}{\partial \sigma_{1}} + \frac{\partial}{\partial \sigma_{2}} \varphi \frac{\partial w}{\partial \sigma_{2}} = \lambda_{1} \varphi \left( \partial_{t} w^{+} \right).
\]

The lowest eigenvalue of (2.21) can be obtained by separation of variables and since \( \varphi \) is independent of \( x \), will correspond to a function of \( \eta \) alone. It is just the lowest eigenvalue of the Sturm-Liouville problem

\[
\begin{cases}
\frac{\partial}{\partial \eta} \varphi \frac{\partial w}{\partial \eta} = \lambda_{1} \varphi \left( \partial_{t} w^{+} \right) \\
w(\eta_{0}) = w(\eta_{1}) = 0.
\end{cases}
\]
For later use we note that the lowest eigenvalue $\mu$ of (2.21) is characterized by

\[
\mu^{-1} = \sup_{v \in H^1_0} \frac{-\int \phi' \nabla z^2}{\int \phi |\nabla z|^2} .
\]

This is a standard result and follows from the same type of arguments as were given in the last proof. An obvious consequence of (2.23) is that

\[
\int_{\Omega} \phi |\nabla z|^2 > \mu \int_{\Omega} (\phi') z^2
\]

for all $z \in H^1_0$. Naturally, the eigenvalue $\mu$ viewed as the lowest eigenvalue of (2.22) has an analogous characterization with $z$ in (2.23) being $z = z(\eta)$ and the integrals being taken from $\eta_0$ to $\eta_1$. As in the nonlinear problem, if $\xi = \xi(\eta)$ denotes an eigenfunction for $\mu$, it has one sign in $\eta_0 < \eta < \eta_1$. Of course $\mu$ depends on $\delta$ and could be denoted $\mu_\delta$. If $\delta = 0$ one can define a weak form of (2.22) by requiring

\[
\int_{\eta_0}^{\eta_1} \phi(\eta) \frac{d\omega}{d\eta} \frac{d\phi}{d\eta} d\eta = -\lambda \int_{\eta_0}^{\eta_1} \phi(\eta) \frac{d}{d\eta} (\phi(\eta) \psi(\eta)) d\eta
\]

for arbitrary $\psi$ in $H^1([\eta_0, \eta_1])$ with $\omega$ a fixed element of the same space. One easily verifies that $w = \xi^\delta$, defined by

\[
\xi^\delta(\eta) = \begin{cases} 1 - \eta/\eta_0; & \eta_0 < \eta < 0 \\ 1 - \eta/\eta_1; & 0 < \eta < \eta_1, \end{cases}
\]

is a solution of (2.25) with $\lambda = \mu_0$ defined by

\[
\mu_0^{-1} = \frac{\phi_0 - \phi_1}{\phi_0/|\phi_0| + \phi_1/|\phi_1|} .
\]

In the literature treating waves in fluids the velocity $c_\delta = (g/\mu_0)^{1/2}$ is called the critical velocity and is regarded as the velocity with which infinitesimal long waves propagate.

We next obtain some quantitative estimates on $\xi$ and $\mu$ when $\delta > 0$. These will be used in this section and in Section 5.
Lemma 2.2. Let $\mu$ be the lowest eigenvalue of (2.22) and $\xi(\eta)$ the associated eigenfunction, normalized so that $\max \xi = 1$. Then

\[
\frac{1}{|\eta_0| + \xi_0 \frac{\partial \xi}{\partial \eta}} < \xi \leq \frac{1}{\eta_0} \quad \text{on } \eta_0 < \eta < 0
\]
\[
\frac{-1}{\eta_1 - \delta} < \xi \leq \frac{1}{\eta_1 - \delta + V_1 \frac{\partial \xi}{\partial \eta}} \quad \text{on } \delta < \eta < \eta_1
\]
\[
\frac{\partial_1 - \xi_0 \frac{\partial \xi}{\partial \eta}}{\eta_1 - \delta} \leq \xi \leq \frac{\partial_0}{\eta_1 \partial_1} \frac{1}{\eta_0} \quad \text{on } 0 < \eta < \delta
\]

and

\[
|\xi - \xi_0| \leq \delta \left( \bar{s} + \frac{1}{\eta_1} \right) \quad \text{for } \eta_0 < \eta < \eta_1
\]

where $\bar{s} = \max |\xi_\eta|$ on $0 < \eta < \delta$.

Proof. From the equation (2.22) satisfied by $(\xi, \mu)$ we conclude that $\xi_\eta$ is constant on $[\eta_0, 0]$ and on $[\delta, \eta_1]$. Let $s_0 > 0$ and $s_1 < 0$ be the slopes on the respective intervals. We can integrate the equation to obtain

(2.28) \[ \phi(\eta) \xi(\eta) - \phi(\eta) \xi(\bar{\eta}) = \mu \int_\eta^{\bar{\eta}} \phi' \xi. \]

If we let $\bar{\eta} = 0$, then since $\phi' \xi < 0$

\[
\xi_\eta \frac{\partial_0}{\phi(\eta)} s_0 \leq \frac{\partial_0}{\eta} s_0
\]

for $\eta > 0$. Since $\xi(\eta_0) = 0$, $\xi(0) = -s_0 \eta_0$ and so

\[
\xi(\eta) \leq -s_0 \eta_0 \frac{\partial_0}{\eta} s_0 \eta \quad \text{for } \eta > 0.
\]

Since $\xi_\eta < 0$ for $\eta > \delta$, the maximum of $\xi$ occurs for some $\bar{\eta}$ in $[0, \delta]$. It follows that

\[-s_0 \eta_0 \leq 1 = \xi(\bar{\eta}) \leq -s_0 \eta_0 + \frac{\partial_0}{\eta} \delta \]

or that

\[
\frac{1}{|\eta_0| + (\xi_0 \frac{\partial \xi}{\partial \eta}) \delta} < s_0 \leq \frac{1}{|\eta_0|}
\]
which is the first claim of the lemma. Similarly from (2.28) one shows
\( \xi > (\ell / \ell_e) \), and proceeds to the second inequality in the lemma. The third
inequality follows immediately from the bounds on \( \xi \). For the final claim
we note that \( \max |\xi(\eta) - \xi^0(\eta)| \) must occur for \( 0 < \eta < \delta \) since \( \xi - \xi^0 \) is linear
outside that interval. Since \( \xi \) and \( \xi^0 \) each assume the value 1 in \([0, \delta]\) a
simple estimate using the derivative bounds for \( \xi \) and \( \xi^0 \) yields the final
assertion.

**Lemma 2.3.** The lowest eigenvalue \( \mu = \mu_0 \) of (2.22) satisfies

\[
\frac{(q_0 - q_1)(1 - \delta/\eta_1)}{\eta_0/\eta_1 + q_2/\eta_1 + (\delta/\eta_1^2)(q_0 - q_1)} < \mu^2 < \frac{\eta_0}{\eta_1 \eta_2} \frac{q_0 - q_1}{(\eta_1 - \delta + (q_0/\eta_0) \delta)^2} + \frac{q_1(\eta_1 - \delta)}{(\eta_1 - \delta + (q_0/\eta_0) \delta)^2}
\]

and consequently \( \lim_{\xi \to \infty} \mu_0 = \mu_0 \). Further \( \mu_0 < \mu(q_0, q_1, \eta_0) \) if \( 0 < \delta < \eta_1/2 \).

**Proof.** We use the variational characterization analogous to (2.23).
The quotient equals \( \mu^{-1} \) with \( z = \xi \) and we can use the last lemma to
estimate the quotient from above. In particular the first two inequalities
of the lemma yield a lower bound for \( \int \xi^0 \) with \( \max \xi = 1 \) and the
numerator is estimated above by \( q_0 - q_1 \). For a lower bound on \( \mu^{-1} \) we merely
use \( \xi^0 \) as a trial function. The final assertions are clear from the ine-
qualities.

We next obtain crude bounds on the value \( \lambda_i \) occurring in Theorem 2.1.
This is an intermediate step to obtaining the more precise bounds given
in Section 5.

**Lemma 2.4.** Let \( \delta_0 = \min (q_1/\eta_0, \eta_0 q_2/\delta_0, \eta_1/2) \). Then there are posi-
tive constants \( l_0, l_1 \) such that for \( 0 < \delta < \delta_0 \), \( \lambda_i \) in Theorem 2.1 satisfies

\[
(2.29) \quad l_0 < \frac{\lambda_i}{\mu} < l_1; \quad i = 1, 2
\]

where \( \mu \) is the lowest eigenvalue of (2.2).

**Proof.** We let \( (\lambda, w) \) stand for either solution found in Theorem 2.1.
After applying the two sides of (2.16) to the vector \( w \) one can solve for \( \lambda \)
and use Corollary 2.1 to obtain

\[
(2.30) \quad \lambda = \frac{\langle A'(w), w \rangle}{\langle B'(w), w \rangle} \sigma_0 \frac{|\nabla w|^2}{\sigma_0} - \frac{\int \sigma^w_{i} w^x}{\sigma_0}.
\]
However, the Rayleigh quotient of integrals in (2.30) is at least as large as the first eigenvalue of

\[(2.31) \quad \sigma_3 \text{div} (\rho \text{grad} w) = \lambda \rho w\]

with \(w\) vanishing at \(\eta = \eta_0, \eta_1\) and \(2k\) periodic in \(x\). Comparing (2.31) with (2.21) we see that the lowest eigenvalue is \(\lambda = \sigma_4 \mu\), \(\mu\) being the lowest eigenvalue of (2.22). Thus \(l_0 = \sigma_3\) will suffice in (2.29).

For an upper bound on \(\lambda\) we use Corollary 2.1 again to obtain

\[\lambda = \frac{\langle A'(w), w \rangle}{\langle B'(w), w \rangle} \leq \frac{2\sigma_4 \sigma_1^{-1} A(w)}{B(w)} = \frac{2\sigma_4 \sigma_1^{-1} R^2}{B(w)}.\]

Now suppose \(\lambda = \lambda_1\), so that \(w > 0\). If \(z = z(x, \eta) > 0\) is any function in \(H^2\) for which \(A(z) = R^2\), then since \(B(z) < B(w)\), by virtue of the variational characterization, we see that \(\lambda < 2\sigma_4 \sigma_1^{-1} R^2 B^{-1}(z)\). We let \(z = \alpha \xi(\eta)\) with \(\xi\) the function from Lemma 2.2 and \(\alpha > 0\) a free parameter. Let \(\hat{\delta}\) denote the largest value of \(|\xi_0|\) for \(0 \leq \eta \leq \delta\); from Lemma 2.2 one can estimate \(\hat{\delta}\). Since

\[\int_0^{\eta_1} \int_0^{\eta_1} 2k\sigma^2 \mu \int_0^{\eta_1} 2k\sigma^2 \mu (q_0 - \xi) < 2k\sigma^2 \mu (q_0 - \xi),\]

if we let \(\alpha^2 = \sigma_0^2 = R^4 (\sigma_2 k \mu (q_0 - q_1)^{-1}\), then according to part 1) of Corollary 2.1 we’ll have \(A(z) < R^2\). Hence for an \(\alpha > \alpha_0\) we’ll have \(A(\alpha \xi) = R^2\) and \(B(z) = 2k\sigma^2 \int (\xi')^2 > 2k\sigma^2 \mu (q_0 - q_1) (1 - 2\delta)\). From Lemma 2.2 and the choice of \(\delta\), \(1 - 2\delta > \frac{1}{2}\). Finally, using the lower bound for \(B(z)\) in the expression estimating \(\lambda\) we find that \(\lambda < 2\mu \sigma_1^{-1} \sigma_4\). One obtains the same estimate for \(\lambda_4\) using \(-z\).

3. - Regularity of \(w\).

The results of this section apply to any solution of \(A'(w) = \lambda B'(w)\), equation (2.16). To further simplify notation we’ll let \(h(x_2) = \lambda (q_0 - q(x_1))\) and have \(h'(x_2) > 0\). We’ll use \(\bar{h}\) for \(\max h = \lambda (q_0 - q_1)\). The equation satisfied by \(w = w(x_1, x_2)\) is:

\[(3.1) \quad -\frac{\partial}{\partial x_1} \rho (x_2) a_1(w_{x_1}, w_{x_2}) - \frac{\partial}{\partial x_2} \rho (x_2) a_2(w_{x_1}, w_{x_2}) = h'w.\]
As noted earlier, since \( \phi \) and \( a \) are assumed to be \( C^\infty \) functions, \( w \) is \( C^\infty \) on \( \bar{\Omega} \). Of course the estimates on the derivatives depend on the size of the derivative of \( \phi \); i.e., they depend upon \( \delta \). Our aim in this section is to establish estimates which do not depend on \( \delta \) or on the period \( 2k \).

Let \( \sigma, x_1', x''_1 \) be real numbers with \( x_1' < x''_1 \) and \( \sigma > 0 \). Suppose \( \zeta \) is a \( C^\infty \) function of \( x_1 \) such that \( \zeta = 1 \) for \( x_1' < x_1 < x''_1 \); \( \zeta = 0 \) for \( x_1 < x_1' - 1/\sigma \) or \( x_1 > x_1' + 1/\sigma \); and \( 0 < \zeta < 1 \) for all other \( x_1 \). We can assume \( |\zeta_{x_1}| < 2\sigma \).

Let \( \Omega'' = \{ (x_1, x_2) \in \Omega | \zeta = 1 \} \) and \( \Omega' = \{ (x_1, x_2) \in \Omega | \zeta > 0 \} \). A typical estimate in this section will be of interior type; i.e., an estimate of a function on \( \Omega'' \) in terms of data on \( \Omega' \).

It will be crucial for our arguments to fully utilize a divergence structure for \( (3.1) \). To that end we write the equation as

\[
(3.2) \quad -\frac{\partial}{\partial x_i} \phi(x_j) a_{ij}(w_{x_i}, w_{x_j}) = \frac{\partial}{\partial x_2} (hw) - hw_{x_i},
\]

using the summation convention. Some of the estimates we make would simplify if \( \Omega'' \) were taken to be a period region \( \Omega_\ast \). However, it will be useful later to have estimates on arbitrary regions of the type \( \Omega'' \). We'll let \( b = \max a_{ij}(p_1, p_2) \) where the maximum is taken over all \( i, j, p_1, p_2 \). Recall that \( a_{ij}(\xi, \xi_i) \geq v \xi, \xi_i \), with \( v \) independent of \( p_1, p_2 \). We use \( C, C_1, \) etc. to denote constants, differing from one context to another, and indicate what parameters they depend on.

**Lemma 3.1.** Suppose \( w \in H_*^1 \cap C^2(\Omega) \) is a solution of \( (3.2) \). Then

\[
(3.3) \quad \int_{\Omega'} |\nabla w_{x_i}|^2 < C \int_{\Omega'} |\nabla w|^2
\]

where \( C = C(v, b, h, \varrho_2, \varrho_1, \sigma) \). As \( \sigma \to 0 \) \( C \) approaches \( h^2/\varrho_1 v^4 \).

**Proof.** Using the cutoff function \( \zeta \) we conclude from \( (3.2) \) that

\[
(3.4) \quad \int_{\Omega'} \frac{\partial}{\partial x_i} \left( \phi(x_1) a_{ij} \right) \frac{\partial}{\partial x_j} \zeta^2 w_{x_i} = -\int_{\Omega'} \left( \frac{\partial}{\partial x_2} (hw) - \phi w_{x_2} \right) \frac{\partial}{\partial x_1} \zeta^2 w_{x_i}.
\]

In integrating a term such as \( (\partial/\partial x_2)(\phi a_{ij})(\partial/\partial x_j) \zeta^2 w_{x_i} \), two integrations by parts will interchange the indices in the derivatives. Two benefits result.

We avoid applying an \( x_2 \) derivative to \( \phi(x_1) \) and can express the resulting integrand as \( \zeta^2 \phi a_{ij}(\partial/\partial x_j) w_{x_i}(\partial/\partial x_1) w_{x_i} \). Effecting such interchanges and car-
rying out the resulting derivatives we can write (3.4) as
\[ \int_{\Omega'} \zeta^2 \frac{\partial^2}{\partial x_i^2} w_{x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} w_{x_i} = \int_{\Omega'} \left[ \zeta^2 h \left( w_{x_i} \frac{\partial}{\partial x_i} w_{x_i} - w_{x_i} \frac{\partial}{\partial x_i} w_{x_i} \right) + 2 \zeta \zeta w_{x_i} \left( h w_{x_i} - \theta a_{11} \frac{\partial}{\partial x_1} w_{x_i} - \theta a_{12} \frac{\partial}{\partial x_2} w_{x_i} \right) \right]. \]

Using the standard inequality \( ab \leq \frac{a^2}{4\varepsilon} + (4\varepsilon)^{-1} b^2 \) and the ellipticity we obtain
\[ e \int_{\Omega'} \zeta^2 |\nabla w_{x_i}|^2 \leq e \left( h + 4b \sigma \theta \right) \int_{\Omega'} \zeta^2 |\nabla w_{x_i}|^2 + \left( \frac{h}{4\varepsilon} + \frac{b\sigma}{\varepsilon} + 2h \right) \int_{\Omega'} |\nabla w|^2. \]

Now for \( \varepsilon \) appropriately chosen we find
\[ \int_{\Omega'} \zeta^2 |\nabla w_{x_i}|^2 < C \int_{\Omega'} |\nabla w|^2 \]
and since \( \zeta^2 = 1 \) on \( \Omega' \), we obtain (3.3) with \( C \) depending on the indicated parameters. The choice \( \varepsilon = \frac{\theta}{4h} \) gives the desired limiting behavior as \( \sigma \to 0 \).

We now have an \( L^2 \) bound on \( \nabla w_{x_i} \) in terms of a controllable quantity; i.e., if we fix the size of \( A \), say \( A(w) = R^2 \), then according to Corollary (2.1), \( \int |\nabla w|^2 \) is at most a constant times \( R^2 \). Next let \( v = w_{x_i} \) and apply \( \partial/\partial x_i \) to both sides of (3.2) to obtain
\[ \frac{\partial}{\partial x_i} \rho(x) a_{ij}(w_{x_i}, w_{x_i}) \frac{\partial v}{\partial x_j} = -\frac{\partial}{\partial x_2} (hv) + hv_{x_i}. \]

It follows from Theorem 8.29 of [15] that \( v \) satisfies a Hölder condition. More precisely
\[ \| v \|_{C^\alpha(\Omega')} < C \left( \| v \|_{L^2(\Omega')} + \| hv \|_{L^2(\Omega')} + \| hv_{x_i} \|_{L^2(\Omega')} \right) \]
where \( \alpha > 0 \) and \( C > 0 \) depend on \( v, b, \theta, \| \theta \|_{C^\alpha(\Omega')} \), and \( \sigma \), and where \( \Omega' \) is defined as above. We have
\[ \| hv \|_{L^2(\Omega')} < \| v \|_{L^2(\Omega')} < \| v \|_{H^1(\Omega')} \]
with \( C \) independent of the length of \( \Omega' \) in the \( x_1 \) direction ([18], Lemma 5.14). Poincare's inequality gives
\[ \pi^2 \int_{\Omega'} v^2 < \int_{\Omega'} |\nabla v|^2 \]
and hence from (3.7) and Lemma 3.1 we obtain

**Lemma 3.2.** If \( w \in H^1_\kappa \) is a solution of (3.2) and \( v = w \), then

\[
\|v\|_{C^0(\Omega')} \leq C \left( \int_{\Omega'} |\nabla w|^2 \right)^{1/2}
\]

with \( C = C(v, \kappa, h, \eta_0, \eta_1, \sigma) \).

We are now in a position to obtain a preliminary \( L^\infty \) bound on \( w \).

**Lemma 3.3.** Let \( w \) be a solution of (3.2), let \( \Omega' = \{(x_1', x_2) \in \Omega \mid |x_1' - x_1| < 2\} \), and let \( R' = \left( \int |\nabla w|^2 \right)^{1/2} \). Let \( r \) be the cutoff parameter in Lemma 2.2. Then there is a constant \( C \), depending on \( \eta_0 \) and the parameters entering Lemma 3.2, and there is a constant \( M < \sqrt{2r} + CR' \) such that

\[
|w(x_1, x_2)| < M.
\]

**Proof.** For \( |x_1| > \min(|\eta_1|, \eta_2) \), \( \rho' = 0 \) and (3.9) follows as did the previous lemma. For the remaining \( x_2 \) we can suppose that the point at which we want to estimate \( w \) is the origin of new coordinates \((\tilde{x}_1, \tilde{x}_2)\), but for ease of writing omit the tildas and use \((x_1, x_2)\) in the proof. We let \( \Omega'' = \{(x_1, x_2) \in \Omega \mid |x_1| < 1\} \). Using a cutoff function \( \zeta \) with \( \zeta(x_1) = 1 \), \( \zeta(s_a) = 0 \) we have

\[
w(x_1, x_2) = \int_{\Omega''} \int_{\eta_2}^{x_2} \frac{\partial}{\partial x_1} \zeta w_{x_1},
\]

and a simple estimate using Lemma 3.1 shows that

\[
m = \|w\|_{L^\infty(\Omega')} \leq CR'.
\]

From Lemma 3.2 \( |v(0, 0)| \) and \( \|v\|_{C^0(\Omega')} \) are likewise bounded by \( C'R' \).

We will use a comparison theorem in \( \Omega = \Omega'' \cap \{x_2 > 0\} \) for the quasilinear operator \( Q \) defined by

\[
Qw = \frac{\partial}{\partial x_1} \hat{\rho}(x_2) a_i(x_1, w_{x_1}, w_{x_2})
\]

where \( \hat{\rho} \) is the density based at the new origin. We'll use only the property \( \hat{\rho}_{x_2} < 0 \) so a more precise description of \( \hat{\rho} \) seems unnecessary. Since \( w \) is a solution of (3.1), \( Qw = \lambda \hat{\rho}_{x_2} w \). Consider a function \( u \) defined by

\[
u(x_1, x_2) = w(0, 0) + A_1 x_1 + A_2 \text{Re}(x_2 + ix_2)^{1+a} + A_2 x_2
\]
where $\alpha$ is the Hölder exponent from (3.8), $A_1 = w_{x_1}(0, 0) = v(0, 0)$, and the constants $A_\alpha$, $A_2$ are to be determined so that Theorem 9.2 of [15] is applicable.

Note that with $\theta = \tan^{-1} x_1/x_2$,

$$
\frac{\partial}{\partial x_2} \text{Re}(x_2 + ix_1)^{1+\alpha} = (1 + \alpha)(x_2^2 + x_1^2)^{1-(\alpha-1)/2} x_2 \cos((1 + \alpha) \theta) -
(1 + \alpha)(x_2^2 + x_1^2)^{1-(\alpha-1)/2} x_1 \sin((1 + \alpha) \theta)
$$

and so

(3.12) \[ \left| \frac{\partial}{\partial x_2} \text{Re}(x_2 + ix_1)^{1+\alpha} \right| < (1 + \alpha) d^\alpha \]

where $d^2 = x_1^2 + x_2^2$. In $\Omega$, $d < \sqrt{2}$. Hence if

(3.13) \[ |A_1| = (1 + \alpha) |A_\alpha| 2^{\alpha/2} > \sqrt{2} \]

then $|u_{x_1}| > \sqrt{2} r$ and from (2.14) we see that $Qu$ takes the simple form

(3.14) \[ Qu = \delta A u + \delta_{x_1} u_{x_1} = \delta_{x_1} \left( A_\alpha \frac{\partial}{\partial x_2} \text{Re}(x_2 + ix_1)^{1+\alpha} + A_2 \right) \]

Here we make essential use of the fact that $u$ is harmonic and must have $|\nabla u| > \sqrt{2} r$ to exploit this property.

Since $\delta_{x_1} < 0$, to have $Qu > Quw$ we require

$$
A_\alpha \frac{\partial}{\partial x_2} \text{Re}(x_2 + ix_1)^{1+\alpha} + A_2 - \lambda w < 0 ,
$$

which will be satisfied if

(3.15) \[ (1 + \alpha) |A_\alpha| 2^{\alpha/2} + A_2 + \lambda m < 0 . \]

Next, consider $u - w$ on the boundary $\partial \Omega$ starting where $x_2 = 0$ and $0 < x_1 < 1$. Using the $C^\alpha$ estimate (3.8) on $v = w_{x_1}$ we have

(3.16) \[ w_{x_1}(x_1, 0) - w_{x_1}(0, 0) > - C \rho x_1^2 \]

and, since $w_{x_1}(0, 0) = A_1$,

$$
u_{x_1}(x_1, 0) - w_{x_1}(0, 0) = A_\alpha (1 + \alpha) \cos \left( (1 + \alpha) \frac{x_1}{2} \right) x_1 .
$$
Then 
\[ w_3(x_1, 0) - u_3(x_1, 0) > \left[ -CR' - A_3(1 + x) \cos(1 + x) \frac{\pi}{2} \right] x_1^n. \]

But \( \cos(1 + x)(\pi/2) < 0 \) so for
\[ 2013 > 0 \text{ for } x_1, I. \]
Since \( w = u \) at \((0, 0)\), \( (3.17) \) will guarantee \( W > U \) where \( x = 0 \) and \(-I < x_1 < 0\). Similarly, \( (3.17) \) implies \( w > u \) for \( x_1 = 0 \) and \(-1 < x_1 < 0\).

The set \( \mathcal{Q} \) has the form \((-1, 1) \times (0, D)\) for some \( D < 1\). We choose \( \bar{x}_1 \in (0, D) \) so that if \( \theta = \tan^{-1}(\bar{x}_1^{-1}) \), then \( \cos((1 + x)\theta) = \delta < 0 \). We will choose \( A_1 < 0 \) in \( (3.11) \) and hence if
\[ A_1 > \frac{2m + \left| A_4 \right|}{\delta}, \]
then \( u < -m < w \) where \( 0 < x_2 < \bar{x}_2 \) and \( x_2 = 1 \) (or \(-1 \) by symmetry). One verifies that on the remainder of the boundary of the rectangle,
\[ d^{1+x} \cos(1 + x)\theta < D^{1+x}. \]
Hence \( u < w \) provided
\[ 2m + A_1 + A_3 D^{1+x} + A_2 \bar{x}_2 < 0 \]
or, since \( D < 1 \), provided
\[ -A_3 (\bar{x}_2)^{-1}(2m + A_1 + A_3). \]

First we choose \( A_3 \) to satisfy \( (3.17) \) and \( (3.18) \). Then we choose \( A_3 < 0 \) so that \( (3.13), (3.15) \) and \( (3.19) \) are satisfied.

Then \( Q_u \geq Q_w \) in \( \mathcal{Q} \) and \( u < w \) on the boundary. The remaining hypotheses of the comparison theorem ([15], p. 207) regard the form of \( Q \) and are readily verified. It follows that \( u < w \) and hence \( w_3(0, 0) > u_3(0, 0) = = A_2 \). In a completely analogous way one shows \( w_3(0, 0) < |A_4| \) and thus \( M = |A_2| \) serves in the lemma. One readily sees that \( M \) depends only on the parameters indicated in the statement of the lemma, recalling that \( \lambda = \tilde{h}(\tilde{q}_0 - \tilde{q}_1)^{-1} \). Moreover as \( R' \) approaches 0, so do \( m \) and \( A_1 \), allowing the choice of \( A_3 \) to approach zero and the choice of \( A_2 \) to approach zero in proportion with \( R' \), except in \( (3.13) \) where \( |A_4| < \sqrt{2r + CR'} \) will be
compatible with satisfying (3.13) for $R' > 0$. This completes the proof of
the lemma.

Again using $\zeta$, $Q'$ and $Q''$ introduced at the beginning of this section
we show

**Lemma 3.4.** There exist positive constants $r_1$ and $R_0$ such that if $A$ is
defined using a cutoff at $r < r_1$ and $w \in H_k^1 \cap C^2_k(\Omega)$ is a solution of (3.1) with
$A(w) = R^2$, $R < R_0$, then $v = w_{z_1}$ satisfies

$$
\int_{\Omega'} |\nabla w_{z_1}|^2 \leq C \int_{\Omega'} |\nabla w|^2
$$

where $Q' \subset Q \subset Q_{\varepsilon}$. Here $R_0$ and $C$ depend on $\varepsilon$, $\varepsilon_1$, $q_1$, $\eta_0$, and $\sigma$. For
all sufficiently small $\sigma$ one can take $C = 4 \hbar \|q_{0, \varepsilon_1} v^*$.\]

**Proof.** Recall, it is assumed that $r_1 < r_0$ from Lemma 2.1. We multiply
equation (3.6) by $(\partial/\partial x_1) \zeta^2 v_{z_1}$ and integrate (all integrals will be over $\Omega'$
otherwise stated) to obtain

$$
\int \frac{\partial}{\partial x_1} \left[ \rho(x_1) \frac{\partial}{\partial x_1} (w_{x_1}, w_{x_2}) \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} \right] = \int \left[ -\frac{\partial}{\partial x_1} \left( \rho v \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} + \rho v_{x_1} \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} \right) \right].
$$

As in the proof of Lemma 3.1 we integrate by parts to obtain

$$
\int \frac{\partial}{\partial x_1} \left[ \rho(x_1) \frac{\partial}{\partial x_1} (w_{x_1}, w_{x_2}) \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} \right] \frac{\partial}{\partial x_1} v_{z_1} = \int \left[ -\frac{\partial}{\partial x_1} \left( \rho v \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} + \rho v_{x_1} \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} \right) \right]
$$
or

$$
(3.21) \int \zeta^2 \rho(x_1) \frac{\partial}{\partial x_1} v_{z_1} \frac{\partial}{\partial x_1} v_{z_1} =
$$

$$
\quad = -\int \rho(x_1) \left( a_{\varepsilon_{1k}} w_{x_2 x_1} v_{z_1} \frac{\partial}{\partial x_1} \zeta^2 v_{z_1} \right) + \int \left[ -\zeta^2 \rho v_{x_1} v_{x_1, z_1} + \zeta^2 \rho v_{x_1} v_{x_1, z_1} + 2 \rho v_{x_1, z_1} \zeta^2 v_{z_1} \right]
$$

where we have used $a_{\varepsilon_{1k}}$ to denote a third derivative of the function $a$.
Using ellipticity we see that the left side of (3.21) is at least $\nu J$, where
$J = \int \zeta^2 \rho |\nabla v_{z_1}|^2$. We will show that the right side of (3.21) can be bounded
by $\frac{1}{2} \nu J$ plus a multiple of $\int |\nabla w|^2$. Then (3.20) will follow easily.

We will use the form of $a_{\varepsilon_{1k}}$ given in Lemma 2.1. That is, if we let
$a_{\varepsilon_{1k}} = a_{\varepsilon_{1k}} (0, 0) + \alpha_{\varepsilon_{1k}}$, we know the values at $(0, 0)$ and that $\alpha_{\varepsilon_{1k}}$ is of
order $r$. We let $\alpha$ be a bound for $|z_{ik}|$ for any choice of indices. Separating out the expressions $a_{ijk}(0,0)$ which do not vanish we find that the right side of (3.21) is at most

$$
(3.22) \quad 2 \varepsilon \int q(x_s) v_{x_i} v_{x_j^2} v_{x_s^2} + \int q(x_s) v_{x_i} \frac{\partial}{\partial x_i} v_{x_s^2} v_{x_s} + \\
+ \int q(x_s) v_{x_i} v_{x_j} \frac{\partial}{\partial x_i} v_{x_s} v_{x_s} + \alpha \int \varepsilon \sum_{i,j,k} \frac{v_{x_i} v_{x_j} v_{x_k}}{2} + \\
\frac{\partial}{\partial x_i} v_{x_s^2} v_{x_s} + 2 \alpha \int \varepsilon \sum_{i,j} \frac{v_{x_i} v_{x_j} v_{x_s^2}}{2} + 2b \int \varepsilon \sum_{i,j} \frac{v_{x_i} v_{x_j} v_{x_s^2}}{2} + \\
+ \int \varepsilon \sum_{i,j} \frac{v_{x_i} v_{x_j} v_{x_s^2}}{2} + \frac{v_{x_i} v_{x_j} v_{x_s^2}}{2} + 2 \alpha \int \varepsilon \sum_{i,j} \frac{v_{x_i} v_{x_j} v_{x_s^2}}{2}.
$$

Let us denote the eight terms in (3.22) by $T_1, ..., T_8$ respectively. To estimate $T_1$ we write

$$
\int q(x_s) v_{x_i} v_{x_j} \frac{\partial}{\partial x_i} v_{x_s^2} v_{x_s} = \int q(x_s) v_{x_i} \left[ \frac{\partial}{\partial x_i} \left( \frac{v_{x_s^2}}{2} + v_{x_s} \right) \right]
$$

from which, after one integration by parts, we find

$$
(3.23) \quad T_1 < 2 \int q(x_s) v_{x_i} v_{x_j}^2 v_{x_i}^2 + 4 \int q(x_s) v_{x_j} v_{x_i} \frac{v_{x_s}^2}{2}.
$$

Since $T_8$ is equal to the first term on the right in (3.23)

$$
(3.24) \quad T_1 + T_8 < \varepsilon \int \varepsilon v_{x_i} v_{x_i}^2 + 4 \frac{\varepsilon}{\varepsilon} \int \varepsilon v_{x_i} v_{x_i}^2 + 4 \int \varepsilon v_{x_i} v_{x_i}^2 + \sigma^2 \int v_{x_i}.
$$

Here and below we use the inequality $ab \leq \varepsilon a^2 + (1/4\varepsilon) b^4$. To estimate $T_8$ we write

$$
\int q(x_s) v_{x_i} \frac{\partial}{\partial x_i} v_{x_s^2} v_{x_s} = \frac{1}{3} \int \varepsilon v_{x_i} \frac{\partial}{\partial x_i} v_{x_s^2} v_{x_s} = -\frac{2}{3} \int \varepsilon v_{x_i} v_{x_s^2}
$$

and so

$$
(3.25) \quad T_8 < 4 \int \varepsilon v_{x_i} v_{x_s^2} < \varepsilon \int \varepsilon v_{x_i} v_{x_s^2} + \frac{4\sigma^2}{\varepsilon} \int v_{x_i}.
$$

If we estimate the terms $T_4$ through $T_8$ in a straightforward way and com-
bine the result with (3.24) and (3.25) we find

\[ \nu J \leq \left( \varepsilon_1 + \varepsilon_\sigma \alpha + \varepsilon_\sigma b \varepsilon_\sigma + \varepsilon_\sigma \frac{\kappa}{\varepsilon_\sigma} \right) J + \]

\[ + \left( 4 \sigma \varepsilon_\sigma + \frac{4 \alpha}{\varepsilon_\sigma} + 4 \alpha \right) \int \zeta^2 \varphi v_{\xi_1}^4 + \]

\[ + \left( \varepsilon_2 + \frac{2 \alpha}{\varepsilon_\sigma} + 4 \alpha \right) \int \zeta^2 \varphi v_{\xi_1}^4 + \]

\[ + \left( \sigma + \frac{4 \sigma^2}{\varepsilon_\sigma} + 2 \alpha + \frac{b \sigma}{\varepsilon_\sigma} + \frac{\kappa}{4 \varepsilon_\sigma} + \frac{\kappa}{4 \varepsilon_\sigma} \right) \int |\nabla \varphi|^2. \]

We allow distinct \( \varepsilon_i \)'s in the inequalities to maintain flexibility and to label terms. Continuing, we have

\[ \int \zeta^2 \varphi v_{\xi_1}^4 = -\int v \frac{\partial}{\partial x_1} (\zeta^2 \varphi v_{\xi_1}^2) = \]

\[ = -\int 2\varphi v_{\xi_1}^2 \varphi v_{\xi_1}^2 + 3\varphi^2 \varphi v_{\xi_1}^4 v_{\xi_1^2} \cdot \]

So, supposing \( \max |v| < \bar{v} \) on \( \Omega' \) (cf. (3.8)), we obtain

\[ \int \zeta^2 \varphi v_{\xi_1}^4 < \bar{v} \int \zeta^2 \varphi v_{\xi_1}^4 + \bar{v} \int \varphi v_{\xi_1}^2 + 3 \varepsilon_\sigma \int \zeta^2 \varphi v_{\xi_1}^4 + \frac{3 \bar{v}}{4 \varepsilon_\sigma} \int \zeta^2 \varphi v_{\xi_1}^4 \]

and thus

\[ (1 - \varepsilon_\sigma \bar{v} - \varepsilon_\sigma \bar{v}) \int \zeta^2 \varphi v_{\xi_1}^4 < \frac{\bar{v}}{\varepsilon_\sigma} \int \varphi v_{\xi_1}^2 + \frac{3 \bar{v}}{4 \varepsilon_\sigma} \int \zeta^2 \varphi v_{\xi_1}^4. \]

Next, we have

\[ \int \zeta^2 \varphi v_{\xi_1}^4 = \int \zeta^2 \varphi w_{\xi_1}^2 \cdot w_{\xi_1^2} = \]

\[ = -\int \varphi \frac{\partial}{\partial x_1} (\zeta^2 \varphi w_{\xi_1}^2) = \]

\[ = -\int \varphi \left( 2 \zeta^2 \varphi w_{\xi_1}^2 + 3 \zeta \varphi w_{\xi_1}^2 \right). \]

Letting \( M \) be an upper bound for \( |w_{\xi_1}| \) (cf. Lemma 3.3) we see that

\[ \int \zeta^2 \varphi v_{\xi_1}^4 < \varepsilon_\sigma M \int \zeta^2 \varphi w_{\xi_1}^2 + \frac{\sigma M}{\varepsilon_\sigma} \int \varphi w_{\xi_1}^2 + 3 \varepsilon_\sigma M \int \zeta^2 \varphi w_{\xi_1}^2 + 3 \frac{M}{4 \varepsilon_\sigma} \int \zeta^2 \varphi w_{\xi_1^2}^2. \]
and thus
\begin{equation}
(1 - \varepsilon_6 \sigma M - 3\varepsilon_9 M) \int \xi^2 \varrho v_{z_i}^4 < \frac{\sigma M}{\varepsilon_6} \int \varrho v_{z_i}^3 + \frac{3M}{4\varepsilon_9} \int \xi^2 \varrho v_{z_i}^2.
\end{equation}

We let \( \varepsilon_6 = \varepsilon_9 = (8\bar{v})^{-1}, \varepsilon_3 = (8\sigma M)^{-1}, \) and \( \varepsilon_3 = (8M)^{-1}. \) Then
\begin{equation}
\int \xi^2 \varrho v_{z_i}^4 < 16\sigma^2 \bar{v}^2 \int \varrho v_{z_i}^2 + 12\bar{v}^2 \int \xi^2 \varrho v_{z_i}^2.
\end{equation}

and
\begin{equation}
\int \xi^2 \varrho v_{z_i}^4 < 16\sigma^2 M^2 \int \varrho v_{z_i}^2 + 12 M^2 \int \xi^2 \varrho v_{z_i}^2.
\end{equation}

Since \( \Omega' \subset \Omega_z, \) if \( A(w) < R_0^2, \) then by Corollary 2.1 \( \|w\|_{H^2}^2 < 2\sigma^{-1} R_0^2 \) and by Lemma 3.2 applied to \( \Omega' \) and \( \Omega_z, \) \( \bar{v} < CR_0. \) Similarly, we can apply Lemma 3.3 to estimate \( |w_{z_i}|. \) For a suitable \( R_0, \sqrt{2r} + CR' < \sqrt{2r} + CR_0 < 2r. \) Then if \( r \) is restricted to satisfy \( r < r_1 \), the bound \( M \) for \( |w_{z_i}| \) becomes \( 2r_1. \) To simplify matters we will assume \( R_0 \) is chosen so that the upper bound \( \bar{v} = 2r_1 \) as well. If (3.29) and (3.30) are used to estimate \( \int \xi^2 \varrho v_{z_i}^2 \) \( (i = 1, 2) \) in (3.26) and the choices \( \varepsilon_3 = 4\sqrt{6}r_1, \varepsilon_3 = \varrho_0 \varrho_1 \delta k, \varepsilon_3 = \sqrt{3}r_1 \) are made, then an inequality
\begin{equation}
\begin{split}
\left( \eta - \frac{\eta}{8} - 16 \sqrt{3}r_1 - 8 \sqrt{6}r_1 \alpha + O(\eta^2 \sigma) O(\varepsilon_4 \sigma) \right) J < \\
< \left( \frac{2\bar{h}_2}{\varrho \nu} + O(\sigma) + O \left( \frac{\sigma}{\varepsilon_4} \right) \int \varrho \eta \right)^2
\end{split}
\end{equation}
results. Since \( \alpha = O(\eta), \) if we choose \( r_1 \) and \( \varepsilon_4 \) appropriately (cf. Remark 2 following Lemma 2.1), we obtain a bound on \( J \) which, combined with Lemma 3.1, yields (3.20). In the limit \( \sigma \to 0 \) we can obtain
\begin{equation}
\frac{\eta}{2} J < \frac{2\bar{h}_2}{\varrho \nu} \int \varrho \eta
\end{equation}
from (3.31) which combined with the limiting constant in Lemma 3.1, produces a constant \( C = \frac{4\bar{h}_2}{\varrho \nu} \) in (3.20). This completes the proof of Lemma 3.4.

We now have \( L^2 \) bounds on \( D_i^r D_j^r w \) for \( i + j < 3 \) and \( j = 0 \) or \( 1 \) and can make the bounds small on \( \Omega' \) by making \( \int \varrho \eta \) small over \( \Omega' \subset \Omega'. \) We next obtain sharper information regarding \( w_{z_i}. \) We'll use \( b' \) for an upper bound on \( |u_{z_i}| \) over all indices and over all \( p_1, p_2. \)
LEMMA 3.5. There are positive constants \( r \) and \( R_0 \) (cf. Lemma 3.4) such that if the cutoff \( r \) satisfies \( r = r \) and \( w \in H^1_k \cap C^2(\Omega) \) is a solution of (3.1) with \( A(w) = R^2 \), \( R < R_0 \), then

\[
\|w_{x_1}\|_{L^{\infty}(\Omega')} \leq C \left( \int_{\Omega'} |\nabla w|^2 \right)^{1/2}
\]

for \( \Omega' \subset \Omega' \subset \Omega_k \). Further if \( \Omega \) denotes either component of \( \{(x_1, x_2) \in \Omega' \mid \varphi_{x_1} = 0\} \) then with \( \alpha \) from Lemma 3.2

\[
\|w_{x_1}\|_{C^1(\Omega')} \leq C \left( \int_{\Omega'} |\nabla w|^2 \right)^{1/2}.
\]

Here \( R_0 \), \( C \), and \( C' \) depend on \( \nu, b, b', \bar{h}, \bar{q}_0, \bar{q}_1, \eta_0, \) and \( \sigma \).

PROOF. Assume \( r < r \) so that Lemma 3.4 applies. We integrate form (3.2) of the equation between points \((s, t)\) and \((s, t + \tau)\) in \( \Omega' \) obtaining

\[
-\int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho(x_2) a_1(\nabla w)|_{x_1 = s} dx_2 - \varrho(x_2) a_2(w_{x_1}(s, x_2), w_{x_2}(s, x_2))|_{t}^{t+\tau} = h(x_2) w_{x_1}(s, x_2) dx_2.
\]

Using the function \( \zeta \) introduced at the outset of this section and assuming \( \zeta(s) = 1 \), \( \zeta(s) = 0 \) we have

\[
-\int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho a_1(\nabla w)|_{x_1 = s} dx_2 = \int_t^{t+\tau} \frac{\partial}{\partial x_1} \zeta^2 \frac{\partial}{\partial x_1} \varrho(x_2) a_1(\nabla w) dx_1 dx_2 =
\]

\[
= \int_t^{t+\tau} \int_{x_1} \zeta^2 \varrho a_{12} w_{x_1 x_2} dx_1 dx_2 + \int_t^{t+\tau} \int_{x_1} \zeta^2 \varrho(a_{12} w_{x_1 x_2} + a_{12} w_{x_1 x_2} w_{x_2}) dx_1 dx_2.
\]

In terms of \( v = w_{x_1} \) we obtain

\[
\int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho a_1(\nabla w)|_{x_1 = s} dx_2 = 0
\]

\[
< \varrho_0 \int_t^{t+\tau} \int_{x_1} \left[ 2 \sqrt{2} \nu b |\nabla v| + \sqrt{2} b |\nabla v| + 2 b \zeta |\nabla v|^2 \right] dx_1 dx_2.
\]
However $|\nabla v|$, $|\nabla v_z|$, and $|\nabla v|^2$ are all in $L^2$ on $\Omega^r = [t, t + \tau] \times [s, s]$ with $L^2$ norms bounded by a multiple of $R' = \left( \int_{\Omega^r} |\nabla v|^4 \right)^{1/4}$ for $\Omega^r \subset \Omega' \subset \Omega_4$; cf. (3.3), (3.20), (3.29), and (3.30) (we assume $R < R_0$, so $(R')^4 < C_R R'$).

Using the Schwarz inequality on the double integral in (3.36) we conclude that

$$\left( \int_0^{t+\tau} \frac{\partial}{\partial x_1} \rho a_s(\nabla w) \right)_{x_1=s} \, dx_1 \leq \tau^{1/2} C_1 R'$$

where $C_1$ arises from the earlier estimates just listed and the constants in (3.36). In an analogous way one shows

$$\int_0^{t+\tau} h(x_1) w_{x_2}(s, x_2) \, dx_2 \leq \tau^{1/2} C_2 R'$$

and hence from (3.34)

$$\left[ \left( \rho a_s(w_z(s, t), w_z(s, t)) + h(x_1) w(s, x_2) \right)_{x_1=s} \right]^{1+\tau} \leq \tau^{1/2} C_2 R'. \tag{3.39}$$

In (3.39) we evaluate derivatives at two points on the line where $x_1 = s$. At one such point, say $(s, t + \tau)$, we may assume $w_z = 0$ for $w$ has a maximum (or minimum) on the line. From (2.17) we see that at $(s, t + \tau)$ $|a_z(w_z, w_z)| \leq \frac{1}{2} w_z^2$. As in the proof of Lemma 3.3 $|w_z| < C_4 R'$ and $|w| < C_5 R'$ for all points on the line $x_1 = s$. Thus from (3.39)

$$|a_z(w_z(s, t), w_z(s, t))| \leq \frac{1}{2} \left[ \frac{\rho_0}{\rho_1} (C_4 R')^2 + 2 C_5 C_6 R' + \tau^{1/2} C_2 R' \right]. \tag{3.40}$$

Since $\xi_2 = 0$ for $\rho_1^2 + \rho_2^2 > 2r^2$ we see from (2.17) that

$$a_s(p_1, p_2) = p_4 (1 + O(r)) + O(p_4^2) \tag{3.41}$$

and by restricting $r$, if necessary, to $r = \hat{r} < r_1$, one can assume $|O(r)| < \frac{1}{4}$. Setting $p_1 = w_z(s, t)$, one can assume $|O(p_1^2)| < C_6(R')^2$, according to Corollary 2.1 and Lemma 3.2. Then, from (3.40) and (3.41)

$$|w_z(s, t)| \leq CR',$n

that is, (3.32) holds.

Now consider a region $\bar{Q} = Q^r \cap \{x_3 < 0\}$ or $Q^r \cap \{x_3 > \delta\}$ where $\theta_{3+} = 0$ (it would be enough to have $\theta_{3+}$ bounded for this argument). Since
$h_{x_1} = 0$ in $\tilde{Q}$, starting with the form (3.1) of the equation for $w$ and proceeding as above we arrive at

\begin{equation}
|a_i(w_{x_i}(s, x_2), w_{x_2}(s, x_2))|_{t_2}^{t+\tau} < \tau^{1/2} C_4 R'.
\end{equation}

Let $w_{x_i}(s, t) = q_i$ and $w_{x_i}(s, t + \tau) = q_i + p_i$. From Lemma 3.2 we conclude that $|p_i| < \tau^a C_4 R'$. We would like to conclude that

$$|w_{x_1}(s, t + \tau) - w_{x_1}(s, t)| = |p_1| < \tau^a C' R'.$$

From part 5) of Lemma 2.1 it follows that

\begin{equation}
(a_i(q_1 + p_1, q_2) - a_i(q_1, q_2) = p_2 + O(r) z \cdot (p_1, p_2)
\end{equation}

where $z$ is a unit 2-vector. From (3.42) and (3.43) we find

\begin{equation}
p_2 + O(r) = O(r) \tau^a C_4 R' + \tau^{1/2} C_4 R'
\end{equation}

and with $|O(r)| < \frac{1}{2}$, as before, conclude that $|p_2| < \tau^a C' R'$ (assume $\alpha < \frac{1}{2}$). Thus $w_{x_1}$ is Hölder continuous with respect to changes in $x_1$.

To show that $w_{x_1}$ is Hölder continuous as $x_1$ changes let

$$T(x_1, x_2) = a_i(w_{x_1}(x_1, x_2), w_{x_2}(x_1, x_2)).$$

Consider (3.1) in a region $\tilde{Q}$ where $h' = 0$ and $\varrho$ is constant. Integrate the equation from $t$ to $t + \tau$ in $x_2$ for $x_1 = s$ and $s + \gamma$, subtract the results, and express the difference as

\begin{equation}
T(s + \gamma, t + \tau) - T(s + \gamma, t) - (T(s, t + \tau) - T(s, t)) = \int_t^{t+\tau+\gamma} \int_s^{s+\gamma} \frac{\partial}{\partial x_1} a_i.
\end{equation}

The type of integral occurring in (3.45) has already been estimated (cf. (3.35)-(3.37)). One finds that

\begin{equation}
|T(s + \gamma, t + \tau) - T(s, t + \tau) - [T(s + \gamma, t) - T(s, t)]| < \tau^{1/2} \gamma^{1/2} C_4 R'.
\end{equation}

Since

$$\frac{\partial}{\partial x_i} a_i(w_{x_i}, w_{x_2}) \frac{\partial}{\partial x_j} w_{x_1} = 0$$

in $\tilde{Q}$, i.e. where $\varrho = \text{constant}$, an interior estimate (cf. [15], Theorem 8.29)
yields
\[ \|w_{z_2}\|_{C^0(\Omega')} \leq C_4 \|w_{z_2}\|_{L^2(\Omega)} , \]
where \( \Omega' \) is a subdomain of \( \Omega' \). One can choose \( \Omega' \) so that the distance from \( \Omega' \) to \( \eta = 0 \) is larger than some positive number \( \delta' \), independently of \( \delta \). Then the constant \( C_4 \) in (3.47) will depend only on the parameters listed in the lemma. Since the cutoff parameter \( r \) is bounded, \( |\text{grad}a_z| \leq \bar{C} \) for some constant \( \bar{C} \). It then follows from (3.47) that
\[ |T(s + \gamma, t) - T(s, t)| \leq \gamma^\alpha C_4 \|w_{z_2}\|_{L^2(\Omega')} , \]
provided the segment from \( (s, t) \) to \( (s + \gamma, t) \) is in \( \Omega' \). Now let \( (s, t + \tau) \) and \( (s + \gamma, t + \tau) \) be unrestricted points in \( \Omega' \) to obtain
\[ |T(s + \gamma, t + \tau) - T(s, t + \tau)| \leq \gamma^\alpha C_4 R' + \tau^{1/2} \gamma^{1/2} C_2 R' \]
from (3.46) and (3.48). Having a bound on the variation of \( a_4(w_{z_1}, w_{z_2}) \) as \( x_1 \) varies, one can obtain an equation analogous to (3.44), but with \( \gamma \) playing the role that \( \tau \) played. The Hölder continuity in \( x_1 \) follows. Hence
\[ \|w_{z_2}\|_{C^0(\Omega')} \leq C'R' , \]
completing the proof of Lemma 3.5.
We will use the Hölder estimates in Section 8. In the next section we will use the following corollary of Lemma 3.5.

**Corollary 3.2.** Let \( \bar{r} \) be as in Lemma 3.5. Then there is a positive constant \( R_1 \) depending on \( r, b, b', h, \varrho_0, \varrho_1, \eta_0, \) and \( \sigma \) such that if \( w \) is a solution of
\[ A'(w) = \lambda B'(w) \]
coming from Theorem 2.2, with \( 0 < R < R_1 \), then
\[ A(w) = F(w) \]
and
\[ F'(w) = \lambda B'(w) \]
(cf. (2.12), (2.13), (2.15)); that is
\[ \frac{\partial}{\partial x} q(\eta) \frac{w_x}{1 + w_q} + \frac{\partial}{\partial \eta} q(\eta) \frac{(2w_q + w_q^3 - w_q^3)}{2(1 + w_q)^3} = \lambda \phi' w . \]
PROOF. Assume \( R_1 < R_0 \) so that Lemma 3.5 applies. From Corollary 2.1 and Lemmas 3.2 and 3.5 \( R_1 \) can, if necessary, be reduced further to guarantee \( |\nabla w| < \tilde{r} \). Then the functions \( a \) and \( f \) in (2.14) agree near \( \nabla w \) and the conclusions follow easily.

For use in § 7 we include the following estimate for \( w \), continuing to use \( \Omega' \subset \Omega' \) introduced early in this section.

**Lemma 3.6.** Let \( w \in C^2_0(\bar{\Omega}) \) be a solution of \( A'(w) = \lambda B'(w) \). Then

\[
\int_{\Omega'} |\nabla w|^2 < C \max_{\Omega'} w^2
\]

where \( C \) depends on \( \tilde{h} \), \( \sigma \), and the measure of \( \Omega' \).

**Proof.** Using a cutoff function introduced at the beginning of this section we start with

\[
\int_{\Omega'} \frac{\partial}{\partial \omega_i} \varphi a_i(w_{x_1}, w_{x_2}) \xi^2 w = \lambda \int_{\Omega'} \xi^2 w^2
\]

or

\[
\int_{\Omega'} \xi^2 \varphi a_i(w_{x_1}, w_{x_2}) w_{x_i} = \lambda \int_{\Omega'} (-\varphi') \xi^2 w^2 - \int_{\Omega'} \varphi a_i \xi^2 \xi_{x_i} w <
\]

\[
< \lambda \int_{\Omega'} (-\varphi') \xi^2 w^2 + \epsilon \int_{\Omega'} \varphi \xi^2 (a^2_1 + a^2_3) + \frac{\sigma^2}{\epsilon} \int_{\Omega'} w^2.
\]

Using parts 2) and 3) of Lemma 2.1 we have

\[
(3.51) \quad \sigma_3 (1 - \epsilon \sigma_3) \int_{\Omega'} \xi^2 \varphi |\nabla w|^2 < \int_{\Omega'} \xi^2 \varphi (1 - \epsilon \sigma_3) a_i(w_{x_1}, w_{x_2}) w_{x_i} <
\]

\[
< \max_{\Omega'} w^2 \left( \lambda (\varphi_0 - \varphi_1) + \frac{\sigma^2}{\epsilon} \right) |\Omega'|
\]

where \( |\Omega'| = \text{measure of } \Omega' \). The desired inequality follows from (3.51). Recall that \( \tilde{h} = \lambda (\varphi_0 - \varphi_1) \).

4. - Symmetrization.

Our aim in this section will be to show that the solutions of \( A'(w) = \lambda B'(w) \) arising in Theorem 2.2 can be assumed to lie in a class of func-
tions with a particular « symmetry » property, provided \( A(w) = R^2 \) is restricted as in Corollary 3.2. We begin with a brief description of Steiner symmetrization (also called decreasing rearrangement) of a function. The reader is directed to [19] for further information. We use the notation \( \mathcal{D} \) from Section 1 and consider a class \( P_k \) of functions defined on \( \mathcal{D} \). Each function \( z = f(x, \psi) \) in \( P_k \) should be continuous and \( 2k \) periodic in \( x \). Further it should be piecewise affine: i.e. for each \( f \) there is a triangulation of \( \mathcal{D} \) such that on each triangle \( f \) has the form \( c_0 + c_1 x + c_2 \psi \), the \( c_i \)'s changing from one triangle to another (to avoid ambiguity below, we'll assume \( c_i \neq 0 \)).

Further each \( f \) is assumed constant for \( \psi = \psi_o \) and for \( \psi = \psi_1 \). The class \( P_k \) is dense in \( H^1 \) (the periodic \( H^1 \) functions) and will be large enough for our application. To define the symmetrization we consider a fixed \( \psi \) and let \( \mu(f, \psi, t) = \text{meas}\{x|f > t\} \) where the measure is Lebesgue measure on \(-k < x < k \) and \( t \) is a real number. We seek a function \( \hat{f}(x, \psi) \) such that for each \( \psi \), \( \hat{f}(-x, \psi) = f(x, \psi) \) on \( |x| < k \), \( \hat{f} \) is nonincreasing in \( x \) for \( 0 < x < k \), and the distribution functions satisfy \( \mu(\hat{f}, \psi, t) = \mu(f, \psi, t) \). If \( t(\mu, \psi), 0 < \mu < k \) is the function inverse to the decreasing function of \( t, \frac{1}{2} \mu(f, \psi, t) \), then one verifies that the function \( \hat{f} \) defined on \( 0 < x < k \) by \( \hat{f}(x, \psi) = t(x, \psi) \), extended evenly to \(-k < x < 0 \) and periodically outside \( \mathcal{D} \), is a function with the desired properties. We call \( \hat{f} \) the symmetrization of \( f \). One shows easily that the function \( \hat{f} \) will again be in \( P_k \) and that

\[
\gamma g + \gamma f = g + \gamma \hat{f}
\]

if \( g = g(\psi) \) and \( \gamma > 0 \). The function \( \hat{f} \) has the further properties:

\[
\int_{\mathcal{D}} G(\psi, f(x, \psi)) \, dx \, d\psi = \int_{\mathcal{D}} G(\psi, \hat{f}(x, \psi)) \, dx \, d\psi
\]

for any continuous \( G(\psi, x) \) and

\[
\int_{\mathcal{D}} |\nabla f|^2 \, dx \, d\psi \geq \int_{\mathcal{D}} |\nabla \hat{f}|^2 \, dx \, d\psi
\]

where the integrals are taken in the \( H^1 \) sense. (The proofs given in [19], Note A, can be applied over one period.)

We will also want to consider the space of continuous, piecewise affine, \( 2k \) periodic functions on \( S = \{(x, y)|x \in R, \ 0 < y < 1\} \) and denote this space by \( P_k \) as well. In particular we’ll use the property corresponding to (4.2) for a function \( \psi = \psi(x, y) \) with integration over \( S_k = S \cap \{|x| < k\} \).
While our aim is to « symmetrize » \( w \) we do it through a symmetrization of \( y = y(x, \psi) \) defined by

\[
y = \hat{y}(\psi) + w(x, \eta(\psi))
\]

where \( \hat{y} \) corresponds to a trivial flow with speed \( c = (g/\lambda)^{1/2} \) (cf. (2.9)). We will define symmetrization for a \( w = w(x, \eta) \) which is \( 2k \) periodic in \( x \) on \( \Omega \); which is 0 on \( \eta = \eta_0, \eta_1 \); and which gives rise to an element \( y(x, \psi) \in P_k \) through the correspondence (4.4). We further require that \( w_{\eta} > -1 \) and that \( w \geq 0 \) (or \( <0 \)) in \( \Omega_k \). We denote the collection of such functions \( w \) by \( P'_k \).

**DEFINITION 4.1.** To each \( w = w(x, \eta) \in P'_k \) we associate a function \( \hat{w} \) as follows:

1. for \( w > 0 \), let \( y = \hat{y}(\psi) + w(x, \eta(\psi)) \)
2. form \( \hat{y} \) and \( \hat{w}(x, \psi) = \hat{y} - \hat{y} \)
3. let \( \hat{w} = \hat{w}(x, \eta(\psi)) \)
4. for \( w < 0 \), \( \hat{w} = -(-w) \).

Note that for fixed \( \eta, w(x, \eta) \) is piecewise linear in \( x \) and \( \hat{w} \) could easily be defined directly. However it is simplest to symmetrize piecewise affine functions. They have the added feature that the class is preserved in the transformation taking \( y(x, \psi) \) to the inverse function \( \psi(x, y) \). To achieve this simplifying feature we introduce the slightly awkward definition of \( P'_k \) and \( \hat{w} \).

Our aim is to show that the functional \( F \) in (2.12) cannot increase if \( w \) is replaced by \( \hat{w} \). We’ll require some lemmas

**LEMMA 4.1.** Suppose \( y = y(x, \psi) \in P_k \) and that for a given \( \psi \) and all \( x \)

\[
s_1 \Delta \psi < y(x, \psi + \Delta \psi) - y(x, \psi) < s_2 \Delta \psi
\]

for any real increment \( \Delta \psi \) satisfying \( 0 < \Delta \psi < \varepsilon_0, \varepsilon_0 > 0 \). Then (4.5) holds with \( \hat{y} \) replacing \( y \).

**PROOF.** Letting \( \mu(y, \psi, t) = \mu(\psi, t) \) we conclude from (4.5) that

\[
\{ x | y(x, \psi + \Delta \psi) > t + s_1 \Delta \psi \} \subset \{ x | y(x, \psi) > t \}
\]

so

\[
\mu(\psi + \Delta \psi, t + s_1 \Delta \psi) > \mu(\psi, t).
\]
By the definition of \( g \), \( g\left(\frac{1}{2} \mu(\psi, t), \psi + \Delta \psi\right) \geq t + s_1 \Delta \psi \). Since \( t = \int \phi(y, \psi, t, \psi) \), and as \( \mu \) takes all values in \([0, 2k]\) as \( t \) varies, we conclude that the first inequality in (4.5) holds for \( g \). The second is shown similarly.

**Lemma 4.2.** Suppose \( y \in P_k \), \( y > y \chi \), and \( y \gamma > s_1 > 0 \) a.e. then \( y(x, \psi) \) and \( \psi(x, y) \) give rise to inverse functions \( \psi(x, y) \) and \( \psi(x, y) \), respectively, which are in \( P_k \) and satisfy \( \psi(x, y) = \psi(x - k, y) \).

**Proof.** From Lemma 4.1 one easily concludes \( g > s_1 \) a.e. and so the inverse functions are defined. Note that if \( y(x, \psi(x, y)) = y \), \( y_x + y_\psi \cdot y_\psi = 0 \) and \( y_\psi \cdot y_\psi = 1 \) so \( y_x \) and \( y_\psi \) are constant on a triangle in the \((x, y)\) space corresponding to an \((x, \psi)\) triangle, so one easily sees that \( \psi \in P_k \). Note, however, that \( y_x \) and \( y_\psi \) have opposite signs. Now suppose \((y', \psi')\) is in the graph of \( \psi(x_0, y) \) for some \( x_0 \) in \([-k, k]\). Then we claim that

\[
\{x | y(x, y') < y'\} = \{x | \psi(x, y') > y'\}.
\]

To see this, suppose \( y = y(x, y') < y' \). Then \( \psi(x, y) = \psi' \) and since \( \psi \) is increasing in \( y \), \( \psi(x, y') > \psi' \), showing an inclusion in one direction. The reverse is similar. Now using the definition of \( \phi \) and \( \psi \) together with (4.6) and its analogue for \( \phi \) we see that

\[
\text{mes} \{x | \phi(x, y') > y'\} = \text{mes} \{x | \psi(x, y') > y'\}
= \text{mes} \{x | y(x, y') < y'\}
= \text{mes} \{x | \phi(x, y') < y'\}
= \text{mes} \{x | \psi(x, y') > y'\}
\]

where we use the fact that

\[
\text{mes} \{x | y(x, y') < y'\} = 2k - \text{mes} \{x | y(x, y') > y'\}
\]

and a similar equality for \( \phi \), valid since their \( x \) derivatives are nonzero. Since \( \phi \) and \( \psi \) have the same measure distribution and \( \phi \) is increasing on \( 0 < x < k \) and periodic, it follows that \( \psi(x, y) = \psi(x - k, y) \).

**Lemma 4.3.** Suppose \( w = w(x, \eta) \in P_k \) and \( |\nabla w| < \tilde{r} \), \( \tilde{r} \) being the cutoff parameter in (2.14). Then \( |\nabla \tilde{w}| < \tilde{r} \) and

\[
F(\tilde{w}) < F(w).
\]
PROOF. Consider a value \( \eta = \bar{\eta} \) such that \( w_\eta(x, \eta) \) exists, except for a finite set of \( x \)'s on the boundaries of a triangulation. The function \( u(x, \psi) = w(x, \eta(x)) \) will satisfy \( |u| < \bar{r} \) at \( \psi = \tilde{\psi} = \psi(\bar{\eta}) \) a.e. in \( x \) with \( \bar{r} = \eta_\psi(\tilde{\psi}) \). The associated \( \tilde{y} = \tilde{\psi} + u \) satisfies \( \tilde{y}_\psi - \bar{r} < y_\psi < \tilde{y}_\psi + \bar{r} \) for \( \psi = \tilde{\psi} \) and hence a.e. in \( x \), \( (\tilde{y}_\psi - \bar{r} - o(1))D\psi < y(\tilde{\psi} + \Delta \psi) - y(\tilde{\psi}) < (\tilde{y}_\psi + \bar{r} + o(1))D\psi \) for an increment \( \Delta \psi \). From Lemma 4.1 it follows that \( \tilde{y}_\psi \), existing a.e. in \( x \) for \( \psi = \tilde{\psi} \), has the same upper and lower bound as \( y_\psi \). Reversing steps we let \( \tilde{u} = \tilde{\psi} - \tilde{\psi} \) and \( \tilde{w}(x, \eta) = \tilde{u}(x, \eta(\tilde{\psi})) \) and readily see that \( |\tilde{w}| < \bar{r} \) at \( \eta = \bar{\eta} \) a.e. in \( x \). In fact the bound holds except on the boundaries of a triangulation.

To show (4.8) we relate \( F(w) \) to the functionals introduced in Section 2. Reintroducing \( \tilde{\psi} \) we see from (2.5), (2.7) and (2.10) that

\[
\begin{align*}
\phi F(w) &= G(w) - \int_{\bar{D}} \int_{\bar{D}} \frac{u^2}{2} = E(u) - \int_{\bar{D}} \int_{\bar{D}} \frac{u^2}{2} = \\
&= N(\tilde{\psi} + u) - N(\tilde{\psi}) - \langle N'(\tilde{\psi}), u \rangle
\end{align*}
\]

with a corresponding identity for \( \tilde{w} \) and \( \tilde{u} \). Thus \( F(\tilde{w}) \leq F(w) \) will follow if we show

\[
N(\tilde{\psi}) - N(\tilde{\psi}) - \langle N'(\tilde{\psi}), \tilde{u} \rangle < N(y) - N(\tilde{\psi}) - \langle N'(\tilde{\psi}), u \rangle.
\]

Since \( u = 0 \) for \( \psi = \psi_0 \) and \( \psi = \psi_1 \), we see looking at (2.5) that

\[
\langle N'(\tilde{\psi}), u \rangle = \int_{\bar{D}} \int_{\bar{D}} \frac{u}{2} \nabla \tilde{\psi} \cdot \nabla \tilde{\psi} = -\int_{\bar{D}} \int_{\bar{D}} \frac{\partial}{\partial \psi} \left( \frac{1}{2} \tilde{\psi} \right) u
\]

with an analogous expression for \( \langle N'(\tilde{\psi}), \tilde{u} \rangle \). From (4.2) we can conclude that \( \langle N'(\tilde{\psi}), u \rangle = \langle N'(\tilde{\psi}), \tilde{u} \rangle \) and thus it will suffice to show \( N(\tilde{\psi}) < N(y) \). Recall that \( N(y) \) is merely the Dirichlet integral in new variables (cf. (2.1), (2.3), (2.4)). Using Lemma 4.2 together with inequality (4.3) we have

\[
N(\tilde{\psi}) = \int_\bar{S} |\nabla \tilde{\psi}|^2 dx dy = \int_\bar{S} |\nabla \tilde{\psi}|^2 dx dy < \int_\bar{S} |\nabla \psi|^2 dx dy = N(y)
\]

completing the proof of the lemma.

In the next result we combine the last lemma with Theorem 2.2 and elements of its proof to improve on Corollary 3.2. In stating subsequent results we will suppress the dependence of constants on quantities related to a choice of cutoff parameter \( r \). Assuming that the choice \( r = \bar{r} \) is made...
(cf. Lemmas 2.1, 3.4, and 3.5) we take the values $a_1, \ldots, a_n$ from Lemma 2.1 and the bounds $v, b, b'$ associated with $a(p_1, p_2)$ as given; $v$ will be approximately 1, and so on. We also suppress $\sigma_i$ coming from the cutoff function $\zeta$, since the bounds in Section 3 are uniform in $\sigma$ for $\sigma$ small. The constant $\lambda$ occurring in Section 3 is merely $\lambda(q_0 - q_1)$ and from Lemmas 2.3 and 2.4 we see that $\lambda$ is uniformly bounded provided $0 < \delta < \delta_0$, $\delta_0(q_0, q_1, \eta_0, \eta_1)$ coming from Lemma 2.4. Further since $\eta_1 - \eta_0 = 1$ we henceforth focus on the parameters $q_0, q_1$, and $\eta_0$ as regards the dependence of various bounds on parameters.

**Theorem 4.1.** There are positive constants $\delta_0, R_1, A^-, A^+$ depending on $q_0, q_1 < q_2$, and $\eta_0$ such that for $0 < \delta < \delta_0$, $0 < R < R_1$ and $k > 0$ the equation (2.13)

$$F'(w) = \lambda B'(w)$$

has solutions $(\lambda_i, w_i)$, $i = 1, 2$ with $A^- < \lambda_i < A^+$, $w_i \in H^1_0(\Omega) \cap C^0(\Omega)$, and $F(w_i) = R^2$. The solution $w_i$ satisfies $w_i = \beta_i > 0$ in $\Omega_\delta$ and has the variational characterization (2.18). The solution $w_i$ satisfies $w_i = \tilde{\beta}_i < 0$ and is characterized by (2.18) with $B(w^-)$ replacing $B(w^+)$. Each solution has the regularity shown in Lemma 3.5.

**Proof.** Let $\delta_0$ and $R_1$ be the constants occurring in Lemma 2.4 and Corollary 3.2, respectively. Let $(\lambda, w)$ with $w > 0$ be obtained from Theorem 2.2. As noted, Lemmas 2.3 and 2.4 yield $A^- < \lambda < A^+$ and $\lambda = \lambda(q_0 - q_1) < A^+(q_0 - q_1)$ with $A^-$ and $A^+$ depending on $q_0, q_1$, and $\eta_0$. Thus in Corollary 3.2, $R_2 = R_2(q_0, q_1, \eta_0)$.

From Corollary 3.2, $|\nabla w| < \tilde{r}$, so $A$ and $F$ agree near $w$. The function $w$ can be approximated arbitrarily closely in $W^{1,\infty}(\Omega)$ by functions in $P_k$. This can easily be seen by going over to $y(x, \psi) = \tilde{y} + w(x, \eta(\psi))$ where one can approximate $y$ (in this case $y \in C^1$) arbitrarily closely by piecewise affine functions on a triangulation. The transformation back to $w$ preserves $W^{1,\infty}$ proximity. For $n = 3, 4, 5, \ldots$ we choose $w_n \in P_k$ so that $\|w_n\|_{W^{1,\infty}(\Omega)} < \tilde{r}$, $|F(w_n) - R^2| < 1/n$, and $w_n - w\|_{W^{1,\infty}} < 1/n$. One computes that $F(t_n w_n)$ has a $t$ derivative

$$\frac{1}{2} \int_{\Omega} \eta^2 |\nabla w_n|^2 \frac{t (2 + t (\partial w_n / \partial \eta)^2)}{1 + t (\partial w_n / \partial \eta)^2}$$

and thus there is a $t = t_n$ with $|t_n - 1|$ of order $1/n$ such that $F(t_n w_n) = R^2$. 


For large $n$ we’ll have $|t_n \nabla w_n| < \tilde{r}$ so $F(t_n w_n) = A(t_n w_n)$. If $B(w) = -\int \phi'(w^2/2) = s$, then since $t_n w_n$ converges to $w$ in $L^2_\Omega$, $s_n = B(t_n w_n)$ converges to $s$ as $n$ approaches $\infty$. From (4.1) and Lemma 4.3, $F(t_n w_n) > F(t_n \hat{w}_n) = A(t_n \hat{w}_n)$. Since $A(t \hat{w}_n)$ approaches $\infty$ as $t$ increases to $\infty$ and is continuous in $t$, there is a $t_n \hat{w}_n$ such that $A(t_n \hat{w}_n) = R^2$. The property (4.2) holds for $w_n$ and $\hat{w}_n$ and so $B(t_n \hat{w}_n) = B(t_n w_n) = s_n$. Hence $t_n \hat{w}_n$ is a maximizing sequence for the original problem (2.18). Referring to the proof of Theorem 2.2 we see that a subsequence will converge in $H^1_\Omega$ to a solution $w_1$ of $A'(w) = \lambda B'(w)$ with a corresponding $\lambda_1$. Since $t_n \hat{w}_n$ is nonnegative and symmetrized, $w_1$ inherits these properties; that is, $(\partial/\partial x) w_1 < 0$ for $0 < x < k$ in the $L^1$ sense. But since it satisfies (3.1), we see as before that $w_1 > 0$ and that it is smooth so $(\partial/\partial x) w_1 < 0$ pointwise. The estimate on $\lambda_1$ follows as before and the regularity in Lemma 3.5 holds since $w_1$ is a solution of (3.1) and $R < R_1$. Likewise, $|\nabla w_1| < \tilde{r}$ as before and so $F'(w_1) = \lambda_1 B'(w_1)$. The treatment of $(\lambda_2, w_2)$ is completely analogous. This completes the proof.

5. - Precise bounds on $\lambda$.

One of the eigenvalues $\lambda_1$ or $\lambda_2$ occurring in Theorem 4.1 can be shown to be strictly less than the « critical » eigenvalue $\mu$. Which eigenvalue it is depends on the sign of

\begin{equation}
\varepsilon = \frac{\varphi_0}{\eta_0} - \frac{\varphi_1}{\eta_1}.
\end{equation}

If $\varepsilon > 0$ we can estimate $\lambda_1$ and if $\varepsilon < 0$, $\lambda_2$. We’ll see in Section 8 that the corresponding function, $w_1$ if $\varepsilon > 0$ and $w_2$ if $\varepsilon < 0$, will have a non-trivial limit as the period approaches $\infty$.

**Lemma 5.1.** Let $\mu$ be the lowest eigenvalue of problem (2.21) and let $(\lambda_i, w_i)$ $i = 1, 2$ be the solutions of $F'(w) = \lambda B'(w)$ from Theorem 4.1. Then if $z_i \in H^1_\Omega$, $|\nabla z_i| < \tilde{r}$, $F(z_i) = R^2$, $z_i > 0$, and $z_i < 0$ it follows that

\begin{equation}
\lambda_i < \frac{R^2}{2B(z_i)} \left( 3 + \frac{(CR)^3}{1 - CR} \right) \frac{\mu}{2}
\end{equation}

where $C = C(\varepsilon, \eta_1, \eta_0)$.

**Proof.** Suppose $i = 1$; the case $i = 2$ is similar. Letting $\lambda_1 = \lambda$,
For \( w_1 = w \), and \( z_1 = z \) we have

\[
\langle F'(w), w \rangle = \int_\Omega \varrho(\eta) \left[ \frac{w_\eta}{1 + w_\eta} w_\eta + \frac{w_\eta}{1 + w_\eta} w_\eta - \frac{1}{2} \frac{w_\eta^2 + w_\eta^3}{(1 + w_\eta)^2} \right] = \\
= \int_\Omega \varrho(\eta) |\nabla w|^2 \left[ \frac{3}{2} \frac{1}{1 + w_\eta} \frac{1}{2} + \frac{1}{2} \frac{w_\eta^3}{(1 + w_\eta)^2} \right] = \\
= 3F(w) - \frac{1}{2} \int_\Omega \varrho(\eta) |\nabla w|^2 + P(w)
\]

in obvious notation. From Corollary 2.1 and Lemma 3.5 \(|w_\eta| < CR\) so

\[
P(w) < \frac{(CR)^2}{1 - CR} \int_\Omega \frac{1}{2} \varrho(\eta) |\nabla w|^2 = \frac{(CR)^2}{1 - CR} F(w).
\]

Equation (2.13) yields \( \langle F'(w), w \rangle = \lambda \langle B'(w), w \rangle = 2\lambda B(w) \) so

\[
\lambda = \frac{\langle F'(w), w \rangle}{2B(w)} \leq \frac{F(w)}{2B(w)} \left( 3 + \frac{(CR)^2}{1 - CR} \right) - \frac{1}{2B(w)} \int_\Omega \frac{1}{2} \varrho(\eta) |\nabla w|^2.
\]

If \( F(x) = R^2 \), then \( A(z) = R^2 \) and \( B(z) < B(w) \), by the characterization in (2.18). The quotient in the second term on the right side of (5.4) is the Rayleigh quotient for a linear eigenvalue problem which has a lowest eigenvalue \( \mu/2 \) (cf. (2.21)). Since \( F(w) = R^2 \), the inequality (5.2) follows from (5.4).

**Lemma 5.2.** Let \((\lambda_i, w_i), i = 1, 2\), be the solutions from Theorem 4.1. Suppose \( \varrho \) defined by (5.1) satisfies \( \varrho \neq 0 \) and let \( \lambda(\varrho) = \lambda_1 \) if \( \varrho > 0 \), \( \lambda_2 \) if \( \varrho < 0 \). Then there are positive constants \( \delta, \tilde{R}, k_1 = k_1(R) \), and \( C_1 \) depending on \( \varrho_0, \varrho_1 \), and \( \eta_0 \) such that if \( 0 < \delta < \tilde{\delta}, 0 < R < \tilde{R} \), and \( k > k_1 \),

\[
\lambda(\varrho) < \mu(1 - C_1 R^{4\alpha})
\]

where \( \mu \) is the lowest eigenvalue of (2.22).

**Proof.** We'll suppose \( \varrho > 0 \), the case \( \varrho < 0 \) being similar. The inequality (5.5) will follow from (5.2) with the use of a suitable function \( x \). We let \( z \in H^1_k \) be defined by

\[
z(x, \eta) = \alpha \xi(\eta) \exp[-\beta|x|]
\]

for \((x, \eta) \in [-k, k] \times [\eta_0, \eta_1] \) where \( \xi \) is the eigenfunction from Lemma 2.2
and $\alpha, \beta$ are constants to be determined. Using $\int q \, dq$ to denote an integral over $[\eta, \eta_i]$ and denoting $1 - \exp[-j\beta k]$ by $t_j(t_j(\beta, k)$ we find

$$B(z) = -\frac{\alpha^2}{2\beta} \int_{\eta_i}^\eta \frac{\xi^2}{2} \exp \left[-2\beta|x|\right] - \frac{\alpha^2 t_2}{2\beta} \int_{\eta_i}^\eta \frac{\xi^2}{2} \, d\eta = \frac{\alpha^2 t_2}{2\beta \mu} \int_{\eta_i}^\eta \xi^2 \, d\eta$$

and

$$F(z) = \frac{1}{2} \int_{\eta_i}^\eta \left( \alpha^2 \xi^2 \exp \left[-2\beta|x|\right] + \alpha^2 \beta^2 \xi^2 \exp \left[-2\beta|x|\right] \right) \cdot \left(1 - \alpha \xi \exp \left[-\beta|x|\right] + \frac{\alpha^2 \beta^2 \exp \left[-2\beta|x|\right]}{1 + \alpha \xi \exp \left[-\beta|x|\right]} \right) = \frac{\alpha^2 t_2}{2\beta} \int_{\eta_i}^\eta \xi^2 \, d\eta + \frac{\alpha^2 t_2}{3\beta} \int_{\eta_i}^\eta \xi^2 \, d\eta - \frac{\alpha^2 \beta^2 t_2}{3} \int_{\eta_i}^\eta \xi^2 \xi \, d\eta + \frac{\alpha^4 t_4}{4\beta} \int_{\eta_i}^\eta \xi^2 \xi \, d\eta (1 + O(\xi)) + \frac{\alpha^4 t_4}{4} \int_{\eta_i}^\eta \xi^2 \xi \, d\eta (1 + O(\xi)).$$

We introduce the further notation $\int q \xi^2 \xi \, d\eta = m_{\delta \gamma}$. Of particular interest here is $m_{\delta \gamma}$. Using Lemma 2.2 we obtain the estimate

$$\int \xi^2 \xi \, d\eta > \frac{\theta_0 |\eta|}{(\eta \eta) + \theta_0 \delta / (\eta \eta)} \left(\frac{\eta_1}{\theta_0 (\eta_1 - \delta)}\right)^3 \delta - \frac{\theta_0 (\eta_1 - \delta)}{(\eta_1 - \delta)^3}.$$

For $\delta = 0$ the right side of (5.8) is just $e$ in (5.1). As $e > 0$, $m_{\delta \gamma} > 0$ for a range of $\delta$, $0 < \delta < \delta (\theta_0, \theta_1, \eta_0) < \delta_0$, $\delta_0$ from Lemma 2.4. For $e < 0$ one obtains an upper bound for $m_{\delta \gamma}$ which reduces to $\delta$ when $\delta = 0$.

Now suppose $\alpha > 0$ and let $\beta = \gamma \sqrt{\alpha}$ where $\gamma$ and $\alpha$ are to be determined (for $m_{\delta \gamma} < 0$ choose $\alpha < 0$ and $\beta = \gamma \sqrt{|\alpha|}$). With the new notation

$$F(z) = \frac{\alpha^2 t_2}{2\gamma} m_{\alpha \gamma} + \frac{\alpha^2 t_2}{3\gamma} t_3 m_{\alpha \gamma} + \frac{\alpha^2 t_2}{3\gamma} t_3 m_{\beta \gamma} - \frac{\alpha^2 t_2}{3} t_3 m_{\gamma \gamma} + \epsilon(\alpha, \gamma)$$

where $\epsilon(\alpha, \gamma) = \const \alpha^2$ for $\alpha < \alpha_0$, $\alpha_0$ a positive constant chosen so that $\alpha_0 |\xi_0| < \frac{1}{2}$. We assume $k > k_1 = (\ln 2)/3\beta$ so that, $1 > t_j > \frac{1}{2}$ for $j = 2, 3$ and choose $\gamma$ so that $\gamma t_2 m_{\alpha \gamma} - \frac{3}{2} \gamma^{-1} t_3 m_{\beta \gamma} = - t_3 m_{\alpha \gamma}$ ($\gamma$ will depend on $k$ in an inessential way). The condition $F(z) = R^2$, i.e.

$$R^2 = \frac{\alpha^2 t_2}{2\gamma} t_2 m_{\alpha \gamma} (1 - \gamma \alpha + O(\xi^2)).$$

will determine $\alpha(R) = (2\gamma/t_2 m_{\alpha \gamma}^{1/2} R^2 + O(R^4))$ in a range $0 < R < R_1$, $R_1$ from Theorem 4.1. If necessary, $R$ can be reduced to guarantee $|\nabla \xi| < \tilde{\xi}$. **INTERNAL WAVES IN FLUIDS WITH RAPIDLY VARYING DENSITY**
In the expression $R^2/2B(z)$ occurring in (5.2) we use (5.10) in the numerator and (5.6) in the denominator (with $\beta = \gamma \sqrt{z}$) obtaining

$$
\lambda < \frac{\mu}{2} \left( 1 - \gamma z + O(z^2) \right) \left( 3 + \frac{(CR)^2}{1 - CR} \right) - \frac{\mu}{2} = \mu \left( 1 - \frac{3}{2} \gamma z(R) + \frac{1}{2} (CR)^2 + O(R^{1/2}) \right).
$$

Choosing $\bar{R}$ smaller, if necessary, we obtain the inequality (5.5) from (5.11). Note that since $\beta = \gamma \sqrt{z}$, the lower bound $k_1$ is of order $R^{-1/2}$. We have noted some differences in the case $e < 0$; otherwise it is done similarly, completing the proof.

6. – A lower bound for the amplitude.

Up to this point we have been concerned with solutions of $F'(w) = \lambda B'(w)$ which ostensibly were functions of two variables. It may be that the solutions obtained are really functions of $\eta$ alone. That such solutions exist can be seen by restricting the variational procedure to functions of $\eta$ alone or, what amounts to the same thing, treating the Sturm-Liouville problem for $w = w(\eta)$:

$$
\frac{d}{d\eta} q(\eta) \frac{1}{2} \left( 2w_\eta + w_\eta^2 \right) = \lambda q' w \\
0 = w(\eta_0), \quad w(\eta_1) = 0
$$

by variational methods. One could also treat (6.1) as bifurcation from the simple eigenvalue $\mu$ in (2.22). Taking this approach one can show that for $k$ small the solutions $w_1, w_2$ in Theorem 4.1 are both functions of $\eta$ alone.

If $w = w(\eta)$ is a solution of (6.1) for which $\int_0^1 qw_\eta^2/(1 + w_\eta^2) = N_1^2$, then viewed as an element of $H_1^2$ it satisfies $F(w) = 2kN_1^2$. Thus if we fix $F(w) = R^2$ and let $k \to \infty$, the corresponding $N_1$ must approach zero. That is, the functions of $\eta$, normalized by $F(w) = R^2$, approach the function $w = 0$ in $H_1^2$ as $k \to \infty$. The next lemma shows that one of the solutions $w_1$ or $w_2$ does not « collapse » in this way as $k \to \infty$.

**Lemma 6.1.** Assume $0 < \delta < \delta$, $0 < R < \bar{R}$, and $k > k_1$, from Lemma 5.2. Let $v$ stand for a solution from Theorem 4.1 which has the same sign as $e$ (cf. 5.1). Then there is a constant $C = C(q_0, q_1, \theta_0)$ such that

$$
\|v\|_{L^w(\omega_0)} > CR^{1/3}.
$$
PROOF. Fix $R$ and let $m = \|w\|_{L^\infty(D_k)}$. From Lemma 3.6 with $\Omega^u \subset \Omega_x$ one obtains
\[ \int_{\Omega} |\nabla w|^2 < C_6 m^2 \]
and from Lemma 3.5 with $\Omega^u \subset \Omega^x$
\[ \|w\|_{L^6(\Omega^x)} < Cm. \]
The same estimate will hold on any translate of $\Omega^u$ in the $x$ direction so $|w_w| < Cm$ on $\Omega_x$. Referring to the proof of Lemma 5.1 and equality (5.3) in particular, we see that
\[ \lambda = \frac{\langle F'(w), w \rangle}{2B(w)} > \frac{\langle g(\eta) |\nabla w|^2 \rangle}{2B(w)} \left( 1 - \frac{3}{2} \frac{Cm}{\mu} \right). \]
Comparing this last inequality with the upper bound (5.5) one concludes that $C_1 R^{4/3} < \frac{3}{2} C \mu$ and so $m(R) = \frac{3}{2} C_1 \left( 1 - \frac{3}{2} \frac{Cm}{\mu} \right)$. We will need some properties of the Green's function for the equation
\[ (7.1) \quad -\frac{\partial}{\partial x} \varphi(\eta) \frac{\partial z}{\partial \eta} - \frac{\partial}{\partial \eta} \varphi(\eta) \frac{\partial z}{\partial \eta} + \lambda \varphi z = g(x, \eta) \]
on the strip $\Omega = \mathbb{R} \times [\eta_0, \eta_1]$ with $z = z(x, \eta)$ vanishing at $\eta = \eta_0$ and $\eta = \eta_1$. We write (7.1) as $Lz = g$.

**Lemma 7.1.** Suppose $0 < \delta < \eta / 2$ and let $\lambda$ be fixed, with $\lambda < \mu$, the lowest eigenvalue of (2.22). Let $(w_n, \gamma_n)$ be eigenfunctions and eigenvalues of
\[ (7.2) \quad \begin{cases} \frac{d}{d\eta} \varphi \frac{d}{d\eta} + \lambda \varphi w = \gamma \varphi \theta \\ w(\eta_0) = w(\eta_1) = 0 \end{cases} \]
with $\int_{\eta_0}^{\eta_1} \varphi \varphi d\eta = 1$ and $\gamma_1 < \gamma_2 < \gamma_3 < \ldots$. Then there are constants $C_i$,\[ \frac{\partial}{\partial x} \varphi(\eta) \frac{\partial z}{\partial \eta} - \frac{\partial}{\partial \eta} \varphi(\eta) \frac{\partial z}{\partial \eta} + \lambda \varphi z = g(x, \eta) \]
on the strip $\Omega = \mathbb{R} \times [\eta_0, \eta_1]$ with $z = z(x, \eta)$ vanishing at $\eta = \eta_0$ and $\eta = \eta_1$. We write (7.1) as $Lz = g$.

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\[ (7.2) \quad \begin{cases} \frac{d}{d\eta} \varphi \frac{d}{d\eta} + \lambda \varphi w = \gamma \varphi \theta \\ w(\eta_0) = w(\eta_1) = 0 \end{cases} \]
with $\int_{\eta_0}^{\eta_1} \varphi \varphi d\eta = 1$ and $\gamma_1 < \gamma_2 < \gamma_3 < \ldots$. Then there are constants $C_i$,\[ \frac{d}{d\eta} \varphi \frac{d}{d\eta} - \frac{\partial}{\partial \eta} \varphi(\eta) \frac{\partial z}{\partial \eta} + \lambda \varphi z = g(x, \eta) \]
on the strip $\Omega = \mathbb{R} \times [\eta_0, \eta_1]$ with $z = z(x, \eta)$ vanishing at $\eta = \eta_0$ and $\eta = \eta_1$. We write (7.1) as $Lz = g$.

**Lemma 7.1.** Suppose $0 < \delta < \eta / 2$ and let $\lambda$ be fixed, with $\lambda < \mu$, the lowest eigenvalue of (2.22). Let $(w_n, \gamma_n)$ be eigenfunctions and eigenvalues of
\[ (7.2) \quad \begin{cases} \frac{d}{d\eta} \varphi \frac{d}{d\eta} + \lambda \varphi w = \gamma \varphi \theta \\ w(\eta_0) = w(\eta_1) = 0 \end{cases} \]
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on the strip $\Omega = \mathbb{R} \times [\eta_0, \eta_1]$ with $z = z(x, \eta)$ vanishing at $\eta = \eta_0$ and $\eta = \eta_1$. We write (7.1) as $Lz = g$.
\[ i = 0, 1, 2, 3 \text{ depending on } \eta_0, \eta_1 \text{ and } \eta_0 \text{ such that} \]

\[ \begin{align*}
\gamma_i &> C_0 (\mu - \lambda) \\
\left| \frac{dw_n}{d\eta} \right|^2 d\eta &< C_1 n^2 \\
|w_n|_{L^\infty} &< C_2 n \\
\left| \frac{dw_n}{d\eta} \right|_{L^\infty} &< C_3 n^2.
\end{align*} \]

**PROOF.** The eigenvalues are characterized by

\[ \gamma_n = \min_{\varepsilon_n} \max_{w \in \varepsilon_n, w \neq 0} \frac{\int (\varrho|dw|d\eta)^2 + \lambda \varrho' w^2) d\eta}{\int \varrho w^2 d\eta} \]

for \( n = 1, 2, \ldots \) where \( \varepsilon_n \) is any \( n \)-dimensional subspace of \( H^1(\eta_0, \eta_1) \). Since \( \mu \) is the lowest eigenvalue of the problem (2.22),

\[ \frac{1}{\mu} \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta > \int (-\varrho') w^2 d\eta. \]

Note that, since \( \delta \eta_1/2 \), Lemma 2.3 yields a bound for \( \mu \) in terms of \( \varrho_0, \varrho_1, \) and \( \eta_0 \). Using (7.5) and the fact that \( \varrho' < 0 \) we get

\[ \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta > \int \left( \varrho \left| \frac{dw}{d\eta} \right|^2 + \lambda \varrho' w^2 \right) d\eta > \left( 1 - \frac{\lambda}{\mu} \right) \int \varrho \left| \frac{dw}{d\eta} \right|^2. \]

Suppose we let \( \tilde{\gamma}_n, n = 1, 2, 3, \ldots \) denote the eigenvalues in (7.2) when \( \lambda = 0 \). Then the characterization (7.4) together with (7.6) yields

\[ \tilde{\gamma}_n > \gamma_n > (1 - \lambda/\mu) \tilde{\gamma}_n. \]

Using the bounds on \( \varrho(\eta) \) one easily shows \( \varrho_1 \varrho_1^{-1}(\eta) < \tilde{\gamma}_n < \varrho_2 \varrho_2^{-1}(\eta) \), using the variational characterization corresponding to \( -w w = = \tilde{\lambda} w \) and recalling that \( \eta_1 - \eta_0 = 1 \). The first inequality in (7.3) follows, setting \( n = 1 \).

We can estimate the numerator in (7.4) by

\[ \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta + \lambda \int \varrho' w^2 d\eta = \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta - \lambda \int 2 \varrho \frac{dw}{d\eta} \frac{dw}{d\eta} d\eta \]

\[ > \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta - \frac{1}{2} \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta - 2 \lambda \int \varrho w^2 d\eta = \]

\[ = \frac{1}{2} \int \varrho \left| \frac{dw}{d\eta} \right|^2 d\eta - 2 \lambda \int \varrho w^2 d\eta. \]
Since the quotient in (7.4) equals $y_n$ with $w = w_n$ and since $\int q|w_n|^2 \, d\eta = 1$, (7.7) provides a bound

$$\int q \left| \frac{d w_n}{d \eta} \right|^2 \, d\eta < \int q \left| \frac{d w_n}{d \eta} \right|^2 \, d\eta < 2\gamma_n + 4\lambda^2$$

which easily gives the second claim in (7.3) since $\lambda < \mu$ and $\mu$ is bounded. Next,

$$w_n(\eta) = \int_{\eta_n}^{\eta} \frac{d w_n}{d \eta} \, d\eta < \left( \int \left| \frac{d w_n}{d \eta} \right|^2 \, d\eta \right)^{1/2}$$

providing an $L^\infty$ bound. Finally, we integrate (7.2) in the form

$$- \frac{d}{d \eta} \left( \Theta \frac{d w_n}{d \eta} \right) + \lambda \left[ \frac{d}{d \eta} (\Theta w_n) - \Theta \frac{d w_n}{d \eta} \right] = \gamma_n \Theta w_n$$

from a point $\bar{\eta}$ where $d w_n/d \eta = 0$ to obtain

$$\left| \Theta(\eta) \frac{d w_n}{d \eta} \right| = \left| \lambda \Theta w_n \right| - \lambda \int_{\eta_n}^{\eta} \frac{d w_n}{d \eta} \, d\eta - \gamma_n \left( \Theta w_n \right)$$

$$< 2\lambda \Theta_0 C_n + \lambda \Theta_0 \left( \int \left( \frac{d w_n}{d \eta} \right)^2 \, d\eta \right)^{1/2} + \gamma_n \Theta_0 \frac{1}{2} \gamma_n^2 < \bar{C}_n n^2$$

from which the last of (7.3) follows, completing the proof.

The estimates in the last lemma are rather crude but will be sufficient for use in estimating the Green’s function

$$G(x-x', \eta, \eta') = \sum_{n=1}^{\infty} \frac{\exp \left[ -\gamma_n^{1/2} |x-x'| \right]}{2\gamma_n^{1/2}} w_n(\eta) w_n(\eta')$$

for the operator $L$ in (7.1).

The next lemma is a technical one required in estimating the decay of $w$. It is tailored to the problem at hand, but could clearly be extended to cover a variety of decay problems for nonlinear elliptic equations in conjunction with estimates like those in the remainder of this section.

**Lemma 7.2.** Let $b = \{b_j\}, n < j < 2k - n$ be a sequence of nonnegative real numbers satisfying:

1) $b_{x+i} = b_{x-i}$, $i = 1, 2, \ldots, k-n$

2) $b_i < \bar{d}_n j^{-2/3}$ for $n < j < k$
and

$$b_{j} < \tau_{j}(b) \quad \text{for } n + 1 < j < k$$

where

$$\tau_{j}(b) = d_{j}(b_{j-1} + b_{j} + b_{j+1})^{2} + d_{2} \left[ \sum_{i=n+1}^{2k-n-1} \exp[-p|i-j|]b_{i} + q \exp[-p(j-n)] \right]^{2}$$

and

$$q = [d_{4}\exp[-pn] + d_{4}n^{-2/3}] .$$

Here $d_{4}, \ldots, d_{4}$ and $p$ are positive constants.

Then for $n$ sufficiently large

$$|g + \tilde{g}| < C \exp[-2pj]$$

$$(7.9) \quad n + 1 < j < k \text{ where } C \text{ depends on } d_{4}, \ldots, d_{4}, p \text{ and } n.$$
j-th component of $Tg$ by

$$d_0^2(j-1)^{-4/3} + d_2\left[2d_0 \sum_{i-n+1}^{k} \exp[-p|i-j|]i^{-2/3} + q \exp[-p(j-n)]\right]^2$$

using the fact that $g_j$ is even about $k$ and that, since $j<k$, the discrete convolution over $[k, 2k-n-1]$ is at most that over $[n+1, k]$. To estimate the sum in (7.10) we note that the sum over $n+1 < i < j-1$ is at most $d_j^{-2/3}$ provided

$$\sum_{i=n+1}^{j-1} \frac{\exp[-p(i-j)]}{(i/j)^{2/3}} < d.$$ 

However, a simple integral comparison shows the last inequality to be true with $d = d(p)$ independent of $j$. As regards the sum in (7.10) over $j < i < k$, it is easily seen to be at most $j^{-2/3}(1 - \exp[-p])^{-1}$. Since $q \to 0$ as $n$ increases it is easy to see that the expression in (7.10) is at most $d_0 j^{-2/3}$ for $n + 1 < j < k$ provided $n$ is sufficiently large.

Next consider $I_3 = \{g \in K | g < q^{2/3} \exp[-2p(j-n)], n+1 < j < k\}$. To see that $I_3$ is invariant under $T$ for $n$ large, we'll need estimates on some sums. Clearly

$$\sum_{i=n+1}^{j-1} \exp[-p(j-i)] \exp[-2p(i-n)] = \exp[-p(j-n)] \sum_{i=n+1}^{j-1} \exp[-p(i-n)]$$

and

$$\sum_{i=j}^{k} \exp[-p(i-j)] \exp[-2p(i-n)] = \exp[-p(j-n)] \sum_{i=j}^{k} \exp[-2p(i-j) - p(i-n)].$$

Since $\exp[-2p] < \exp[-p]$, each sum is at most $\exp[-p(j-n)] \cdot (\exp[p] - 1)^{-1}$. Using the symmetry of $g_j$ about $k$ and these estimates it is easy to show that for $n + 2 < j < k$

$$(Tg)_j < 9d_1 q^3 \exp[4p] \exp[-4p(j-n)] + d_2 q^2 \exp[-2p(j-n)][4q^{2/3}(\exp[p] - 1)^{-1} + 1]^2.$$ 

Choosing $n$ sufficiently large and thus $q$ sufficiently small we see that $(Tg)_j < q^{2/3} \exp[-2p(j-n)]$. When $j = n + 1$, $(Tg)_j$ involves $g_n = d_0 n^{-2/3}$. 

Specifically

\[(Tg)_{n+1} < d_1(n^{-2/3} + q^{3/2} \exp[-2p] + q^{3/2} \exp[-4p]) + \]

\[+ d_2 q^{3/2} \exp[-2p] [4q^{1/2} \exp[p] - 1]^{-1} + 1\]

which is less than \(q^{3/2} \exp[-2p]\) for \(n\) sufficiently large, since \(q\) is of order \(n^{-2/3}\) for \(n\) large. Thus \(T(I_2) \subset I_2\). Moreover, for \(n\) large \(I_2 \subset I_1\).

There must be a fixed point of \(T\) in \(I_2\). Suppose \(b\) in the statement of the lemma (restricted to \([n+1, k]\)) is not in \(I_2\). By hypothesis \(Tb > b\) and thus the order interval \(I_4 = (b + K) \cap I_1\) is invariant under \(T\). Hence \(T\) has a fixed point in \(I_4\). But \(I_2 \cap I_3 = \emptyset\) and \(I_2 \cup I_3 \subset I_3\) where \(T\) is a contraction and hence has a unique fixed point. The contradiction implies that \(b \in I_2\) which means (7.9) holds for a suitable \(C\).

Before making use of Lemma 7.2 we’ll need a crude bound on the decay of the solution \(w\) occurring in Lemma 6.1.

**Lemma 7.3.** Let \(w = w_1\) or \(w_2\) in Theorem 4.1. Then

\[(7.11) \quad |w(x, \eta)| \leq \frac{CR^{1/3}}{x^{1/3}}\]

on \(0 < x < k\) with \(C = C(\theta_0, \theta_1, \eta_0)\).

**Proof.** Let \(m(x) = \max |w(x, \eta)|\) for \(\eta_0 \leq \eta \leq \eta_1\). Since \(w_\eta < CR\) from Lemma 3.5, \(|w(x, \eta)| > m(x)/2\) on an \(\eta\) interval of length at least \(m/CR\).

Since \(|w|\) is nonincreasing in \(x\) for \(0 < x < k\),

\[(7.12) \quad \|w\|_{L^2(U_{\delta})}^2 \geq 2x \left(\frac{m^2}{2}\right) \frac{m}{CR}.
\]

But from the Poincaré inequality and Corollary 2.1

\[(7.13) \quad \int_{U_{\delta}} |w^2| \leq \pi^{-2} \int_{U_{\delta}} |\nabla w|^2 \leq \frac{2\pi^{-2}}{\sigma_1 q_1} R^4.
\]

The inequality (7.11) follows from (7.12) and (7.13).

If we combine Lemmas 6.1 and 7.3 we easily obtain

**Corollary 7.1.** Suppose \(0 < \delta < \delta\) and \(0 < R < \bar{R}\) from Lemma 5.2. Let \(w\) stand for a \(2k\) periodic solution from Theorem 4.1, which has the same sign as \(e\) (cf. 5.1). Then there is a constant \(\bar{k}(R)\) such that for \(k > \bar{k}(R)\), \(w\) is not a function of \(\eta\) alone. Here \(\bar{k}(R) < CR^{-2}\) with \(C = C(\theta_0, \theta_1, \eta_0)\).
PROOF. We assume \( \tilde{k} > k_1 \) from Lemma 5.2 and recall from the proof of that lemma that \( k_1 \leq C'R^{-2/3} \). If \( w \) is independent of \( x \) then from Lemmas 6.1 and 7.3, \( CR^{4/3} < C'R^{2/3} k^{-1/3} \) so \( k < k_n(R) = (C/C)^4 R^{-4} \). The choice \( k(R) = \max\{k_1(R), k_2(R)\} \) meets the requirements for \( k \).

**Lemma 7.4.** The solution \( w \) from Theorem 4.1 having the same sign as \( e \) satisfies

\[
|w(x, \eta)| < C \exp[-px]
\]

\[
|\nabla w(x, \eta)| < C' \exp[-px]
\]

for \( 0 < x < k \) where \( p \) is any real constant satisfying \( p < \gamma_1^{1/2} \) (cf. (7.2)) and \( C, C' \) are constants depending on \( \varrho_0, \varrho_1, \eta_0, R \) and \( p \).

**Proof.** We use the equation (3.49) for \( w \) and write it as

\[
Lw = \text{div } V
\]

where \( L \) is the elliptic operator in (7.1) and

\[
V = -\varrho \left( \frac{w_x w_\eta - 4w_x w_\eta + 2w_x w_\eta^2 + w_\eta^2 - w_\eta}{2(1 + w_\eta)^2} \right).
\]

Let \( \zeta = \zeta(x) \) be a \( C^\infty \) function taking values in \([0,1]\) which is supported on \([-\frac{1}{2}, \frac{3}{2}]\) and which equals 1 on \([-\frac{1}{2}, \frac{1}{2}]\). For \( 0 < j < 2k \), let \( \zeta_j \) denote the extension of \( \zeta(x-j) \) to a \( 2k \) periodic function of \( x \) and let \( S_j \) denote the support of \( \zeta(x-j) \) as a function on \( \Omega_k \).

In analogy with (7.6) one sees from (2.24) that for \( w \in H_k^1 \cap C^2_k(\Omega) \)

\[
\int_{\Omega_k} (Lw) w > \left( 1 - \frac{2}{p} \right) \int_{\Omega_k} \varrho |\nabla w|^2.
\]

Since \( \tilde{L} \) is coercive and uniformly elliptic we can uniquely solve

\[
L\theta = \zeta_j \text{ div } V
\]

and

\[
L\chi = (1 - \zeta_j) \text{ div } V
\]

for \( \theta, \chi \) in \( H_k^1 \cap C^2_k \) (cf. [20]). By uniqueness, \( w = \theta + \chi \) and we can estimate \( w \) by estimating \( \theta \) and \( \chi \).
We begin with

\[ \int_{\Omega} (L \theta) \theta = \int_{\Omega} (\zeta_i \operatorname{div} V) \theta = V \cdot \operatorname{grad} \zeta_i \theta. \]

Using (7.17) and the form of $V$ together with standard inequalities we find

\[ \left(1 - \frac{\lambda}{\mu}\right) \int_{\Omega} \theta |\nabla \theta|^2 \leq C_0 \int_{\Omega} |\nabla w|^4 + \varepsilon \int_{\Omega} |\nabla \theta|^4 \]

where $C_0$ depends on $\bar{e}_0$ (we are assuming $|\nabla w| < \bar{e} < 1$). Choosing $\varepsilon$ to be $(\mu - \lambda)/2\mu$ in (7.20) we get

\[ \int_{\Omega} \theta |\nabla \theta|^2 \leq \frac{2\mu C_0}{\mu - \lambda} \int_{\Omega} |\nabla w|^4. \]

Let $B_i = \{(x, \eta) \in \Omega | j \leq x < j + 1\}$. According to Lemmas 3.2 and 3.5 we can estimate $|\nabla w|$ on $S_i$ in terms of $\int |\nabla w|^4$ over the set $B_{i-1} \cup B_i \cup B_{i+1} \subset S_i$. We define

\[ b_j = \int_{B_i} |\nabla w|^2 \]

and conclude from (7.21) that

\[ \int_{\Omega} |\nabla \theta|^2 \leq C_1 (b_{i-1} + b_i + b_{i+1})^2. \]

To estimate $\chi$ we write

\[ \chi(x, \eta) = \int_{\Omega} G(x - x', \eta, \eta')(1 - \zeta_i(x')) \operatorname{div} V(x', \eta') \, dx' \, d\eta' \]

and restrict $x$ to the interval $[j, j + 1]$. Due to the presence of the factor $1 - \zeta_i$, we will have $|x - x'| > \frac{1}{4}$ when the integrand is nonzero. Given the estimates of Lemma 7.1 it is an easy matter to see from (7.8) that the sum for $G$ as well as the sums corresponding to any compound derivative of $G$ containing at most one each with respect to $\eta$ and $\eta'$ converge absolutely for $|x - x'| > \frac{1}{4}$. If we integrate by parts to remove derivatives from $V$, we can show simply that

\[ |\nabla \chi| \leq C_2 \sum_{i=-\infty}^{\infty} \exp \left[-p|i - j|\right] b_i. \]
The periodicity of $\omega$ allows the further estimate

\begin{equation}
(7.25) \quad \sum_{i \neq j} \exp[-p|i-j|] b_i < \sum_{i=1}^{2k} \sum_{N=-\infty}^{\infty} \exp[-p|i+2kNd-j|] b_i = \frac{1}{1-\exp[-2kp]} \sum_{i=1}^{2k} \exp[-p|i-j|] b_i.
\end{equation}

In the sequel we will absorb the factor $(1-\exp[-2kp])^{-1}$ into the constants appearing.

From Lemmas 7.3 and 3.6 we know that $b_i < \min(C'_0 m^2, C'_0 i^{-2/3})$ where $m = \|w\|_{L^2}$ and $C'_0, C'_0$ depend on $\xi_0, \xi_1, \eta_0$, and $R$. Suppose $C'_0 n_0^{-2/3} < C'_0 m^2$ and that $n > n_0$. Then

\begin{equation}
(7.26) \quad \beta_j = \sum_{i=0}^{n} \exp[-p(j-i)] b_i < C'_0 \left[ m^2 \sum_{i=n_0}^{n-1} \exp[-p(j-i)] + \sum_{i=n_0}^{n} \exp[-p(j-i)] i^{-2/3} \right] < C'_0 \exp[-p(j-n)][m^2 \exp[-p(n-n_0 + 1)] + C'_0 n^{-2/3}]
\end{equation}

where we have estimated the second term in the square bracket using a comparison with an integral as in the proof of Lemma 7.2. Since $w$ is even in $x$ and $2k$ periodic, $b_i = b_{2k-i}$. Thus for $j < k$,

\begin{equation}
(7.27) \quad \sum_{i=2k-n}^{2k} \exp[-p(i-j)] b_i < \beta_j.
\end{equation}

If we combine (7.23)-(7.27), recalling that $b_j < 2 \left( \int_{B_j} |\nabla \theta|^2 + \int_{B_j} |\nabla \chi|^2 \right)$ we obtain

\begin{equation}
\begin{aligned}
b_j &< C_1 (b_{j-1} + b_j + b_{j+1}) + C_2 \left[ \sum_{i=n_0 + 1}^{2k-n-1} \exp[-p|i-j|] b_i + q \exp[-p(j-n)] \right]^2
\end{aligned}
\end{equation}

where $q = [C_4 \exp[-pn] + C_4 n^{-2/3}]$.

We can now appeal to Lemma 7.2 to conclude that there is a constant $\tilde{C} = \tilde{C}(\xi_0, \xi_1, \eta_0, R, p)$ such that

\begin{equation}
\left( \begin{array}{c} \nabla w \end{array} \right)_{B_j} = b_j < \tilde{C} \exp[-2pj]
\end{equation}

for $n < j < k$ and $n$ sufficiently large. The estimates (7.14) now follow from inequality (3.10) and from Lemmas 3.2 and 3.5.
8. - Steady waves and limiting forms.

To avoid excessive length in the statements of the theorems in this section we briefly summarize what we know at this point and recall notation that will be used in the statements. A density stratification (2.11) is given which is \( \rho_0 \) for \( y \) below \( |\eta_0| \) and drops to a value \( \rho_0 < 0 \) over a transition width \( \delta \). With this density the problem (2.16) \( A'(w) = \lambda B'(w) \), \( A(w) = R^2 \) has \( 2k \) periodic solutions \( w_0 > 0 \) and \( w_0 < 0 \) with corresponding eigenvalues \( \lambda_1, \lambda_2 \). The restrictions \( 0 < \delta < \delta_0, 0 < R < R_1 \) in Theorem 4.1 guarantee that the pairs \( (w_0, \lambda_0) \) are actually solutions of the *physical* problem \( F'(w) = 2B'(w) \) and that they are symmetrized. Letting \( e = \rho_0 \eta_0^{-2} - \rho_1 \eta_1^{-2} \) (cf. (5.1)) we distinguished the solution \( w \) having the same sign as \( e \). With the restrictions \( 0 < \delta < \delta_0, 0 < R < R_1 \), and \( k > k(R) \) from Lemma 5.2 and Corollary 7.1, we saw in §§ 5-7 that the associated \( \lambda \) is strictly below the critical value \( \mu = \mu_0 \) that \( \|w\|_{L^\infty} \) has a positive lower bound, that \( w \) has nontrivial dependence on \( x \), and, indeed, that it has exponential decay in \( x \) as does its gradient. All of these estimates were independent of \( \delta \) and \( k \) for fixed \( R \). Of course, \( \delta, R, \) and \( k \) depend on \( \rho_0, \rho_1, and \eta_0 \).

With \( \sigma = g/\lambda, w \) is a critical point of \( G \) in (2.10) or equivalently \( u = u(x, \psi) = w(x, \eta(\psi)) \) is a critical point of \( E(u) \) in (2.7) where \( \tilde{g}(\psi) \) is the inverse function to \( \tilde{g}(\psi) \) defined by (1.7) using \( \rho_0 \) and \( c \). The functions \( g(\psi) \) and \( H(\psi) \) are defined by (1.8)-(1.10). Since \( \tilde{g} \) is a critical point of \( \Phi \) defined by (2.3), according to (2.6) \( y = \tilde{g}(\psi) + u(x, \psi) \) is a critical point of \( \Phi \); that is, \( y \) satisfies equation (1.11). Since \( \rho_0 \) is \( C^\infty \) for \( \delta > 0 \), all solutions are \( C^\infty \). The solution \( y \) corresponds to a periodic internal wave. According to Corollary 7.1 it is a wave with vertical component \( V \neq 0 \). For \( e > 0 \) it is a wave of elevation; for \( e < 0 \) it is a wave of depression. This is easily seen since for \( e > 0 \) (\( < 0 \)) and \( 0 < x < k \) we have \( y < 0 \) (\( > 0 \)) on a streamline. Alternatively, \( w \) measures the spatial deviation of the streamline with label \( \eta \).

We can briefly summarize the results so far obtained as follows.

**Theorem 8.1.** If the hypotheses of Theorem 2.1 are further restricted to require \( \delta > 0 \) and \( k < \infty \) then the problem (1.11)-(1.12) has a nontrivial solution \( (\lambda, y) \), with \( y \in C^\infty \), satisfying properties 1)-6) of Theorem 2.1.

We'll use \( \Omega^- \) and \( \Omega^+ \) to denote the subsets of \( \Omega \) where \( \eta < 0 \) or \( \eta > 0 \), respectively. We let \( \Omega^\pm = \Omega^\infty \) and define \( D^\pm, D^- \) analogously using \( \psi < 0 \) or \( \psi > 0 \).

**Theorem 8.2.** Suppose \( e = \rho_0 \eta_0^{-2} - \rho_1 \eta_1^{-2} \neq 0, 0 < R < R, and k > k(R) \).
Then for \( \delta = 0 \) there is a solution \((\lambda, y)\) of problem (1.11)-(1.12) satisfying properties 1)-6) of Theorem 2.1 with \(|Vw|\) taken in the \( W^{1,\infty}(\Omega_k) \) sense. Here the base flow corresponds to

\[
\hat{y}(\psi) = \begin{cases} 
\frac{\psi - \psi_0}{\sqrt{\varepsilon} e^{\lambda \sqrt{\varepsilon}}}, & 0 < \psi < 0 \\
\frac{\psi - \psi_0}{\sqrt{\varepsilon} e^{\lambda \sqrt{\varepsilon}}} - \frac{\psi_0}{\sqrt{\varepsilon} e^{\lambda \sqrt{\varepsilon}}}, & 0 < \psi < \psi_1.
\end{cases}
\]  

Also, \( w \) satisfies

7) For each \( \beta < 1 \) and some \( \alpha > 0 \), \( w \) is a limit in \( H^1_k \cap C^0(\Omega) \) and in \( C^{1+\alpha} \) on each compact subset of \( \Omega_k \) of solutions of \( F'(w) = \lambda B'(w) \) with \( \delta \)'s approaching 0. Further \( w \in C^{1+\alpha}_k(\Omega^\pm) \) and \( \partial w/\partial x \in C^\alpha_k(\Omega) \) with bounds independent of \( k \).

**Proof.** We suppose \( e > 0 \) (the case \( e < 0 \) being similar), fix \( R \) in \((0, \hat{R})\) and fix \( k > \hat{k}(R) \). Let \( \delta_j \) be a sequence of transition widths converging to zero and \((\lambda_j, w_j)\) the associated solutions in Theorem 4.1 with \( w_j > 0 \). To avoid any confusion suppose \( j = 3, 4, 5, \ldots \). We will return to the associated \( y_j(x, \psi) \) below. First we establish limit properties for a subsequence of \( w_j \). By the results earlier in this section properties 1)-6) of Theorem 2.1 are satisfied by \((\lambda_j, w_j)\) uniformly in \( j \). By Corollary 2.1 and Lemma 3.5 \( w_j \) is uniformly bounded in \( C^{1+\alpha} \) on \( \Omega_k - \Sigma_j \) where \( \Sigma_j := \{(x, \eta)|0 < \eta < \delta_j\} \) and uniformly bounded in \( W^{1,\infty} \). Thus a subsequence \( w_{j,1}, w_{j,2}, w_{j,3}, \ldots \) converges to \( w \) weakly in \( H^1_k \), strongly in \( L^2_k \) and by Arzela-Ascoli, uniformly in \( C^1(\Omega_k - \Sigma_j) \) and in \( C^\alpha(\Omega_k) \) for a chosen \( \beta < 1 \). We can also assume that the corresponding eigenvalues converge to \( \lambda \). A further subsequence \( w_{j,1}, w_{j,2}, \ldots \) converges in \( C^\beta(\Omega_k - \Sigma_k) \) as well, and continuing we find a diagonal sequence \( w_{j,s} \), which, in addition to the given convergence, converges in \( C^{1+\alpha} \) on every compact subset of \( \Omega_k^+ \) or \( \Omega_k^- \) (cf. [17], p. 283). For notational convenience suppose the original \( w_j \) is the sequence with these convergence properties. Since \( w_j \) has a uniform \( C^{1+\alpha} \) bound on \( \Omega_k - \Sigma_j \), the limit \( w \) has an extension to \( C^{1+\alpha}(\Omega^\pm_k) \) with a norm independent of \( k \). Since \( (\partial/\partial x)w \) has a \( C^\alpha \) bound uniformly in \( j \) and \( k \) in all of \( \Omega_k \) (cf. Lemma 3.2) the limit function \( w \) will have a derivative \( \partial w/\partial x \) in \( C^\alpha(\Omega) \) with the same bound. We use this \( w \) in part 7) and form \( y = \hat{y} + w(x, \eta(\psi)) \) from it. To complete 7) we note that the convergence of \( w_j \) in \( H^1_k \) follows from the \( C^1 \) convergence of \( w_j \) on each compact subset of \( \Omega^\pm_k \) and from the uniform boundedness of \( \nabla w_j \) on all of \( \Omega_k \) (cf. Lemmas 3.2, 3.5). In fact, properties 1), 3), 5), and 6) follow easily from the nature of the convergence and the bounds existing for \( \lambda_j \) and \( w_j \). Likewise, 2) will be satisfied independently of \( \hat{y} \) since \( w \) inherits periodicity.
To see that part 4) holds we first note that \( w \) satisfies the linear equation (2.19) \((\partial / \partial x_i) \bar{a}_{ij} (\partial w / \partial x_j) = 0\) in \( \Omega^\pm \) with \( \bar{a}_{ij} \) smooth (cf. the proof of Theorem 2.2). We're assuming \( \epsilon > 0 \) so the limiting \( w \) satisfies \( w > 0 \). Since \( w = 0 \) where \( \eta = \eta_0 \) or \( \eta_1 \), and is periodic in \( x \), the maximum of \( w \) in each of \( \Omega^\pm \) must occur where \( \eta = 0 \). Since \( w = \bar{w} \), the maximum occurs at \( x = 0 \) and must be positive, for \( w \neq 0 \). From the strong maximum principle it follows that \( w > 0 \) in both \( \Omega^\pm \). We'll show that \( w > 0 \) where \( \eta = 0 \), as well, so that 2) holds. If \( w(x_0, 0) = 0 \) for some \( x_0 \geq 0 \) then since \( w = \bar{w} \), \( w_\epsilon(x_0, 0) = 0 \) follows. To see that \( w \) and \( w_\epsilon \) cannot both vanish at \((x_0, 0)\) we integrate the equation \( F'(w) = \lambda B'(w) \) (equivalently (3.49)) over a rectangle \( D = \{(x, \eta) \mid |x - x_\epsilon| < \epsilon_1, \ |\eta| < \epsilon_2 \} \), assuming for the moment that \( w \) is smooth on \( D \) (i.e. consider \( w = w_j \)). We find

\[
\int_{-\epsilon}^{\epsilon} \frac{w_x}{1 + w_\eta} \bigg|_{x = -\epsilon}^{x = \epsilon} \, dx + \int_{-\epsilon}^{\epsilon} \left( \frac{1}{2} \left( \frac{2w_\eta + w_\eta^2}{1 + w_\eta^2} \right) \right) \bigg|_{x = -\epsilon}^{x = \epsilon} \, dx = \lambda \int_{x = -\epsilon}^{x = \epsilon} (w_\eta) \bigg|_{x = -\epsilon}^{x = \epsilon} - \lambda \int_D w_\eta \, dx.
\]

The identity (8.2), which holds for \((\lambda_j, w_j)\) will hold for the limiting quantities \((\lambda, w)\) described above since all integrands are bounded and converge uniformly in closed subsets of \( D \cap \Omega^\pm \). Since the integrands in (8.2) are continuous in \( \bar{D}^+ \) and \( \bar{D}^- \) separately, we can let \( \epsilon_1 \to 0 \) and then \( \epsilon_2 \to 0 \) to obtain

\[
\left[ \frac{\partial}{\partial x} \left( \frac{w_\eta + \frac{1}{2} w_\eta^2}{1 + w_\eta^2} \right) \right]_{x = \epsilon} = \lambda [\partial w]_{x = \epsilon}
\]

where \([g]_{x = \epsilon} = g(x_\epsilon, 0 +) - g(x_\epsilon, 0 -)\). In general the term \( w_\eta \) would appear in (8.3), but we're assuming it vanishes at \((x_\epsilon, 0)\). Since we are assuming \( w(x_\epsilon, 0) = 0 \), a further consequence of the strong maximum principle is that \( w_\eta(x_\epsilon, 0 +) > 0 \) and \( w_\eta(x_\epsilon, 0 -) < 0 \). Of course, \(|w_\eta| < \epsilon \sqrt{2} \). Since \( w(0, 0) = 0 \), \([\partial w]_{x = 0} = 0 \) and the equation resulting from (8.3) cannot be satisfied. Thus \( w(x, 0) > 0 \) for \( 0 < x < k \).

We must still show that we obtain a weak solution of (1.11) by adding to \( w \) a suitable \( \hat{y} \), i.e. the limit of the trivial flow solutions associated with \( w_j, \lambda_j \). Letting \( \epsilon_j = (g/\lambda_j)^{1/2} \) and letting \( \hat{y}(\psi) \) be the function \( \hat{y} \) corresponding to a trivial flow with speed \( \epsilon_j \) and transition width \( \delta_j \), we know that \( y_j(x, \psi) = \hat{y}(\psi) + w_j(x, \eta_j(\psi)) \), with \( \eta_j(\psi) = \hat{y}(\psi) - \hat{y}(0) \), is a solution of (1.11)-(1.12). Letting \( \epsilon = \lim_{j \to \infty} \epsilon_j \) one shows easily using (1.7) that \( \hat{y}_j \) converge uniformly to \( \hat{y} \) given in (8.1) and that the convergence is in \( C^1 \) on
any closed set not containing \( \psi = 0 \). The functions \( \eta_i(\psi) \) converge in the same manner to

\[
\eta(\psi) = \begin{cases} 
\frac{\psi}{\partial \phi^i \phi^j}; & \psi < 0 \\
\frac{\psi}{\partial \phi^i \phi^j}; & 0 < \psi < \psi_1.
\end{cases}
\]

Since the \( C^{1+\alpha} \) bounds on \( y_j \) are uniform in \( j \) on each compact subset of \( \Omega^\pm \), \( y_j \) converges to a function \( y = y(x, \psi) \) in \( C^1 \) on such subsets and the resulting \( y \) has extensions to \( C^{1+\alpha} \) of the closed regions \( \Omega^{\pm} \). Similarly \( y \in C^\beta_0(\Omega) \) for any \( \beta < 1 \) and \( \partial y / \partial x \in C^\beta_0(\Omega) \). Clearly (1.12) holds. We have \( \partial y_j / \partial \psi > (\partial y_j / \partial \psi)(1 - \rho) \) (cf. (2.8)) so \( \partial y / \partial \psi > 0 \) for \( \psi \neq 0 \). Since the associated functions \( \psi_i \) and \( H_j \) converge uniformly on compact subsets of \( \Omega^\pm \), it follows that \( y = y(x, \psi) \) is a weak solution of (1.11) in \( \mathcal{D}^\pm \). In fact, since \( \psi_i, H_j \) and their limits are zero wherever \( \phi \) is constant, \( w \) satisfies an analytic equation and thus is an analytic function of \( x \) and \( \psi \) (cf. [21], p. 505).

Since \( \hat{y} \) is linear in \( \Omega^\pm_\phi \), \( y \) is also analytic in \( \mathcal{D}^\pm \). One can also see this from the fact that the inverse function \( \psi(x, y) \) is harmonic where \( \phi \) is constant (cf. (1.5)) and hence \( \psi \) is analytic for \( y \neq y(x, 0) \).

What remains to be shown is the continuity of pressure (item ii) in Definition (1.1)) which is the condition connecting the two regimes on either side of \( \psi = 0 \). Suppose for the moment that we have a solution \( y = y(x, \psi) \) of (1.11) which is in \( C^0(\Omega) \). The pressure is defined by equation (1.4) and according to (1.3) can be expressed as

\[
p = H(\psi) - \frac{1}{2} |\nabla \psi|^2 - g\psi(\phi) =
\]

\[
= H(\psi) - \frac{1}{2} \frac{1 + y^2}{y^2} - g\psi(\phi).
\]

Let \( \mathcal{D} = \{ (x, \psi) | |x - x_0| < \varepsilon_1, |\psi| < \varepsilon_2 \} \). In analogy with the derivation of (8.3) from (3.49) one can integrate (1.11) over \( \mathcal{D} \), pass to the limit using smooth solutions and then let \( \varepsilon_1 \to 0 \) and \( \varepsilon_2 \to 0 \) to obtain

\[
[p]_{x_0} = 0
\]

for all \( x_0 \), using (8.5). Thus \( p \) is continuous and the proof is complete.

Referring again to the discussion at the beginning of this section for notation we have the following result, showing that a system of two layers of constant but differing density will support a solitary wave.
THEOREM 8.3. For $\delta = 0$, each $e \neq 0$, and each $R$, $0 < R < \hat{R}$ there is a solution $(\lambda, y)$ of problem (1.11)-(1.12) satisfying properties 1), 3), 4), 5), and 6) of Theorem 2.1 with $k = +\infty$ and $\nabla w$ taken in the sense of $W^{1,\infty}$. The base flow corresponds to $y$ in (8.1) and $w$ satisfies

7) For each $\beta < 1$ and each bounded set $\bar{Q}$, $w$ is a limit in $C^\beta(\bar{Q}) \cap C^1(\bar{Q}^\pm)$ of periodic solution with periods increasing to $\infty$. Further $w \in C^{\alpha+\varepsilon}(\bar{Q}^\pm)$ and $\partial w/\partial x \in C^{\alpha}(\bar{Q})$ for some $\alpha > 0$.

PROOF. As before, we do the case $e > 0$ and fix an $R$, $0 < R < \hat{R}$. For $k = 3, 4, 5, \ldots$ let $w_k$ denote the solution from Theorem 8.2 having period $2k$. From part 7) of that result and the Arzela-Ascoli Theorem it follows that a subsequence $w_{k,1}, w_{k,2}, \ldots$ converges in $C^0(\bar{Q}) \cap C^1(\bar{Q}^\pm)$ to a function $w$. A further subsequence has the same type of convergence on $Q_2$, and so on. The diagonal sequence $w_{k,k}$ will converge to a function $\bar{w}$ on $\Omega$ in $C^\beta(\bar{Q}) \cap C^1(\bar{Q}^\pm)$ for each bounded set $\bar{Q}$. By restricting $k$ to some subset of the integers we can call the convergent sequence $w_k$ and can suppose a sequence is chosen so that $\lambda_k$ converges to a number $\lambda$. The bounds on $\lambda_k$ persist, so $\lambda$ satisfies property 1) with $\delta = 0$. Likewise, properties 5), 6) and the remainder of 7) follow from the convergence and uniform estimates. Of course, all the properties shown so far are shared by the solution $w = 0$. That $w$ is nontrivial follows from the lower bound in Lemma 6.1. Alternatively, since the exponential decay of $\nabla w_k$ is uniform in $k$, the contribution to the integral giving $F(w_k)$, coming from $\Omega_k - \Omega_N$, can be made arbitrarily small for a suitably large $N$ and all $k > N$. Then since the gradient converges in $L^\infty$ on each $\Omega_N$, it follows that $F(w) = R^2$, property 3), where the integral is taken on all of $\Omega$. Property 4) is shown as in the previous proof.

The associated functions $\hat{y}_k(x)$ and $\eta_k(x)$ are given by (8.1) and (8.4), respectively, and change with $k$ only in that the speed $c = \rho_k = (g/\lambda_k)^{1/2}$ changes. The convergence of $y_k = \hat{y}_k + w_k(x, \eta_k(x))$ to function $y(x, \eta)$ in $\mathcal{D}$, in $C^0(\mathcal{D}) \cap C^1(\mathcal{D}^\pm)$ for each bounded $\mathcal{D}$, follows from the known convergence of the component functions. The property of being a weak solution of (1.11) in $\mathcal{D}^\pm$ is maintained by the $C^1$ convergence as is the pressure continuity. The boundary condition (1.12) is clearly satisfied by $y$ so the proof is complete.

Rather than let $\delta \to 0$ and then let $k \to \infty$ one can first let $k \to \infty$ for a fixed $\delta > 0$, obtaining a solitary wave corresponding to a smooth density. We state a theorem dealing with this limit below. A subsequent limit in $\delta$ can be taken and might produce a solution of (1.11) different from that in Theorem 8.3 (we do not show uniqueness) but we conjecture that it yields
the same solution. Consequently we don’t state a theorem covering the latter limit.

**Theorem 8.4.** For each $\epsilon \neq 0$, for each $\delta$, $0 < \delta < \delta$ and for each $R$, $0 < R < \bar{R}$ there is a solution $(\lambda, y)$ of problem (1.11)-(1.12) satisfying properties 1), 3), 4), 5), and 6) of Theorem 2.1 with $\kappa = \infty$. Also

7) $w$ is in $C^\infty$ and is the limit in $C^\infty$ on each bounded set, of solutions from Theorem 8.1, with periods increasing to $\infty$. The function $y$ is $C^\infty$ in $\Omega$.

**Proof.** Most of the proof is analogous to that of Theorem 8.3 and we don’t repeat the similar parts. The fact that $w$ is in $C^\infty$ follows from standard regularity results (cf. [16], Chapter 4). The $C^{1,1}$ bounds on $w$ on bounded sets depend on the smoothness of $f(p_1, p_2)$ and $g'$ on the ellipticity constant and on $\lambda$ and hence will be independent of the period $2k$. We can thus use a diagonal process to choose a sequence $\omega_k$, $k$ in some subset of the positive integers, so that $\omega_k$ converges in $C^l(\Omega)$ for each $l > 0$ and $n > 0$. Since $\eta$ is $C^\infty$, $y$ is $C^\infty$ in $\Omega$, completing the proof.

We conclude this section with a few comments on the results. The restriction on energy $0 < R < \bar{R}$ in the results of this section is largely due to the regularization of $f(p_1, p_2)$ by $a(p_1, p_2)$ in § 2. Otherwise the techniques are capable of producing «finite amplitude» solutions as is done in [10]. In none of the permanent wave problems can one expect to have solutions of arbitrarily large energy. In the recent work of Amick and Toland [22] (cf. also [23]) one can reasonably assume that the finite range of possible energies is exhausted along a branch of solutions they obtain. One end of the branch is the bifurcation point $c = c_0$, $w = 0$ in our notation and the other «end» is the limiting case of zero particle velocity at the crest of the wave, the Stokes wave.

We saw in Theorems 8.1 and 8.2 that the wave is one of elevation if $\epsilon_0 \eta_0^{-2} > \epsilon_1 \eta_1^{-2}$ and is of depression if the reverse inequality holds. This condition was obtained by Keulegan [2], Long [3], and Benjamin [1] in asymptotic analyses of the problem. The exponential behavior of the wave tails is observed in the asymptotic analyses just cited and in the exact treatment ([8], [9]) of both internal and surface waves. In the works [8] and [9] which use an asymptotic analysis as the basis for an existence result using the implicit function theorem, the exponent in the exponential decay is quite precise. In our notation it is a multiple of $\lambda = c_0^{-2} - c^2$, measuring the deviation from the critical wave speed. Then waves are asymptotically equal to $A \exp [-\sqrt{\lambda} |x|]$ multiplied by a function not depending on $A$. In Lemma 7.4 we showed the exponential decay to be of order $\exp [-p |x|]$. 


where $p$ was chosen to satisfy $p < \gamma_1^{1/2}$, $\gamma_1$ being the first eigenvalue of (7.2). The eigenvalue $\gamma_1$ is bounded below by a multiple of $\mu - \lambda = g(c_0^{-2} - \sigma^{-2})$, so we obtain the correct order of decay.

It is interesting to examine the velocity $(U, V)$ near the fluid interface corresponding to $\psi = 0$ (equivalently $\eta = 0$) in the case $\delta = 0$. Since $y(x, 0)$ is the height of the streamline where $\psi = 0$ and $y_x = w_x$, the continuity of $w_x$ expresses the fact that fluid particles on either side of the streamline are moving in the same direction. Underlying all the estimates in Section 3 is the idea of controlling the $L^\infty$ norm of $w_n$. No stronger estimate is possible, for when $\delta = 0$ the strong maximum principle shows $w_n(0, 0 -) > 0$ and $w_n(0, 0 +) < 0$ when $w > 0$, and thus no further regularity could have been attained, uniformly in $\delta$. The discontinuity in $w_n$ corresponds to a discontinuity in $U$ across the separating streamline. To see this we recall that

$$\frac{\sigma^{1/2} U}{\hat{y}} = \frac{\partial \hat{\psi}}{\partial y} = \frac{1}{\partial y / \partial \hat{\psi}} = \frac{1}{\hat{y} + w_n \hat{\psi}} = \frac{\sigma^{1/2} c}{1 + w_n}$$

using (1.3), (1.6) and (2.8), and so $U = c(1 + w_n)^{-1}$. Thus for a wave of elevation, corresponding to $w > 0$, we see that at the crest, $U$ is smaller than $c$ just below the streamline $\psi = 0$ and larger than $c$ just above it.

Internal waves which exhibit the characteristics of solitary waves are observed, both in the laboratory and in the field. Walker [24] has generated such waves in a two-fluid system, though with a free surface rather than a rigid top. His article also contains references to field observations in geophysical two-layer systems.

REFERENCES


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