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An Estimate of the Gap of the First Two Eigenvalues in the Schrödinger Operator.

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1. Introduction.

We shall consider the following Dirichlet eigenvalue problem on a smooth bounded domain $\Omega \subset \mathbb{R}^n$,

$$
\begin{align*}
-\Delta u + Vu &= \lambda u \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where $V$ is a nonnegative function defined on $\Omega$. As is well-known, the eigenvalues of problem (1.1) can be interpreted as the energy levels of a particle travelling under an external force field of a potential $q$ in $\mathbb{R}^n$, where

$$
q(x) = \begin{cases}
V(x) & x \in \Omega \\
+\infty & x \notin \Omega
\end{cases}
$$

and the corresponding eigenfunctions are wave functions of the Schrödinger equation $-\Delta u + qu = \lambda u$. Furthermore, the set of eigenvalues $\{\lambda_n\}$ of (1.1) are nonnegative and can be arranged in a nondecreasing order as follows,

$$
0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots
$$

It is a significant problem to find a lower bound for $\lambda_1$ in terms of the geometry of $\Omega$. This subject has been studied extensively by many authors. A rather precise bound in the case $V \equiv 0$ was worked out not only for a bounded domain in $\mathbb{R}^n$ but actually valid for a general Riemannian manifold with certain curvature conditions; we refer to [4] for these recent developments. Nevertheless, very little is known about the obvious interesting question of how big the gap is between $\lambda_2$ and $\lambda_1$. There are both physical...
and mathematical interests in finding out a lower bound for $\lambda_2 - \lambda_1$ in terms of the geometrical invariants of $\Omega$ and the given potential function $V$. Our main result is the following.

**Theorem (1.1).** Let $\Omega$ be a smooth convex bounded domain in $\mathbb{R}^n$ and $V: \overline{\Omega} \rightarrow \mathbb{R}$ a nonnegative convex smooth potential function.

Suppose $\lambda_2$ and $\lambda_1$ are the first and second nonzero eigenvalues of (1.1), then the following pinching inequality holds

$$\frac{n^2}{4d^2} < \lambda_2 - \lambda_1 < \left( \frac{4n\pi^2}{D^2} + \frac{4(M - m)}{n} \right)^{\frac{1}{2}},$$

where $d = \text{diameter of } \Omega$, $D = \text{the diameter of the largest inscribed ball in } \Omega$, $M = \sup_{\partial \Omega} V$, and $m = \inf_{\Omega} V$.

In the last section, we demonstrate how to make use of the main theorem here to obtain a similar theorem when $\Omega = \mathbb{R}^n$.

In Appendix B, we give a short proof of a theorem of Brascamp and Lieb on the log concavity of the first eigenfunction. A similar method of gradient estimate was used by Li and the third author in [4].

2. A gradient estimate.

Let $f_1$ and $f_2$ be the first and second eigenfunctions of (1.1). It is a known fact that $f_1$ must be a positive function (a theorem of Courant [3]), and thus $u = f_2/f_1$ is a well-defined smooth function on $\Omega$. Using the Hopf lemma and the Malgrange preparation theorem, one can actually verify that $u$ is smooth up to the boundary $\partial \Omega$ (for a short proof of the case we need, see § 6). In this section, the following gradient estimate will be established, which is the key step to derive the lower bound for $\lambda_2 - \lambda_1$.

**Theorem 2.1.** With the same conditions stated in Theorem (1.1), we have the following estimate for the gradient of $u$,

$$|\nabla u|^2 + \lambda(\mu - u)^2 \leq \sup_{\partial \Omega} \lambda(\mu - u)^2,$$

where $\lambda = \lambda_2 - \lambda_1$, $\mu$ is a constant not less than $\sup_{\partial \Omega} u$.

We proceed to give the proof by dividing our argument into two propo-
sitions. In the sequel of this, we denote by $G = |\nabla u|^2 + \lambda(\mu - u)^2$, which is a smooth function on $\Omega$ as $u$ is.

**Proposition 2.2.** With the same conditions in Theorem (1.2), if $G$ attains its maximum in an interior point of $\Omega$, we have the following inequality

$$G \leq \sup_{\partial \Omega} \lambda(\mu - u)^2.$$  

**Proof.** By direct computation, we have

$$G_i = \sum_{j=1}^{n} 2u_j u_{ji} - 2\lambda(\mu - u)u_i$$  

$$\Delta G = \sum_{i=1}^{n} G_{ii} = 2 \sum_{i,j=1}^{n} u_{ij}^2 + \sum_{i,j=1}^{n} 2u_i u_{ij} + 2\lambda \sum_{i=1}^{n} u_i^2 - 2\lambda(\mu - u)\left(\sum_{i=1}^{n} u_{ii}\right).$$

It is by straightforward computation that

$$\Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_i)$$

$(\log f_i)$ is well-defined since $f_i > 0$ on $\Omega$.

We substitute (2.3) into (2.2) and obtain

$$\Delta G = \left\{2 \sum_{i,j=1}^{n} (u_{ij})^2 + 2\lambda^2(\mu - u)\right\}$$

$$+ \left\{4\lambda(\mu - u)(\nabla u \cdot \nabla \log f_i)\right\} - \left\{4(\nabla u) \cdot (\nabla u \cdot \nabla \log f_i)\right\}.$$

Suppose $G$ attains its maximum in an interior point $p \in \Omega$. If $(\nabla u)(p) \neq 0$, then we can choose a coordinate such that $u_i(p) \neq 0$, $u_i(p) = 0$ for $2 < i < n$. Furthermore, since $\nabla G(p) = 0$, one easily deduces from (2.1) that relative to the above coordinate the following is true

$$u_{11}(p) = \lambda(\mu - u)(p)$$

$$u_{1i}(p) = 0, \quad 2 < i < n.$$  

Putting (2.5) and (2.6) into (2.4), we find a simplification for $\Delta G(p)$ with respect to this particular coordinate system,

$$\Delta G(p) = \left\{2 \sum_{i,j=1}^{n} u_{ij}^2 + 2\lambda^2(\mu - u)u\right\} - \left\{4u_i^2(\log f_i)_i\right\} < 0.$$
Since both $V$ and $Q$ are convex by assumption, according to a result of Brascamp and Lieb [1], $\log f_i$ is concave, in particular $(\log f_i)_u(p)<0$. Consequently, the second term of (2.7), namely $-4w^{2}_i(\log f_i)_u$, is nonnegative. Therefore, we have
\[
(2.8) \quad \left\{ \sum_{i,j=1}^{n} u^{2}_{ij} + \lambda^2(\mu - u)u \right\}(p) < 0.
\]
Furthermore, $u^2_{ij}(p) > 0 \; \forall i, j$ implies
\[
\{u^2_{11} + \lambda^2(\mu - u)u \}(p) < 0.
\]
Again from (2.5), this leads to
\[
(2.9) \quad \mu(\mu - u(p)) < 0.
\]
We can assume that $\sup u$ is positive. On the other hand, $\sup u$ is greater than $u(p)$ as $u_i(p) \neq 0$. If $\mu > \sup u > 0$, it gives rise to a contradiction of (2.9).

Our argument above shows that $\nabla u(p) = 0$ and establishes the inequality $G < \sup_{\partial} \lambda(\mu - u)\mu$ as desired.

**Proposition 2.3.** Let us assume equation (1.1) satisfying all the conditions in Theorem (2.1). If $G$ attains its maximum on $\partial \Omega$, then we have the same estimate
\[
G < \sup_{\partial} \lambda(\mu - u)\mu.
\]

**Remark.** We recall a differential geometric description of convexity here which will be used later. Suppose $H = (h_{\alpha\beta})_{2 \leq \alpha, \beta \leq n}$ is the second fundamental form of $\partial \Omega$ relative to a unit normal of $\partial \Omega$ pointing outward to $\Omega$. It is known that $\partial \Omega$ is convex iff $H$ is positive definite.

**Proof of Proposition 2.3.** Suppose $G$ attains its maximum on $\partial \Omega$ at a point $p$. We can choose an orthonormal frame $\{l_1, l_2, ..., l_n\}$ around $p$ such that $l_1$ is perpendicular to $\partial \Omega$ and pointing outward. We also use the notation $\partial / \partial x_1$ to denote the restriction of $l_1$ on $\partial \Omega$, that is the normal unit vector field along $\partial \Omega$.

A simple computation shows
\[
(2.10) \quad \frac{\partial G}{\partial x_1}(p) = 2 \sum_{i=1}^{n} u_i u_{i1} - 2\lambda u_i(\mu - u)
\]
\[
= 2w^{2}_{i1} + 2 \sum_{i=2}^{n} u_i u_{i1} - 2\lambda u_i(\mu - u) \geq 0.
\]
Consider the equation $\Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_1)$, where both $\Delta u$ and $u$ are smooth up to the boundary and thus attain finite values on $\partial \Omega$. Hence, $(\nabla u \cdot \nabla \log f_1) = (1/f_1) \left[ u_i (f_1) + \sum_{2 \leq i \leq n} u_i (f_1) \right]$ achieves finite values on $\partial \Omega$ as well. Nevertheless, since $f_1 \equiv 0$ on $\partial \Omega$, we have $(f_1)_i = 0 \ \forall 2 < i < n$ ($l_i$, $2 < i < n$, is in the tangential direction). This implies that $(1/f_1) u_i (f_1)_i$ must be finite. By the Hopf lemma, $(f_1)_i = \partial f_1/\partial x_i \neq 0$ on $\partial \Omega$, we get the important observation that

$$u_i \equiv 0 \quad \text{on} \ \partial \Omega.$$ 

Using (2.11) one can rewrite (2.10) as follows

$$\frac{\partial G}{\partial x_i} (p) = 2 \sum_{i=2}^n u_i u_{ii} > 0.$$ 

From the definition of second fundamental form of a hypersurface in $\mathbb{R}^n$, one can derive

$$u_i = -\sum h_{ij} u_j + \sum b_{ij} u_i, \quad 2 \leq i, j \leq n$$ 

where $(b_{ij})$ is a skew symmetric matrix i.e. $b_{ij} = -b_{ji}$. Putting (2.13) into (2.12), we have

$$\frac{\partial G}{\partial x_i} (p) = -2 \sum_{i,j=2}^n u_i h_{ij} u_j > 0.$$ 

This contradicts the convexity of $\partial \Omega$. Thus $u_i (p) = 0$ for all $2 \leq i \leq n$, and yields our inequality $G < \sup \lambda (\mu - u)^2$.

Theorem 2.1 follows from the above two propositions.

3. - Lower bound.

In this section, we shall derive our lower bound $\pi^2/4d^2 < \lambda_3 - \lambda_1$.

Recall our basic estimate (Theorem 2.1) which says that for $\mu > \sup u$:

$$|\nabla u|^2 < \lambda \left\{ \sup_{\partial} (\mu - u)^2 - (\mu - u)^2 \right\}.$$ 

In particular, we have

$$|\nabla u|^2 < \lambda \left\{ (\sup u - \inf u)^2 - (\sup u - u)^2 \right\}.$$
Furthermore,

\[(3.3) \quad \sqrt{\lambda} \geq \frac{|\nabla u|}{\sqrt{\left(\sup u - \inf u\right)^2 - (\sup u - u)^2}}.\]

Let \( A = \sup u - \inf u \) and \( W = \sup u - u \). One can rewrite (3.3) as

\[(3.4) \quad \sqrt{\lambda} \geq \frac{|\nabla W|}{\sqrt{A^2 - W^2}}.\]

Let \( q_1, q_2 \) be two points of \( \bar{\Omega} \) such that \( u(q_1) = \sup u \), \( u(q_2) = \inf u \), and \( \sigma \) is the line segment joining them. \( \sigma \) lies in \( \Omega \) since it is convex by assumption. We integrate both sides of (3.3) along \( \sigma \) from \( q_1 \) to \( q_2 \) and obtain

\[
\int_{q_1}^{q_2} \frac{|\nabla u| ds}{\sqrt{\left(\sup u - \inf u\right)^2 - (\sup u - u)^2}} \leq \int_{q_1}^{q_2} \sqrt{\lambda} ds .
\]

Changing variables, we have

\[
\int_{0}^{\sigma} \frac{|dW|}{\sqrt{A^2 - W^2}} \leq \int_{q_1}^{q_2} \sqrt{\lambda} ds .
\]

By elementary calculus, one has

\[
\frac{2}{\pi} \leq \sqrt{\lambda} \ell(\sigma) \leq \sqrt{\lambda} d ,
\]

where \( \ell(\sigma) = \text{length of } \sigma \), \( d = \text{diameter of } \Omega \). This proves \( \lambda_2 - \lambda_1 > \pi^2/4d^2 \) as has been claimed.
4. -- Upper bound.

The major step to establish our upper bound $\lambda_2 - \lambda_1 < 4\pi^2/D^2 + \frac{4(M-m)}{n}$ is the following.

**Lemma 4.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ and $V$ a bounded nonnegative potential defined on $\bar{\Omega}$. Suppose $\lambda_1, \lambda_2$ are the first and second nonzero eigenvalues of the Dirichlet boundary problem

\[
\begin{align*}
\begin{cases}
\Delta f - Vf &= -\lambda f \\
f &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

then

\[
\lambda_2 - \lambda_1 < \frac{4}{n}(\lambda_1 - m).
\]

where $m = \inf_{\bar{\Omega}} V$.

Some results of this sort in the case of $V \equiv 0$ were given by Payne, Pólya and Weinberger [6].

**Proof.** Let $f_1$ be the first eigenfunction of (4.1). Take a trial function $f = x_i f_1 - a f_1$, where $x_i$ is any fixed coordinate function for some $1 \leq i \leq n$ and $a$ is a constant chosen to satisfy $\int_{\partial \Omega} f f_1 = 0$. The following computation shows that

\[
\begin{align*}
-\Delta f + Vf &= -2 \frac{\partial f_1}{\partial x_i} + (x_i - a)(-\Delta f_1 + Vf_1) \\
&= -2 \frac{\partial f_1}{\partial x_i} + \lambda_i(x_i - a)f_1 \\
&= -2 \frac{\partial f_1}{\partial x_i} + \lambda_i f.
\end{align*}
\]

Multiplying both sides of (4.1) by $f$, integrating over $\Omega$ and then dividing by $\int_{\partial \Omega} f^2$, we have

\[
\frac{\int_{\Omega}(-\Delta f + Vf)}{\int_{\partial \Omega} f^2} = \frac{-2 \int_{\partial \Omega} (\partial f_1/\partial x_i) \cdot f}{\int_{\partial \Omega} f^2} + \lambda_1.
\]

The following formula is well-known,

\[
\lambda_2 = \inf_{\{\psi \neq 0 \text{ on } \partial \Omega\}} \frac{\int_{\Omega}(-g\Delta g + Vg^2)}{\int_{\partial \Omega} g^2} = \inf_{\{\psi \neq 0 \text{ on } \partial \Omega\}} \frac{\int_{\partial \Omega} \nabla g^2 + \int_{\Omega} Vg^2}{\int_{\partial \Omega} g^2}.
\]
(4.3) together with (4.2) and the fact that \(f_i f_i\) imply

\[
\lambda_2 < -2 \left( \frac{\int (\partial f_i / \partial x_i) \cdot f}{\int f^2} \right) + \lambda_1,
\]

(4.4)

\[
\lambda_2 - \lambda_1 < -2 \left( \frac{\int (\partial f_i / \partial x_i) \cdot f}{\int f^2} \right).
\]

(4.5)

Substituting \(f = x_i f_i - a f_i\) and integrating by parts, gives

\[
\int_{\partial} f \frac{\partial f_i}{\partial x_i} = \int (x_i f_i - a f_i) \frac{\partial f_i}{\partial x_i} = \frac{1}{2} \int (x_i - a) \frac{\partial (f_i)}{\partial x_i}
\]

\[
= \frac{1}{2} \int_{\partial} x_i \frac{\partial (f_i)}{\partial x_i}
\]

\[
= -\frac{1}{2} \int_{\partial} f_i^2.
\]

We can always normalize \(f_i\) such that \(\int f_i^2 = 1\). Combining (4.5) and (4.6), we have

\[
\lambda_2 - \lambda_1 < \frac{1}{\int f_i^2}.
\]

(4.7)

Again from (4.6) \(\int f (\partial f_i / \partial x_i) = -\frac{1}{2}\); moreover, the Schwarz lemma says that

\[
\left( \int \frac{\partial f_i}{\partial x_i} \right) \left( \int f i \right) = \frac{1}{4}.
\]

(4.8)

This implies that

\[
\left( \int |\nabla f_i|^2 \right) \left( \int f_i^2 \right) \geq \frac{n}{4}
\]

(4.9)

since \(|\nabla f_i|^2 = \sum_{i=2}^{n} (\partial f_i / \partial x_i)^2\). Bringing (4.7) and (4.9) together, we have

\[
\lambda_2 - \lambda_1 < \frac{4}{n} \int |\nabla f_i|^2.
\]

(4.10)

Since \(-\Delta f_i + V f_i = \lambda_i f_i\) and \(V > m\), it is easy to see that \(\int |\nabla f_i|^2 < \lambda_i - m\).
Using this fact, one can conclude from (4.10) that

\[ \lambda_2 - \lambda_1 < \frac{4}{n} (\lambda_1 - m). \]

This completes the proof.

**Remark.** It is in general true that \( \lambda_{i+1} - \lambda_i < \left( \sum_{i=1}^{n} \lambda_i \right)/i \), where \( 1 < i < n - 1 \).

**Proof of Upper Bound of \( \lambda_2 - \lambda_1 \).** Recall the identity (4.3)

\[ \lambda_1 = \inf_{f = 0 \text{ on } \partial \Omega} \frac{\int_{\Omega} |\nabla f|^2 + \int_{\Omega} \nu f^2}{\int_{\partial \Omega} f^2}. \]

Let us choose \( g \) vanishing on \( \partial \Omega \) s.t. \( \int_{\partial \Omega} |\nabla g|^2 \right| g^2 = \mu_1 \), where \( \mu_1 \) is the first- non-zero eigenvalue of the Dirichlet problem (1.1) on \( \Omega \) with \( V = 0 \). Clearly we have

\[ \lambda_1 \leq \frac{\int_{\Omega} |\nabla g|^2 + \int_{\Omega} \nu g^2}{\int_{\partial \Omega} g^2} \leq \frac{\int_{\Omega} |\nabla g|^2}{\int_{\partial \Omega} g^2} + \gamma = \mu_1 + M. \]

Using a theorem of Cheng [2], we have

\[ \mu_1 \leq \frac{n^2 \pi^2}{D^2}, \quad \text{when } n = \dim \Omega \]

and \( D = \) the diameter of the largest inscribed ball in \( \Omega \). With Lemma 4.1, we can now establish our upper bound for \( \lambda_2 - \lambda_1 \) asserted in Theorem 1.1.

\[ \lambda_2 - \lambda_1 < \frac{4}{n} (\lambda_1 - m) < \frac{4}{n} \left[ \mu_1 + M - m \right] < \frac{4}{n} \left[ \frac{n^2 \pi^2}{D^2} + M - m \right] < \frac{4n \pi^2}{D^2} + \frac{4}{n} (M - m). \]

5. - Gap of eigenvalues over \( \mathbb{R}^n \).

In this section, we extend the estimate for eigenvalues of bounded domain to eigenvalues of \( \mathbb{R}^n \). We need the following well-known fact.

**Proposition 5.1.** Let \( \lambda_2(R) \) be the second eigenvalue of \( A - V \) defined on the ball \( B(R) \) with Dirichlet boundary condition. Then \( \lambda_2(R) \) is a con-
tinuous piecewise smooth function of $R$ when $R > 0$. When it is smooth,

\begin{equation}
\frac{d}{dR} \lambda_2(R) = -\int_{\partial B(R)} \left( \frac{\partial \varphi}{\partial r} \right)^2 \end{equation}

where $\varphi$ is a normalized second eigenfunction of $\Delta - V$ defined on $B(R)$.

**Proof.** Let $\varphi(x; r_2)$ be the normalized second eigenfunction of $\Delta - V$ defined on the ball $B(r_2)$ with Dirichlet boundary condition. In polar coordinates, $\varphi$ is a function of the form $\varphi(\theta, r_2; r_2)$ where $\theta \in S^{n-1}$, the unit sphere, and $0 < r_1 < r_2 < \infty$.

It is well-known that we can assume $\varphi$ to be piecewise smooth as a function of $r_2$. At the points where $u$ is smooth, we can differentiate the equation for $\varphi$ and obtain

\begin{equation}
\int_{B(r_2)} \varphi \Delta \left( \frac{\partial \varphi}{\partial r_2} \right) = \int_{B(r_2)} \varphi \left( V - \lambda_2 \right) \left( \frac{\partial \varphi}{\partial r_2} \right) - \int_{\partial B(r_2)} \frac{d \lambda_2}{d r_2} \varphi^2. \end{equation}

Integrating by parts, we derive

\begin{equation}
\frac{d \lambda_2}{d r_2} = \int_{\partial B(r_2)} \frac{\partial \varphi}{\partial r_1} \frac{\partial \varphi}{\partial r_2}. \end{equation}

Notice that $\varphi(\theta, r, r) = 0$ for all $r$. Hence

\begin{equation}
0 = \frac{d}{d r} \varphi(\theta; r, r) = \frac{\partial \varphi}{\partial r_1} (\theta; r, r) + \frac{\partial \varphi}{\partial r_2} (\theta, r, r). \end{equation}

Putting this into (5.3) we have

\begin{equation}
\frac{d \lambda_2}{d r_2} (R) = -\int_{\partial B(r_2)} \left( \frac{\partial \varphi}{\partial r_2} \right)^2. \end{equation}

**Proposition 5.2.** Let $\varphi$ be an eigenfunction of $\Delta - V$ defined on the ball $B(R) \subset \mathbb{R}^n$ with Dirichlet boundary condition and eigenvalue $\lambda$. Then

1. If $2n - 2 > k > 2$,

\begin{equation}
\int_{\partial B(R)} \left( \frac{\partial \varphi}{\partial r} \right)^2 < R^{-k+n-1} \left[ -k \int_{\partial B(R)} (V - \lambda) \varphi^2 - \int_{\partial B(R)} \frac{\partial V}{\partial r} \varphi^2 \right]. \end{equation}
(ii) If $k > 2n - 2$ and $k > 2$,

\[
\int_{\partial B(0)} \left( \frac{\partial \varphi}{\partial r} \right)^2 \leq \frac{k - 2n + 2}{2} R^{-k+n-1} \int_{B(0)} \varphi^2 \Delta r^{k-n} + (2n - 2 - 2k) R^{k+n-1} \int_{B(0)} r^{k-n}(V - \lambda) \varphi^2 - R^{-k+n-1} \int_{B(0)} r^{k+n+1} \frac{\partial V}{\partial r} \varphi^2.
\]

**Proof.** Let $d\theta$ be the volume element of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ and $A_\theta$ be the spherical Laplacian. Then

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r} + \frac{A_\theta \varphi}{r^2} = (V - \lambda)\varphi
\]

and

\[
\frac{d}{dr} \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta = -2 \int_{S^{n-1}} \frac{\partial \varphi}{\partial r} \frac{\partial^2 \varphi}{\partial r^2} d\theta
\]

\[
= -2 \int_{S^{n-1}} (n - 1) \frac{n-1}{r} \left( \frac{\partial \varphi}{\partial r} \right)^2 d\theta - 2 \int_{S^{n-1}} \Delta_\theta \varphi d\theta + 2 \int_{S^{n-1}} (V - \lambda) \frac{\partial \varphi}{\partial r} d\theta
\]

\[
= -2(n-1)r^{-1} \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 d\theta + r^{-1} \int_{S^{n-1}} |\nabla \varphi|^2 d\theta + \int_{S^{n-1}} (V - \lambda) \frac{\partial \varphi}{\partial r} d\theta
\]

\[
= -2(n-1)r^{-1} \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 d\theta + r^{-1} \int_{S^{n-1}} |\nabla \varphi|^2 d\theta + \frac{d}{dr} \int_{S^{n-1}} (V - \lambda) \varphi^2 d\theta - \int_{S^{n-1}} \frac{\partial V}{\partial r} \varphi^2 d\theta.
\]

Multiplying this equation by $r^k$ (with $k > 2$) and integrating from 0 to $R$, we have

\[
\int_0^R \frac{d}{dr} \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta dr = -2(n-1) \int_0^R \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 d\theta dr
\]

\[
+ \int_0^R \frac{d}{dr} \int_{S^{n-1}} |\nabla \varphi|^2 d\theta dr + \int_0^R \frac{d}{dr} \int_{S^{n-1}} (V - \lambda) \varphi^2 d\theta dr - \int_0^R \frac{d}{dr} \int_{S^{n-1}} \left( \frac{\partial V}{\partial r} \right) \varphi^2 d\theta dr.
\]

Integrating by parts, we have the following

\[
\int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 (R, \theta) d\theta = k \int_0^R \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta dr
\]

\[
+ \int_0^R \frac{d}{dr} \int_{S^{n-1}} \left( \frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta dr.
\]
Putting (5.11), (5.12) and (5.13) into (5.10), we have

\begin{equation}
0 = \int_0^R (k - 2) r^{k-3} \int_{S^{k-1}} |\nabla \phi|^2 \, d\theta \, dr + \int_0^R \left[ \frac{d}{dr} \int_{S^{k-1}} |\nabla \phi|^2 \, d\theta \right] \, dr
\end{equation}

\begin{equation}
0 = k \int_0^R r^{k-1} \int_{S^{k-1}} (V - \lambda) \phi^2 \, d\theta \, dr + \int_0^R \left[ \frac{d}{dr} \int_{S^{k-1}} (V - \lambda) \phi^2 \, d\theta \right] \, dr.
\end{equation}

Hence,

\begin{equation}
R^k \int_{S^{k-1}} \left( \frac{\partial \psi}{\partial r} \right)^2 \, d\theta \, dr = (k - 2n + 2) \int_0^R \left( \frac{\partial \psi}{\partial r} \right)^2 \, d\theta \, dr
\end{equation}

\begin{equation}
- (k - 2) \int_0^R r^{k-3} \int_{S^{k-1}} |\nabla \phi|^2 \, d\theta \, dr - k \int_0^R \int_{S^{k-1}} (V - \lambda) \phi^2 \, d\theta \, dr - \int_0^R \int_{S^{k-1}} \frac{\partial V}{\partial r} \phi^2 \, d\theta \, dr.
\end{equation}

Hence,

\begin{equation}
R^{k-n+1} \int_{\partial B(R)} \left( \frac{\partial \psi}{\partial r} \right)^2 \leq (k - 2n + 2) \int_{B(R)} \left( \frac{\partial \psi}{\partial r} \right)^2 - k \int_{B(R)} \frac{V}{\partial r} \int_{\partial B(R)} (V - \lambda) \phi^2
\end{equation}

By the divergence theorem,

\begin{equation}
0 = \int_{\partial B(R)} r^{k-n} \phi \frac{\partial \phi}{\partial r} = \int_{B(R)} r^{k-n} |\nabla \phi|^2 + \int_{B(R)} \phi \nabla r^{k-n} \cdot \nabla \phi + \int_{B(R)} r^{k-n} \Delta \phi.
\end{equation}

Hence,

\begin{equation}
\int_{B(R)} r^{k-n} \left( \frac{\partial \phi}{\partial r} \right)^2 \leq \int_{B(R)} r^{k-n} |\nabla \phi|^2 = -\int_{B(R)} \phi \nabla r^{k-n} \cdot \nabla \phi - \int_{B(R)} r^{k-n} \phi \nabla \phi
\end{equation}

\begin{equation}
= \frac{1}{2} \int_{B(R)} \phi \Delta r^{k-n} - \int_{B(R)} r^{k-n} (V - \lambda) \phi^2.
\end{equation}

The proposition follows from (5.15) and (5.17).

It is straightforward to derive from Theorem 1.1 and the last two propositions the following theorem.

**Theorem 5.1.** Let $V$ be a $C^1$-function defined on $\mathbb{R}^n$ with $n > 4$. Let $\lambda_2(q)$ be the second eigenvalue of the operator $-\Delta + V$ defined on the ball
$B(\rho)$ with Dirichlet boundary condition. Suppose that $V$ is convex in the ball $B(R)$, then

(i) $\lambda_2 - \lambda_1 > \frac{\pi^2}{4R^2} - \frac{1}{k - n} R^{-k+n} \cdot \sup \left\{ -k |x|^{k-n} (\lambda_d(|x|) - V(x)) - |x|^{k-n+1} \frac{\partial V}{\partial r} \right\}_+.$

where $2n - 2 > k > n$ and $\{f\}_+$ stands for the positive part of $f$.

(ii) When $k > 2n - 2$, $k > n$ and $k > 2$,

$\lambda_2 - \lambda_1 > \frac{\pi^2}{4R^2} - \frac{1}{k - n} R^{-k+n} \cdot \sup_{|x| < R} \left\{ \frac{k - 2n + 2}{2} A_{r^{k-n}} + (2n - 2 - 2k) |x|^{k-n} (V(x) - \lambda_d(x)) - r^{k-n+1} \frac{\partial V}{\partial r} \right\}_+.$

Remark. If $\lim_{|x| \to \infty} V(x) = \infty$ and $\partial V/\partial r > 0$, $k - n > 2$ and $R$ large, we can obtain a positive lower estimate for $\lambda_2 - \lambda_1$. Note also that

$\lambda_d(R) < \lambda_d(1) < \frac{n + 4}{n} \lambda_1(1) - \frac{4}{n} \inf_{B(1)} V < n(n+4)\pi^2 + \frac{n + 4}{n} \sup_{B(1)} V - \frac{4}{n} \inf_{B(1)} V.$

Hence $(\lambda_d(R) - V(x))_+$ can be estimated easily if $\lim_{|x| \to \infty} V(x) = \infty$.

6. Appendix.

A) Here we shall give a quick argument to verify the «standard» fact that $u = f_1/\hat{f}_1$ is smooth up to the boundary $\partial \Omega$. In the whole discussion, we assume $\Omega$ to be smooth convex. Our conditions in Theorem 2.1 allow us to apply the classical Hopf lemma to $f_1$.

Let us choose local coordinates $\{x_1, x_2, \ldots, x_n\}$ on a sufficiently small open set $U$ such that $U \cap \partial \Omega = U \cap \{x_1 = 0\}$. Since $f_1$ is identically equal to zero on $\partial \Omega$ and $f > 0$ in $\Omega$, by the Hopf lemma we have $\partial f_1/\partial x_1 < 0$ on $\partial \Omega$. Furthermore, $f_1$ is smooth up to the boundary, thus one can consider $\hat{f}_1$ as a smooth function which is defined on $U$ restricted to $U \cap \bar{\Omega}$. Using the Malgrange preparation theorem [5], together with the fact that $\partial f_1/\partial x_1 \neq 0$ on $\partial \Omega$, we have locally

(6.1) $f_1 = g_1 \cdot x_1,$

where $g_1$ is a unit which is smooth on $\bar{\Omega} \cap U$. 

Moreover, $f_2$ is identically zero on $\partial \Omega$; applying the Malgrange's theorem again, one can write locally

$$f_2 = g_2 \cdot x_1 \cdot h_2,$$

where $g_2$ is a unit which is smooth in $\partial \Omega \cap U$, and $h_2$ is also a smooth function in $\partial \Omega \cap U$. Now it is clear

$$u = \frac{f_2}{f_1} = \frac{g_2 \cdot h_2}{g_1}$$

must be smooth on $U \cap \partial \Omega$.

B) Here we give a proof of a theorem of Brascamp and Lieb.

Let $\phi_1$ be the first positive eigenfunction of the operator $\Delta - V$ on a convex domain $\Omega$ with Dirichlet condition. Then $u = \log f$ satisfies the equation

$$\Delta u = (V - \lambda) - |\nabla u|^2.$$ 

By convexity of $\Omega$, it is easy to see that $u$ is concave in a neighborhood of $\partial \Omega$. If we consider the Hessian of $u$ as a function of the frame bundle of $\Omega$, it achieves a maximum in the interior of $\Omega$. At such a point,

$$0 > \Delta u_{ii} = (V_{ii}) - 2 \sum_j u^2_{ij}$$

and

$$u_{ij} = 0 \quad \text{for } i \neq j,$$

Hence

$$u^2_{ii} \geq \frac{1}{2} \min_{\Omega} V_{ii}.$$

By using (6.5), we can prove the concavity of $u$ by the method of continuity. In fact, we can find family $\Omega_t$ and $V_t$ so that $\partial \Omega_t = \Omega$ and $V_t = V$. Furthermore, we may assume $\Omega_0$ is a ball in $\Omega$ and $V_0$ is a quadratic function so that by computation, the theorem is valid in this case. In fact, we can let $V_t = tV + (1-t) V_0$ and $\Omega_t = \{(1-t)x_0 + t x_1 : x_1 \in \Omega_t \text{ and } x_0 \in \Omega_0\}$. Then $\min_{\Omega_t} (V_{ii}) > 0$.

If for $t < 1$, $u_t$ is not concave, $(u_{ii})_{ii}$ at the maximum point will be positive by (6.5). This is not possible if we have a sequence $t \to t$ with $\max (u_{ii})_{ii} < 0$. Hence we have proven the log concavity of $f_t$.

The proof actually shows that $(\log f_1)_{ii} < -\frac{1}{2} \min_{\Omega_1} V_{ii}$.
REFERENCES


