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Propagation of Singularities for a First Order Semi-Linear System in $\mathbb{C}^{n+1}$.

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Introduction.

We consider the non-characteristic Cauchy problem with singular data in the complex domain. We are interested in the singularities of the solutions in the holomorphic category. This has been intensively studied for linear equations. Few are, however, known for non-linear equations.

Let us first observe the simplest ones:

$$\frac{\partial}{\partial t} u = u^2 , \qquad u|_{t=0} = x^{-k} \to u = (x^k - t)^{-1} , \tag{1}$$

$$\frac{\partial}{\partial t} u = u^3 , \qquad u|_{t=0} = x^{-k} \to u = (x^{2k} - 2t)^{-\frac{1}{2}} . \tag{2}$$

For $k > 0$ the solution u of (1) (resp. (2)) is singular along the non-characteristic surface $\{x^k - t = 0\}$ (resp. $\{x^{2k} - 2t = 0\}$), which differs for each k. That is, singularities propagate along various non-characteristic surfaces issuing from $\{x = 0\}$.

The following example, due to T. Ishii, is also interesting: Let P be a linear differential operator of order m with holomorphic coefficients and $q > 1$ be an integer. Assume that $\mu = -m/(q-1) \notin \mathbb{Z}$ and that $\mu \notin \mathbb{Z}/2$ or m is odd. Then for any non-characteristic surface K: $\varphi = 0$ for P the equation $Pu = u^q$ has a solution of the form $u = \varphi^\mu \times$ holo. ft., which ramifies along K. Therefore, if the initial surface $\{t = 0\}$ is non-characteristic for P, then for any non-characteristic surface K issuing from $\{t = x_1 = 0\}$ there is a solution to the Cauchy problem for $Pu = u^q$ whose singularities propagate along K.

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Thus singularities do not necessarily propagate along characteristics even for semi-linear equations. Then, when do they do so? Note that in the above examples the solutions are unbounded near the non-characteristic surfaces. Furthermore, in (1) and (2) for $-k > 0$, $\notin \mathbb{Z}$ (then the initial data ramify but are bounded) the solutions $u = x^{-k}/(1 - x^{-k}t)$ and $u = x^{-k}/(1 - 2x^{-2k}t)^{\frac{1}{2}}$ are singular only along the characteristic surface $\{x = 0\}$ in a neighborhood of the origin.

Motivated by this, we shall prove in this paper that if the initial data are bounded, then singularities of the solutions propagate (locally) along characteristics for certain first order semi-linear systems with two characteristics in $\mathbf{C}^{n+1}$ where the space dimension $n \geqq 1$.

In Section 1, we state the main result, Theorem 1 and some remarks. In Section 2, we discuss the continuous deformation of contour satisfying certain conditions. When $n = 1$, this is done easily. In Section 3, we prove Theorem 1.

We finally remark that J. Rauch and M. Reed [1], [2] discussed singularities in detail for hyperbolic semi-linear systems in two variables in the real category.

1. – Statements of results.

Let $\Omega \subset \mathbf{C}^{n+1}$ (resp. $\Omega' \subset \mathbf{C}^n$) be an open connected neighborhood of the origin of $\mathbf{C}^{n+1}$ (resp. of $\mathbf{C}^n$). Let $a_i(t, x)$, $b_i(t, x) \in \mathcal{O}(\Omega)$, $i = 1, \ldots, n$ ($t \in \mathbf{C}$, $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$) and

$$X_a = \partial/\partial t + \sum_{i=1}^{n} a_i(\partial/\partial x_i) \qquad \text{and} \qquad X_b = \partial/\partial t + \sum_{i=1}^{n} b_i(\partial/\partial x_i)$$

be holomorphic vector fields on Ω. Here $\mathcal{O}(\Omega)$ is the set of all single-valued holomorphic functions in Ω. For a connected set U we denote by $\mathcal{R}(U)$ the universal covering of U. We identify many-valued holomorphic functions on U with single-valued ones on $\mathcal{R}(U)$.

Given ramified data $u^0, v^0 \in \mathcal{O}\big(\mathcal{R}(\Omega' - \{x_1 = 0\})\big)$, we consider the Cauchy problem for the system

$$(3) \qquad \begin{cases} X_a u = f(t, x, u, v)\,, \\ X_b v = g(t, x, u, v)\,, \end{cases} \qquad (u, v)|_{t=0} = (u^0, v^0)\,,$$

where $f, g \in \mathcal{O}(\Omega \times \mathbf{C}^2)$. Since we consider local problems near the origin, we may assume that $\Omega' = \Omega \cap \{t = 0\}$. Let p^0 be an arbitrary point in $\Omega' - \{x_1 = 0\}$. Fix branches of u^0 and v^0 at p^0. Then, by the Cauchy-

Kovalevskaja Theorem, eq. (3) has a unique solution (u, v) holomorphic near p^0. Our problem is the analytic continuation of the germs u, v at p^0 in a neighborhood of the origin of $\mathbf{C}^{n+1}$.

THEOREM 1. *Assume that*

(A.1) $a_1(0, 0) \neq b_1(0, 0)$,

(A.2) *there is a constant* R_0 *such that*

$$|u_0(\tilde{x})|\,, \quad |v^0(\tilde{x})| \leqq R_0 \qquad \text{for } \forall \tilde{x} \in \mathcal{R}(\Omega' - \{x_1 = 0\})\,. \tag{4}$$

Then there exists an open ball $\omega \in \mathbf{C}^{n+1}$ *centered at the origin such that*

$$u, v \in \mathcal{O}\left(\mathcal{R}(\omega - (K_a \cup K_b))\right). \tag{5}$$

Namely, for an arbitrary point $p^0 \in \omega \cap \{t = 0, x_1 \neq 0\}$ *the germs* u, v *at* p^0 *satisfying* (3) *can be analytically continued to holomorphic functions on* $\mathcal{R}(\omega - (K_a \cup K_b))$. *Here* K_a *and* K_b *are characteristic surfaces issuing from* $\{x_1 = 0\}$. ω *depends on* R_0 *in* (4) *but not on* u^0 *and* v^0.

With an obvious change of notations Theorem 1 is valid for vector-valued u and v. For example, let $n = 1$ and $A = A(t, x)$ be a holomorphic $m \times m$ matrix-valued function on Ω. Consider the Cauchy problem for the unknown vector $w = (w_1, \ldots, w_m)$:

$$\left[\frac{\partial}{\partial t} + A\frac{\partial}{\partial x}\right] w = f(t, x, w)\,, \qquad w(0, x) = w^0(x)\,, \tag{6}$$

where $w^0 = (w_1^0, \ldots, w_m^0)$ and $w_i^0 \in \mathcal{O}\left(\mathcal{R}(\Omega' - \{x = 0\})\right)$ satisfy (A.2). Suppose that (i) $\det(\tau + A) = (\tau + a(t, x))^M (\tau + b(t, x))^N$, (ii) $a(0, 0) \neq b(0, 0)$ and (iii) all the $(m-1)$-th minors of $\tau + A$ are divisible by $(\tau + a)^{M-1} \cdot (\tau + b)^{N-1}$. Then, we can reduce eq. (6) to the diagonal form, eq. (3) with vector-valued u, v where $X_a = \partial/\partial t + a(\partial/\partial x)$ and $X_b = \partial/\partial t + b(\partial/\partial x)$. Hence w_i's are holomorphic on $\mathcal{R}(\omega - (K_a \cup K_b))$ with some ω.

In the above, if (iii) does not hold, then w may be singular along non-characteristic surfaces. For instance, the pair $u = x^{-\frac{1}{4}} \tan(x^{-\frac{1}{4}} t)$, $v = x^{\frac{1}{2}}$ is the solution of

$$\begin{cases} u_t = 2v_x + u^2\,, \\ v_t = 0\,, \end{cases} \qquad (u, v)|_{t=0} = (0, x^{\frac{1}{2}})\,. \tag{7}$$

Obviously u is singular along the non-characteristic surfaces

$$\{x = (\tfrac{1}{2}\pi + k\pi)^{-4} t^4\}\,, \qquad k \in \mathbf{Z}$$

(and $\{x = 0\}$) issuing from $\{x = 0\}$.

We next see hypothesis (A.1). If it is violated, we can not take ω in Theorem 1, in general. Consider the system

$$(8)\qquad \begin{cases} u_t + 2tu_x = 0\,, \\ v_t = 2\,uv^2\,, \end{cases} \qquad (u, v)|_{t=0} = (x^{\frac{1}{2}}, 1)\,.$$

Then

$$u = (x - t^2)^{\frac{1}{2}}\,, \qquad v = \left\{1 - \left(t(x - t^2)^{\frac{1}{2}} + x \arcsin (x^{-\frac{1}{2}} t)\right)\right\}^{-1}$$

is the solution of (8). In a small neighborhood of the origin, we may assume $|t(x - t^2)^{\frac{1}{2}}| \ll 1$. But, however small x may be, we can make the second term arbitrarily large if we go round $t = \pm x^{\frac{1}{2}}$ many times. Hence to avoid the singularities $\{t(x - t^2)^{\frac{1}{2}} + x \arcsin (x^{-\frac{1}{2}} t) = 1\}$ we must shrink ω to 0 as we turn round $x - t^2 = 0$.

The same phenomenon occurs for simple characteristic systems with more than two characteristics. For instance, consider the simple characteristic 3×3 system

$$(9)\qquad \begin{cases} u_t + u_x = 0\,, \\ v_t - v_x = 0\,, \\ w_t = uvw^2\,, \end{cases} \qquad (u, v, w)|_{t=0} = (x^{\frac{1}{2}}, x^{\frac{1}{2}}, 1)\,.$$

The solution of (9) is given by $u = (x - t)^{\frac{1}{2}}$, $v = (x + t)^{\frac{1}{2}}$ and $w = \left(1 - \frac{1}{2} \cdot \left(t(x^2 - t^2)^{\frac{1}{2}} + x^2 \arcsin (x^{-1} t)\right)\right)^{-1}$. By the same reason as above, ω shrinks to 0 as we turn round $t \pm x = 0$.

We remark that the obstruction in the first case is different from that in the last two, where, though we can not take ω, it seems that singularities essentially propagate along characteristics.

2. – Deformation of contours in characteristic curves.

We put $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. In what follows μ denotes a or b. Let $x_\mu(t, y)$ $(\mu = a, b)$ be the solution of the canonical system

$$(10)\qquad \frac{dx}{dt} = \mu(t, x)\,, \qquad x(0, y) = y\,,$$

and $y_\mu(t, x)$ be the solution of the eiconal equation

$$(11)\qquad \frac{\partial y}{\partial t} + \langle \mu, \operatorname{grad}_x y \rangle = 0\,, \qquad y(0, x) = x\,.$$

Then we have

$$x_\mu(t, y_\mu(t, x)) = x , \qquad y_\mu(t, x_\mu(t, y)) = y . \tag{12}$$

Letting Ω, Ω' and r_0 be small, if necessary, we may assume that

$$x_\mu \in \mathcal{O}(\{|t| < r_0\} \times \Omega') , \qquad y_\mu \in \mathcal{O}(\Omega) , \tag{13}$$

$$y_\mu(t, x) \in \Omega' \quad \text{and} \quad |t| < r_0 \qquad \text{for } \forall (t, x) \in \Omega . \tag{14}$$

We put for $p \in \Omega$

$$\Gamma_\mu(p) = \{(t, x) \in \Omega \colon y_\mu(t, x) = y_\mu(p)\} , \tag{15}$$

that is, $\Gamma_\mu(p)$ is the (complex) μ-characteristic curve through p. If $p = (0, y)$, $y \in \Omega'$ (we have assumed $\Omega' = \Omega \cap \{t = 0\}$) we shall write $\Gamma_\mu(y)$ instead of $\Gamma_\mu((0, y))$. We denote, for simplicity, by h_μ the first component of y_μ, that is, h_μ is the solution of

$$\frac{\partial h_\mu}{\partial t} + \langle \mu, \ \mathrm{grad}_x h_\mu \rangle = 0 , \qquad h_\mu(0, x) = x_1 . \tag{16}$$

Put for $z \in \mathbf{C}$

$$K_\mu(z) = \{(t, x) \in \Omega \colon h_\mu(t, x) = z\} , \tag{17}$$

which is a (complex) regular hypersurface in Ω for sufficiently small $z \in \mathbf{C}$. K_μ in Theorem 1 is just $K_\mu(0)$.

LEMMA 1. *Give $y \in \mathbf{C}^n$ and $z \in \mathbf{C}$ and consider the equation for the unknown scalar t:*

$$h_b(t, x_a(t, y)) = z \qquad \big(\text{resp. } h_a(t, x_b(t, y)) = z\big) . \tag{18}$$

Then there exist positive constants r, ϱ_0, C_0 with $C_0 \varrho_0 < r$ such that for any $|z| < \varrho_0$, $|y| < \varrho_0$, eq. (18) *has a unique solution $t = t_a(z, y)$ (resp. $t_b(z, y)$) in $|t| < r$ satisfying*

$$t_\mu \in \mathcal{O}(\{|z| < \varrho_0, \ |y| < \varrho_0\}) , \tag{19}$$

$$\sup\{|t_\mu(z, y)| \colon |z| \leqq \varrho, \ |y| \leqq \varrho\} < C_0 \varrho (< r) \qquad \text{for } 0 < \forall \varrho < \varrho_0 . \tag{20}$$

Here C_0 is independent of ϱ.

REMARK. Letting r and ϱ_0 be sufficiently small, we may assume that the set $\{(t, x_\mu(t, y)) : |t| < r, |y| < \varrho_0\}$ is contained in Ω.

PROOF. From (10) and (16) it follows that

$$\frac{\partial}{\partial t} h_b(t, x_a(t, y))|_{t=0} = \langle a(t, x_a) - b(t, x_a x_a), \operatorname{grad}_x b_a\rangle|_{t=0} = a_1(0, y) - b_1(0, y.)$$

$$\frac{\partial}{\partial t} h_a(t, x_b(t, y))|_{t=0} = \dots = b_1(0, y) - a_1(0, y) .$$

By (A.1) the last terms do not vanish at $y = 0$, and $t = 0$ satisfies (18) for $(z, y) = (0, 0)$. Therefore the implicit function theorem implies (19). (20) is also clear, because t_μ is Lipschitz continuous. ■

We put

$$p_\mu(z, y) = \big(t_\mu(z, y),\ x_\mu(t_\mu(z, y), y)\big) , \tag{21}$$

which are holomorphic in $\{|z| < \varrho_0, |y| < \varrho_0\}$. Then in $\{(t, x): |t| < r\}$, $\Gamma_a(y)$ (resp $\Gamma_b(y)$) intersects $K_b(z)$ (resp. $K_a(z)$) only at $p_a(z, y)$ (resp. $p_b(z, y)$). Therefore for $|z| < \varrho_0$, $|y| < \varrho_0$ we have

$$\begin{cases} (t, x) = p_a(z, y) & \text{iff } |t| < r ,\ (t, x) \in K_b(z) \cap \Gamma_a(y) , \\ (t, x) = p_b(z, y) & \text{iff } |t| < r ,\ (t, x) \in K_a(z) \cap \Gamma_b(y) . \end{cases} \tag{22}$$

For ϱ, $0 < \varrho < \varrho_0$, we take an open ball ω sufficiently small so that

$$|y_\mu(t, x)| \leqq \varrho , \quad |t| \leqq C_0 \varrho \qquad \text{for } \forall (t, x) \in \omega , \tag{23}$$

where C_0 is the constant in (20). Later we will make ϱ smaller a finite number of times. Accordingly we must shrink ω to satisfy (23).

Let $\gamma \in C^0(I; \omega - (K_a \cup K_b))$ be a continuous path in $\omega - (K_a \cup K_b)$ starting from $\gamma(0) = p^0 \in \omega \cap \{t = 0, x_1 \neq 0\}$ where I is the closed unit interval. Let γ^0 be the t-component of γ. Then by (23) we have

$$\gamma^0 \in C^0(I; \bar{D}(C_0 \varrho)) , \qquad \gamma^0(0) = 0 , \tag{24}$$

where $\bar{D}(C_0 \varrho)$ is the closed disc in $\mathbb{C}$ of radius $C_0 \varrho$. By (23) $|y_a(\gamma(s))| < \varrho$, this and (20) imply that $t_a\big(0, y_a(\gamma(\cdot))\big)$ is a path in the open disc $D(C_0 \varrho)$:

$$t_a\big(0, y_a(\gamma(\cdot)\big) \in C^0 (I; D(C_0 \varrho)) . \tag{25}$$

If $t_a\big(z, y_a(\gamma(s))\big) = 0$ (resp. $= \gamma^0(s)$), then

$$z = h_b\Big(0, x_a\big(0, y_a(\gamma(s))\big)\Big) = h_b\big(0, y_a(\gamma(s))\big) = h_a(\gamma(s))$$

$$\Big(\text{resp. } z = h_b\Big(\gamma_0(s), x_a\big(\gamma_0(s), y_a(\gamma(s))\big)\Big) = h_b(\gamma(s))\Big).$$

Therefore $\gamma(s) \notin K_a \cup K_b$ implies that

$$t_a\big(0, y_a(\gamma(s))\big) \neq 0\,, \quad \gamma^0(s)\,, \ \forall s \in I. \tag{26}$$

In view of (24), (25) and (26), we can find a family of smooth paths $\tau_s \in C^\infty(I; \bar{D}(C_0\varrho))$, $s \in I$, a continuous deformation of the constant map 0 along γ^0 avoiding $t_a\big(0, y_a(\gamma(s))\big)$, that is,

$$\text{(27)} \quad \begin{cases} (a) & \tau_0 \equiv 0\,, \\ (b) & \tau_s(0) = \gamma^0(s)\,, \qquad \tau_s(1) = 0\,, \qquad \forall s \in I\,, \\ (c) & \tau_s \in C^\infty\Big(I; \bar{D}(C_0\varrho) - \Big\{t = t_a\big(0, (\gamma(s))\big)\Big\}\Big)\,, \qquad \forall s \in I\,, \\ (d) & \tau = \tau_\bullet(\cdot) \in C^0(I^2)\,. \end{cases}$$

Further, since τ_s is a path in $\bar{D}(C_0\varrho) - \{\text{one point}\}$, we can make the length of τ_s shorter than $4C_0\varrho$. Hence modifying the arc length parameter, we may assume that

$$\left|\frac{\partial}{\partial s_1}\tau_{s_0}(s_1)\right| \leqq 4C_0\varrho\,, \qquad \forall (s_0, s_1) \in I^2\,. \tag{28}$$

Then it is clear that the family of contours, α_s, $s \in I$, defined by

$$\alpha_s(\cdot) = \alpha(s, \cdot) = \Big(\tau_s(\cdot), x_a\big(\tau_s(\cdot), y_a(\gamma(s))\big)\Big) \tag{29}$$

satisfy

$$\text{(30)} \quad \begin{cases} (a) & \alpha_0 \equiv p^0\,, \\ (b) & \alpha_s(0) = \gamma(s)\,, \quad \alpha_s(1) = \big(0, y_a(\gamma(s))\big)\,, \qquad \forall s \in I\,, \\ (c) & \alpha_s \in C^\infty\big(I; \Gamma_a(\gamma(s)) - (K_a \cup K_b)\big)\,, \qquad \forall s \in I\,, \\ (d) & \alpha = \alpha_\bullet(\cdot) \in C^0(I^2)\,. \end{cases}$$

Since y_μ, x_μ and t_μ are holomorphic, we can find a constant $C_1 > 0$ such that

$$(31) \quad \begin{cases} (a) \quad \|\mathrm{grad}_{t,x}\, y_\mu(t, x)\| \leqq C_1\,, & \forall (t, x) \in \Omega\,, \\ (b) \quad \left|\left(1, \dfrac{\partial}{\partial t} x_\mu(t, y)\right)\right| \leqq C_1\,, & \forall |t| < r_0\,,\ y \in \Omega'\,, \\ (c) \quad \left|\dfrac{\partial}{\partial z} t_\mu(z, y)\right| \leqq C_1\,, & \forall |z| < \varrho_0\,,\ |y| < \varrho_0\,. \end{cases}$$

LEMMA 2. *Suppose that* (28) *and* (31) *hold. Then we have*

$$(32) \qquad \left|\frac{\partial \alpha}{\partial s_1}(s_0, s_1)\right| \leqq 4 C_0 C_1 \varrho\,,$$

$$(33) \qquad |y_b(\alpha(s_0, s_1)) - y_b(\gamma(s_0))| \leqq 4 C_0 C_1^2 \varrho s_1\,,$$

in particular,

$$(34) \qquad |h_b(\alpha(s_0, s_1))| \leqq \varrho(1 + 4 C_0 C_1^2)\,.$$

PROOF. (32) easily follows from (28), (29) and (31-b).

Since $\alpha(s, 0) = \gamma(s)$, using the integral formula, we have

$$y_b(\alpha(s_0, s_1)) - y_b(\gamma(s_0)) = \int_0^{s_1} \left\langle \mathrm{grad}_{t,x}\, y_b, \frac{\partial \alpha}{\partial s_1} \right\rangle ds_1\,.$$

Now (31-a) and (32) imply (33).

By (23), $|y_b(\gamma)| \leqq \varrho$, hence

$$|h_b(\alpha)| \leqq |y_b(\alpha)| \leqq |y_b(\alpha) - y_b(\gamma)| + |y_b(\gamma)| \leqq 4 C_0 C_1^2 \varrho + \varrho\,,$$

which proves (34). ■

We take $\varrho > 0$ so that $\varrho < \varrho_0/(1 + 4C_0 C_1^2)$. Then we define a family of smooth contours β_s, $s \in I$, by

$$(35) \qquad \beta_{s_0}(s_1) = \beta(s_0, s_1) = p_b\big(h_b(\alpha(s_0, 1 - s_1)), y_b(\gamma(s_0))\big)\,,$$

that is, $\beta(s_0, s_1)$ is the intersection of $\Gamma_b(\gamma(s_0))$ and $K_a\big(h_b(\alpha(s_0, 1 - s_1))\big)$. Since $|y_b(\gamma)| \leqq \varrho < \varrho_0$, $|h_b(\alpha)| < \varrho_0$ by (34) and $p_b(z, y) \in \mathcal{O}(\{|z| < \varrho_0, |y| < \varrho_0\})$, (35) is well-defined.

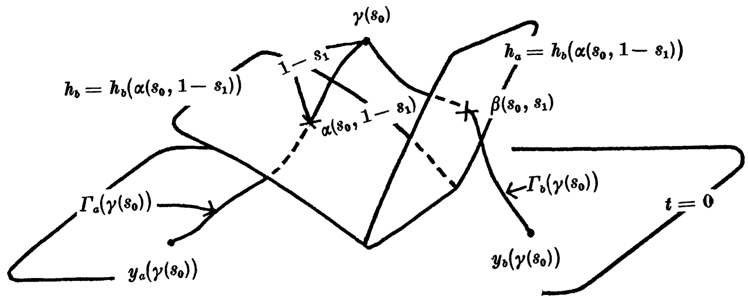

Figure 1

Note that

$$h_a(\beta(s_0, s_1)) = h_b(\alpha(s_0, 1-s_1)) \ . \tag{36}$$

LEMMA 3. *The family of smooth contours β_s, $s \in I$, satisfy*

$$\begin{cases} (a) & \beta_0 \equiv p_0 \, , \\ (b) & \beta_s(0) = \gamma(s) \, , \quad \beta_s(1) = \big(0, y_b(\gamma(s))\big) \, , \qquad \forall s \in I \, , \\ (c) & \beta_s \in C^\infty\big(I; \Gamma_b(\gamma(s)) - (K_a \cup K_b)\big) \, , \qquad \forall s \in I \, , \\ (d) & \beta = \beta_\bullet(\cdot) \in C^0(I^2) \, , \end{cases} \tag{37}$$

$$\left| \frac{\partial}{\partial s_1} \beta(s_0, s_1) \right| \leqq 4C_0 C_1^4 \varrho \, , \tag{38}$$

$$|y_a\big(\beta(s_0, s_1) - y_a(\gamma(s_0))\big)| \leqq 4C_0 \, C_1^5 \varrho s_1 \, . \tag{39}$$

PROFF. To prove (37-*a*), it is enough to show $p^0 \in K_a(h_b(p^0)) \cap \Gamma_b(\gamma(0))$ because of (22) and $\alpha_0 \equiv p^0$. Obviously $p^0 \in \Gamma_b(\gamma(0)) = \Gamma_b(p^0)$. Recall that $h_a = h_b = x_1$ on $t = 0$ and that $p^0 \in \{t = 0\}$. Therefore $h_b(p^0) = h_a(p^0)$, which shows $p^0 \in K_a(h_b(p^0))$.

By (30-*b*) we have

$$h_b(\alpha(s, 0)) = h_b(\gamma(s)) = h_a\big(0, y_b(\gamma(s))\big)$$

$$\big(\text{resp. } h_b(\alpha(s, 1)) = h_b\big(0, \gamma_a(\gamma(s))\big) = h_a(\gamma(s))\big) \, ,$$

hence

$$\big(0, y_b(\gamma(s))\big) \in K_a\big(h_b(\alpha(s, 0))\big) \ \Big(\text{resp. } \gamma(s) \in K_a\big(h_b(\alpha(s, 1))\big)\Big) \cap \Gamma_b(\gamma(s)) \, .$$

This proves (37-b).

From the definition of β, we can easily see (37-c, d).

We differentiate β using (21) and (35):

$$\frac{\partial\beta}{\partial s_1} = -\frac{\partial t_b}{\partial z}\left\langle \mathrm{grad}_{t,x} h_b, \frac{\partial\alpha}{\partial s_1}\right\rangle\left(1, \frac{\partial x_b}{\partial t}\right).$$

This equality, (31) and (32) yield (38).

We can prove (39) in the same way as (33). ■

We introduce the following notation:

$$(40)\quad \begin{cases} \sigma_k = (s_0, s_0, s_1, \dots, s_k)\,, \\ \bar{\sigma}_k = s_1 + s_2 + \dots + s_k\,, \qquad (\bar{\sigma}_0 = 0)\,, \qquad k = 0, 1, 2, \dots\,, \\ \tilde{\sigma}_k = s_k + s_{k-2} + \dots + \begin{cases} s_1 & k\text{: odd}\,, \\ s_2 & k\text{: even}\,, \end{cases} \qquad (\tilde{\sigma}_0 = 0)\,. \end{cases}$$

We now define inductively a countable number of maps $\{\alpha^k(\sigma_k), \beta^k(\sigma_k)\}_{k-1}^{\infty}$, $0 \leqq s_i \leqq 1$, $\bar{\sigma}_k \leqq 1$, as follows:

$$(41)\quad \begin{cases} \alpha^1(s_0, s_1) = \alpha(s_0, s_1)\,, \qquad \beta^1(s_0, s_1) = \beta(s_0, s_1)\,, \\ \alpha^{k+1}(\sigma_{k+1}) = p_a\big(h_b(s_0, \tilde{\sigma}_{k+1})), y_a(\beta^k(\sigma_k))\big)\,, \\ \beta^{k+1}(\sigma_{k+1}) = p_b\big(h_a(\beta(s_0, \tilde{\sigma}_{k+1})), y_b(\alpha^k(\sigma_k))\big)\,. \end{cases}$$

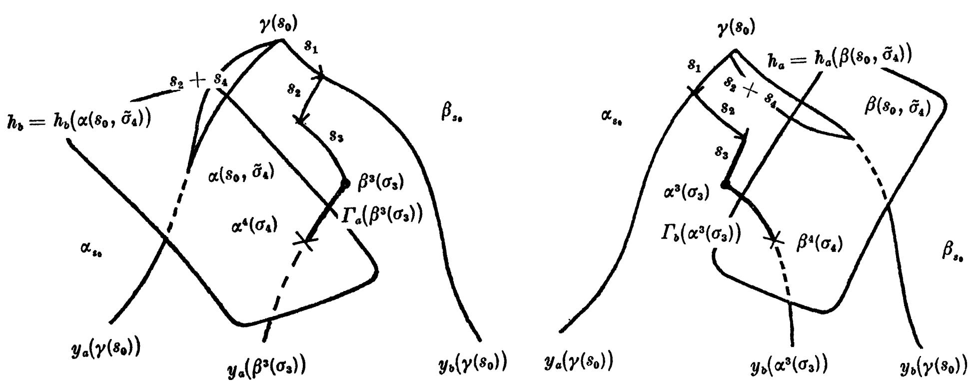

Figure 2

LEMMA 4. *Let* $C_3 = \max(4C_0C_1, 4C_0C_1^7)$ *and* ϱ *be*

$$(42)\qquad 0 < \varrho < \varrho_0/(1 + C_1C_3)\,.$$

Then α^k and β^k are well-defined by (41), *and satisfy the following for* $k=1, 2, 3, \dots$:

$$(43)\quad \begin{cases} (a) & \begin{cases} \alpha^k(\sigma_{k-1}, \cdot) \in C^\infty\big([0, 1-\tilde\sigma_{k-1}];\ \Gamma_a(\beta^{k-1}(\sigma_{k-1})) - (K_a \cup K_b)\big)\,, \\ \beta^k(\sigma_{k-1}, \cdot) \in C^\infty\big([0, 1-\tilde\sigma_{k-1}];\ \Gamma_b(\alpha^{k-1}(\sigma_{k-1})) - (K_a \cup K_b)\big)\,, \end{cases} \\ (b) & \begin{cases} \alpha^k(\sigma_{k-1}, 0) = \beta^{k-1}(\sigma_{k-1})\,, \\ \beta^k(\sigma_{k-1}, 0) = \alpha^{k-1}(\sigma_{k-1})\,, \end{cases} \\ (c) & \begin{cases} \alpha^k(\sigma_{k-1}, 1-\tilde\sigma_{k-1}) = \big(0, y_a(\beta^{k-1}(\sigma_{k-1}))\big)\,, \\ \beta^k(\sigma_{k-1}, 1-\tilde\sigma_{k-1}) = \big(0, y_b(\alpha^{k-1}(\sigma_{k-1}))\big)\,, \end{cases} \end{cases}$$

$$(44)\qquad \left|\frac{\partial}{\partial s_k}\alpha^k(\sigma_k)\right|, \qquad \left|\frac{\partial}{\partial s_k}\beta^k(\sigma_k)\right| \leqq C_3\varrho\,,$$

$$(45)\qquad |y_b(\alpha^k(\sigma_k)) - y_b(\gamma(s_0))|\,, \quad |y_a(\beta^k(\sigma_k)) - y_a(\gamma(s_0))| \leqq C_1C_3\varrho\tilde\sigma_k\,,$$

where we have put $\alpha^0 = \beta^0 = \gamma$.

PROOF. We prove Lemma 4 by induction on k. For $k=1$ we have already done it (see Lemmas 2, 3 and (30)). Let us assume that the assertions hold for $k = 1, 2, \dots l$. Then (42) and (45) imply

$$|y_b(\alpha^l(\sigma_l))|,\ |y_a(\beta^l(\sigma_l))| < \varrho_0\,.$$

Especially for $k=1$ we have $|h_b(\alpha)|$, $|h_a(\beta)| < \varrho_0$. Therefore α^{l+1} and β^{l+1} are well-defined. We can easily see (43-a) and (44). So we verify (43-c) for α^{l+1}. Let $\sigma_{l+1} = (\sigma_l, 1-\tilde\sigma_l)$. Then $\tilde\sigma_{l+1} = 1-\tilde\sigma_l$. Using this equality and (36) we find

$$h_b(\alpha(s_0, \tilde\sigma_{l+1})) = h_b(\alpha(s_0, 1-\tilde\sigma_l)) = h_a(\beta(s_0, \tilde\sigma_l)) = h_a(\beta^l(\sigma_l)) = h_b\big(0, y_a(\beta^l(\sigma_l))\big)\,.$$

Therefore

$$\big(0, y_a(\beta^l(\sigma_l))\big) \in K_b\big(h_b(\alpha(s_0, \tilde\sigma_{l+1}))\big) \qquad \big(\cap\, \Gamma_a(\beta^l(\sigma_l))\big)\,,$$

which shows $\alpha^{l+1}(\sigma_l, 1-\tilde\sigma_l) = \big(0, y_a(\beta^l(\sigma_l))\big)$.

Analogously we can verify (43-c) for β^{l+1} and (43-b). Finally we see (45). By (43-b) we find

$$y_b(\alpha^{l+1}(\sigma_l, 0)) = y_b(\beta^l(\sigma_l)) = y_b(\beta^l(\sigma_{l-1}, 0)) = y_b(\alpha^{l-1}(\sigma_{l-1}))\,,$$

hence

$$|y_b(\alpha^{l+1}(\sigma_{l+1})) - y_b(\gamma(s_0))|$$
$$\leq |y_b(\alpha^{l+1}(\sigma_{l+1})) - y_b(\alpha^{l+1}(\sigma_l, 0))| + |y_b(\alpha^{l-1}(\sigma_{l-1})) - y_b(\gamma(s_0))| .$$

The first term on the right is estimated by $C_1 C_3 \varrho s_{l+1}$ and the second term by $C_1 C_3 \varrho \tilde{\sigma}_{l-1}$, hence the left side is estimated by $C_1 C_3 \varrho \tilde{\sigma}_{l+1}$. The estimate for β^{l+1} is exactly the same.

Thus we have completed the induction. ∎

3. – Proof of Theorem 1.

We solve eq. (3) by successive approximation:

$$\begin{cases} X_a u_{j+1} = f(t, x, u_j, v_j) , \\ X_b v_{j+1} = g(t, x, u_j, v_j) , \end{cases} \qquad (u_{j+1}, v_{j+1})|_{t=0} = (u^0, v^0) , \ j = 0, 1, 2, \ldots , \tag{46}$$

$$\begin{cases} X_a u_0 = 0 , \\ X_b u_0 = 0 , \end{cases} \qquad (u_0, v_0)|_{t=0} = (u^0, v^0) . \tag{47}$$

The solution of (47) is given by

$$u_0(t, x) = u^0(y_a(t, x)) , \qquad v_0(t, x) = v^0(y_b(t, x)) . \tag{48}$$

Since for any path γ in $\Omega - (K_a \cup K_b)$, $y_\mu \circ \gamma$ are paths in $\Omega' - \{x_1 = 0\}$, u_0 and v_0 can be analytically continued to holomorphic functions on

$$\mathfrak{R}(\Omega - (K_a \cup K_b)) .$$

Let p^0 be an arbitrary point in $\Omega' - \{x_1 = 0\}$. Then by the Cauchy-Kovalevskaja Theorem we obtain successively u_j, v_j, $j = 1, 2, \ldots$, which are holomorphic near p^0. They are expressed also by integrals on characteristic curves:

$$\begin{cases} u_{j+1}(t, x) = u_0(t, x) + \displaystyle\int_{(\tau,\xi)=\hat{\alpha}} f(\tau, \xi, u_j(\tau, \xi), v_j(\tau, \xi))\, d\tau , \\ v_{j+1}(t, x) = v_0(t, x) + \displaystyle\int_{(\tau,\xi)=\hat{\beta}} g(\tau, \xi, u_j(\tau, \xi), v_j(\tau, \xi))\, d\tau , \end{cases} \tag{49}$$

where $\hat{\alpha}$ (resp. $\hat{\beta}$) is a contour joining $(0, y_a(t, x))$ $\big($resp. $(0, y_b(t, x))\big)$ and (t, x) in $\Gamma_a(t, x)$ (resp. $\Gamma_b(t, x)$), and on $\hat{\alpha}$ and $\hat{\beta}$ u_j and v_j are holomorphic. If (t, x) is sufficiently close to p^0, we can take as $\hat{\alpha}$ or $\hat{\beta}$ the line $\big(st, x_\mu(st, y_\mu(t, x))\big)$, $s \in I$.

Let γ be a path in Ω starting from p^0. If there exists a family of smooth contours $\hat{\alpha}_s$ such that

$$(50)\quad \begin{cases} (a) & \hat{\alpha}_0 \equiv p^0\,, \\ (b) & \hat{\alpha}_s \text{ connects } \big(0, y_a(\gamma(s))\big) \text{ and } \gamma(s) \text{ in } \Gamma_a(\gamma(s)), \\ (c) & \hat{\alpha}_s \text{ depends on } s \text{ continuously}, \\ (d) & u_j \text{ and } v_j \text{ are analytically continuable along the path } \hat{\alpha}_s^{-1} * \gamma|_s, \\ & \qquad\qquad \text{for every } s \in I\,, \end{cases}$$

then we can continue u_{j+1} along γ to the terminal point. Here $\hat{\alpha}_s^{-1}$ is the inverse path of $\hat{\alpha}_s$, $\gamma|_s$ is the restriction of γ on $[0, s]$ and $*$ stands for the path product. The same assertion is true for v_{j+1} if we replace a with b in (50).

Let ω be as in Section 2 and $\gamma \in C^0(I; \omega - (K_a \cup K_b))$ with $\gamma(0) = p^0 \in \omega \cap \{t = 0, x_1 \neq 0\}$. Let α^k, β^k, $k = 1, 2, \ldots$, be the maps in Lemma 4. Let $\sigma = (s_0, s_1, s_2, \ldots)$ be a sequence of positive numbers satisfying

$$(51)\quad \begin{cases} (a) & 0 \leqq s_i \leqq 1\,, \qquad i = 0, 1, 2, \ldots\,, \\ (b) & \text{all but a finite number of } s_i \text{ are equal to } 0\,, \\ (c) & \sum_{i=1}^{\infty} s_i = 1\,. \end{cases}$$

Then for each $\sigma = (s_0, s_1, \ldots)$ satisfying (51) we define a path $\gamma_\sigma(s)$, $0 \leqq s \leqq s_0 + 1$, as follows:

$$(52)\qquad \gamma_\sigma = \cdots * \alpha^3(\sigma_2, \cdot)|_{s_3} * \beta^2(\sigma_1, \cdot)|_{s_2} * \alpha^1(s_0, \cdot)|_{s_1} * \gamma|_{s_0}\,,$$

that is,

$$(53)\qquad \gamma_\sigma(s) = \begin{cases} \gamma(s)\,, & 0 \leqq s \leqq s_0\,, \\ \alpha^{k+1}(\sigma_k, s - \hat{\sigma}_k)\,, & \hat{\sigma}_k \leqq s \leqq \hat{\sigma}_{k+1}\,, \qquad k\colon \text{even}\,, \\ \beta^{k+1}(\sigma_k, s - \hat{\sigma}_k)\,, & \hat{\sigma}_k \leqq s \leqq \hat{\sigma}_{k+1}\,, \qquad k\colon \text{odd}\,, \end{cases}$$

where $\sigma_k = (s_0, s_1, \ldots, s_k)$ and $\hat{\sigma}_k = s_0 + s_1 + \cdots + s_k$.

Lemma 5. *Let γ_σ be as above. Then for any σ satisfying* (51), *u_j and v_j, $j=0, 1, 2, \dots$, are all analytically continuable along γ_σ.*

Proof. We prove the lemma by induction on j. Since γ_σ are paths in $\Omega-(K_a \cup K_b)$, the assertion is true for $j=0$. Let us assume that the assertion holds for $j=0, 1, \dots, l$. To see u_{l+1} and v_{l+1} are analytically continuable along γ_σ, it is enough to construct deformations along γ_σ satisfying (50). Define a family of contours $\hat{\alpha}_s$ with parameter s as follows:

$$(54) \qquad \hat{\alpha}_s(\eta)=\begin{cases} \alpha^1(s, 1-\eta), \quad \eta \in [0, 1], & \text{if } 0 \leqq s \leqq s_0, \\ \alpha^{k+1}(\sigma_k, 1-\bar{\sigma}_k-\eta), \quad \eta \in [0, 1+s_0-s], & \\ \qquad\qquad \text{if } \hat{\sigma}_k \leqq s \leqq \hat{\sigma}_{k+1}, \ k\text{: even}, & \\ \alpha^{k+2}(\sigma_k, s-\hat{\sigma}_k, 1+s_0-s-\eta), \quad \eta \in [0, 1+s_0-s], & \\ \qquad\qquad \text{if } \hat{\sigma}_k \leqq s \leqq \hat{\sigma}_{k+1}, \ k\text{: odd}. & \end{cases}$$

Then it easily follows from Lemma 3 that $\hat{\alpha}_s$ satisfies (50-a, b, c). Since for each s there is a sequence $\tilde{\sigma}$ satisfying (51) such that $\hat{\alpha}_s^{-1} * \gamma_\sigma|_s = \gamma_{\tilde{\sigma}}$, the inductive hypothesis assures (50-d).

For v_{l+1} the deformation

$$(55) \qquad \hat{\beta}_s(\eta)=\begin{cases} \beta^1(s, 1-\eta), \quad \eta \in [0, 1], & \text{if } 0 \leqq s \leqq s_0, \\ \beta^{k+2}(\sigma_k, s-\hat{\sigma}_k, 1+s_0-s-\eta), \quad \eta \in [0, 1+s_0-s], & \\ \qquad\qquad \text{if } \hat{\sigma}_k \leqq s \leqq \hat{\sigma}_{k+1}, \ k\text{: even}, & \\ \beta^{k+1}(\sigma_k, 1-\bar{\sigma}_k-\eta), \quad \eta \in [0, 1+s_0-s], & \\ \qquad\qquad \text{if } \hat{\sigma}_k \leqq s \leqq \hat{\sigma}_{k+1}, \ k\text{: odd}, & \end{cases}$$

satisfies (50) (where a is replaced by b).

Thus we have completed the proof. ∎

Fix a positive number R larger than R_0 in (A.2). Let M and L be the maximum and the Lipschitz constant, respectively, on $\Omega \times \{|u|, |v| \leqq R\}$:

$$(56) \qquad |\varphi(t, x, u, v)| \leqq M,$$

$$(57) \qquad |\varphi(t, x, u, v)-\varphi(t, x, u', v')| \leqq L(|u-u'|+|v-v'|),$$

where $\varphi=f, g$.

LEMMA 6. *Let* ϱ *be*

$$0 < \varrho \leqq (R - R_0)/C_3 M .$$

Then for all γ_σ *and* $j = 0, 1, 2, \ldots$, *we have*

$$|u_j(\gamma_\sigma(s))| , \qquad |v_j(\gamma_\sigma(s))| \leqq R . \tag{58}$$

PROOF. Induction on j. For $j = 0$, (A.2) implies (58). Note that (44) implies

$$\left| \frac{\partial}{\partial \eta} (t^k \text{component of } \hat{\alpha}_s(\eta)) \right| \leqq C_3 \varrho .$$

Then (49) with (56) yield

$$\begin{aligned} |u_{j+1}(\gamma_\sigma(s))| &\leqq |u_0(\gamma_\sigma(s))| + \int_{(\tau,\xi)=\hat{\alpha}_s(\eta)} |f(\tau, \xi, u_j, v_j)| \, |d\tau| \\ &\leqq R_0 + M \int_{\alpha_s(\eta)} |d\tau| \leqq R_0 + M C_3 \varrho \delta \leqq R , \end{aligned} \tag{59}$$

where $\delta = 1$ if $s \leqq s_0$ and $\delta = 1 + s_0 - s$ if $s \geqq s_0$.
Similarly we can obtain the estimate for v_{j+1}. ■

LEMMA 7. *For all* γ_σ *and* $j = 1, 2, 3, \ldots$, *the following estimates hold*:

$$|w_j(\gamma_\sigma(s)) - w_{j-1}(\gamma_\sigma(s))| \leqq \frac{M}{2L} (2LC_3\varrho)^j (1 + s_0 - s)^j / j! , \tag{60}$$

where $w_j = u_j, v_j$, $j = 0, 1, 2, \ldots$.

PROOF. We prove the lemma by induction on j. For $j = 0$ see (59). Suppose (60) is true for $1, 2, \ldots, j$. Then

$$|u_{j+1}(\gamma_\sigma(s)) - u_j(\gamma_\sigma(s))| \leqq \int_{\hat{\alpha}_s} |f(u_j, v_j) - f(u_{j-1}, v_{j-1})| \, |d\tau| .$$

By Lemma 6 we can use (57), hence

$$\leqq L \int_{\hat{\alpha}_s} |u_j(\hat{\alpha}_s(\eta)) - u_{j-1}(\hat{\alpha}_s(\eta))| + |v_j(\hat{\alpha}_s(\eta)) - v_{j-1}(\hat{\alpha}_s(\eta))| \, |d\tau| .$$

Since $\hat{\alpha}_s(\eta) = \gamma_{\tilde{\sigma}}(1 + \tilde{s}_0 - \eta)$ with some $\tilde{\sigma} = (\tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \ldots)$, the inductive hypothesis implies

$$\leqq L\int_0^{1+s_0-s} \frac{M}{L}(2LC_3\varrho)^j(\eta^j/j!)\,C_3\varrho\,d\eta = \frac{M}{2L}(2LC_3\varrho)^{j+1}(1+s_0-s)^{j+1}/(j+1)!\,.$$

The estimate for v_{j+1} is exactly the same. Thus we have completed the proof. ■

REMARK. The essential point of the proof is that we can take the constant C_3 in (44) which is independent of k and ϱ.

The constants M, K, C_3 and ϱ are independent of γ in

$$C^0(I;\,\omega-(K_a\cup K_b))\,.$$

Therefore for any point $\tilde{p}$ in $\mathfrak{R}(\omega-(K_a\cup K_b))$ we have

$$|w_j(\tilde{p})-w_{j-1}(\tilde{p})|\leqq\frac{M}{2L}(2LC_3\varrho)^j/j!\,.$$

We write

$$\operatorname*{Lim}_{j\to\infty} w_j(\tilde{p}) = \sum_{j=1}^{\infty}\big(w_j(\tilde{p})-w_{j-1}(\tilde{p})\big)+w_0(\tilde{p})\,.$$

The right side convergence absolutely and uniformly on $\mathfrak{R}(\omega-(K_a\cup K_b))$, hence the limit functions u, v, the solution of (3), are holomorphic on $\mathfrak{R}(\omega-(K_a\cup K_b))$. Furthermore, ω depends on ϱ, and ϱ on C_3, M, R_0, R but not on u^0 and v^0.

Thus we have completed the proof of Theorem 1.

Lastly we remark that if the space dimension n is equal to one, then $\Gamma_\mu(p)$ coincides with $K_\mu(h_\mu(p))$, hence for any (s_0, s_1), $\alpha^2(s_0,\cdot,s_1)$ is a contour in $\Gamma_b(\alpha(s_0,s_1))$ (of course $\alpha^2(s_0,s_1,\cdot)$ is a contour in $L_a(\beta(s_0,s_1))$). So we can prove Theorem 1 using only α^2.

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