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Boundary value problems for harmonic and minimal maps of surfaces into manifolds

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1. Statement of results.

Let $M$, $g$ and $N$, $h$ be smooth Riemannian manifolds, with $M$ compact and possibly with boundary. Let $\varphi : M \to N$ be a $C^\infty$ map. Its energy is defined by $E(\varphi) = \int_M e(\varphi) V$, where $V$ is the volume element associated with $g$ and the energy density is given by $e(\varphi) = \frac{1}{2} |d\varphi|^2$. A map $\varphi$ is called harmonic iff it is an extremal of $E$. Such a map satisfies the equation $\tau(\varphi) = 0$, where $\tau$ is its tension (see [5] for details).

The existence problem is primarily concerned with the presence of a harmonic map in the various homotopy classes of maps from $M$ to $N$. We again refer to [5] for a list of known results and a detailed bibliography.

In the present paper, we first study this existence problem in the case of maps from a surface with boundary to a manifold, for homotopy classes relative to Dirichlet and Neumann problems. $M$, $g$ will denote a compact surface with boundary, which is necessarily a connected sum of tori, projective planes and half spheres, whose boundary circles $C_i$ $(i = 1, \ldots, b)$ constitute the boundary $\partial M$. The interior of $M$ is denoted by $M^\circ$. We shall always suppose $\partial N$ empty but note that W. H. Meeks and S.-T. Yau have shown how to replace such an $N$ by a convex manifold with boundary (see [13] for definition and proof).

For the Dirichlet problem, we shall use the following

(1.1) Definition. Let $\varphi_0$ and $\varphi_1$ be two continuous maps from $M$ to $N$ such that $\varphi_0|_{\partial M} = \varphi_1|_{\partial M} = \varphi$. $\varphi_0$ and $\varphi_1$ are homotopic relatively to the Dirichlet problem induced by $\varphi$ iff they are $C^k$ homotopic through a family $\varphi_t$ with $\varphi_0|_{\partial M} = \varphi$. If $\varphi_0$ and $\varphi_1$ are homotopic and $C^k$ (resp. $C^k$ in $M^\circ$ and $C^j$ on $M$, $j < k$), then

the homotopy can be realised by a $C^k$ map (resp. $C^k$ in $M^0 \times [0, 1]$ and $C^l$ on $M \times [0, 1])$.

In other words, the homotopy classes of—say—$C^k$ maps relative to a given $\varphi$ are the connected components of the space of $C^k$ extensions, with respect to any $C^l$ topology, $0 \leq l < k$.

Following the basic method of C. B. Morrey [14], [15, p. 378-389], we use the Sobolev class of maps $L^2(M, N)$, defined by means of any Riemannian embedding of $N$ in $R^p$ as $L^2(M, N) = \{ \Phi : M \to R^p : \Phi(M) \subset N \text{ almost everywhere and } \Phi \in L^2(M, R^p) \}$.

$\Phi$ is here seen as the composition of the map $\varphi : M \to N$ with the embedding.

To state the first result, recall finally that a manifold is called homogeneously regular [14] iff there exist two positive constants $c$ and $C$ such that any point $w$ of $N$ is in the domain of a coordinate chart $F : U \to R^n$ whose image is the unit ball and such that $F(w) = 0$ and

$$c|dF(u)X|^2 < h_u(X, X) < C|dF(u)X|^2 \quad \text{for } u \in U$$

and $X \in T_uN$. This condition is always satisfied if $N$ is compact.

(1.2) Theorem. Let $M, g$ be a compact surface with boundary and $N, h$ a homogeneously regular manifold, such that $\Pi_2(N) = 0$, where $\Pi_2(N)$ denotes the second homotopy group. Let $\varphi : \partial M \to N$ be the restriction to $\partial M$ of a continuous and $L^1$ map from $M$ to $N$. Then the Dirichlet problem

$$\tau(\varphi) = 0 \quad \text{in } M^0$$

$$\varphi = \psi \quad \text{on } \partial M$$

admits a solution $\varphi$ in every relative homotopy class, smooth in $M^0$, continuous on $M$ and minimising the energy in the class. If $\varphi \in C^{s+\alpha}(\partial M, N)$ ($\alpha$-Hölder continuous derivatives up to order $s$) with $0 < \alpha < \infty$, then $\varphi \in C^{s+\alpha}(M, N)$.

A solution of the Dirichlet problem for maps from a plane domain to a manifold, without homotopy restriction, was obtained by C. B. Morrey [14]. The existence of harmonic elements in the homotopy classes of maps from a surface without boundary to $N$ with $\Pi_2(N) = 0$ was proven in [11], using basically Morrey's method, and by J. Sacks and K. Uhlenbeck [16] [17] by a completely different approach. [19] provides a variation of the proof of [11] which will be used here.

(1.3) Remark. The hypothesis $\Pi_2(N) = 0$ cannot be simply omitted.
Indeed, [10] and [11] give an example of a homotopy class relative to a Dirichlet problem for maps from the 2-disk to the 2-sphere without harmonic representative. For a list of some manifolds satisfying $\Pi_2(N) = 0$, see [5, (10.9)].

Consider now the Neumann problem defined by $\partial_\nu \varphi = 0$ along $\partial M$, where $\partial_\nu$ is the normal derivative. If $\varphi_0$ and $\varphi_1$ satisfy this condition and are freely homotopic, then they are also relatively homotopic in the sense that they are homotopic through a family $\varphi_t$ with $\partial_\nu \varphi_t = 0$ along $\partial M$. Indeed, it suffices to modify the parametrisation of the maps of the homotopy. We state:

\begin{equation}
(1.4) \text{THEOREM. Let } M, g \text{ be a compact surface with boundary and } N, h \text{ a compact manifold with } \Pi_4(N) = 0. \text{ Any homotopy class of maps from } M \text{ to } N \text{ contains a smooth solution of the Neumann problem } \tau(\varphi) = 0, \partial_\nu \varphi |_{\partial M} = 0, \text{ minimizing } E \text{ in the class.}
\end{equation}

Recall that by transport of the loops, any continuous map $\varphi : M \to N$ induces on the first homotopy groups a conjugacy class of homomorphisms $[\varphi_*]$, the conjugacy coming from the changes of base point. The preceding result is a corollary of:

\begin{equation}
(1.5) \text{THEOREM. Let } M, g \text{ be a surface with boundary and } N, h \text{ a compact manifold. For any conjugacy class of homomorphisms } \gamma : \Pi_1(M) \to \Pi_1(N), \text{ there is a solution } \varphi \text{ of the Neumann problem such that } [\varphi_*] = \gamma.
\end{equation}

\text{REMARK. We shall show by means of examples that a homotopy class relative to the Dirichlet or Neumann problem can contain more than one harmonic map. However, if the sectional curvature of } N \text{ is negative, the harmonic maps are essentially unique in their classes.}

Classically, the existence of harmonic maps is used to obtain minimal surfaces i.e. (in this framework) maps from a surface $M$ to $N$ which minimise or extremise the area

$$A(\varphi) = \int_\varphi \left| \frac{\det \varphi^* h}{\det g} \right|^\frac{1}{2} V_\varphi.$$

Here, the integral is independent of the choice of $g$, which will in general not be specified a priori.

Without going back to the history of the method (see e.g. [3] and [15]), we note that it was applied by C. B. Morrey to maps from a plane domain to $N$ [14], and by J. Sacks and K. Uhlenbeck [18] and R. Schoen and S.-T. Yau [19] to maps of closed surfaces.

Recall [14, chap. 1] [15, (9.1)] that for two $C^0$ paths $\varphi_0 : I_0 \to N$, $\varphi_1 : I_1 \to N$ (where $I_0$ and $I_1$ are intervals), the distance between $\varphi_0$ and $\varphi_1$ is defined as
the infimum over all homeomorphisms $H: I_1 \rightarrow I_0$ of $\max_t \text{dist} \left( \gamma_1(t) - \gamma_0(H(t)) \right)$. A curve in the sense of Fréchet is an equivalence class of paths at zero distance from each other. One can define the distance between two such curves as the distance defined above for any of their parametrisations, and if a sequence of curves converges to a curve in this sense, there exists a sequence of paths parametrising them converging uniformly to a parametrisation of the limit.

A closed simple curve is called a Jordan curve.

(1.6) **Definition.** Let $\Gamma_i$ be $b$ disjoint Jordan curves in $N$ and $\varphi_0, \varphi_1: M \rightarrow N$ two continuous maps mapping $C_i$ in a monotone way on $\Gamma_i$. We say that $\varphi_0$ and $\varphi_1$ are relatively homotopic if they are homotopic through a family $\varphi_t$ with $\varphi_{t_{C_i}}$ monotone with image in $\Gamma_i$.

Again, homotopic maps with higher differentiability are homotopic through maps with the same differentiability.

(1.7) **Theorem.** Let $M$ be a compact orientable surface with boundary $C_1 \cup ... \cup C_b$ and $N$, $h$ a homogeneously regular manifold such that $\Pi_2(N) = 0$. Let $\Gamma_1, ..., \Gamma_b$ be disjoint Jordan curves in $N$ such that there exists a continuous map $\varphi: M \rightarrow N$ of finite energy, mapping each $C_i$ monotonically on $\Gamma_i$ and such that $[\varphi_*]: \Pi_1(M) \rightarrow \Pi_1(N)$ is injective. Then $\varphi$ is relatively homotopic to a minimal map $\varphi$, smooth in $M^s$ and continuous on $M$, mapping $C_i$ on $\Gamma_i$ in a monotone way and minimising the area among all such maps. If each $\Gamma_i$ is a $C^{s+s}$ curve ($2 < s < \infty$), then $\varphi \in C^{s+s}(M, N)$.

(1.8) **Remark.** The hypothesis that $[\varphi_*]$ be injective is used to prevent standard degeneracies of the surface when its area decreases to an infimum, like a change of genus or the splitting into more than one connected components. The hypothesis $d < d^*$ of [3] for maps from a surface to $R^s$ or of [14] for maps of plane domains to $N$ has the same purpose, but is of a different nature since it restricts the possible types of surfaces $M$ in relation with the given $\Gamma_i$'s. Note also that our hypothesis excludes the case $N = R^s$.

Theorem (1.7) is related to the Dirichlet problem for harmonic maps. One might wonder whether minimal maps could also be solutions of the Neumann problem. We shall easily observe that it is not so. Indeed:

(1.9) **Proposition.** Let $\varphi: M \rightarrow N$ be a minimal map such that $\partial_\nu \varphi = 0$ along an arc $\sigma$ of $\partial M$ ($\sigma$ not reduced to a point). Then $\varphi$ is constant.

It is a pleasure to record my thanks to J. Eells and V. L. Hansen for useful conversations, in particular during the 1980 Complex Analysis Seminar at the I. C. T. P. (Trieste).
2. – The Dirichlet problem for harmonic maps.

We first justify the differentiability statements made in the definition of relative homotopy (1.1).

In the case of manifolds without boundary, it is shown in [4] chapter 16, section 26, that two $C^k$ maps which are $C^0$ homotopic are in fact $C^k$ homotopic.

In the case of definition (1.1), the proof shows that for $\varepsilon > 0$ small enough, two $C^k$ maps $\varphi_0$ and $\varphi_1$ homotopic relatively to their boundary restriction $\psi$ are $C^k$ homotopic through maps $\varphi_i$ such that $\text{dist}(\varphi_i(x), \psi(x)) < \varepsilon$ for $x \in \partial M$. Using the techniques of [4, (16.26.3 and 4)], one can then $C^k$ deform the maps $\varphi_i$ in a tubular neighbourhood of $\partial M$ to make them coincide with $\psi$ on $\partial M$.

When considering two maps $\varphi_0$ and $\varphi_1$ which are $C^k$ in $M^0$ and $C^l$ on $M$, one can first $C^k$ deform $\varphi_1$ in such a way that it coincides with $\varphi_0$ in a tubular neighbourhood of $\partial M$ (using a bump function and [4, (16.26.4)]). The problem is then reduced to that of $C^k$ maps on a slightly smaller manifold with boundary.

We now turn to the proof of theorem (1.2).

(2.1) LEMMA. Let $M$ be a compact surface with boundary, $N$ a manifold with $\pi_2(N) = 0$ and $\varphi_0$ and $\varphi_1$ two continuous maps from $M$ to $N$ equal on $\partial M$. If for any path $\beta$ in $M$ with endpoints on $\partial M$, $\varphi_0(\beta)$ is relatively homotopic to $\varphi_1(\beta)$ (i.e. homotopic with endpoints fixed), then $\varphi_0$ and $\varphi_1$ are relatively homotopic (as in definition (1.1)).

In fact, we shall establish a more technically stated result:

(2.2) LEMMA. With the same hypothesis on $M$ and $N$, fix a point $P \in C_1$. The maps $\varphi_0$ and $\varphi_1$, equal on $\partial M$, are relatively homotopic if a) $\varphi_{0*} = \varphi_{1*}$: $\pi_1(M, P) \to \pi_1(N, \varphi_0(P))$ (we don’t have conjugacy classes here since we can fix a base point $P$ with $\varphi_0(P) = \varphi_1(P)$), and

b) for a set of non-intersecting paths $\beta_i (2 < i < b)$ from $P$ to points of the $C_i$’s, $\varphi_0(\beta_i)$ and $\varphi_1(\beta_i)$ are relatively homotopic.

When $\partial M = \emptyset$, this reduces to a well-know property (see e.g. [20, proof of theorem 11, chapter 8, section 1]).

Proof of lemma (2.2). It is well known that $M$ can be represented by a region $R$ bounded by a polygon, with some sides identified two by two, the others corresponding to the boundary curves (see e.g. [21, chap. 5]).
In the present situation, we can write the surface symbol (list of sides of the polygon, with identification of pairs of sides with the same name) as

\[ C_1 \beta_2 C_2 \beta_2^{-1} \ldots \beta_b C_b \beta_b^{-1} a_1 a_1^{-1} a_2 a_2 b_1 b_1^{-1} c_1 c_1^{-1} \ldots b_p b_p^{-1} c_p c_p^{-1} , \]

so that all vertices except those adjacent to \( C_i \) \((2 < i < b)\) correspond to the point \( P \), the curves \( \beta_i \) provide the slits from \( P \) to \( C_i \), the \( a_i \)'s represent the cross caps and the \( b_i \), \( c_i \)'s the tori.

Still denoting by \( \varphi_0 \) and \( \varphi_1 \) the maps induced from \( R \) to \( N \), we have \( \varphi_0(P) = \varphi_1(P) \) and \( \varphi_0|_{C_i} = \varphi_1|_{C_i} \). Moreover, \( \varphi_0 \) and \( \varphi_1 \) restricted to any side of the polygon \( \partial R \) are homotopic with fixed endpoints.

We can then deform continuously \( \varphi_1 \) to make it equal to \( \varphi_0 \) on \( \partial R \). Indeed, in a tubular neighbourhood \( T \) of \( \partial R \) (with corners, which don’t matter since we work in the \( C^0 \) framework), we can first change the parametrisation of \( \varphi_1 \) in such a way that the half of \( T \) closer to \( \partial R \) (say \( S \)) has image on \( \partial R \). The homotopy from \( \varphi_1 \) to \( \varphi_0 \) defined on \( \partial R \) can then be used to define a map from \( S \) to \( N \), connecting \( \varphi_1|_{\partial R} \) and \( \varphi_0|_{\partial R} \).

\( \varphi_0 \) and \( \varphi_1 \) are then equal on the one-skeleton of a cellular decomposition with only one 2-cell. By a result of W. D. Barcus and M. G. Barratt [2], used in its simplest case \( \Pi_4(N) = 0 \), there is only one homotopy class of extensions of such maps to the 2-cell, so that \( \varphi_0 \) and \( \varphi_1 \) are relatively homotopic.

**Proof of Theorem (1.2).** We shall use a direct method of the calculus of variations, in the framework of the space \( L^2_1(M, N) \).

Recall [15, lemma 9.4.10.d] that if \( \varphi \in L^2_1(M, N) \), in any chart \( (x, y) \) of \( M \), \( \varphi|_{x=x_0} \) is continuous in \( y \) except maybe for \( x_0 \) in a set of measure zero.

R. Schoen and S.-T. Yau have shown in [19] that the image of a closed contractible curve of \( M \) by \( \varphi \) is contractible if this image is continuous. They use this to define the action (which will mean here: up to homotopy) of an \( L^2_1 \) map on a closed curve \( a \) by the common image (up to homotopy) of almost all curves parallel to \( a \) in a tubular neighbourhood.

Since our maps are fixed on \( \partial M \), this defines also an action of \( \varphi \) on any of the curves \( \beta_i \) that we consider, as the common homotopy class of images of curves composed of a segment of \( C_i \), a curve parallel to \( \beta_i \) and a segment of \( C_i \), the segments being necessary to build curves with same endpoints.

The proof follows then as in [14] and [19]: consider a minimising sequence for \( E \) in the class of \( L^2_1 \) maps with value \( \psi \) on \( \partial M \) and inducing on loops \( a \), and paths \( \beta \), the action induced by the given homotopy class. By [15, lemmas 9.4.14-15-16], a subsequence \( \varphi_s \) converges in \( L^2_1(M, N) \) to a map \( \varphi \), equal to \( \psi \) on \( \partial M \), with \( E(\varphi) < \lim E(\varphi_s) \). Moreover, on almost each curve (in the sense of measure as above), a subsequence converges uniformly
[12, proof of lemma 1.4] and \( \varphi \) induces therefore the same action on \( \alpha \) and \( \beta \). Hence, it minimises the energy in every small disk of \( M^0 \) (for the Dirichlet problem induced by its trace on the boundary of the disk), and is \( C^\infty \) in \( M^0 \) and \( C^0 \) on \( M \), by the regularity results of C. B. Morrey [14]. By lemma (2.2), it is in the given homotopy class.

The regularity results along \( \partial M \) are local and were obtained in [9] (see [8] for complete statements).

(2.3) REMARK. As in theorem (1.5) for the Neumann problem, we could state an existence theorem for \( \Pi_2(N) \neq 0 \), the homotopy classes being replaced either by the data of \( [\varphi_\#] \), or by that of the action on \( \Pi_1(M) \) and the curves \( \beta \).

3. - The Neumann problem for harmonic maps.

The proof of theorem (1.5) now reduces to known ingredients. We write it in a way avoiding any use of boundary estimates.

Consider the class of \( L^2 \) maps from \( M \) to \( N \) inducing the given class of homomorphisms \( \gamma \) on \( \Pi_1 \), without any restriction on the boundary. \( N \) being compact, a minimising sequence for \( E \) in that class is a bounded set in \( L^2(M, N) \) and we obtain as above a minimum \( \varphi \) of the energy, smooth and harmonic in \( M^0 \).

Consider now any point \( P \in \partial M \) and a coordinate (half) disk around \( P \). By a conformal change of metric, we can see it as a flat half disk \( D^+ = \{ (x, y) : x^2 + y^2 < 1, x > 0 \} \), in such a way that \( D^+ \cap \partial M = D^+ \cap \{ x = 0 \} \) and \( P = (0, 0) \).

By conformal invariance of the energy [5, (10.2)], \( \varphi \) minimises \( E \) (for the flat metric) among all maps coinciding with \( \varphi \) on \( D^+ \cap \{ x^2 + y^2 = 1 \} \).

We then extend \( \varphi \) to a map \( \bar{\varphi} \) defined on the whole unit disk \( D \) in \( \mathcal{R}^2 \) by setting \( \bar{\varphi} = \varphi \) on \( D^+ \) and \( \bar{\varphi}(x, y) = \varphi(-x, y) \) on \( D \setminus D^+ \). By standard patching properties, \( \bar{\varphi} \in L^2(D, N) \).

Moreover, \( \bar{\varphi} \) minimises \( E \) among all maps \( \psi \) such that \( \psi|_{\partial D} = \bar{\psi}|_{\partial D} \). Indeed, if it didn’t, there would exist a map \( \psi \) with \( \psi|_{\partial D} = \bar{\psi}|_{\partial D} \) and \( E(\psi) < E(\bar{\varphi}) \). The same would be true for \( \psi \) restricted to \( D^\circ \) or to \( D \setminus D^+ \), say to the former. The map equal to \( \varphi \) on \( M \setminus D^+ \) and \( \psi \) on \( D^+ \) would then be in the given class of \( L^2(M, N) \) and have energy smaller than the minimum—a contradiction.

\( \bar{\varphi} \) is therefore smooth and harmonic in \( D^\circ \). Since the normal derivative is the same along \( \partial M \) for the original metric or the conformally equivalent one (the angles and the tangent being preserved), and since \( \bar{\varphi}(x, y) = \bar{\varphi}(-x, y) \) we get \( \partial_n \bar{\varphi} = 0 \) along the boundary and \( \varphi \) is a solution of the Neumann problem.
If $\Pi_1(N) = 0$, the homotopy classes are in bijective correspondence with the conjugacy classes of homomorphisms on $\Pi_1$ and theorem (1.5) implies theorem (1.4).

**Remark.** The fact that $\partial_x \varphi|_{\partial M} = 0$ for the minimising map can also be deduced from the variation formula [5, (12.11)]

$$\nabla_x E(\varphi) = -\int_M \langle \tau(\varphi), \chi \rangle V_\varphi + \int_{\partial M} \langle \partial_x \varphi, \chi \rangle V_\varphi|_{\partial M}$$

since $\chi$ is not restricted in any way on $\partial M$.

4. – On the unicity of harmonic maps.

We first show by an example that the solutions of the Dirichlet and Neumann problems need not be unique in their relative homotopy classes. By using a somewhat artificial metric, we obtain in fact a continuous family of solutions.

Let $M$ be the cylinder $[0, \Pi] \times \mathbb{R}/2\Pi\mathbb{Z}$ endowed with its flat metric and Euclidean coordinates $(x, y)$. For $0 < b < a$, denote by $N_{a,b}$ the cylinder $[-a + b, a - b] \times \mathbb{R}/2\Pi\mathbb{Z}$ equipped in coordinates $(u, v)$ with the metric

$$h = \begin{pmatrix} 1 & 0 \\ 0 & a^2 - u^2 \end{pmatrix}$$

and consider $N_{a,b}$ as a band isometrically embedded in a torus $N$.

As noted in [12, § 3], a map $\varphi: M \to N_{a,b}$ of the form $\varphi(x, y) = (u, v) = (F(x), y)$ is harmonic iff $d^2 F/dx^2 + F = 0$ and the solutions are of course given by $\varphi(x, y) = (c \cdot \cos(x + d), y)$ where $c$ takes any value in $[-a + b, a - b]$.

For $d = -\Pi/2$, these solutions satisfy $\varphi(0, y) = \varphi(\Pi, y) = (0, y)$, so that they are all solutions of the same Dirichlet problem.

For $d = 0$, they are all solutions of the Neumann problem.

In both cases, the hypothesis of the existence theorems (1.2) or (1.4) are satisfied.

On the other hand, we note the

(4.1) Proposition. If the sectional curvature of $N$ is non-positive, the Dirichlet problem of theorem (1.2) admits only one solution in each homotopy class. If the curvatures of $N$ is strictly negative, the Neumann
problem of theorem (1.4) admits only one solution with image not reduced to a point or a geodesic in each homotopy class.

As noted in [5], the first statement follows immediately from the proof of the corresponding theorem of P. Hartman in the boundaryless case [6]. For the Neumann problem, the statement can be reduced to the boundaryless case by doubling the surface $M$ and defining an extension of the map on the double as in § 3.

5. - Minimal surfaces of given boundary.

Since the energy is invariant by conformal transformations of the surface $M$, it suffices to define on $M$ a conformal—or a complex—structure in order to define its harmonic maps. To obtain a minimal map, we shall use the following version of a classical result, which can be proven exactly as in the boundaryless case [17].

(5.1) **LEMMA.** Let $\varphi: M \rightarrow N$, $h$ be a map extremising the energy for all variations of the conformal structure on the surface $M$ and all deformations of the map in a relative homotopy class (in the sense of def. (1.6)). Then $\varphi$ is harmonic and conformal in $M^o$ and is therefore a minimal branched immersion.

To get such an extremal, we shall use three more lemmas.

Analogously to lemma (2.2), we have:

(5.2) **LEMMA.** Let $\varphi_0$ and $\varphi_1: M \rightarrow N$ be two continuous maps, mapping each $C_i$ monotonically on $\Gamma_i$, such that $\varphi_1$ restricted to the one-skeleton of a cellular decomposition is homotopic to $\varphi_0$ restricted to the same space, in such a way that along the homotopy $\varphi_1(C_i) \subset \Gamma_i$. Then $\varphi_0$ and $\varphi_1$ are relatively homotopic in the sense of definition (1.6).

From now on, we shall suppose that the genus $p$ of $M$ (number of tori in the connected sum decomposition) is greater than zero or that $b \geq 3$. The case $p = 0$, $b = 1$ is that of a disk, and existence in the unique homotopy class (since $\Pi_1(N) = 0$) is classical. The case $p = 0$, $b = 2$ can be proven as those that we consider, by using a flat torus instead of the closed surfaces here below.

(5.3) **LEMMA.** Any orientable surface with boundary, with $p \geq 1$ or $b \geq 3$, equipped with a conformal structure can be doubled as a closed Riemann surface $\bar{M}$, endowed with a conformal involution $i$ interchanging $M$ and the closure
of $\overline{M} \setminus M$ and whose fixed points constitute $\partial M$. When $\overline{M}$ is equipped with its Poincaré metric (unique metric of curvature $-1$ compatible with the conformal structure), it is an isometry and the $C_i$'s are closed geodesics.

See e.g. [1, chap. II, § 1.2].

(5.4) LEMMA. Let $N$ be a homogeneously regular Riemannian manifold and $\Gamma_i$, $(1 \leq i \leq b)$ a family of disjoint Jordan curves in $N$. There is a strictly positive number $K$ smaller than or equal to the following quantities:

1) the length of all non-homotopically trivial curves;
2) the length of all curves with an endpoint on $\Gamma_i$ and the other on $\Gamma_j$, for $i \neq j$;
3) the length of all curves with endpoints on $\Gamma_i$ and not homotopic to a point through curves with endpoints on $\Gamma_j$.

Indeed, since $N$ is homogeneously regular, any non-homotopically trivial curve has length greater than or equal to $2\sqrt{c}$, and in the two other classes of curves described in the lemma, a direct method of the calculus of variations yields a minimising geodesic of positive length.

PROOF OF THEOREM (1.7). Consider a minimising sequence $(t_k, \varphi_k)$ for $E$, where each $t_k$ is a conformal structure on $M$ and $\varphi_k$ a harmonic map with respect to $t_k$, such that $\varphi_{t_k|\Sigma}$ converge as Fréchet curves to $\Gamma_i$, and belonging to the given relative homotopy class, up to a small deformation in a neighbourhood of $\partial M$, bringing the image of $C_i$ by $\varphi_k$ on $\Gamma_i$.

We want to show that both $(t_k)$ and $(\varphi_k)$ admit converging subsequences. By continuity of $E$ with respect to $(t_k)$ [18, lemma 4.2] and lower semicontinuity of $E$ with respect to $(\varphi_k)$, the limit will realise a minimum of $E$.

For each $k$, consider first the double of $M$, $t_k$ with its conformal structure $\tilde{t}_k$ and Poincaré metric as in lemma (5.3). Define an $L^2$ map $\tilde{\varphi}_k$ on $\overline{M}$ by $\tilde{\varphi}_k = \varphi_k$ on $M$ and $\tilde{\varphi}_k = \varphi_k \circ i$ on $\overline{M} \setminus M$. $\tilde{\varphi}_k \in L^2(\overline{M}, N)$ and $E(\tilde{\varphi}_k)$ is bounded independently of $k$.

By $\tilde{\varphi}_k$, the image of almost all non-homotopically trivial curve is continuous, $L^2$ and of length bounded below by $K > 0$. Indeed, observe that any such curve in $\overline{M}$ has one of the following properties:

1) it is contained and non-homotopically trivial in $M^* \setminus M$ or $\overline{M} \setminus M$,
2) it intersects two $C_i$'s,
3) it is homotopically equivalent to a multiple of $C_i$,
4) it contains an arc with endpoints on $C_i$, non homotopic to a point through arcs with endpoints on $C_i$. 
In each case, the injectivity of [\varphi_{k}] and lemma (5.4) imply that the length of the image of the curve is \(\geq K\).

From lemma 3.1 and the proof of theorem 3.1 of [19], it follows that the sequence of conformal structures \(I_k\) is contained in a compact subset of the moduli space of \(\hat{M}\), so that a subsequence converges to a conformal structure on \(\hat{M}\), inducing one on \(M\) by restriction.

From now on, we shall calculate the energies with respect to this limit \(t\), which, by compacity, will simply modify the inequalities in consideration by fixed constants.

The sequence \((\varphi_k)\) is a bounded set in \(L^2(M, N)\), so that a subsequence converges weakly to an \(L^2\) limit \(\varphi\,\), with \(E(\varphi) \leq \liminf E(\varphi_k)\), and such that \(\varphi(C_i) \subset \Gamma\). On almost each curve, a subsequence converges uniformly.

Following the proofs of lemmas 9.3.2 and 9.3.3 of [15], we shall show that the restriction of the maps to each \(C_i\) converge uniformly to a continuous and monotone limit.

For a boundary curve \(C_i = C\) parametrised by a coordinate \(\eta + 2\Pi k\), and a Jordan curve \(\Gamma_i = \Gamma\), there exist by the properties recalled in § 1 two sequences of maps \(f_k: R \to R\) and \(F_k: R \to N\) such that \(\varphi_{k\eta}(\eta) = F_k \circ f_k(\eta)\), \((F_k)\) is a sequence of continuous periodic maps converging uniformly to \(F\), \((f_k)\) a sequence of monotone continuous functions such that \(f_k(\eta) \to \eta\) is periodic of period \(2\Pi\) and a subsequence \((f_k)\) converges in \(L^2\) and almost everywhere to a function \(f\) in such a way that \(\varphi_0 = F \circ f\).

We first show that \(f\) is continuous.

Suppose, on the contrary, that for a value of \(\eta\, f^+(\eta) \neq f^-(\eta)\) (where \(f^+\) and \(f^-\) denote right and left limits).

Let \(D\) denote the disk of radius \(r\) centred at \(\eta\, \gamma^\prime\) its boundary in \(M\) and \(\gamma^\prime\) the arc \(C \setminus D\). For any \(\varphi_k\) and a.e. \(r\) small enough, \(\varphi_k(\gamma^\prime) \cup \gamma^\prime\) has length \(\geq K\) since it is homotopically non-trivial in \(N\).

If \(f^+(\eta) - f^-(\eta) = 2\Pi\), \(Ff^+(\eta) \neq Ff^-(\eta)\) and for suitable \(r\), the length of the arc \(\varphi_k(\gamma^\prime)\) must tend to a positive value.

If \(f^+(\eta) - f^-(\eta) = 2\Pi\), \(Ff^+(\eta) = Ff^-(\eta)\), but as the length of \(\varphi_k(\gamma^\prime)\) tends to zero, the length of \(\varphi_k(\gamma^\prime)\) tends again to a positive value. More precisely, for any \(\delta > 0\) small enough, there is an integer \(k_0\) such that for \(k > k_0\), the length of \(\varphi_k(\gamma^\prime)\) is greater than \(\delta > 0\) for a.e. \(r\) such that \(\delta^* < r < \delta\). Therefore, in polar coordinates \((r, \theta)\) in \(D\), we have, using the flat metric:

\[
E(\varphi_k) > \int_0^\delta \int_{\gamma^\prime} \left| \frac{\partial \varphi_k}{\partial \theta} \right|^2 d\theta \, dr > \frac{1}{2\Pi} \left( \int_{\gamma^\prime} \left| \frac{\partial \varphi_k}{\partial \theta} \right|^2 d\theta \right) dr > \frac{d^2}{2\Pi} \log \frac{1}{\delta}.
\]

Since \(E(\varphi_k)\) is bounded, this is impossible for \(\delta \to 0\) and the limit \(f\) is continuous.
The sequence \((f_k)\) of continuous and monotone functions converges a.e. to a continuous and monotone function \(f\). This convergence must be uniform.

Indeed, \(\forall \varepsilon > 0, \exists \delta \) such that \(0 < \mu < \delta\) implies \(0 < f(\eta + \mu) - f(\eta) < \varepsilon\). We can then choose \([\frac{4\mu}{\delta}]\) numbers \(\eta_i\) in an interval containing \([0, 2\mu]\) (in which we want to show convergence) such that \(\eta_i < \eta_{i+1} < \eta_i + \delta\) and \(f_k(\eta_i) \to f(\eta_i)\). For the given \(\varepsilon\), there is therefore a number \(k_0\) such that \(k > k_0\) implies \(|f_k(\eta_i) - f(\eta_i)| < \varepsilon\). Then, for any \(\eta_i\), say such that \(\eta_i < \eta < \eta_{i+1}\), we have
\[
f_k(\eta) - f(\eta) < f_k(\eta_{i+1}) - f(\eta_i)
\]
and similarly for \(f(\eta) - f(\eta_i)\).

The sequence \((\varphi_k)\) converges therefore uniformly on each \(C_i\). With the interior convergence as above, this implies that the characterising properties of lemma (5.2) are preserved at the limit \(\varphi\). As \(\varphi\) minimises \(E\) among all maps satisfying those properties, it is smooth in \(M^0\) and continuous on \(M\). By lemma (5.2), it belongs to the given homotopy class.

The regularity results along \(\partial M\) are due to E. Heinz and S. Hildebrandt [7].

To prove that \(\varphi\) minimises the area, we can essentially follow [15] or [19]. Let \(e : M \to R^2\) be a Riemannian embedding and \(\varepsilon e\) its composition with the multiplication by \(\varepsilon\) in \(R^2\).

Suppose that there is a map \(\psi\), relatively homotopic to \(\varphi\) with \(A(\psi) < A(\varphi)\). The map \(\Psi \oplus \varepsilon e : M \to N \times R^2 \subset R^2 \times R^2\) is an embedding for \(\varepsilon > 0\), and for the induced conformal structure \(\tilde{i}\) on \(M\) we have \(E_i(\Psi \oplus \varepsilon e) = A(\Psi \oplus \varepsilon e)\) [5, (10.3)]. For any \(\delta < A(\varphi) - A(\psi)\), we can choose \(\varepsilon\) such that \(A(\Psi \oplus \varepsilon e) < A(\psi) + \delta\) so that \(E_i(\psi) < E_i(\Psi \oplus \varepsilon e) < A(\psi) + \delta < A(\varphi) = E_i(\varphi)\), contradicting the minimising property of \(\varphi\) for the energy.

(5.5) RemarK. If \(\Pi_2(N) \neq 0\), the hypothesis of lemma (5.2) describe classes of maps containing a minimal surface, as in remark (2.3).


Suppose that \(\varphi : M \to N\) is smooth, harmonic, conformal and satisfies \(\partial_n \varphi = 0\). Then, along \(\sigma\) the tangential derivative is zero by conformality and \(\varphi\) is constant. Steps 2 and 3 of the proof of theorem (3.2) of [11] show then precisely that \(\varphi\) must be constant, as successive derivations of the conformality conditions show that all its derivatives vanish at a point.
BIBLIOGRAPHY