

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MARCO FORTI

FURIO HONSELL

Set theory with free construction principles

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 10,
n° 3 (1983), p. 493-522

http://www.numdam.org/item?id=ASNSP_1983_4_10_3_493_0

© Scuola Normale Superiore, Pisa, 1983, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Set Theory with Free Construction Principles.

MARCO FORTI - FURIO HONSELL (*)

0. – Introduction.

In the history of set theory, the Axiom of Foundation has been omitted or replaced by less restrictive axioms on several occasions.

D. Scott introduced, in an unpublished paper, the following axiom:

For any extensional relation R on A and for any not R -well founded $x \in A$ there is no set containing all possible images of x under isomorphisms between (A, R) and a transitive structure (T, \in) .

This axiom postulates the existence of a proper class of «Mostowski collapses» for any model of the Axiom of extensionality. Actually the uniqueness of a «Mostowski collapse» cannot be consistently postulated for non-well-founded structures.

A. Levy [9] refers to and uses this axiom to prove some of his results on Formulas' Hierarchy theory.

P. Hájek [7] and M. Boffa ([1], [2]) considered axioms similar to Scott's and proved them to be consistent relative to Gödel-Bernays set theory.

E. De Giorgi introduced and discussed with the authors, during his 1980 seminar «Foundations of Mathematics» at the Scuola Normale Superiore in Pisa, a «Free Construction Principle» for sets, namely:

FCP *It is always possible to define a set E giving a priori (through a parametrisation) the intersections of its elements with E and $V \setminus E$.*

In this paper the authors study a list of axioms derived from the **FCP**, which give rise to a hierarchy of Universes of set theory with respect to their richness in standard representatives of binary structures (see 1.6 and 1.7).

(*) Ricerca parzialmente finanziata dal GNSAGA del CNR.

Pervenuto alla Redazione il 9 Giugno 1982 ed in forma definitiva il 10 Marzo 1983.

These axioms are proved to be consistent relative to Gödel-Bernays set theory (without Foundation, of course) by the method of the admissible models (introduced and developed in §3), which generalizes that of the permutations of the universe (see [10]). Moreover their relative strength is exhaustively analyzed, by determining all the implications holding among them.

All the axioms introduced up to now in order to enrich the \in -relation (like those previously referred to) find an equivalent axiom in the list of axioms derived from **FCP**.

In particular the authors obtain a new consistency proof for each of them.

In *Section 1* the axioms are given and the main theorem is stated.

In *Section 2* a generalized notion of a well-founded set is given and the corresponding constraints for the membership relation are investigated.

In *Section 3* the theory of f -admissible relations and the technique of f -admissible models is developed.

In *Section 4* the main theorem is proved and some interesting consequences derived.

The main results of this paper were the object of a communication at the « Convegno di logica matematica » in Siena (14-17/IV/1982) (see [5]).

The authors are deeply indebted with E. De Giorgi for a lot of suggestions and useful discussions.

1. - Definitions and statement of results.

Since we shall deal with proper classes, we shall work throughout the paper inside a Gödel-Bernays-type set theory, whose axioms are **ABC** of [6]. We list them for sake of completeness.

- A** (General axioms).
- A₁** *Any set is a class.*
- A₂** *Any element of a class is a set.*
- A₃** *Extensionality* (classes having the same elements are equal).
- A₄** *Pairing* (the set $\{x, y\}$ exists for any pair of elements x, y).
- B** (Class-formation axioms).

Given any class X there exist the following classes:

- B₁** $\{(x, y) | x \in y\}$ (*membership relation*).
- B₂** $V \setminus X = \{x | x \notin X\}$ (*complement*).
- B₃** $X \cap Y$ for any class Y (*intersection*).
- B₄** $\text{dom } X = \{x | \exists y (x, y) \in X\}$ (*domain*).

- B₅** $X \times V = \{(x, y) | x \in X\}$ (*cartesian product*).
- B₆** $X^{-1} = \{(y, x) | (x, y) \in X\}$ (*inversion of pairs*).
- B₇** $\{(y, z, x) | (x, y, z) \in X\}$ }
B₈ $\{(x, z, y) | (x, y, z) \in X\}$ } (*rearranging of triples*).
- C** (Set-formation axioms).
- C₁** *Infinity* (there exists an infinite set).
- C₂** *Union* (the union $\cup x$ of any set x is a set).
- C₃** *Power-set* (the subsets of any set x constitute a set $\mathfrak{P}(x)$).
- C₄** *Replacement* (the range $\hat{F}(x)$ of the restriction of any function F to a set x is a set).

We omit axiom **D** of foundation and, whenever a choice axiom is needed we use the axiom **N** of von Neumann $V \simeq \text{Ord}$, which in the absence of foundation is stronger than global choice **E** (see, e.g., [1], [3]).

We shall use freely standard facts and notations of set theory, with the following particular conventions:

Given a function $f: A \rightarrow B$ we denote by $\hat{f}: \mathfrak{P}(A) \rightarrow \mathfrak{P}(B)$ the function such that $\hat{f}(E) = \{f(x) | x \in E\}$.

Given a set (or a class) A we denote:

- by $TC(A)$ the transitive closure of A ;
- by $\mathbf{\Pi}(A)$ the cumulative structure inductively defined starting from A ;
- by δ_A the identity map (or the equality relation) on A .

We denote by \mathbf{Aut} the class of the autosingletons, i.e. $\{x | x = \{x\}\}$.

Any binary structure (A, R) can be viewed as arising from a function $f: A \rightarrow \mathfrak{P}(A)$ defined by $f(a) = \{x \in a | xRa\}$.

A homomorphism g from (A, R) onto a transitive structure (E, \in_E) verifies $g(x) = \hat{g}(f(x))$ for any $x \in A$.

Bearing this in mind, we give the following definitions in order to state the basic axioms we are going to deal with.

DEFINITION 1.1. Let $f: A \rightarrow B$ be given. A function $g: A \rightarrow E$ is

- f-inductive* iff $\forall x \in A, g(x) = \hat{g}(f(x) \cap A)$,
- f-constructive* iff $\forall x \in A, g(x) = \hat{g}(f(x) \cap A) \cup (f(x) \setminus A)$.

A set E is *f-inductive* (resp. *f-constructive*) iff it is the range of an *f-inductive* (resp. *f-constructive*) function.

- $C \subseteq A$ is *f-transitive* iff $f(x) \cap A \subseteq C$ for all $x \in C$.
- $S \subseteq A$ is *f-saturated* iff $\forall x \in A(f(x) \cap A \subseteq S \rightarrow x \in S)$.

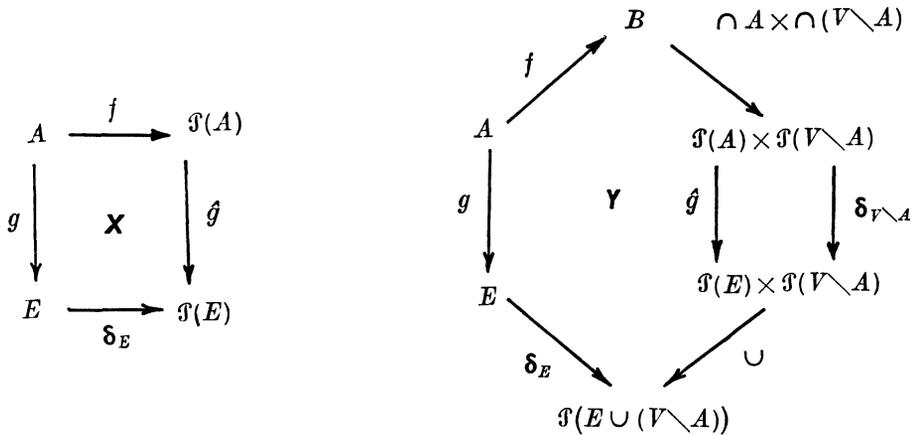
The notion of *f-constructive* map is a strengthening of the notion of homomorphism: namely an *f-constructive* map g can be extended, when $f(A) \not\subseteq \mathcal{T}(A)$, to a homomorphism defined on the whole $A \cup \bigcup f(A)$ inducing the identity outside A .

The notions of *f-transitive* and *f-saturated* classes are dual to each other. They reduce to the usual notions of transitive class and of class «closed under subset formation» respectively, when f is δ_V .

A first formalization of the **FCP** yields the following axioms:

- X:** For any function f there is an *f-inductive* function g ,
- Y:** For any function f there is an *f-constructive* function g .

Note that in terms of commutative diagrams the axioms above can be rephrased by asserting the existence of a function g which makes the following diagrams commute



In order to simplify the use of the axioms **X** and **Y** we make the following two remarks:

1.2. As far as *f-inductive* functions are concerned, it is possible, without loss of generality, to restrict to the case where $\text{Range } f \subseteq \mathcal{T}(\text{Dom } f)$, and we shall assume this whenever it is useful.

1.3. The role of the set A is merely that of a set of names for the elements of E , which thus comes to be constructed according to the prescriptions coded through f .

In fact it is possible to replace A by means of any bijection σ , provided $\hat{\sigma}(A)$ and $(\cup \text{Range } f) \setminus A$ are disjoint. Actually a function g on A is f -constructive iff g' is f' -constructive, where

$$g' \circ \sigma = g \quad \text{and} \quad f'(\sigma(x)) = \hat{\sigma}(f(x) \cap A) \cup (f(x) \setminus A).$$

We shall make use of this possibility in order to avoid undesired clashing.

We remark that the restriction of an f -constructive (f -inductive) function to an f -transitive subset C is obviously $f|_C$ -constructive ($f|_C$ -inductive).

A natural strengthening of the axioms \mathbf{X} and \mathbf{Y} arises by adding the corresponding « extension property », namely

Y* (resp. **X***): *For any function f and any f -transitive $C \subseteq \text{Dom } f$ there is an f -constructive (resp. f -inductive) function g which induces on C a given $f|_C$ -constructive (resp. $f|_C$ -inductive) function.*

Actually \mathbf{Y} already captures the desired strength. To show this, let us introduce the particular instance of \mathbf{X}^* given by

X^t: *For any function f inducing the identity on a transitive $T \subseteq \text{Dom } f$ there is an f -inductive function inducing the identity on T .*

Now we can prove the following

LEMMA 1.4. *Both $\mathbf{X}^t \rightarrow \mathbf{Y}$ and $\mathbf{Y} \rightarrow \mathbf{Y}^*$ hold in **ABC**. Hence the axioms \mathbf{X}^t , \mathbf{X}^* , \mathbf{Y}^* are all equivalent to \mathbf{Y} .*

PROOF. (i) $\mathbf{X}^t \rightarrow \mathbf{Y}$. Given a function $f: A \rightarrow B$, put $T = TC((\cup B) \setminus A)$; then Remark 1.3 allows us to assume w.l.o.g. that $A \cap T = \emptyset$. We put $A' = A \cup T$ and $f' = f \cup \delta_T$; we then apply \mathbf{X}^t getting an f' -inductive function g' extending δ_T . We claim that $g = g'|_A$ is f -constructive. In fact for $x \in A$ we have

$$\begin{aligned} g(x) &= \hat{g}'(f'(x) \cap A') = \hat{g}'(f(x) \cap A) \cup \hat{g}'(f(x) \cap T) \\ &= \hat{g}(f(x) \cap A) \cup (f(x) \cap T) = \hat{g}(f(x) \cap A) \cup (f(x) \setminus A). \end{aligned}$$

(ii) $\mathbf{Y} \rightarrow \mathbf{Y}^*$. Given $f: A \rightarrow B$, $C \subseteq A$ f -transitive, and $h: C \rightarrow D$ $f|_C$ -constructive, we may assume $(A \setminus C) \cap D = \emptyset$ by Remark 1.3.

Define f' on $A \setminus C$ by $f'(x) = \hat{h}(f(x) \cap C) \cup (f(x) \setminus C)$ and take g' f' -constructive.

Put $g = g' \cup h$; a direct computation yields, for $x \in C$

$$\begin{aligned} g(x) &= h(x) = \hat{h}(f(x) \cap C) \cup (f(x) \setminus C) \\ &= \hat{g}(f(x) \cap C) \cup (f(x) \setminus C) = \hat{g}(f(x) \cap A) \cup (f(x) \setminus A) \end{aligned}$$

since C is f -transitive.

For $x \in A \setminus C$ one gets:

$$\begin{aligned} g(x) &= g'(x) = \hat{g}'(f(x) \cap (A \setminus C)) \cup (f(x) \setminus (A \setminus C)) \\ &= \hat{g}'\left(\left(\hat{h}(f(x) \cap C) \cup (f(x) \setminus C)\right) \cap (A \setminus C)\right) \\ &\quad \cup \left(\left(\hat{h}(f(x) \cap C) \cup (f(x) \setminus C)\right) \setminus (A \setminus C)\right) \\ &= \hat{g}'\left(\hat{h}(f(x) \cap C) \cap (A \setminus C)\right) \cup \hat{g}'(f(x) \cap (A \setminus C)) \\ &\quad \cup \left(\hat{h}(f(x) \cap C) \setminus (A \setminus C)\right) \cup \left((f(x) \setminus C) \setminus (A \setminus C)\right). \end{aligned}$$

Recalling that by assumption the range of h is disjoint from $A \setminus C$ we get

$$\begin{aligned} g(x) &= \hat{g}'(f(x) \cap (A \setminus C)) \cup \hat{h}(f(x) \cap C) \cup (f(x) \setminus A) \\ &= \hat{g}(f(x) \cap A) \cup (f(x) \setminus A). \end{aligned}$$

Hence g is an f -inductive function extending h . Q.E.D.

The axioms considered up to now are still too weak, in the sense that the complexity of the relation coded by f may be utterly flattened when standardized by an f -constructive function. This fact can be overcome requiring g to be an isomorphism, provided the relation $R = \{(x, y) \mid x \in f(y)\}$ induced by $f: A \rightarrow \mathfrak{F}(A)$ is extensional. This occurs iff f is injective.

We are thus led to the axioms:

X_i: For any injective $f: A \rightarrow \mathfrak{F}(A)$ there is an injective f -inductive function g .

X_i^t: For any injective $f: A \rightarrow \mathfrak{F}(A)$ inducing the identity on a transitive $T \subseteq A$ there is an injective f -inductive function g inducing the identity on T .

X_i and **X_i^t** are respectively equivalent to the axioms **U_i** and **S** introduced by Boffa ([2]).

It is worth mentioning that the corresponding axiom

Y_i : *For any injective function f there is an injective f -constructive function g*

is inconsistent (see p. 504).

As pointed out in [1], in extending the existence of Mostowski's isomorphism to extensional non-well-founded structures (as X_i postulates), uniqueness must fail in a strong sense (and this is the very ground of the formulation of Scott's axiom).

Dropping out injectivity (as in X or Y), that is allowing non-injective homomorphisms onto transitive structures, uniqueness can again be consistently postulated. This leads to the axioms

X_1 (resp. Y_1): *For any function f there is an unique f -inductive (resp. f -constructive) function g .*

All axioms stated up to now deal with functions which are sets.

They all have a class-version obtained by allowing f and g to be (possibly proper) classes. We shall refer to these class-versions by means of a bar over the corresponding set version name.

The relations between class- and set-version of each axiom and among the axioms will be studied in detail in section 4.

However we note that if f is a proper class the restriction in Remark 1.3 becomes essential, and $A \cap TC(\cup B \setminus A) = \emptyset$ in the proof of Lemma 1.4 cannot be any more assumed w.l.o.g. But the corresponding implications continues to hold if N is assumed:

LEMMA 1.5. *Both $\bar{X}^t \rightarrow \bar{Y}$ and $\bar{Y} \rightarrow \bar{Y}^*$ hold in **ABCN**. Hence the axioms \bar{X}^t , \bar{X}^* , \bar{Y}^* are all equivalent to \bar{Y} .*

PROOF. First of all we claim that given $f: A \rightarrow B$, the class A can be viewed as an increasing union of f -transitive subsets A_α ($\alpha \in \text{Ord}$).

In fact, given a subset $D \subseteq A$, the f -transitive closure of D can be defined in a natural way by:

$$f\text{-}TC(D) = \bigcap \{C \mid C \supseteq D, C \text{ } f\text{-transitive}\} = \bigcup_{n \in \omega} C_n$$

where $C_0 = D$, $C_{n+1} = \bigcup f(C_n) \cap A$.

Now assuming a fixed wellordering of A , we can define inductively

on Ord:

$$\begin{aligned}
 A_0 &= f\text{-TC}(\{\min A\}), \\
 A_{\alpha+1} &= f\text{-TC}(A_\alpha \cup \{\min A \setminus A_\alpha\}), \\
 A_\lambda &= \bigcup_{\alpha < \lambda} A_\alpha \quad \text{for limit } \lambda.
 \end{aligned}$$

To prove $\bar{X}^t \rightarrow \bar{Y}$, by Lemma 1.4 we may use axiom Y^* to get an increasing sequence $\{g_\alpha | \alpha \in \text{Ord}\}$ of $f|_{A_\alpha}$ -constructive functions. Their union is the required f -constructive function on A .

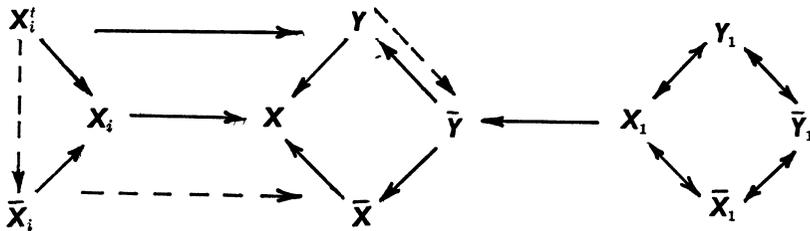
Similarly given $C \subseteq A$ f -transitive and $h: C \rightarrow D$ $f|_C$ -constructive, we put $C_\alpha = A_\alpha \cap C$ and define the increasing sequence of functions g_α on A_α as follows:

- g_0 any $f|_{A_0}$ -constructive function extending $h|_{C_0}$,
- $g_{\alpha+1}$ the first $f|_{A_{\alpha+1}}$ -constructive function extending $g_\alpha \cup h|_{C_{\alpha+1}}$ in the fixed wellordering of the universe,
- $g_\lambda = \bigcup_{\alpha < \lambda} g_\alpha$ for limit λ .

All the steps can be accomplished since Y^* holds, and as above the union $g = \bigcup_{\alpha \in \text{Ord}} g_\alpha$ is the required f -constructive function extending h . Q.E.D.

The principal results we obtain in this paper are summarized by the following

MAIN THEOREM. *In **ABCN** all the implications of the following diagram hold:*



*All axioms are consistent relatively to **ABCN**, and no arrow can be added except compositions. Continuous arrows already hold in **ABC**.*

Note that in the diagram X_i^t replaces axiom Y_i which is inconsistent (see p. 504).

Axiom $\bar{\mathbf{X}}_i^t$ is not included since it is also inconsistent.

For let $f = \delta_{\text{Aut}} \cup \{(\emptyset, \emptyset)\}$: the range of any f -inductive function is Aut , hence no injective f -inductive g can be given.

What our axioms say about homomorphisms between binary structures (i.e. pairs (A, R) with $R \subseteq A^2$) and transitive structures is expressed by the following theorem, whose proof is a straightforward rewriting of the definitions.

THEOREM 1.6.

(i) \mathbf{X} holds iff any binary structure has a homomorphism onto a transitive structure.

(ii) \mathbf{X}_1 holds iff any binary structure has a unique homomorphism onto a transitive structure (which, in general, cannot be an isomorphism).

(iii) \mathbf{X}_i holds iff any extensional structure has an isomorphism onto a transitive structure (and uniqueness necessarily fails).

(iv) \mathbf{Y} (resp. \mathbf{X}_i^t) holds iff given a binary structure which induces δ_x on a transitive $T \subseteq A$, the homomorphism of (i) (resp. isomorphism of (iii)) can be taken to be the identity on T . **Q.E.D.**

REMARK 1.7. If homomorphisms onto standard (not necessarily transitive) structures are required, the axioms can be weakened to postulate the existence of weakly- f -inductive functions (i.e. functions $g: A \rightarrow E$ s.t.

$$g(x) \cap E = \hat{g}(f(x) \cap A) \quad \text{for all } x \in A).$$

Then we would have

\mathbf{X}^{weak} (resp. $\mathbf{X}_i^{\text{weak}}$) iff any binary structure has a homomorphism (resp. isomorphism) onto a standard structure.

The weak versions are indeed weaker than the ordinary versions: however the analysis of this topic falls outside the scope of this paper.

2. - f -founded sets.

In this section we give a suitable generalization of the notion of well-foundedness, namely that of f -founded element relatively to a fixed map $f: A \rightarrow B$.

The interest in f -founded sets lies in the fact that the strongest form of our **FCP** holds in **ABC** for the collection of f -founded elements without further assumptions (Theorem 2.3).

Fix a function $f: A \rightarrow B$ throughout the section.

LEMMA 2.1. *The following conditions are equivalent for any $x \in A$:*

- (i) x belongs to any f -saturated subset of A ,
- (ii) $\forall D \subseteq A$ if $x \in D$ then $\exists y \in D$ s.t. $f(y) \cap D = \emptyset$.

The elements verifying the above conditions are called the f -founded elements of A and they will be denoted by A_f .

PROOF. The two conditions are equivalent since both define the *least* f -saturated subset of A , as shown below.

Put $A_f = \{x \in A \mid x \text{ satisfies (ii)}\}$.

Assume $f(x) \cap A \subseteq A_f$ and $x \in D \subseteq A$. If $f(x) \cap D \neq \emptyset$ pick $z \in f(x) \cap D \subseteq A_f$: since $z \in D$ there exists $y \in D$ s.t. $f(y) \cap D = \emptyset$.

Hence A_f is f -saturated.

It remains to show that A_f is included in any f -saturated $S \subseteq A$.

If S is f -saturated and $x \in A_f \setminus S = D$, then by definition there exists $y \in D$ s.t. $f(y) \cap D = \emptyset$; but $f(y) \cap A = f(y) \cap A_f \subseteq S$, hence $y \in S$, contradiction. Q.E.D.

The property (ii) is essentially that used by Boffa in [2] to define the sets «well-founded over \emptyset ».

The f -founded elements have the following useful properties.

LEMMA 2.2.

- (i) x is f -founded iff $f(x) \cap A \subseteq A_f$.

Hence A_f is f -transitive and f -saturated.

- (ii) $A_f = \emptyset$ iff $\forall x \in A$ $f(x) \cap A \neq \emptyset$.
- (iii) If $x \in A_f$, then $x \notin f(x)$.
- (iv) If T is f -transitive, then $A_f \cap T = T_{f|_T}$.

PROOF.

- (i) A_f being f -saturated, $f(x) \cap A \subseteq A_f$ implies $x \in A_f$.

The converse follows provided $\{x \in A_f \mid f(x) \cap A \subseteq A_f\} = S$ is f -saturated, by property (i) of Lemma 2.1. But this is trivial, since

$f(x) \cap A \subseteq S \subseteq A_f$ implies, by the first part, that $x \in A_f$, hence $x \in S$.

(ii) Obviously \emptyset is f -saturated iff $\forall x \in A \ f(x) \cap A \neq \emptyset$.

(iii) Assume $x \in A_f \cap f(x)$. Then $A_f \setminus \{x\}$ is f -saturated, for $f(y) \cap A \subseteq A_f \setminus \{x\} \subseteq A_f$ implies $y \in A_f$ and $y \neq x$.

(iv) Let $x \in T_{f|x}$ and $x \in D \subseteq A$: then there is $y \in D \cap T$ such that $f(y) \cap D \cap T = \emptyset$, whence $f(y) \cap D = \emptyset$ for $f(y) \subseteq T$. Thus $T_{f|x} \subseteq A_f \cap T$; the converse inclusion is trivial. Q.E.D.

As remarked at the beginning of this section, the interest in f -founded elements lies in the following theorem, which places in our framework all the assertions of Mostowski's collapsing theorem.

THEOREM 2.3. *For any $f: A \rightarrow B$ there is an unique $f|_{A_f}$ -constructive function on A_f .*

PROOF. Recall that the union of f -constructive functions on f -transitive subsets which agree on the intersection of their domains is again an f -constructive function.

Let

$$\mathcal{C} = \{C \subseteq A_f \mid C \text{ } f\text{-transitive and there is a unique } f|_C\text{-constructive function } g_C\}.$$

Now $C_0 = \bigcup \mathcal{C}$ belongs to \mathcal{C} , since the $f|_C$ -constructive functions g_C for $C \in \mathcal{C}$ can be glued together by uniqueness. The thesis then follows if C_0 is f -saturated, for then $A \in \mathcal{C}$. Assume $f(x) \cap A \subseteq C_0$: then $x \notin f(x)$ (since $x \in A_f$) and we can define

$$g(x) = \hat{g}_{C_0}(f(x) \cap A) \cup (f(x) \setminus A)$$

obtaining an $f|_{C_0 \cup \{x\}}$ -constructive function, which is unique by definition. As $C_0 \cup \{x\}$ is f -transitive, this implies $x \in C_0$, which is thus f -saturated. Q.E.D.

Images of f -founded elements are « weakly-founded » in the following sense:

PROPOSITION 2.4. *Let g be the f -constructive function on the f -founded elements of $\text{Dom } f$. Then $\text{Range } g \subseteq \mathbf{II}(\cup \text{Range } f \setminus \text{Dom } f)$.*

PROOF. Let $S = \{x \in A_f \mid g(x) \in \mathbf{II}(\cup \text{Range } f \setminus A)\}$ and let $f(x) \cap A \subseteq S$. Then $g(x) = \hat{g}(f(x) \cap A) \cup f(x) \setminus A \subseteq \mathbf{II}(\cup \text{Range } f \setminus A)$, hence S is f -saturated and thus must be A_f itself. Q.E.D.

One can use the uniqueness of the f -constructive function on the f -founded elements to get the inconsistency of axiom \mathbf{Y}_i (as asserted at p. 499).

For instance consider on $A = \{1, 2, 3\}$ the injective function

$$f = \{(1, 0), (2, \{1\}), (3, \{0\})\}:$$

then $A_f = A$ and the f -constructive g satisfies $g(2) = g(3) = \{0\}$; therefore no injective f -constructive function can be given on A .

We conclude the section with some remarks concerning the case when f and A are proper classes.

A_f can again be defined by means of property (ii) of Lemma 2.1, where this time D ranges over the *subsets* of A . Assuming this definition, one can prove that all the properties of f -founded elements continue to hold. Namely:

PROPOSITION 2.5.

- (i) A_f is the least f -saturated subclass of A and is f -transitive.
- (ii) There exists a unique $f|_{A_f}$ -constructive function.

PROOF. Let us begin by proving that property (ii) of Lemma 2.1 holds for any *sub-class* $D \subseteq A$.

Let $x \in D$ and pick $y \in f\text{-TC}(\{x\}) \cap D$ such that

$$f(y) \cap f\text{-TC}(\{x\}) \cap D = \emptyset.$$

Since $f(y) \subseteq f\text{-TC}(\{x\})$ we have $f(y) \cap D = \emptyset$, as required. Now we can proceed in exactly the same way as in the proof of Lemma 2.1 to prove part (i) of Proposition 2.5.

The proof of part (ii) can be done by looking at the f -transitive closure T_x of any $x \in A_f$. Since the elements of T_x are all $f|_{T_x}$ -founded (Lemma 2.2 (iv)) we may apply Theorem 2.3 and get a unique $f|_{T_x}$ -constructive g_x on T_x ; moreover the g_x 's can be glued together to give an $f|_{A_f}$ -constructive g since they are mutually compatible by uniqueness. Q.E.D.

3. - f -admissible relations and models.

This section is devoted to the development of the techniques which will be used in the last section to define the particular inner models involved in our consistency proofs.

At first we study those equivalence relations on A which carry over the structure induced by a fixed map $f: A \rightarrow \mathfrak{F}(A)$ to the quotient, making it extensional (f -admissible relations). These equivalences give rise to models of **ABC** whenever the considered map is surjective (Theorem 3.8). A new interpretation of our axioms is given in terms of these equivalence relations (Theorem 3.9).

Fix a function $f: A \rightarrow \mathfrak{F}(A)$ throughout the section.

DEFINITION 3.1. Given a *reflexive* and *symmetric* relation R on A define the relation \tilde{R} on A by

$$(*) \quad x \tilde{R} y \text{ iff } \begin{cases} \forall s \in f(x) \exists t \in f(y) & s R t, \\ \forall t \in f(y) \exists s \in f(x) & t R s. \end{cases}$$

Then

$$\begin{aligned} R \text{ is } f\text{-conservative} & \text{ iff } R \subseteq \tilde{R}, \\ R \text{ is } f\text{-compatible} & \text{ iff } R \supseteq \tilde{R}, \\ R \text{ is } f\text{-admissible} & \text{ iff } R = \tilde{R}. \end{aligned}$$

The properties we list below are easy and completely standard, and are therefore stated without proof.

3.2.1. \sim a is monotone operator which preserves reflexivity and symmetry (and also transitivity).

In particular \tilde{E} is an equivalence if E is.

3.2.2. f -conservative (f -compatible) relations are closed under arbitrary union (intersection).

3.2.3. Since the f -admissible relations are the fixed points of the monotone operator \sim on the complete lattice $\mathcal{R}(A)$ of reflexive and symmetric relations on A , they form a complete lattice.

3.2.4. Note that the equivalences $\mathcal{E}(A)$ on A are a complete lattice (although not a complete sublattice of $\mathcal{R}(A)$), hence, as before, the f -admissible equivalences are a complete lattice.

f -admissible relations can be obtained by means of the following construction.

Given an f -conservative (resp. f -compatible) relation $R \in \mathcal{R}(A)$ define inductively

$$\begin{aligned} R_0 &= R, \\ R_{\alpha+1} &= \check{R}_\alpha, \\ R_\lambda &= \bigcup_{\alpha < \lambda} R_\alpha \quad (\text{resp. } R_\lambda = \bigcap_{\alpha < \lambda} R_\alpha). \end{aligned}$$

The sequence R_α is increasing (resp. decreasing) and definitively constant; its limit will be denoted by \check{R} (resp. \check{R}) and is f -admissible by definition.

LEMMA 3.3. *Let R be an f -conservative (resp. f -compatible) relation. \check{R} (resp. \check{R}) is the least f -admissible relation including R (resp. the largest f -admissible relation included in R). If R is an equivalence so is \check{R} (resp. \check{R}).*

In particular \mathfrak{D}_A (resp. \check{A}^2) is the least (resp. largest) f -admissible relation, and it is an equivalence.

We shall denote \mathfrak{D}_A by \equiv , (resp. \check{A}^2 by \approx).

PROOF. By induction on α it follows that all R_α 's are f -conservative (resp. f -compatible) and are equivalences if R is.

If S is f -admissible and $R \subseteq S$ (resp. $R \supseteq S$), then by induction $R_\alpha \subseteq S$ (resp. $R_\alpha \supseteq S$) for any α , hence $\check{R} \subseteq S$ (resp. $\check{R} \supseteq S$). Q.E.D.

LEMMA 3.4. *If R is f -conservative, S is f -compatible and $R \subseteq S$, then $\check{R} \subseteq \check{S}$.*

Hence \equiv , is the intersection of all f -compatible and \approx , the union of all f -conservative relations.

PROOF. Assume by induction that $R_\alpha \subseteq S$ and let $x R_{\alpha+1} y$: then $\forall s \in f(x) \exists t \in f(y) s R_\alpha t$ and conversely; by induction hypothesis and f -compatibility of S it follows $x S y$, whence $R_{\alpha+1} \subseteq S$. Since at limits the passage is trivial, we get $\check{R} \subseteq S$ and, by Lemma 3.3, $\check{R} \subseteq \check{S}$. Q.E.D.

The relations introduced up to now behave well under restriction to f -transitive subsets. Namely

LEMMA 3.5. *Let $C \subseteq A$ be f -transitive.*

(i) *If R is f -conservative (resp. f -compatible) then $R \cap C^2$ is $f|_C$ -conservative (resp. $f|_C$ -compatible).*

Conversely if $S \subseteq C^2$ is $f|_C$ -conservative (resp. $f|_C$ -compatible) then S is the restriction to C of the f -conservative (resp. f -compatible) relation $S \cup \mathfrak{D}_A$ (resp. $S \cup (A^2 \setminus C^2)$).

(ii) If $S \subseteq C^2$ is $f|_C$ -admissible, it is the restriction to C of all f -admissible relations between $\overset{\vee}{S \cup \delta_A}$ and $\overset{\circ}{(S \cup (A^2 \setminus C^2))}$. In particular $\equiv_f \cap C^2 = \equiv_{f|_C}$ and $\approx_f \cap C^2 = \approx_{f|_C}$.

PROOF. The restriction to C is a monotone operator from $\mathcal{R}(A)$ onto $\mathcal{R}(C)$.

Let us denote by \sim^C the operator tilde on $\mathcal{R}(C)$: in general $\tilde{R} \cap C^2 \supseteq \widetilde{R \cap C^2}$ holds, but if C is f -transitive then equality holds, and tilde and restriction commute.

The property(*) being already decided within any f -transitive subset $C \supseteq \{x, y\}$, all the assertions (i) of the lemma are straightforward consequences of the definitions.

As far as (ii) is concerned, in the same order of ideas, we note that, S being $f|_C$ -admissible, the equalities

$$(S \cup \delta_A)_\alpha \cap C^2 = S = (S \cup (A^2 \setminus C^2))_\alpha \cap C^2$$

follow by simple ordinal induction.

Hence both $\overset{\vee}{S \cup \delta_A}$ and $\overset{\circ}{S \cup (A^2 \setminus C^2)}$ induce on C the relation S , and the last assertions follow at once. Q.E.D.

We single out the following consequence of the above lemma, which will have great importance in the sequel:

COROLLARY 3.5.1. *If f restricted to an f -transitive $C \subseteq A$ is injective, then \equiv_f induces equality on C .*

In particular if $T \subseteq A$ is transitive and $f|_T = \delta_T$, then $\equiv_f \cap T^2 = \delta_T$.

PROOF. The conclusion follows from the preceding lemma, observing that δ_C is $f|_C$ -admissible by injectivity of $f|_C$. Q.E.D.

As stated at the outset of the section the interest in f -admissible equivalences is in the following

THEOREM 3.6. *Let E be an f -admissible equivalence. There is a unique function \bar{f} s.t. the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{f} & \mathfrak{F}(A) \\ \pi \downarrow & & \downarrow \bar{\pi} \\ \bar{A} & \xrightarrow{\bar{f}} & \mathfrak{F}(\bar{A}) \end{array} \quad (\text{where } \pi: A \rightarrow \bar{A} = A/E \text{ is the canonical map}).$$

Moreover \bar{f} is injective.

Conversely any f -inductive function g defines an f -admissible equivalence $E_g = \{(x, y) | g(x) = g(y)\}$.

In particular if $x \equiv_f y$ then $g(x) = g(y)$ and if $g(x) = g(y)$ then $x \approx_f y$.

PROOF. The function $\bar{f}(\bar{a}) = \{\bar{b} | b \in f(a)\}$ is well defined and injective by f -admissibility of E , and verifies trivially the desired properties.

Conversely f -inductivity of g implies $g(x) = g(y)$ iff $\hat{g}(f(x)) = \hat{g}(f(y))$, which means that E_g is f -admissible. Q.E.D.

We shall use in our constructions f -admissible relations on a proper class A : it is apparent that Definition 3.1 makes sense when A and f are proper classes. Unfortunately the inductive construction of Lemma 3.3 cannot be done directly in this case. In order to obtain the desired extensions of the properties proved in the set-case, the following « approximation » technique will be useful.

We say that $C \subseteq \mathcal{P}(A)$ is an f -cover of A iff $\forall x, y \in A \exists C \in C$ s.t. $x, y \in C$ and $\forall C \in C$ C is f -transitive.

LEMMA 3.7. Let C be an f -cover of a set A . Then R is f -admissible iff $R \cap C^2$ is $f|_C$ -admissible for any $C \in C$.

Moreover if R is f -conservative (resp. f -compatible), then

$$\overset{\vee}{R} \cap C^2 = \check{R} \cap C^2 \quad (\text{resp. } \overset{\circ}{R} \cap C^2 = \mathring{R} \cap C^2)$$

and

$$\check{R} = \bigcup_{C \in C} \overset{\vee}{R} \cap C^2 \quad (\text{resp. } \mathring{R} = \bigcup_{C \in C} \overset{\circ}{R} \cap C^2).$$

In particular $\equiv_f = \bigcup_{C \in C} \equiv_{f|_C}$ and $\approx_f = \bigcup_{C \in C} \approx_{f|_C}$.

PROOF. By definition of f -cover the equality $R = \bigcup_{C \in C} R \cap C^2$ holds for any binary relation $R \subseteq A^2$.

Since the C 's are f -transitive, Lemma 3.5 yields at once that $R \cap C^2$ is $f|_C$ -admissible whenever R is f -admissible.

Conversely, since $\check{R} \cap C^2 = \widetilde{R \cap C^2}$, one gets

$$\check{R} = \bigcup_{C \in C} \check{R} \cap C^2 = \bigcup_{C \in C} \widetilde{R \cap C^2} = \bigcup_{C \in C} R \cap C^2 = R$$

provided all the relations $R \cap C^2$ are $f|_C$ -admissible.

Now assume R f -conservative: then both $\overset{\vee}{R} \cap C^2$ and $\check{R} \cap C^2$ define the least $f|_C$ -admissible relation extending $R \cap C^2$, whence they must coincide.

The other assertions of the lemma are proved similarly. Q.E.D.

We remark that to get the stronger property that a relation R on A is an f -admissible *equivalence* iff its restrictions $R \cap C^2$ to an f -cover are f -admissible equivalences it is necessary to strengthen the definition of f -cover by requiring that

$$\forall x, y, z \in A \exists C \in \mathcal{C} \quad x, y, z \in C.$$

For any class A a canonical f -cover can be defined by taking the f -transitive closures of the doubletons

$$\mathcal{C} = \{f\text{-TC}(\{x, y\}) \mid x, y \in A\}$$

(recall that $f\text{-TC}(\{x, y\}) = \bigcup_{n \in \omega} Z_n$ where $Z_0 = \{x, y\}$, $Z_{n+1} = \bigcup f(Z_n)$). Now we may glue up along \mathcal{C} and define

$$\equiv_f = \bigcup_{C \in \mathcal{C}} \equiv_{f|_C}, \quad \approx_f = \bigcup_{C \in \mathcal{C}} \approx_{f|_C}$$

and more generally, for any f -conservative (resp. f -compatible) R

$$\check{R} = \bigcup_{C \in \mathcal{C}} \overbrace{R \cap C^2}^{\vee} \quad (\text{resp. } \mathring{R} = \bigcup_{C \in \mathcal{C}} \overbrace{R \cap C^2}^{\circ}).$$

The above definitions do not depend on the particular choice of the f -cover \mathcal{C} , as follows immediately from Lemma 3.7.

Incidentally the canonical f -cover defined above is the finest f -cover of A .

Assuming these definitions it is straightforward to check that all the assertions of Lemmas 3.3, 3.4, 3.5 continue to hold when A is a proper class.

To extend Lemma 3.7 to a proper class A we need only the existence of a map π verifying $\pi(x) = \pi(y)$ iff $x E y$, so as to be able to dispose of a « quotient class » \bar{A} . When this is available all the assertions of the theorem hold without modification.

When the equivalence classes of E are proper, since we do not assume the regularity axiom, we need some form of choice to define \bar{A} . E.g. if a well ordering of A is given we can take

$$\bar{A} = \{x \in A \mid \forall y \in A (y < x \rightarrow y \notin x)\},$$

thus π chooses the first element in each equivalence class.

We shall use f -admissible equivalences on a proper class A to define inner models, by means of the following construction.

THEOREM 3.8. *Assume **ABCN**. Let E be an f -admissible equivalence on A such that the function \bar{f} given by Theorem 3.6 is surjective.*

Define the model \mathfrak{A} as follows

sets ^{\mathfrak{A}} are the elements of \bar{A} ;

classes ^{\mathfrak{A}} are the elements of \bar{A} and the proper subclasses of \bar{A} ;

$x \in^{\mathfrak{A}} y$ iff $y \in \bar{A}$ and $x \in \bar{f}(y)$ or y is proper and $x \in y$.

Then $\mathfrak{A} \models \mathbf{ABCN}$.

PROOF. By our hypotheses $\bar{A} \simeq \mathfrak{F}(\bar{A})$, whence \bar{A} is proper and in 1 – 1 correspondence with V .

We may then assume that $\bar{A} = V$.

With this assumption the model \mathfrak{A} becomes isomorphic to the inner model defined by Boffa in Lemma 6 of [2] § 5 (p. 21-23), when the relation R is chosen as $x R y$ iff $x \in f(y)$ for $x, y \in V$. **Q.E.D.**

Observe that any f -admissible model preserves the well-founded sets of the universe in the following sense.

COROLLARY 3.8.1. *There is a unique isomorphism between $(\mathbf{\Pi}, \in)$ and $(\mathbf{\Pi}^{\mathfrak{A}}, \in^{\mathfrak{A}})$.*

PROOF.

$$\begin{aligned} \mathbf{\Pi}^{\mathfrak{A}} &= \{x \in \bar{A} \mid \mathfrak{A} \models \forall y (x \in y \rightarrow \exists w \in y (w \cap y = \emptyset))\} \\ &= \{x \in \bar{A} \mid \forall y \in \bar{A} (x \in \bar{f}(y) \rightarrow \exists w \in \bar{f}(y) (\bar{f}(w) \cap \bar{f}(y) = \emptyset))\}. \end{aligned}$$

Since \bar{f} is surjective, the last property is precisely (ii) of Lemma 2.1. Therefore $\mathbf{\Pi}^{\mathfrak{A}} = \bar{A}_{\bar{f}}$.

Now Theorem 2.3 and Corollary 3.5.1 give a unique isomorphism g between $(\bar{A}_{\bar{f}}, \in^{\mathfrak{A}})$ and a transitive well-founded class M .

Let $x \in \mathbf{\Pi}$ be an element of least rank not in M .

Then $\forall t \in x \exists a_t \in \bar{A}_t$ s.t. $g(a_t) = t$ and $g(\bar{f}^{-1}(\{a_t \mid t \in x\})) = x$. (Note that $\bar{f}^{-1}(\{a_t \mid t \in x\})$ belongs to $\bar{A}_{\bar{f}}$ since this class is \bar{f} -saturated). **Q.E.D.**

In order to relate the axioms of free construction to f -admissible equivalences we give the following definition.

An f -admissible equivalence E is *induced* iff there is an f -inductive function g s.t.

$$g(x) = g(y) \leftrightarrow x E y .$$

THEOREM 3.9. *Assume **ABCN**. Then*

- (i) **X** holds iff $\forall f \approx_f$ is induced.
- (ii) **X_i** holds iff $\forall f \equiv_f$ is induced (actually all *f*-admissible relations are induced).
- (iii) **X₁** holds iff $\forall f$ only \approx_f is induced.

*The same holds for the class versions of the axioms, allowing *f* to be a proper class.*

PROOF.

(i) **X** holds iff $\forall f$ some *f*-admissible equivalence is induced, hence the left-pointing implication is trivial. For the converse, given $f: A \rightarrow \mathcal{F}(A)$ consider the corresponding $\bar{f}: A/\approx_f \rightarrow \mathcal{F}(A/\approx_f)$. It is easy to check that any \bar{f} -inductive function inducing $\approx_{\bar{f}}$ composed with the canonical map from *A* onto A/\approx_f is an *f*-inductive function inducing \approx_f on *A*.

(ii) When *f* is injective \equiv_f is δ_A . The left-pointing arrow follows immediately; the right-pointing arrow follows as above considering A/\equiv_f (or A/E for any *f*-admissible *E*).

(iii) By part (i) \approx_f is induced for any *f* when **X** holds. But then if **X₁** holds, no other *f*-admissible equivalence can be induced, otherwise there would be more than one *f*-inductive function.

Conversely assume two *f*-inductive functions g_1 and g_2 are given: let E_1, E_2 be their ranges (which are transitive sets). Obviously $\delta_{E_1 \cup E_2}$ is a $\delta_{E_1 \cup E_2}$ -inductive function inducing $\delta_{E_1 \cup E_2}$.

But $\delta_{E_1 \cup E_2}$ is not $\approx_{\delta_{E_1 \cup E_2}}$. In fact the relation *R* on $E_1 \cup E_2$ obtained from

$$\{(x_1, x_2) \mid \exists a \in A \ g_1(a) = x_1, \ g_2(a) = x_2\}$$

by symmetrization and reflexivization is *f*-conservative. Hence \check{R} is a $\delta_{E_1 \cup E_2}$ -admissible relation properly including $\delta_{E_1 \cup E_2}$.

The extension of the preceding proofs to the case of proper classes can be done directly provided a class of representatives \bar{A} is available whenever required. \bar{A} can be defined as before (p. 509) using axiom **N**. This is the only use of **N** in the proof. **Q.E.D.**

We have seen that if some *f*-admissible relation is induced for any *f*, then \approx_f is certainly induced, and that if \equiv_f is always induced, then so is any *f*-admissible relation.

It is interesting to notice that the stronger assertion

« any f -admissible relation including an induced one is itself induced »

can fail also if \mathbf{X} is assumed.

We only sketch the idea of the proof.

Suppose that the transitive class M is a model of \mathbf{ABCNX}_1 . Then M has a unique autosingleton x . Suppose that in the universe V there is another autosingleton y and a set $z = \{x, y, z\}$. Put $N = \mathbf{\Pi}(M \cup y \cup z)$. It can be shown that N is a transitive model of \mathbf{ABCNX} . The function

$$f = \{(0, \{0\}), (1, \{1\}), (2, \{0, 1, 2\})\} \quad (f: A = \{0, 1, 2\} \rightarrow \mathfrak{F}(A))$$

has exactly 3 f -admissible equivalences, namely $\equiv_f = \delta_A$, $\approx_f = A^2$ and $E = \delta_A \cup \{(0, 1), (1, 0)\}$. But only \equiv_f and \approx_f are induced. In fact let g be an f -inductive function such that 0 and 1 have the same image (which must be an autosingleton a); then $g(2) = b = \{a, b\}$, which in the class N is possible only for $b = a$.

The consistency of assuming such a class M can be proved from the consistency of \mathbf{ABCNX}_i^t in the following way. By our main theorem $\mathbf{X}_i^t \rightarrow \bar{\mathbf{X}}_i$ and by the above theorem in any model of $\mathbf{ABCN}\bar{\mathbf{X}}_i$ there is a δ_V -inductive function φ inducing the relation \approx_{δ_V} . Its range M is indeed a transitive model of \mathbf{ABCNX}_1 . At this stage a proof of this fact would require several technicalities which will become clear in the next section; the reader will then be able to fill in the proof.

Since the existence of sets like y and z is an immediate consequence of \mathbf{X}_i^t (see also Boffa [2]), the class N is definable in any model of \mathbf{ABCNX}_i^t : this concludes our sketch.

4. – Proof of the main theorem.

We split the theorem in several assertions which we prove separately. We assume \mathbf{ABCN} throughout the section.

We begin by proving the relative consistency of \mathbf{X}_1 by means of a suitable model \mathfrak{M} defined through a function F which codifies all possible f -inductive problems.

Put $A = \{(x, f, z) | x \in z, f: z \rightarrow \mathfrak{F}(z)\}$ and define $F: A \rightarrow \mathfrak{F}(A)$ by

$$F((x, f, z)) = \{(t, f, z) | t \in f(x)\}.$$

Let M be a class of representatives of the equivalence classes mod \approx_F and let $\pi: A \rightarrow M$ be the canonical map.

Let $G: M \rightarrow \mathfrak{F}(M)$ be the unique function s.t. $G \circ \pi = \hat{\pi} \circ F$.
Then

LEMMA 4.1. G is surjective.

PROOF. First of all we claim that $(x, f, z) \approx_F (x', f', z')$ whenever there exist C f -transitive $\subseteq z$, C' f' -transitive $\subseteq z'$ and a bijection $\varphi: C \rightarrow C'$ s.t. $\varphi(x) = x'$ and $\forall t, u \in C, t \in f(u)$ iff $\varphi(t) \in f'(\varphi(u))$.

In fact the relation

$$R = \left\{ (v, w) \mid v = w \text{ or } \exists t \in C (v = (t, f, z) \text{ and } w = (\varphi(t), f', z') \text{ or conversely}) \right\}$$

is F -conservative, hence included in \approx_F .

Now let $y \in \mathfrak{F}(M)$: by the above claim we may choose, without loss of generality, a set Y of representatives of the equivalence classes of y such that, for any pair $(x, f, z), (x', f', z')$ of them, $z \cap z' = \emptyset$. Put

$$u = \{x \mid (x, f, z) \in Y\}, \quad w = \{u\} \cup \bigcup \{z \mid (x, f, z) \in Y\}$$

and define $g: w \rightarrow \mathfrak{F}(w)$ by putting $g(u) = u$ and, for any z belonging to a triple $(x, f, z) \in Y$, $g|_z = f$. It is easy to verify that

$$G\pi((u, g, w)) = \hat{\pi}(F(u, g, w)) = y,$$

since by the claim above $(x, f, z) \approx_F (x, g, w)$ holds for any $(x, f, z) \in Y$.
Q.E.D.

Let \mathfrak{M} be the \approx_F -admissible model, existing by Theorem 3.8, in virtue of Lemma 4.1. Then

THEOREM 4.2.

$$\mathfrak{M} \models \mathbf{ABCNX}_1.$$

Hence \mathbf{X}_1 is consistent relative to \mathbf{ABCN} .

PROOF. We need only to prove that \mathbf{X}_1 holds in \mathfrak{M} .

The idea of the proof below that \mathbf{X} holds in \mathfrak{M} is the following: given any (f -inductive problem)³ we lift it up in the universe V so as to obtain a real k -inductive problem. Our choice of the function F ensures that the canonical image of the triples coding the problem k is precisely the (solution)³ of the initial (problem)³. The uniqueness of the (solution)³ follows from the choice of \approx_F , the largest F -admissible relation.

Let $a \in M$ and let f be a function ^{\mathfrak{M}} from a to $\mathfrak{F}^{\mathfrak{M}}(a)$. The elements ^{\mathfrak{M}} of $a = \pi(u, h, v)$ can all be assumed to be of the form $\pi(t, h, v)$ with $t \in h(u)$.

For $t \in h(u)$ define

$$k(t) = \{s \in h(u) \mid \pi(s, h, v) \in^{\mathfrak{M}} f^{\mathfrak{M}}(\pi(t, h, v))\}.$$

Note that by the very definition of $k(t)$,

$$f^{\mathfrak{M}}(\pi(t, h, v)) = \{\pi(s, h, v) \mid s \in k(t)\}^{\mathfrak{M}}.$$

Now let g be the function ^{\mathfrak{M}} defined by

$$g = \left\{ \left(\pi(t, h, v), \pi(t, k, h(u)) \right)^{\mathfrak{M}} \mid (t, h, v) \in^{\mathfrak{M}} a \right\}^{\mathfrak{M}}.$$

g is well defined, since $(t', h, v) \approx_F (t, h, v)$ implies $k(t) = k(t')$. g is f -inductive ^{\mathfrak{M}} , since

$$\begin{aligned} g^{\mathfrak{M}}(\pi(t, h, v)) &= \pi(t, k, h(u)) = \{\pi(r, k, h(u)) \mid r \in k(t)\}^{\mathfrak{M}} \\ &= \hat{g}^{\mathfrak{M}}(\{\pi(r, h, v) \mid r \in k(t)\}^{\mathfrak{M}}) = \hat{g}^{\mathfrak{M}}(f^{\mathfrak{M}}(\pi(t, h, v))). \end{aligned}$$

Hence

$$\mathfrak{M} \models \mathbf{X}.$$

Finally assume two f -inductive ^{\mathfrak{M}} functions ^{\mathfrak{M}} g_1, g_2 exist for $(f: a \rightarrow \mathfrak{F}(a))^{\mathfrak{M}}$.

In analogy with the proof of Theorem 3.9 (iii) we define on A the relation R by

$x R y$ iff $x = y$ or $\exists t \in G(a)$ s.t. $\pi(x) = g_1^{\mathfrak{M}}(t)$ and $\pi(y) = g_2^{\mathfrak{M}}(t)$ or conversely.

By f -inductivity ^{\mathfrak{M}} of g_1 and g_2 , R is F -conservative, hence included in \approx_F . g_1 and g_2 must then agree on each argument. Q.E.D.

The arrows constituting the right diamond of the diagram in the main theorem are proved in the following lemma.

LEMMA 4.3. *The following assertions hold in **ABC**:*

- (i) \mathbf{X}_1 implies \mathbf{X}_1^t
- (ii) \mathbf{X}_1^t implies \mathbf{Y}_1
- (iii) \mathbf{Y}_1 implies $\bar{\mathbf{Y}}_1$.

Hence $\mathbf{X}_1, \bar{\mathbf{X}}_1, \mathbf{Y}_1, \bar{\mathbf{Y}}_1$ are all equivalent in **ABC**.

PROOF.

(i) Is trivial, since if f is the identity on a transitive set T , then δ_T is the unique f -inductive function on T .

(ii) By Lemma 1.4 $\mathbf{X}_1^t \rightarrow \mathbf{Y}$. As far as the uniqueness is concerned we remark that the translation of an f -constructive problem into an f' -inductive one carried out in the proof of Lemma 1.4 works in such a way that two different f -constructive functions g_1, g_2 can be extended to two different f' -inductive functions. Thus not $\mathbf{Y}_1 \rightarrow$ not \mathbf{X}_1^t .

(iii) Given $f: A \rightarrow B$ take for any $x \in A$ an f -transitive set $C_x \ni x$. Use \mathbf{Y}_1 to get the unique $f|_{C_x}$ -constructive g_x on C_x . Then on $C_x \cap C_y$ the functions g_x and g_y agree by uniqueness. Hence $g = \bigcup_{x \in A} g_x$ is the unique f -constructive function on A . Q.E.D.

Since each axiom with the subscript 1 trivially implies the same axiom without subscript, we have

COROLLARY 4.3.1. *All axioms $\mathbf{X}, \bar{\mathbf{X}}, \mathbf{Y}, \bar{\mathbf{Y}}$ are consistent relatively to \mathbf{ABCN} .*

The axioms $\mathbf{X}_i, \bar{\mathbf{X}}_i$ and \mathbf{X}_i^t are equivalent (in \mathbf{ABC}) to the axioms \mathbf{U}_i, \mathbf{U} and \mathbf{S} of Boffa ([2]), which are shown to be consistent (ibid.). However we give a new proof of the consistency of \mathbf{X}_i^t , by means of a suitable admissible model, instead of using forcing techniques.

The idea of the definition of the model is to fix a class B (to be regarded as a class of atoms) such that the \in -relation between objects of $V \setminus B = A$ is so rich as to ensure a solution for any \mathbf{X}_i^t -problem up to « atoms » from B . One has to check afterwards that the least extensionalization of $\in \cap A^2$ is still universal enough to satisfy \mathbf{X}_i^t .

Assume \mathbf{ABCNY}_1 . Then there is a unique autosingleton which we call b .

Let $B = \{x | b \subseteq x\}$ and $A = V \setminus B$; define $F: A \rightarrow \mathcal{P}(A)$ by $F(x) = x \cap A$.

Let W be a class of representatives of the equivalence classes mod \equiv_F , and assume, for sake of simplicity, that $W \subseteq A$ and $\pi|_W = \delta_W$.

Let $G: W \rightarrow \mathcal{P}(W)$ be the unique function such that $G \circ \pi = \hat{\pi} \circ F$.

LEMMA 4.4.

- (i) $\forall x, y \in W, x \in G(y)$ iff $\exists s \in y \cap A \ \pi s = x$.
- (ii) $x \subseteq A$ implies $x \in A$ and $F(x) = x$.
- (iii) $x \subseteq W$ implies $G(\pi(x)) = x$, hence G is surjective.

PROOF.

- (i) $\pi(x) = x$ and $G(y) = G(\pi(y)) = \hat{\pi}(F(y)) = \hat{\pi}(y \cap A)$ and (i) follows.

(ii) By definition of A .

(iii) $G(\pi(x)) = \mathcal{A}(x \cap A) = \mathcal{A}(x) = x$. Q.E.D.

Let \mathfrak{B} be the $\equiv_{\mathcal{F}}$ -admissible model.

THEOREM 4.5.

$$\mathfrak{B} \models \mathbf{ABCN}\mathbf{X}_i^t.$$

Hence \mathbf{X}_i^t is consistent relative to \mathbf{ABCN} .

PROOF. We need only prove that \mathbf{X}_i^t holds in \mathfrak{B} .

For any $x \in W$ define $x' = \{y \in W \mid y \in^{\mathfrak{B}} x\}$: then $\pi(x') = x$.

Suppose $\mathfrak{B} \models (f: a \rightarrow \mathcal{F}(a))$ injective inducing δ_i on t transitive $\subseteq a$.

Let $u' = a' \setminus t'$ and fix a bijection between u' and a subset of $(B \setminus b) \setminus TC(a')$ (denote by \bar{x} the image of $x \in u'$ under this bijection).

Define the function h on u' by $h(x) = (f^{\mathfrak{B}}(x))' \cup \{\bar{x}\}$ and let k be the h -constructive function. Then by definition

$$k(x) = \hat{k}((f^{\mathfrak{B}}(x))' \cap u') \cup ((f^{\mathfrak{B}}(x))' \cap t') \cup \{\bar{x}\}.$$

We claim:

- (i) $k(x) \neq b$ for all $x \in u'$,
- (ii) $k(x) \notin B$ (applying (i)),
- (iii) k is injective (since $k(x) \ni \bar{x}$ and $k(x) \not\ni \bar{z}$ if $z \neq x$),
- (iv) $k(x) \notin TC(a')$ (since $\bar{x} \notin TC(a')$).

Now define the function g on a by

$$g^{\mathfrak{B}}(x) = \begin{cases} x & \text{if } x \in^{\mathfrak{B}} t, \\ \pi k(x) & \text{if } x \notin^{\mathfrak{B}} t. \end{cases}$$

We go to prove that g is f -inductive $^{\mathfrak{B}}$, i.e. $g^{\mathfrak{B}}(x) = \{g^{\mathfrak{B}}(y) \mid y \in^{\mathfrak{B}} f^{\mathfrak{B}}(x)\}^{\mathfrak{B}}$. Since $g|_t^{\mathfrak{B}}$ is $\delta_i^{\mathfrak{B}}$ and t is transitive $^{\mathfrak{B}}$, this holds for $x \in^{\mathfrak{B}} t$. Let $x \in^{\mathfrak{B}} (a \setminus t)^{\mathfrak{B}}$. Then

$$z \in^{\mathfrak{B}} g^{\mathfrak{B}}(x) \Leftrightarrow z \in^{\mathfrak{B}} \pi k(x) \Leftrightarrow \exists s \in k(x) \cap A \quad \pi(s) = z$$

$$\Leftrightarrow \left\{ \begin{array}{l} \text{either } \exists y \in (f^{\mathfrak{B}}(x))' \setminus u': \pi(y) = y = z \\ \text{or } \exists y \in (f^{\mathfrak{B}}(x))' \cap u': \pi k(y) = z \end{array} \right\} \Leftrightarrow \exists y \in (f^{\mathfrak{B}}(x))' \quad g^{\mathfrak{B}}(y) = z$$

as required.

It remains to show that g is injective^{3B}.

Let $C = \hat{k}(u') \cup C_{t'}$ where $C_{t'}$ is the F -transitive closure of t' . If $x \in C_{t'}$ then $F(x) \subseteq C_{t'}$ by definition; if $x = k(z)$ for $z \in u'$, then

$$F(x) = \hat{k}(h(z) \cap u') \cup (h(z) \cap t') \subseteq C.$$

Hence C is F -transitive, actually the F -transitive closure of $\hat{k}(u') \cup t'$.

We claim that $\hat{\pi}(C) = (\hat{g}^{3B}(a))'$.

By definition $x \in (\hat{g}^{3B}(a))'$ implies $x = \pi(z)$ for some $z \in t' \cup \hat{k}(u') \subseteq C$.

Let $D = \{x \in C \mid \pi(x) \in (\hat{g}^{3B}(a))'\}$. Since $D \supseteq \hat{k}(u') \cup t'$, if we can prove that D is F -transitive, then it will follow that $D = C$.

Let $x \in D$ and $y \in F(x)$: then $\pi(y) \in^{3B} \pi(x) \in^{3B} \hat{g}^{3B}(a)$ which is transitive^{3B}, hence $y \in D$.

In the same way we obtain that $\hat{\pi}(C_{t'}) = t'$.

To prove the injectivity^{3B} of g we only need to prove that \equiv_F does not identify different elements of $t' \cup \hat{k}(u')$ and by Lemma 3.5 we may consider $\equiv_{F|_C}$.

Since $C_{t'} \subseteq TC(t')$, we obtain from (iv) above that $\hat{k}(u') \cap C_{t'} = \emptyset$.

Define $R = \delta_{\hat{k}(u')} \cup \equiv_{F|_{C_{t'}}}$: it suffices to show that R is $F|_C$ -compatible, since R induces equality on $\hat{k}(u') \cup t'$ and so will \equiv_F do.

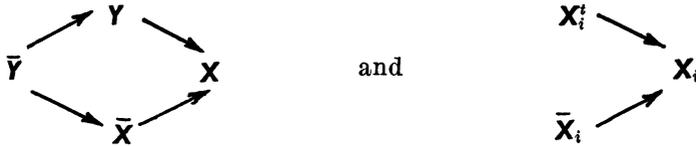
Assume $x = k(z)$, $y \in C_{t'}$ and $x \tilde{R} y$: then by (iv) it must be the case that $(f^{3B}(z))' \subseteq t'$ and $\forall s \in (f^{3B}(z))' \exists v \in y \cap C_{t'} s \equiv_F v$ and conversely.

But this means that $f^{3B}(z) = f^{3B}(\pi y)$ contrary to the injectivity^{3B} of f .

Hence \tilde{R} , as R , does not identify elements of $\hat{k}(u')$ with elements of $C_{t'}$.

Since $\equiv_{F|_{C_{t'}}}$ is obviously $F|_{C_{t'}}$ -compatible, it remains to show that $x_1 = k(z_1)$, $x_2 = k(z_2)$ and $x_1 \tilde{R} x_2$ implies $z_1 = z_2$. But this follows as above from the injectivity^{3B} of f . Q.E.D.

In order to prove the remaining implications of the theorem, we begin by observing that the following are trivial:



Moreover $X_i^t \rightarrow \bar{X}_i$ is quoted by Boffa as a theorem of J. Coret. The proof runs with an argument similar to that of Lemma 1.5; however we refer to [2] for a detailed proof.

Next we prove

LEMMA 4.6. Y implies \bar{Y} (in **ABCN**).

PROOF. Given $F: A \rightarrow \mathfrak{F}(A)$ let $\langle C_\alpha | \alpha \in \text{Ord} \rangle$ be an increasing continuous sequence of F -transitive subsets of A whose union is A (existing by **N**).

By Lemma 1.4 and axiom **N** there is a sequence $\langle g_\alpha | \alpha \in \text{Ord} \rangle$ of $F|_{C_\alpha}$ -constructive functions such that $g_\alpha|_{C_\beta} = g_\beta$ for all $\beta < \alpha$.

Then $G = \bigcup_{\alpha \in \text{Ord}} g_\alpha$ is the required F -constructive function on A . Q.E.D.

Finally we have

LEMMA 4.7.

- (i) \mathbf{X}_i implies \mathbf{X} .
- (ii) \mathbf{X}_i^t implies \mathbf{X}^t .
- (iii) $\bar{\mathbf{X}}_i$ implies $\bar{\mathbf{X}}$.

PROOF. Given $f: A \rightarrow \mathfrak{F}(A)$ let $\pi: A \rightarrow \bar{A}$ be the canonical map onto a class of representatives for the equivalence \equiv_f .

Let \bar{f} be the injective function given by Theorem 3.6.

It is easy to check that if \bar{g} is \bar{f} -inductive, then $\bar{g} \circ \pi = g$ is f -inductive, whence (i) and (iii).

If moreover f induces the identity on a transitive subset $T \subseteq A$, Corollary 3.5.1 allows us to assume that the elements of T are their own representatives in \bar{A} . Now \bar{f} induces the identity on T and, if \mathbf{X}_i^t holds, also \bar{g} can be chosen inducing δ_T .

Observe that no form of choice is needed in (i) and (ii), whereas to get (iii) we need some assumption giving \bar{A} and π when A is proper.

Q.E.D.

We have now proved all the arrows of the main theorem, and we start to show that no «new» arrow can be added.

We begin by remarking that $\bar{\mathbf{Y}}$ cannot imply either \mathbf{X}_1 or \mathbf{X}_1^t . In fact $\bar{\mathbf{Y}}$ follows from both \mathbf{X}_1 and \mathbf{X}_1^t , which are incompatible since $\mathbf{X}_1 \rightarrow (|Aut| = 1)$ whereas $\mathbf{X}_1^t \rightarrow (Aut \text{ is proper})$.

An analysis of the diagram shows that it suffices to prove that neither $\bar{\mathbf{X}}_i \rightarrow \mathbf{Y}$ nor $\mathbf{X}_i \rightarrow \bar{\mathbf{X}}$ hold.

For each of the implications above we find a set-theoretical sentence which follows from the premise, but fails in a suitable transitive inner model of the consequence.

LEMMA 4.8. Let σ be the sentence $\forall x \forall y \exists z (z = \{x, y, z\})$. Then

- (i) **ABCY** implies σ ,
- (ii) **ABCN $\bar{\mathbf{X}}_i$** does not imply σ .

Hence $\bar{\mathbf{X}}_i$ cannot imply \mathbf{Y} .

PROOF. (i) holds trivially (take $t \neq x, y$ and a $\{(t, \{x, y, t\})\}$ -constructive g).

The model \mathfrak{N} of $\mathbf{ABCN}\bar{\mathbf{X}}_i$ constructed by Boffa in his proof of Theorem 8 part (ii) of [2] does not satisfy σ . In fact, take $y = 0$ and $x = t$, where t is the « adjoined autosingleton » in Boffa's proof.

The very same argument used by Boffa to show that in \mathfrak{N} there is no set $y = \{t, y\}$ can also be used to prove that there is no set $z = \{t, 0, z\}$.
 Q.E.D.

The construction of the model \mathfrak{M} carried out in [4], which was successful for proving that \mathbf{X}_i does not imply $\bar{\mathbf{X}}_i$ cannot be employed to prove that $\mathbf{X}_i \leftrightarrow \bar{\mathbf{X}}$, since actually $\mathfrak{M} \models \bar{\mathbf{X}}$. We therefore need a more sophisticated construction.

The idea is like that of [4], but the diagonalization must be carried out over an ordinal-indexed, complexity-increasing collection of non-well-founded structures (namely the « double ordinals »), which can be discerned by means of homomorphisms. The « weak ordinals » of [4] are discernible only under isomorphism and actually collapse under non-injective homomorphisms.

For $\alpha \in \text{Ord}$ we define $f_\alpha: \alpha \cup (\alpha \times 1) \rightarrow \mathfrak{P}(\alpha \cup (\alpha \times 1))$ by $f_\alpha(\gamma) = \gamma$ and $f_\alpha(\gamma, 0) = \gamma \cup ((\gamma + 1) \times 1)$ for $\gamma \in \alpha$.

We say that g is an *ordinal-doubling function* iff it is f_α -inductive for some α , and that x is a *double ordinal* iff $x \notin \text{Ord}$ and it is in the range of some ordinal-doubling function.

Let g be an ordinal-doubling function: if $\gamma \in \text{dom } g$, then by induction $g(\gamma) = \gamma$ and $g((\gamma, 0)) \cap \text{Ord} = \{\gamma\}$.

In particular g is injective.

For any double ordinal x call *degree* of x the ordinal $\gamma = \gamma(x)$ such that $\{\gamma\} = x \cap \text{Ord}$. Remark that $x = g((\gamma, 0))$ for *any* ordinal-doubling function g having x in its range.

For any ordinal-doubling function g , $b = g((0, 0))$ verifies $b = \{0, b\}$. It follows inductively that to any double ordinal x belongs exactly one set $b = \{0, b\}$ (namely $g(0, 0)$ if $x = g((\gamma, 0))$): this set b will be called the *basis* of x and denoted $\mathbf{b}(x)$.

Now we can state

LEMMA 4.9. *Let τ be the sentence*

$$\exists b \forall \alpha \exists x (x \text{ is a double ordinal, } \mathbf{b}(x) = b \text{ and } \gamma(x) = \alpha)$$

- (i) $\mathbf{ABC}\bar{\mathbf{X}}$ implies τ .
- (ii) $\mathbf{ABCN}\mathbf{X}_i$ does not imply τ .

PROOF.

(i) Since any $(\bigcup_{\alpha \in \text{Ord}} f_\alpha)$ -inductive function has in its range double ordinals with the same basis and arbitrary degrees, τ follows directly from \bar{X} .

(ii) Assuming $\mathbf{ABCN}X_i$, $\{b|b = \{0, b\}\}$ is a proper class, which we index with Ord once for all.

We call *index* of a double ordinal the index of its basis (in the fixed indexing).

We call a set x *controlled* iff all double ordinals belonging to $TC(x)$ have degree smaller than their index.

Let C be the (transitive) class of controlled sets.

Let \mathfrak{C} be the transitive inner model whose classes are the subclasses of C and whose sets are the elements of C .

That $\mathfrak{C} \models \mathbf{ABCN}$ needs only a straightforward verification. As double ordinals are absolute for \mathfrak{C} , τ fails in \mathfrak{C} . It remains to show that $\mathfrak{C} \models \mathbf{X}_i$.

Let $f: A \rightarrow \mathfrak{F}(A)$ be an injective function in \mathfrak{C} . Let us assume, for convenience, that $TC(A) \cap \text{Ord} = \emptyset$.

Let $|A| = \mu$; take a cardinal $\nu > \mu^+$ and define on

$$A' = A_f \cup ((A \setminus A_f) \times \nu)$$

the function f' by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in A_f, \\ (f(y) \cap A_f) \cup \{(t, \gamma) | t \in f(y) \setminus A_f\} & \text{if } x = (y, \gamma) \in (A \setminus A_f) \times \nu. \end{cases}$$

Now f' is injective, for A_f is f -saturated.

Working outside \mathfrak{C} we pick an f' -inductive injective function g' .

For any $\delta \in \nu$ the function g_δ defined on A by

$$g_\delta(x) = \begin{cases} g'(x) & \text{if } x \in A_f, \\ g'((x, \delta)) & \text{if } x \in A \setminus A_f, \end{cases}$$

is f -inductive.

By Proposition 2.4 $g'(A_f) \subseteq \mathbf{II}(\emptyset)$, hence $g'|_{A_f} \in C$.

If x is any double ordinal in the range of g_δ , which is a transitive set of power not exceeding μ , then $\gamma(x) < \mu^+$.

Moreover in the range of any other g_γ , ($\gamma < \nu$) there is a double ordinal of the same degree.

Since the respective bases must all be different, a trivial cardinality argument implies that some g_γ is controlled. Q.E.D.

Note that actually we have proved the somewhat stronger assertion that $\mathbf{ABCX}_i\bar{X}$ is an extension of $\mathbf{ZF}_0\mathbf{X}_i$ which is not conservative with respect to \mathbf{ZF} -sentences.

Our main theorem is now completely proved.

As «the combinatorial power of various axioms of choice is best seen in the absence of foundation» ([3]), the relationship between choice axioms and «free construction» axioms is an interesting field of research, almost unexplored up to now.

We shall not deal with this topic here. As an instance of possible results in this area we show that if axiom \mathbf{X}_1 is assumed then axiom \mathbf{N} of von Neumann is equivalent to global choice \mathbf{E} , as it occurs when the axiom of foundation holds. It is well known, on the other hand, that \mathbf{ABCE} alone cannot prove \mathbf{N} (see [1]).

Assume \mathbf{ABCEX}_1 and consider the function F defined at the beginning of the construction of the model \mathfrak{M} of Theorem 4.2. Let F' be the restriction of F to $A \cap \mathbf{II}(\emptyset)$. Then the unique F' -inductive function H gives a projection of $P = A \cap \mathbf{II}(\emptyset)$ onto V . In fact given $a \in V$ take $b = TC(a) \cup \{a\}$ and consider the δ_b -inductive problem. By Remark 1.3 one can find a well-founded function $f: b' \rightarrow \mathfrak{F}(b')$ which codes through a bijection $\sigma: b \rightarrow b'$ an isomorphic problem.

Then $H(\sigma(a), f, b') = a$ by uniqueness. Since \mathbf{E} is sufficient to have $P \simeq \text{Ord}$, V itself is equipotent to Ord .

Let us conclude with some remarks concerning the broken arrows of our main theorem.

As pointed out previously, to derive $\bar{\mathbf{X}}$ from $\bar{\mathbf{X}}_i$ the weak axiom «for any equivalence E on A there is a function π such that $\pi(x) = \pi(y)$ iff $x E y$ » is sufficient.

As far as the implications $\mathbf{X}_i^! \rightarrow \bar{\mathbf{X}}_i$ and $\mathbf{Y} \rightarrow \bar{\mathbf{Y}}$ (or even $\mathbf{Y} \rightarrow \bar{\mathbf{X}}$) are concerned, the role of some strong class-form of choice axiom seems to be essential.

Actually it can be proved assuming an inaccessible cardinal, that both fail in a suitable model of \mathbf{ABC} .

We conjecture, however, that $\mathbf{ABC}\bar{\mathbf{Y}}$ (resp. $\mathbf{ABCX}_i^!\bar{\mathbf{X}}_i$) are conservative extensions of $\mathbf{ZF}_0\mathbf{Y}$ (resp. $\mathbf{ZF}_0\mathbf{X}_i^!$).

Added in proof.

The referee has pointed out that R. Hinnion introduces in [A], [B] the notions of «final equivalence» and «contraction» which are substantially the same as our « f -conservative» and « f -admissible» equivalences; he also proves the counterparts of our Lemmas 3.3 and 3.5 and of Theorem 3.6. See

- [A] R. HINNION, *Contraction de structures et application à NFU*, C. R. Acad. Sc. Paris, T. 290 (1980), Sér. A, pp. 677-680.
- [B] R. HINNION, *Extensional quotients of structures and applications to the study of the axiom of extensionality*, Bull. Soc. Math. Belg., Vol. XXXIII, Fas. II, Sér. B (1981).

REFERENCES

- [1] M. BOFFA, *Sur la théorie des ensembles sans axiome de fondement*, Bull. Soc. Math. Belg., **21** (1969), pp. 16-56.
- [2] M. BOFFA, *Forcing et négation de l'axiome de fondement*, Mem. Acad. Sc. Belg., Tome XL, fasc. 7 (1972).
- [3] U. FELGNER, *Choice functions on sets and classes*, in *Sets and classes; on the work by Paul Bernays* (G. H. Müller, ed.), New York - Amsterdam, 1976, pp. 217-255.
- [4] M. FORTI - F. HONSELL, *Comparison of the axioms of global and local universality*, Zeitschr. für Math. Log. u. Grund. der Math. (1982, to appear).
- [5] M. FORTI - F. HONSELL, *Formalizzazioni del principio di libera costruzione*, in *Atti degli incontri di logica matematica* (C. Bernardi, cur.), Siena 1982, pp. 209-213.
- [6] K. GÖDEL, *The consistency of the axiom of choice and of the generalized continuum hypothesis*, Ann. of Math. Stud., vol. 3, Princeton, 1940.
- [7] P. HÁJEK, *Modelle der Mengenlehre in denen Mengen gegebener Gestalt existieren*, Zeitschr. f. Math. Log. u. Grund. der Math., **11** (1965), pp. 103-115.
- [8] F. HONSELL, *Modelli della teoria degli insiemi, principi di regolarità e di libera costruzione*, Pisa, tesi di laurea (1981).
- [9] A. LEVY, *A hierarchy of formulas in set theory*, Mem. Amer. Math. Soc., **57** (1965).
- [10] L. RIEGER, *A contribution to Gödel's axiomatic set theory*, I, Czech. Math. J., **7** (1957), pp. 323-357.

Dipartimento di Matematica
via Buonarroti, 2
56100 Pisa

Scuola Normale Superiore
56100 Pisa