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Nonsteady flow of water and oil through inhomogeneous porous media

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 12, n° 3 (1985), p. 335-392

<http://www.numdam.org/item?id=ASNSP_1985_4_12_3_335_0>
Nonsteady Flow of Water and Oil through Inhomogeneous Porous Media (*).

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1. - Formulation of the problem.

The flow of two immiscible fluids through a porous medium is described by (see e.g. [2] (9.3.25) and [3] (6.36), (6.52))

\[
\partial_t s_i - \nabla \cdot (k_i (\nabla p_i + e_i)) = 0, \quad i = 1, 2
\]

with

\[ s_i = s_i(x, p_1 - p_2) \quad \text{and} \quad k_i = k(x, s_i), \]

and

\[
s_1(x, p_1 - p_2) + s_2(x, p_1 - p_2) = s_0(x).
\]

The differential equation (1.1) is considered in \( \Omega_T := \Omega \times ]0, T[ \), where \( \Omega \subset \mathbb{R}^n \) is the porous medium. \( s_i, i = 1, 2, \) is the fluid content of the \( i \)-th fluid and \( s_0 \) the porosity, that is, the relative volume of the pores, which for inhomogeneous media depends on \( x \). \( k_i \) is the permeability depending on \( x \) and \( s_i \), the hydrostatic pressure is given by \( p_i \), and \( e_i \) is the gravity term. Although we restrict ourselves to scalar functions \( k_i \) all our results remain valid for symmetric matrices \( k_i \), that is, if we consider anisotropic media.

In the following we often suppress the argument \( x \) in the functions \( k_i \) and \( s_i \).

(*) Supported by Deutsche Forschungsgemeinschaft SFB 72.

The content $s_i$ as a function of the capillary pressure $p_1 - p_2$ as well as the permeability $k_i$ as function of $s_i$ are obtained by experiments, see [2; fig. 9.2.14, 9.2.15] and [3; fig. 6.6]. For the definition of $p_i$ see [3; fig. 6.7]. The qualitative behavior of these functions is shown in fig. 1-3.

We refer to [2; fig. 9.2.7a), 9.2.10, 9.3.1], [3; fig. 6.9, 6.13, 6.16, 6.17], [4; fig. 6-13], and [5; fig. 6]. Because of this behavior of the coefficients the system (1.1) is a degenerate elliptic-parabolic equation. Since $p_i$ is not determined by its differential equation when $s_i = 0$ we have to add the condition

\[(1.3) \quad P_{\text{min}} < p_1 - p_2 < P_{\text{max}}, \]

\[s_i \quad 0 \quad s_a \quad k_i \]

Figure 1

\[s_i \quad p_{\text{min}} \quad 0 \quad p_{\text{max}} \quad p_1 - p_2 \]

Figure 2

\[p_1 - p_2 = s_i'(s) \quad k_i(s_a - s) \quad s_a \quad s_i(s) \]

Figure 3
where \( p_{\text{min}} \) and \( p_{\text{max}} \) are given with \( -\infty < p_{\text{min}} < 0 < p_{\text{max}} < \infty \). For example, \( p_{\text{min}} = -\infty \) in fig. 2 and \( p_{\text{min}} > -\infty \) in fig. 3. In particular, if \( s_2 = 0 \) then \( p_2 = p_1 - p_{\text{max}} \) and \( p_1 \) is determined by an elliptic equation. If \( 0 < s_2 < s_0 \) then \( p_{\text{min}} < p_1 - p_2 < p_{\text{max}} \) and \( p_1 \) and \( p_2 \) satisfy an elliptic-parabolic system.

We consider three types of boundary conditions for each fluid, that is, for \( i = 1, 2 \) the boundary \( \partial \Omega \) is divided into three sets \( I_i^a, I_i^b, I_i^v \) with Dirichlet condition

\[
(1.4) \quad p_i = p_i^0 \quad \text{on} \quad I_i^a \times [0, T],
\]

where \( p_i^0 \) is the trace of a function in \( \Omega \), also denoted by \( p_i^0 \), with \( p_{\text{min}} < p_i^0 - p_i^v < p_{\text{max}} \). We assume Neumann conditions

\[
(1.5) \quad k_i (\nabla p_i + e_i) \cdot v = 0 \quad \text{on} \quad I_i^v \times [0, T],
\]

where \( v \) is the exterior normal to \( \partial \Omega \), and overflow conditions

\[
\begin{align*}
&k_i (\nabla p_i + e_i) \cdot v = 0 \quad \text{if} \quad p_1 - p_2 < p_{\text{max}} \quad \text{on} \quad I_i^a \times [0, T], \\
&k_i (\nabla p_i + e_i) \cdot v < 0 \quad \text{if} \quad p_1 - p_2 = p_{\text{max}} \quad \text{on} \quad I_i^a \times [0, T], \\
&k_i (\nabla p_2 + e_i) \cdot v = 0 \quad \text{if} \quad p_1 - p_2 > p_{\text{min}} \quad \text{on} \quad I_i^v \times [0, T], \\
&k_i (\nabla p_2 + e_i) \cdot v < 0 \quad \text{if} \quad p_1 - p_2 = p_{\text{min}} \quad \text{on} \quad I_i^v \times [0, T].
\end{align*}
\]

We assume that \( I_i^a \subset I_i^b \) and \( I_i^v \subset I_i^a \). As initial condition we pose

\[
(1.7) \quad s_i (p_1 - p_2) (x, 0) = s_i^0 (x) \quad \text{for} \quad x \in \Omega
\]

with given functions \( s_i^0 \) satisfying \( s_i^0 + s_2^0 = s_0 \).

The differential equation (1.1) together with the boundary conditions (1.4)-(1.6) has the following weak formulation. Let

\[
\mathcal{K} := \{(v_1, v_2); v_i = p_i^0 \text{ on } I_i^a \times [0, T], \quad v_1 - v_2 < p_{\text{max}} \text{ on } I_i^a \times [0, T], \quad v_1 - v_2 > p_{\text{min}} \text{ on } I_i^v \times [0, T]\}.
\]

Then \( (p_1, p_2) \) is a weak solution, if \( (p_1, p_2) \in \mathcal{K} \) with \( p_{\text{min}} < p_1 - p_2 < p_{\text{max}} \).
and for all \((v_1, v_2) \in \mathcal{K}\) the inequality

\[
(1.8) \quad \sum_{i=1,2} \int_0^T \int_B \left( \partial_t s_i (p_1 - p_2) (v_i - p_i) + k_i (s_i (p_1 - p_2)) (\nabla p_i + e_i) \nabla (v_i - p_i) \right) \geq 0
\]

is satisfied. This weak formulation may be inaccurate in two points. First \(\partial_t s_i\) needs not to be a function, and secondly \(\nabla p_i\) may explode near the set \(\{k_i = 0\}\) and therefore it may be well defined in the sense of distribution. Because of this we did not specify the topology in the above definition of the set \(\mathcal{K}\). On the other hand using \(v_i = p_i^0\) as test function in (1.8) and using the fact that \(s_i\) are monotone increasing, we see that

\[
(1.9) \quad \sum_{i=1,2} \int_0^T \int_B k_i (s_i (p_1 - p_2)) |\nabla p_i|^2
\]

determines the natural topology of the problem.

Therefore let us assume that

\[
k_i (x, s_i (x, p)) > ck_i^* (s_i^*(p)) \quad \text{for } x \in \Omega
\]

with \(c > 0\) and some functions \(k_i^*, s_i^*\) which behave like the functions in figs. 1-3. We introduce the transformation

\[
\begin{align*}
\begin{cases}
u_1 := p_2 + & \int_0^{p_1 - p_2} \sqrt{\frac{k_i^* (s_i^*(\min (\xi, 0)))}{k_i (s_i^*(0))}} \, d\xi, \\
u_2 := p_1 - & \int_0^{p_1 - p_2} \sqrt{\frac{k_i^* (s_i^*(\max (\xi, 0)))}{k_i (s_i^*(0))}} \, d\xi.
\end{cases}
\end{align*}
\]

Then

\[
(1.10) \quad \sum_i |\nabla u_i|^2 \leq C \sum_i k_i (s_i (p_1 - p_2)) |\nabla p_i|^2,
\]

so that we expect a solution \(u_i\) in \(L^2(0, T; H^1 (\Omega))\). If in addition

\[
k_i (x, s_i (x, p)) \leq CK_i^* (s_i^*(p))
\]
then both sides of (1.11) are equivalent, so that the above space is the
natural space to consider. The variational inequality (1.8) can be trans-
formed in terms of $u_i$ where the elliptic part becomes

$$
(1.12) \quad \begin{pmatrix}
  k_1(s_1(p_1-p_2)) & \nabla P_1 \\
  k_2(s_1(p_1-p_2)) & \nabla P_2
\end{pmatrix} = K(s_1(p_1-p_2)) \begin{pmatrix}
  \nabla u_1 \\
  \nabla u_2
\end{pmatrix}.
$$

In the set $\{p_1 > p_2\}$ the matrix $K$ is given by

$$
K(s_1) = \begin{pmatrix}
  k_1(s_1) & 0 \\
  -k_2(s_1)\left(\sqrt{\frac{k^*_s(s_2^0)}{k^*_s(s_2)}} - 1\right) & k_2(s_1)\left(\sqrt{\frac{k^*_s(s_2^0)}{k^*_s(s_2)}} - 1\right)
\end{pmatrix}
$$

and in $\{p_1 < p_2\}$ by

$$
K(s_1) = \begin{pmatrix}
  k_1(s_1)\sqrt{\frac{k^*_s(s_2^0)}{k^*_s(s_2)}} & -k_2(s_1)\sqrt{\frac{k^*_s(s_2^0)}{k^*_s(s_2)}} - 1 \\
  0 & k_2(s_2)
\end{pmatrix}.
$$

Therefore the equation in $u_i$ is still degenerate elliptic-parabolic (in the
case that $k_i$ and $k_i^*$ are equivalent). One could avoid this by replacing
$\sqrt{k^*_i}$ essentially by $k_i^*$ in the definition of $u_i$, but this would be no advantage
for the existence proof, since in any case the quality of the weak solution
$u_i$ is related to the natural topology (see Remark 2.5).

To illustrate the behavior of the solution let us consider special traveling
solutions, that is, solutions of the form $p_i(c, t) = \bar{p}_i(x - t)$. As data
we choose $k_i(x) = x^\alpha$, $P_{\text{min}} > -\infty$, and $s_i(P_{\text{min}} + x) = x^\beta$
for small $x > 0$ with $\alpha$, $\beta$ positive. Then for $N = 1$ in a neighborhood of 0 there is a special
solution with

$$
(\bar{p}_1 - \bar{p}_2)(x) = \begin{cases}
  P_{\text{min}} + ce^{1/(1+\alpha\beta)} + o(x^{1/(1+\alpha\beta)}) & \text{for } x < 0, \\
  P_{\text{min}} & \text{for } x > 0.
\end{cases}
$$

$\bar{p}_2$ as a solution of an elliptic equation is Lipschitz continuous. $\nabla u_i$ is of
class $L^r$ near zero if and only if $r < 2(1+\alpha\beta)/\alpha\beta$.

Another special solution with unbounded pressure is given by $\bar{p}_2(x) =$
$-2x$ and

$$
\bar{p}_1(x) = \begin{cases}
  \cot x - 2x & \text{for } x < 0, \\
  -\infty & \text{for } x > 0.
\end{cases}
$$
The data are $p_{\min} = -\infty$, $s_0 = 1$, $s_1(z) = 1/(1 + z^2)$ for $z < 0$, $k_1(z) = z$, $k_2(z) = z$, and $e_1 = 2$, $e_2 = 1$. Here $\nabla u_1$ is in $L^\infty$ near zero.

Using the transformation (1.10) we prove in section 2 the existence of a weak solution $u$ for the transformed system. This solution satisfies $p_{\min} \leq u < u_{\max}$, where $u_{\min}$ and $u_{\max}$ are the transformed values of $p_{\min}$ and $p_{\max}$.

Therefore the pressure can be recovered using (1.10). If in addition we know that

$$s_i(p_1 - p_2) \in C^0(\Omega \times [0, T]),$$

then $\nabla p_1$ is defined in the sense of distribution in the open set $\{p_1 - p_2 < p_{\max}\}$ and in $L^1_\text{loc}(\{p_1 - p_2 < p_{\max}\})$ satisfying the first equality in (1.12). As a consequence weak solutions as defined in 2.3 satisfy the original variational inequality (1.8) in integrated form. We shall prove (1.13) in sections 3-5 under certain assumptions on the coefficients. For the proof we use a different transformation of the differential equation (see [10], [15; Appendix A])

$$v(x, t) := \frac{s_i(x, (p_1 - p_2)(x, t))}{s_0(x)},$$

$$u(x, t) := p_2(x, t) + \int_0^{(p_1 - p_2)(x, t)} \frac{k_1(x, s_1(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))} d\xi$$

$$= p_1(x, t) - \int_0^{(p_1 - p_2)(x, t)} \frac{k_2(x, s_2(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))} d\xi.$$

In terms of these new variables the system (1.1) reads

$$0 = \nabla \cdot (k(\nu) \nabla u + e(\nu)), \quad (\text{define } \nu := -(k(\nu) \nabla u + e(\nu))),$$

$$s_0 \partial_t v = \nabla \cdot (a(\nu) \nabla v + b(\nu) + a(\nu)v).$$

Using the notation

$$k_1(x, z) := k_1(x, s_0(x)),$$

$$k_2(x, z) := k_2(x, s_0(x)(1 - z))$$
the coefficients in (1.16) and (1.17) are given by

$$
egin{align*}
    & k(x, z) := \tilde{k}_1(x, z) + \tilde{k}_2(x, z), \\
    & c(x, z) := \tilde{k}_1(x, z) c_1 + \tilde{k}_2(x, z) c_2 \left(1 - k(x, z) \int_0^{s_1(x, z) s_2(x, z)} \nabla z \left( \frac{k_1(x, s_1(x, z)) + k_2(x, s_2(x, z))}{2} \right) d\xi \right), \\
    & a(x, z) := \frac{\tilde{k}_1(x, z) \tilde{k}_2(x, z)}{k(x, z)} s_3(x) \partial_z s_3^{-1}(x, s_3(x) z), \\
    & b(x, z) := \frac{\tilde{k}_1(x, z) \tilde{k}_2(x, z)}{k(x, z)} \left( e_1 - e_2 - a(x, z) \nabla z \left( \frac{s_3(x)}{s_3(x)} \right) \right) \left( x, s_3^{-1}(x, s_3(x) z) \right), \\
    & d(x, z) := \frac{\tilde{k}_2(x, z)}{k(x, z)} \\
    & \text{ or } -\frac{\tilde{k}_1(x, z)}{k(x, z)}.
\end{align*}
$$

(1.19)

Here $s_3^{-1}$ denotes the inverse of $s_3$ with respect to the $z$ variable.

Therefore the system is separated in an elliptic equation for $u$ and a parabolic equation for $v$. Since $p_1 < u < p_2$ in $\{p_1 > p_2\}$ (and $p_2 < u < p_1$ in $\{p_1 > p_2\}$) the quantity $u$ can be considered as a mean pressure. Equation (1.16) then can be interpreted as equation of continuity with pressure $u$ and velocity $\nu$ for an "idealized" incompressible fluid replacing the mixture of the two fluids.

In [10] the existence of a classical solution for the system (1.16) is proved in the case that the equation for the saturation is strictly parabolic, that is, $0 < a < a(x, z) < C$. Also the overflow condition is not included. Some of the arguments are restricted to the two dimensional case, for higher dimensions it is required that $k$ is a small perturbation of a continuous function depending only on $x$. In addition this paper contains a uniqueness and a stability result.

Recently independent to our work in [9] the problem was solved for the original system with Dirichlet and Neumann data. The main assumption is that the initial and boundary data stay away from one side of the degeneracy, so that the solution contains only one pure fluid besides the mixture. Then under certain condition on $k_i$ and $s_i$ one of the pressures is of class $L^p(0, T; H^{1,p}(Q))$ for $p < 2$.

In [6], [7] and [15] the problem is treated numerically.
2. Existence of a weak solution.

In this section we state the assumptions on the data and introduce the notion of a weak solution. Using the transformation (1.10) we prove the existence of such a solution. For this we approximate the equation by nondegenerate ones, that is, we approximate \( k_i \) by strictly positive functions. Using the technique of [1] we obtain the convergence of the approximate solutions. In addition we have to choose the approximations such that in the limit the solution \( u_i \) satisfies the inequality \( u_{\text{min}} < u_i - u_2 < u_{\text{max}} \).

Throughout this paper we denote by \( C \) large and by \( c \) small positive constants.

2.1. Assumptions on the differential equation. The water content \( s_i(x, z) \) is measurable in \( x \) and continuous in \( z \) and

\[
\begin{align*}
  s_1(x, z) &= 0 \quad \text{for} \quad z < p_{\text{min}}, \\
  s_2(x, z) &= 0 \quad \text{for} \quad z > p_{\text{max}}, \\
  s_1(x, z_1) &< s_1(x, z_2) \quad \text{and} \quad s_2(x, z_1) > s_2(x, z_2) \quad \text{for} \quad p_{\text{min}} < z_1 < z_2 < p_{\text{max}}.
\end{align*}
\]

Here \( -\infty < p_{\text{min}} < 0 < p_{\text{max}} < \infty \). By \( u_{\text{min}}, u_{\text{max}} \) we denote the transformed values according to (1.10), that is,

\[
\begin{align*}
  u_{\text{min}} &:= \int_0^{p_{\text{min}}} \frac{k_i^* s_1^*(\xi)}{k_i^* s_1^*(0)} \, d\xi, \\
  u_{\text{max}} &:= \int_0^{p_{\text{min}}} \frac{k_i^* s_2^*(\xi)}{k_i^* s_2^*(0)} \, d\xi.
\end{align*}
\]

Furthermore, for all \( x \) and \( z \)

\[
s_1(x, z) + s_2(x, z) = s_0(x)
\]

with a measurable function \( s_0 \) satisfying \( c_0 < s_0(x) < C_0 \). The conductivity \( k_i(x, z) \) is measurable in \( x \) and continuous in \( z \) with

\[
k_i(x, 0) = 0 \quad \text{and} \quad k_i(x, z) > 0 \quad \text{for} \quad z > 0.
\]

Moreover

\[
c_0 k_i^*(s_0^*(z)) < k_i(x, s_i(x, z)) < C_i(z)
\]

with

\[
\begin{align*}
  C_i(z) &\to 0 \quad \text{as} \quad z \uparrow p_{\text{min}}, \\
  C_i(z) &\to 0 \quad \text{as} \quad z \uparrow p_{\text{max}}.
\end{align*}
\]
Here \( k^*_i \) and \( s^*_i \) are continuous functions (which are independent of \( x \)) with the same properties as \( k_i \), \( s_i \). Since \( s_i \) is strictly monotone in \([P_{\min}, P_{\max}]\) the elements \( k_{ij} \) of the matrix \( K \) defined in (1.12) can be written as

\[
k_{ij} = k_{ij}(x, s_i(x, p_1 - p_2)) .
\]

2.2. ASSUMPTION ON THE DATA. The porous medium \( \Omega \subset \mathbb{R}^d \) is an open connected bounded set with Lipschitz boundary. For \( i = 1, 2 \) the boundary \( \partial \Omega \) consists of three measurable sets \( I^0_{1i} \), \( I^0_{2i} \) and \( I^0_\cdot \) with \( I^0_{11} \subset I^0_{21} \) and \( I^0_{22} \subset I^0_{11} \). The boundary data \( p^0_i \) are in \( L^\infty(\Omega \times ]0, T[) \) with \( p_{\min} < p^0_1 < p^0_2 < p_{\max} \) and

\[
p_i^0 \in L^2(0, T; H^1(\Omega)) , \quad \partial_t p_i^0 \in L^1(0, T; L^2(\Omega)) \cap L^\infty(\Omega \times ]0, T[) \quad \text{for some } r > 1 .
\]

The initial data \( s^0_i \) are nonnegative measurable functions with \( s^0_i + s^0_\cdot = s_0 \) satisfying \( \Psi(s^0_i) \in L^1(\Omega) \), where \( \Psi \) is defined in 2.4. They are in the range of \( s_i \), hence there is a measurable function \( p^0 \) with \( p_{\min} < p^0 < p_{\max} \) and

\[
s_i(x, p^0(x)) = s^0_i(x) \quad \text{for } i = 1, 2 .
\]

2.3. WEAK SOLUTIONS. We consider the following sets of functions

\[
\mathcal{K} := \{(v_1, v_2) \in L^2(0, T; H^1(\Omega)) ; \quad v_i = p_i^0 \text{ on } I^0_i \times ]0, T[ , \quad v_1 - v_2 < p_{\max} \text{ on } I^0_1 \times ]0, T[ , \quad v_1 - v_2 > p_{\min} \text{ on } I^0_2 \times ]0, T[ ,
\]

and

\[
\mathcal{K}^* := \{(v_1, v_2) \in L^2(0, T; H^1(\Omega)) ; \quad u_{\min} < v_1 - v_2 < u_{\max} ,
\]

and \( v_i \) on \( I^0_i \times ]0, T[ \) equals the transformation of some \((p_1, p_2)\) according to (1.10) with \( p_i = p^0_i \).

We call \( p_1, p_2 : \Omega \times ]0, T[ \rightarrow \mathbb{R} \) a weak solution of the differential equation (1.1) with boundary conditions (1.4)-(1.6) and initial condition (1.7), if

\[
P_{\min} < p_1 - p_2 < P_{\max} \quad \text{if the transformed function } (u_1, u_2) \text{ obtained by (1.10) belongs to } \mathcal{K}^* , \quad \text{and if for all } (v_1, v_2) \in \mathcal{K} \text{ with } \partial_t v_i \in L^2(\Omega \times ]0, T[) \text{ and for}
\]

\[
Here and s* are continuous functions (which are independent of x) with the same properties as si . Since si is strictly monotone in [P_{\min}, P_{\max}] the elements k_{ij} of the matrix K defined in (1.12) can be written as

k_{ij} = k_{ij}(x, s_i(x, p_1 - p_2)).

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p_i^0 \in L^2(0, T; H^1(\Omega)) , \quad \partial_t p_i^0 \in L^1(0, T; L^2(\Omega)) \cap L^\infty(\Omega \times ]0, T[) \quad \text{for some } r > 1 .
\]

The initial data \( s^0_i \) are nonnegative measurable functions with \( s^0_i + s^0_\cdot = s_0 \) satisfying \( \Psi(s^0_i) \in L^1(\Omega) \), where \( \Psi \) is defined in 2.4. They are in the range of \( s_i \), hence there is a measurable function \( p^0 \) with \( p_{\min} < p^0 < p_{\max} \) and

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s_i(x, p^0(x)) = s^0_i(x) \quad \text{for } i = 1, 2 .
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\]

and

\[
\mathcal{K}^* := \{(v_1, v_2) \in L^2(0, T; H^1(\Omega)) ; \quad u_{\min} < v_1 - v_2 < u_{\max} ,
\]

and \( v_i \) on \( I^0_i \times ]0, T[ \) equals the transformation of some \((p_1, p_2)\) according to (1.10) with \( p_i = p^0_i \).

We call \( p_1, p_2 : \Omega \times ]0, T[ \rightarrow \mathbb{R} \) a weak solution of the differential equation (1.1) with boundary conditions (1.4)-(1.6) and initial condition (1.7), if

\[
P_{\min} < p_1 - p_2 < P_{\max} \quad \text{if the transformed function } (u_1, u_2) \text{ obtained by (1.10) belongs to } \mathcal{K}^* , \quad \text{and if for all } (v_1, v_2) \in \mathcal{K} \text{ with } \partial_t v_i \in L^2(\Omega \times ]0, T[) \text{ and for}
\]
almost all $0 < t < T$ the following inequality holds:

\begin{equation}
\int_0^t \left( \Psi(s_i(p_1-p_2)(t)) - \Psi(s_{i1}) \right) + \int_0^t \sum_i \left( \sum_j \frac{k_{ij}(s_i(p_1-p_2)) \nabla u_{ij}}{k_i(s_i(p_1-p_2))} + \sum_j k_{ij}(s_i(p_1-p_2)) \nabla u_{ij} \cdot e_i \right) \\
< \sum_i \left( \int_0^t (s_i(p_1-p_2)(t) v_i(t) - s_{i1} v_i(0)) - \int_0^t s_i(p_1-p_2) \partial_i v_i \\
+ \int_0^t \int_{\Omega} \nabla v_i \left( \sum_j k_{ij}(s_i(p_1-p_2)) \nabla u_{ij} + k_i(s_i(p_1-p_2)) e_i \right) \right).
\end{equation}

Here $(k_{ij})_{ij}$ is the matrix in (1.12) with the convention that

$$k_{ij}(0) = 0 \quad \text{and} \quad \frac{k_{ij}}{\sqrt{k_i}}(0) = 0.$$

Note that $k_{ij}$ may be unbounded. The function $\Psi$ is defined as follows:

**2.4. Definitions.** We set

$$\Psi(x, z) := \sup_{p_{\min} \leq p \leq p_{\max}} \int_0^z (x - s_i(x, \xi)) \, d\xi.$$ 

Then

$$\Psi(x, s_i(x, z)) = \int_0^z (s_i(x, z) - s_i(x, \xi)) \, d\xi,$$

hence formally $\partial_i \Psi(s_i(p_1-p_2)) = \partial_i s_i(p_1-p_2)(p_1-p_2)$, and therefore the parabolic part in the variational inequality (2.1) formally equals

$$\sum_i \int_0^t \int_{\Omega} \partial_i s_i(p_1-p_2)(p_1-v_i).$$

If $(u_1, u_2)$ is obtained by (1.10) we have

$$u_1 - u_2 = \Psi(p_1-p_2),$$
2.5. REMARK. The second term on the left in the variational inequality (2.1) represents the natural topology and gives an estimate for the weak solution. In the case that \( k^*_i(s^*_i(z)) \) tends to zero faster than \( a(x, s_i(x, z)) \) as \( \varepsilon \rightarrow \min (P^{\text{mag}}) \) this estimate is stronger than the statement \( u_i \in L^2(0, T; H^\alpha(\Omega)) \). If \( \nabla p_i \) (in the sense of distribution) is a measurable function we can replace

\[
\sum_i k_{ij}(s_i(p_1 - p_2)) \nabla u_j \quad \text{by} \quad k_i(s_i(p_1 - p_2)) \nabla p_i .
\]

Thus we obtain the original variational inequality (1.8) with integrated parabolic part.

2.6. EXISTENCE THEOREM. Suppose in addition to 2.1 and 2.2 that the sets \( \Gamma_1^\eta \cap \Gamma_2^\eta \) and \( \Gamma_2^\eta \cap \Gamma_1^\eta \) are empty and that one of the following conditions is satisfied:

1) \( \mathcal{K}^{\alpha-1}(\Gamma_1^\eta \cap \Gamma_2^\eta) > 0 \),

2) \( \mathcal{K}^{\alpha-1}(\Gamma_1^\eta) > 0 \), \( p_{\min} > -\infty \), and \( u_{\max} < \infty \),

3) \( \mathcal{K}^{\alpha-1}(\Gamma_2^\eta) > 0 \), \( p_{\max} < +\infty \), and \( u_{\min} > -\infty \).

Then there exists a weak solution.

REMARK. The last condition in 2) and 3) can always be achieved by a suitable choice of \( k^*_i \), for example, if \( k^*_i(z) \) is replaced by

\[
\min \left( k^*_i(z), |s^*_{i-1}(z)|^{-\alpha} \right) \quad \text{with} \quad \alpha > 2 .
\]

PROOF. We approximate the conductivity \( k_i \) by positive functions

\[
k_{\varepsilon i} := \max (\varepsilon^2, k_i)
\]
and define $k_{ef}^*$ similarly. The water content we approximate by adding a penalizing term

$$s_{1i}(x, z) := s_1(x, z) + \varepsilon z, \quad s_{2i}(x, z) := s_2(x, z) - \varepsilon z.$$ 

Here

$$s_1(x, z) = \begin{cases} 0 & \text{for } z < p_{\min}, \\ s_0(x) & \text{for } z > p_{\max}, \end{cases}$$

and similarly for $s_2$. The approximating system with these coefficients is (nondegenerate) elliptic-parabolic and the existence of a solution for the corresponding variational inequality (1.8) can be shown similar to \cite[Theorem 1.7, Theorem 3.2]{1}. The difference is that here the variational inequality is only on the lateral boundary. But since it is convenient, although not necessary, for our convergence considerations let us include the approximation of the time derivative by backward difference quotients $\partial_t^{-h}$ in the proof here. Thus we start with solutions $(p_{h1}, p_{h2}) \in \mathcal{K}_h$ of the variational inequality

$$\sum_i \left( \int_{\Omega} \partial_t^{-h} s_{ei}(p_{h1} - p_{h2})(p_{h1} - v_i) + \int_{\Omega} (p_{h1} - v_i) k_{ei}(s_i(p_{h1} - p_{h2}))(\nabla p_{h1} + e_i) \right) < 0$$

at all times $0 < t < T$ for every $(v_1, v_2) \in \mathcal{K}_h$. Here $\mathcal{K}_h$ consists of all functions $(v_1, v_2) \in L^2(0, T; H^{1,2}(\Omega))$ which are time independent in each interval $[j-1, j]h$, $jh[ and satisfy the boundary conditions

$$v_i = p^0_i \quad \text{on } I^0_i \times [0, T],$$

$$v_1 - v_2 < p_{\max} \quad \text{on } I^0_1 \times [0, T],$$

$$v_1 - v_2 > p_{\min} \quad \text{on } I^0_2 \times [0, T].$$

The discrete Dirichlet data $p^0_{hi}$ are defined by

$$p^0_{hi}(t) := \int_{(j-1)h}^{jh} p^0(t) \, dr \quad \text{for } (j-1)h < t < jh,$$

and the approximate initial condition is

$$s_{1i}(p_{h1} - p_{h2})(t) = s^0_i \quad \text{for } -h < t < 0.$$
which is a condition on $p_{h(1)} - p_{h(2)}$ for $s_{e_1}$ are strictly monotone. The existence of a solution $p_{h(1)}$ of these inductively defined elliptic variational inequalities is assured since $\mathcal{E}^{s_{e_1}}(F^{p_1} \cap F^{p_2}) > 0$.

In order to obtain an a priori estimate set $v_i = p_{h(i)}^\rho$ in the time interval $]0, T[$. Then for the parabolic part

$$
\int_0^T \int_\Omega \frac{\partial}{\partial t} s_{e_1}(p_{h(1)} - p_{h(2)}) \left((p_{h(1)} - p_{h(2)}) - (p_{h(1)}^\rho - p_{h(2)}^\rho)\right) \, dx \, dt \\
> \int_0^T \int_\Omega \frac{\partial}{\partial t} s_{e_1}(p_{h(1)} - p_{h(2)}) \, dx \, dt \\
- \int_0^T \int_\Omega \left(s_{e_1}^\rho - s_{e_1}(\xi)\right) \, dx \, dt \\
- \frac{1}{t-h} \int_0^T \int_\Omega s_{e_1}(p_{h(1)} - p_{h(2)}) (p_{h(1)}^\rho - p_{h(2)}^\rho) + \int_0^{t-h} \int_\Omega s_{e_1}^\rho (p_{h(1)}^\rho - p_{h(2)}^\rho) \, dx \, dt \\
+ \int_0^T \int_\Omega s_{e_1}(p_{h(1)} - p_{h(2)}) \frac{\partial}{\partial t} s_{e_1}(p_{h(1)}^\rho - p_{h(2)}^\rho) \, dx \, dt \\
> \frac{\alpha}{2q} \int_0^T \int_\Omega |p_{h(1)} - p_{h(2)}|^2 - \frac{1}{t-h} \int_0^T \int_\Omega |p_{h(1)} - p_{h(2)}| \left|\frac{\partial}{\partial t} s_{e_1}(p_{h(1)} - p_{h(2)})\right| \, dx \, dt - C .
$$

Here we used the fact that

$$
\int_0^T \int_\Omega \left(s_{e_1}^\rho - s_{e_1}(\xi)\right) \, dx \, dt < \int_0^T \int_\Omega \left(s_{e_1}^\rho - s_{e_1}(\xi)\right) \, dx \, dt < \int_0^T \left|\nabla s_{e_1}(p_{h(1)} - p_{h(2)})\right|^2 < \infty .
$$

Hence including the elliptic part we obtain

$$
\epsilon \sup_{0 \leq t \leq T} \int_\Omega |p_{h(1)} - p_{h(2)}|^2 + \sum_{i=1}^T \int_\Omega k_{e_i} \left(s_{e_i}(p_{h(1)} - p_{h(2)})\right) |\nabla p_{h(i)}|^2 < C .
$$

Therefore if $u_{h(i)}$ are the transformed functions defined as in (1.10) with respect to the coefficients $k_{e_i}$ we conclude (see (1.11)) that $\nabla u_{h(i)}$ are bounded in $L^2(\Omega \times [0, T])$. Next we have to estimate $u_{h(i)}$ itself.

In case 1) the functions $u_{h(i)}$ have fixed bounded values on $(F^{p_1} \cap F^{p_2})$.
\( x \in ]0, T[ \) since \( p_{ih}^D \) are uniformly bounded functions. Therefore \( u_{hei} \) are bounded in \( L^2(0, T; H^{1,1}(\Omega)) \).

In case 2) we have \( I_1^D \subset I_2^D \cap I_2^0 \), therefore

\[
\begin{align*}
p_{hei} &= p_{h1}^D \quad \text{and} \quad p_{hei} - p_{hei}^D \geq p_{min} \quad \text{on} \quad I_1^D \times ]0, T[. \end{align*}
\]

In the part where \( p_{hei} - p_{hei}^D \leq 0 \) we conclude

\[
\begin{align*}
p_{h1}^D < p_{hei} < p_{h1}^D - p_{min},
\end{align*}
\]

that is, \( u_{hei} \) are bounded. In the remainder \( p_{hei} - p_{hei}^D > 0 \), hence

\[
\begin{align*}
u_{hei} &= p_{h1}^D \quad \text{and} \quad u_{hei} = p_{h1}^D - (u_{hei} - u_{hei}^D)
\end{align*}
\]

with \( u_{hei} - u_{hei}^D > 0 \). Thus if \( \max (u_{hei} - u_{hei}^D, 0) \) is bounded in \( L^2(\Omega \times ]0, T[) \), it is bounded in \( L^2(0, T; H^{1,1}(\Omega)) \) and therefore also in \( L^2(\Omega \times ]0, T[) \). Consequently \( u_{hei} \) are bounded in \( L^2((I_1^D \cap I_2^0) \times ]0, T[) \) and therefore again bounded in \( L^2(0, T; H^{1,1}(\Omega)) \). To prove an estimate for \( \max (u_{hei} - u_{hei}^D, 0) \) we note that in the set \( \{p_{hei} \geq p_{hei}^D\} \)

\[
0 < u_{hei} - u_{hei}^D = \int_0^{p_{hei} - p_{hei}^D} \sqrt{\frac{k^*_2(s^*_2(\xi))}{k^*_2(s^*_2(0))}} \, d\xi.
\]

Since \( k^*_2(s^*_2(\xi)) = 0 \) for \( \xi > p_{max}^* \) and \( k^*_2(s^*_2(0)) > 0 \), for amall \( \varepsilon \) this is estimated by

\[
\begin{align*}
&\min(p_{hei} - p_{hei}^D, p_{max}^*) \int_0^{p_{hei} - p_{hei}^D} \sqrt{\frac{k^*_2(s^*_2(\xi))}{k^*_2(s^*_2(0))}} \, d\xi + \int_0^{p_{hei} - p_{hei}^D} \varepsilon \sqrt{\frac{k^*_2(s^*_2(0))}{k^*_2(s^*_2(\xi))}} \, d\xi
\end{align*}
\]

which tends to \( u_{max} \) in \( L^\infty(0, T; L^2(\Omega)) \) by the above energy estimate. This proves the desired estimate.

In addition this argument shows that whenever \( u_{max} < \infty \) we have

\[
\max (u_{hei} - u_{hei}^D, 0) \to 0 \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)).
\]

Similarly, whenever \( u_{min} > - \infty \)

\[
\min (u_{hei} - u_{hei}^D, u_{hei}^D, 0) \to 0 \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)).
\]
We conclude that for a subsequence \( h \to 0, \varepsilon \to 0 \)
\[
\begin{align*}
u_{n+1} & \to \nu \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \\
\text{and} \\
\ell_{\min} & < \ell_1 < \ell_2 < \ell_{\max}.
\end{align*}

As a consequence we can go back with the transformation (1.10) and define \( p_1 \) and \( p_2 \) pointwise, satisfying the inequality \( p_{\min} < p_1 < p_2 < p_{\max} \).

The next step is to prove compactness of the functions \( s_{\varepsilon_1}(\ell_{\ell_1} - \ell_{\ell_2}) \), which essentially follows as in [1]. Indeed, if we choose in the equation for \( p_{n+1} \) in the time interval \([j-1)n, jn]\ the time independent function
\[
v_i = p_{n+1} = \eta^2(u_{n+1}(t)-u_{n+1}(m)h)
\]
where \( \eta \in C^\infty_c(\Omega), j > m, \) and \( (j-1)n < t < jn \), we get
\[
\begin{align*}
\int_\Omega \eta^2(s_{\varepsilon_1}(p_{n+1}-p_{n+2})(t)-s_{\varepsilon_1}(p_{n+1}-p_{n+2})(t-mh))
\cdot((u_{n+1}-u_{n+2})(t)-(u_{n+1}-u_{n+2})(t-mh))
\end{align*}
\]
\[
= -\sum_{i=1}^{j-1} \int_\Omega \eta^2(u_{n+1}(t)-u_{n+1}(t-mh))h(s_{\varepsilon_1}(p_{n+1}-p_{n+2}))\cdot(u_{n+1}-u_{n+2})(t-mh) dt.
\]
Since \( u_{n+1}, \nabla u_{n+1}, \) and \( k_{\varepsilon_1}(s_{\varepsilon_1}(p_{n+1}-p_{n+2})) \) \( \nabla p_{n+1} \) are bounded in \( L^2(\Omega \times [0, T]) \) we conclude integrating over \( t \)
\[
\text{(2.3)} \quad \int_0^T \int_\Omega \eta^2(s_{\varepsilon_1}(p_{n+1}-p_{n+2})(t)-s_{\varepsilon_1}(p_{n+1}-p_{n+2})(t-mh))
\cdot((u_{n+1}-u_{n+2})(t)-(u_{n+1}-u_{n+2})(t-mh)) dt \leq C mh.
\]
Since the functions involved are step functions in time the estimate remains valid if we replace \( mh \) by any positive number. Since \( \nabla u_{n+1} \) are in \( L^1(\Omega \times [0, T]) \) we also have
\[
\text{(2.4)} \quad \int_0^T \int_\Omega \eta^2((u_{n+1}-u_{n+2})(x+\xi, t)-(u_{n+1}-u_{n+2})(x, t)) dx dt \leq C |\xi|.
\]
For small \( \varepsilon > 0 \) define values \( p_{\min} < p_{\min}(x) < p_{\max}(x) < p_{\max} \) by
\[
s_1(x, p_{\min}(x)) = \varepsilon \quad \text{and} \quad s_1(x, p_{\max}(x)) = 0.
\]
Then the truncated functions
\[
b^\varepsilon(x, z) := s_1(x, \varphi^\varepsilon(x, z)) \quad \text{with} \ \varphi^\varepsilon(x, z) := \max(p_{\min}(x), \min((p_{\max}(x), z))
\]
satisfy
\[ |\beta^p(p_{ht1} - p_{ht2}) - \beta^e(p_{ht1} - p_{ht2})| < \varepsilon + \varepsilon|p_{ht1} - p_{ht2}|, \]
which is small in \( L^1(\Omega \times [0, T]) \) if \( \varepsilon \) and \( \varrho \) are small.

Therefore it suffices to show that \( \beta^p(p_{ht1} - p_{ht2}) \) are precompact in \( L^1(\Omega \times [0, T]) \) if \( \varrho > 0 \) is fixed. For \( \delta > 0 \) there is a small constant \( c(x, \delta) \) and a constant \( C(\delta) \) such that if
\[ |\beta^p(x, \xi) - \beta^e(x, \xi)| > \delta \]
then
\[ c(x, \delta) < |\psi^p(\xi) - \psi^e(\xi)| < C(\delta)|\psi^p(\xi) - \psi^e(\xi)| \]
uniformly in \( \varepsilon \), where \( \psi^p \) is defined as in (2.2) according to \( k^p_{di} \). Note that \( \psi^p \) is monotone. This yields
\[ |\psi^p(\xi) - \psi^e(\xi)| > c(x, \delta). \]

Then the sets
\[ E^\delta := \{ x \in \Omega; c(x, \delta) > \sigma \} \]
for fixed \( \delta > 0 \) define a monotone covering of \( \Omega \) and therefore for \( \eta \in C_c^\infty(\Omega \times [0, T]) \) by the estimates (2.3) and (2.4) on the time and space differences
\[
\begin{align*}
&\int_0^T \int_{\Omega} \eta^p(x, t)|\beta^p(p_{ht1} - p_{ht2})(x + \xi, t + r) - \beta^e(p_{ht1} - p_{ht2})(x, t)| \, dx \, dt < C\mathcal{L}^n(\Omega \setminus E^\delta) \\
&+ \int_0^T \int_{E^\delta} \eta^p(x, t-r)|\beta^p(p_{ht1} - p_{ht2})(x + \xi, t) - \beta^e(p_{ht1} - p_{ht2})(x, t)| \, dx \, dt \\
&+ \int_0^T \int_{E^\delta} \eta^p(x, t)|\beta^p(p_{ht1} - p_{ht2})(x, t + r) - \beta^e(p_{ht1} - p_{ht2})(x, t)| \, dx \, dt \\
&< C(\mathcal{L}^n(\Omega \setminus E^\delta) + \delta) \\
&+ C \frac{C(\delta)}{\sigma} \int_0^T \int_{E^\delta} \eta^p(x, t-r)|(u_{ht1} - u_{ht2})(x + \xi, t) - (u_{ht1} - u_{ht2})(x, t)| \, dx \, dt \\
&+ C(\delta) \frac{C(\delta)}{\sigma} \int_0^T \int_{E^\delta} \eta^p(x, t-r)|\beta^p(p_{ht1} - p_{ht2})(x, t + r) - \beta^e(p_{ht1} - p_{ht2})(x, t)| \\
&\cdot |(u_{ht1} - u_{ht2})(x, t + r) - (u_{ht1} - u_{ht2})(x, t)| \, dx \, dt. 
\end{align*}
\]
Since the last integral is dominated by the left side of (2.3), and using (2.4) we get an estimate by
\[\langle G^a(\Omega \setminus E^a_r) + \delta + \frac{G(\delta)}{\sigma} (|\bar{z}| + |r|)\rangle,\]
which proves the desired compactness.

Therefore \(s_t(p_{he1} - p_{he2})\) has a strong limit in \(L^1(\Omega \times ]0, T[)\) and by the standard monotonicity argument, that is, using the fact that for \(v \in L^2(0, T; H^{1/2}(\Omega))\) with \(p_{\text{min}} \leq v \leq p_{\text{max}}\)
\[\left(s_t(\psi^{-1}(v)) - s_t(p_{he1} - p_{he2})\right)(v - (u_{he1} - u_{he2})) \geq 0,
\]
this limit equals \(s_t(p_1 - p_2)\).

We also have to prove that the weak limit \(u_t\) is admissible, that is, of class \(K^a\), which is not obvious since the strong convergence of the functions \(u_{he t}\) is not yet known. But since \(b^l(p_{he1} - p_{he2}) \to b^l(p_1 - p_2)\) almost everywhere in \(\Omega \times ]0, T[\), as just proved, also \(q^l(p_{he1} - p_{he2}) \to q^l(p_1 - p_2)\) almost everywhere, consequently \(u_{he1} - u_{he2} \to u_1 - u_2\) almost everywhere in \(\Omega \times ]0, T[\). But since \(u_{he t}\) are bounded in \(L^2(0, T; H^{1/2}(\Omega))\) this implies that \(u_{he1} - u_{he2} \to u_1 - u_2\) almost everywhere \(\partial \Omega \times ]0, T[\). Now on \(I^a_1 \times ]0, T[\)
\[p_{he1} = p^D_{h_1} \quad \text{and} \quad p_{he1} - p_{he2} \geq p_{\text{min}}\]
since \(I^D_1 \subset I^D_2 \cap I^D_{\frac{3}{2}}\), that is, \((u_{he1}, u_{he2})\) lies on a curve \n\{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2; \quad \varepsilon_1 + \varepsilon_2 = \gamma(\varepsilon_1 - \varepsilon_2), \quad u_{\text{min}} < \varepsilon_1 - \varepsilon_2 < u_{\text{max}}\}\n
with continuous functions \(\gamma_\varepsilon\) converging uniformly to some \(\gamma\). Hence
\[u_{he1} + u_{he2} = \gamma_\varepsilon(u_{he1} - u_{he2})\]
and for \(\varepsilon \to 0\) we obtain
\[u_1 + u_2 = \gamma(u_1 - u_2)\]
that is, \((u_1, u_2)\) is admissible.

Finally we have to show that \(u_t\) satisfies the variational inequality in 2.3. For this approximate any function \(v_t\) as in 2.3 in the corresponding norms by functions \(v_{hi} \in K_h\) and write the equation for \(p_{he t}\) in the form (three
positive terms on the left side are omitted)\n
\[
\left(2.5\right) \int_{t-h}^{t} \int_{\Omega} \left( (\Psi(s_{1}(p_{h} - p_{h})) - \Psi(s_{1})) \right) + \sum_{i} \int_{0}^{t} \int_{\Omega} k_{si}(s_{i}(p_{h} - p_{h})) |\nabla p_{h}|^{2} + \sum_{i} \int_{0}^{t} \int_{\Omega} k_{si}(s_{i}(p_{h} - p_{h})) \nabla p_{h} \cdot e_{i} \nabla p_{h} \cdot e_{i}
\]

\[< \int_{t-h}^{t} \int_{\Omega} s_{e_{i}}(p_{h} - p_{h})(v_{h_{1}} - v_{h_{2}}) - \int_{t-h}^{t} \int_{\Omega} s_{e_{i}}(v_{h_{1}} - v_{h_{2}}) \nabla p_{h} \cdot e_{i}
\]

\[= \int_{0}^{t} \int_{\Omega} k_{si}(s_{i}(p_{h} - p_{h})) \nabla v_{h_{1}} \cdot e_{i} + k_{si}(s_{i}(p_{h} - p_{h})) \nabla p_{h} \cdot \nabla v_{h_{1}}.
\]

Since \(s_{i}(p_{h} - p_{h})\) converges almost everywhere the first integral on the left and all terms on the right except the last one converge to the desired limit. Next we look at the last terms on both sides. Let \(\phi > 0\) and \(\varepsilon < \phi\). Then

\[
\int_{0}^{t} \int_{\Omega} k_{si}(s_{i}(p_{h} - p_{h})) \nabla v_{h_{1}} \cdot e_{i} \nabla v_{h_{1}}
\]

\[= \sum_{i} \int_{0}^{t} \int_{\Omega} k_{si}(s_{i}(p_{h} - p_{h})) \nabla v_{h_{1}} \cdot e_{i} + \mathcal{R}_{k_{si}}\]

with

\[
k_{si}(x, z) := \begin{cases} \max\left(k_{si}(x, z) - \phi, 0\right), & \text{for } z > 0, \\ k_{si}(x, z), & \text{for } z = 0 \end{cases}
\]

where \((k_{x/y})_{ij}\) is the matrix in (1.12) corresponding to the functions \(k_{xi}\) and \(k_{si}^{*}\). If \(k_{x/y}(x, s_{i}(x, z)) > 0\) then by 2.1

\[
\phi < k_{x}(x, s_{i}(x, z)) < C_{1}(z)
\]
hence \( z > p_{\text{min}} + c(\varrho) \) (\( z > -C(\varrho) > -\infty \)), if \( p_{\text{min}} = -\infty \) and therefore \( k_{ij}^\varrho(\varrho(\varrho_{h_1} - \varrho_{h_2})) \) are bounded functions uniformly in \( h \) and \( \varrho \), and converge almost everywhere as \( h \to 0 \), \( \varrho \to 0 \). We conclude

\[
\sum_j \int_0^t \int_{\Omega} k_{ij}^\varrho(\varrho(\varrho_{h_1} - \varrho_{h_2})) \nabla u_{h_1j} \cdot \nabla v_{h_1} \to \sum_j \int_0^t \int_{\Omega} k_{ij}^\varrho(\varrho(\varrho_1 - \varrho_2)) \nabla u_j \cdot \nabla v_1 .
\]

As we will see in a moment

\[
\sum_j k_{ij}(\varrho_1 - \varrho_2) \nabla u_j
\]

is bounded in \( L^2(\Omega \times [0, T]) \), hence as \( \varrho \downarrow 0 \) this converges to the desired limit.

For the remainder we have

\[
|\mathcal{R}_{\varrho}(\varrho)| < \delta \int_0^t \int_{\Omega} k_{ij}(\varrho(\varrho_{h_1} - \varrho_{h_2})) |\nabla \varrho_{h_1}|^2 + \frac{C_0}{\delta}
\]

for any small \( \delta > 0 \) which finally will tend to zero. The first term on the right can be absorbed by the second integral on the left in the variational inequality (2.5), and the second term is small for small \( \varrho \). It remains to consider

\[
\mathcal{J}_{\varrho} := \sum_j \int_0^t \int_{\Omega} k_{ij}(\varrho(\varrho_{h_1} - \varrho_{h_2})) |\nabla \varrho_{h_1}|^2
\]

\[
= \sum_j \int_0^t \int_{\Omega} k_{ij}(\varrho(\varrho_{h_1} - \varrho_{h_2})) \nabla u_j \cdot \nabla v_i,
\]

From what has been shown up to now it follows that

\[
(1 - \delta) \mathcal{J}_{\varrho} < \sum_j \int_0^t \int_{\Omega} k_{ij}(\varrho(\varrho_{h_1} - \varrho_{h_2})) \nabla u_{h_1j} \left( |\nabla v_{h_1}| + |v_i| \right) \left( |v_{h_1}| + |v_i| \right) + C_0 \left( v_1, v_2 \right) + \frac{C_0}{\delta}.
\]

Since

\[
\left| \sum_j k_{ij}(\varrho(\varrho_{h_1} - \varrho_{h_2})) \nabla u_{h_1j} \right| > \left| \sum_j k_{ij}(\varrho(\varrho_{h_1} - \varrho_{h_2})) \nabla u_{h_2j} \right|
\]
we obtain

$$
\int_0^t \int_D \left|\sum_{i} k_{\text{eff}}(s_i (p_{he1} - p_{he2})) \nabla u_{\text{eff}}\right|^2 \, dx < C,
$$

and since \( k_{\text{eff}}(s_i (p_{he1} - p_{he2})) \) converges almost everywhere also

$$
\int_0^t \int_D \left|\sum_{i} k_{\text{eff}}(s_i (p_1 - p_2)) \nabla u_j\right|^2 \, dx < C,
$$

which was used above. Moreover for fixed \( \varepsilon_0 > 0 \) and \( \varepsilon < \varepsilon_0 \)

$$
\left|\sum_{i} \frac{k_{\text{eff}}(s_i (p_{he1} - p_{he2})) \nabla u_{\text{eff}}}{k_{\text{eff}}(s_i (x))} \right|^2 \geq \left|\sum_{i} \varphi_{\text{eff}}^n(p_{he1} - p_{he2}) \nabla u_{\text{eff}}\right|^2
$$

with

$$
\varphi_{\text{eff}}^n(x) := \frac{\sqrt{k_{\text{eff}}(s^0_i (x))} k_{\text{eff}}(s_i (x)) / \sqrt{k_{\text{eff}}(s^0_i (x))}}{\sqrt{k_{\text{eff}}(s_i (x))}}.
$$

Since the numerator of \( \varphi_{\text{eff}}^n \) is uniformly bounded and the denominator strictly positive, \( \varphi_{\text{eff}}^n(p_{he1} - p_{he2}) \) converges almost everywhere, which yields

$$
\lim_{(h,\varepsilon) \to 0} \inf \sum_{i} \int_0^T \int_D \left|\sum_{i} \varphi_{\text{eff}}^n(p_1 - p_2) \nabla u_j\right|^2
$$

and as \( \varepsilon_0 \to 0 \) this converges to the desired integral.

The variational inequality (2.1) contains all information about the solution. In particular we shall show that weak solutions in the sense of 2.3 are weak solutions of the differential equation (1.1).

2.7. LEMMA. Let \( p_1, p_2 \) be a weak solution as in (2.3). Then

$$
\partial_t s_i(p_1 - p_2) \in L^2(0, T; \tilde{H}^{-1}(\Omega)^*)
$$

with initial values \( s_i^0 \), that is,

$$
(2.6) \quad \int_0^T \langle \partial_t s_i(p_1 - p_2), \zeta \rangle + \int_0^T \int_D (s_i(p_1 - p_2) - s_i^0) \partial_t \zeta = 0, \quad i = 1, 2,
$$
for every $\xi \in L^2(0, T; \dot{H}^{1,2}(\Omega))$ with $\partial_t \xi \in L^1(\Omega \times ]0, T[)$ and $\xi(T) = 0$. And the differential equation

\begin{equation}
\partial_t s_i (p_1 - p_2) - \nabla \cdot \left( \sum_j k_j (s_i (p_1 - p_2)) \nabla u_i + k_i (s_i (p_1 - p_2)) e_i \right) = 0, \quad i = 1, 2,
\end{equation}

holds in $L^2(0, T; \dot{H}^{1,2}(\Omega)^*)$.

**Proof.** Formally this follows from (2.1) by setting $v_i = p_i \pm \xi$, for $\xi$ as in the statement of the lemma. But since the space and time behavior of $p_i$ is not good enough to use it as a test function we have to approximate these functions.

For this choose sequences $u^\varepsilon_{\min}, u^\varepsilon_{\max}$ and $u^\varepsilon_{\max}, u^\varepsilon_{\max}$ as $\varepsilon \downarrow 0$ and define the truncated functions

\begin{equation}
\begin{aligned}
u^\varepsilon_1 &= \frac{u^\varepsilon_1 - u^\varepsilon_2}{2} + \frac{1}{2} \max\left(u^\varepsilon_{\min}, \min\left(\frac{u^\varepsilon_{\max}}{2}, u^\varepsilon_1 - u^\varepsilon_2\right)\right), \\
u^\varepsilon_2 &= \frac{u^\varepsilon_1 - u^\varepsilon_2}{2} - \frac{1}{2} \max\left(u^\varepsilon_{\min}, \min\left(\frac{u^\varepsilon_{\max}}{2}, u^\varepsilon_1 - u^\varepsilon_2\right)\right).
\end{aligned}
\end{equation}

Then $u^\varepsilon_i \in L^2(0, T; H^{1,2}(\Omega))$ and also the corresponding pressure values $p_i^\varepsilon$ defined by (1.10), that is,

\begin{equation}
\begin{aligned}
u^\varepsilon_i &= p^\varepsilon_i \quad \text{and} \quad u^\varepsilon_2 = p^\varepsilon_1 - \psi(p^\varepsilon_1 - p^\varepsilon_2) \quad \text{in} \{u^\varepsilon_2 > 0\}, \\
u^\varepsilon_i &= p^\varepsilon_2 + \psi(p^\varepsilon_1 - p^\varepsilon_2) \quad \text{and} \quad u^\varepsilon_2 = p^\varepsilon_2 \quad \text{in} \{u^\varepsilon_2 < 0\},
\end{aligned}
\end{equation}

are of this class. Similarly we define $p^\varepsilon_i$ starting from the transformed functions $u^\varepsilon_i$ of $p_i^\varepsilon$. Then the functions

$$w_i := p^\varepsilon_i + (p^\varepsilon_1 - p^\varepsilon_i)$$

satisfy

$$w_i = p^\varepsilon_i \quad \text{on} \ I_i \times ]0, T[$$

and in $\Omega \times ]0, T[$

\begin{equation}
w_1 - w_2 < p^\varepsilon_1 - p^\varepsilon_2 + p^\varepsilon_{\max} - (p^{\varepsilon_1} - p^{\varepsilon_2}) < p_{\max},
\end{equation}

where $\psi(p_{\max}) = u^\varepsilon_{\max}$. Similarly $w_i - w_i > p_{\min}$. In particular, $w_i$ are of class $K$. 
As test function in (2.1) we use

\[
\begin{align*}
v^e_1 := & \frac{w^e_1 + w^e_2}{2} + \frac{1}{2} \max \left( p_{\min}, \min \left( p_{\max}, w^e_1 - w^e_2 \right) \right) \pm \zeta_1, \\
v^e_2 := & \frac{w^e_1 + w^e_2}{2} - \frac{1}{2} \max \left( p_{\min}, \min \left( p_{\max}, w^e_1 - w^e_2 \right) \right) \pm \zeta_1.
\end{align*}
\]

Here \( \zeta_1 \) are as above such that \( \zeta(t) = 0 \) for \( t \) near \( T \), and for small \( h > 0 \) and \( 0 < \tau < h, 0 < \varepsilon \) and \( \delta \) the function \( w^e_1 \) is defined by

\[
w^e_1(t) := p^e_1(t) + (p^e_1 - p^e_2)(jh - r)
\]

\[
\quad \quad \quad + \max \left( 0, 1 - \frac{(j + 1)h - r - 1}{\varepsilon} \right) \left( p^e_1 - p^e_2 \right)(j + 1)(h - r) - (p^e_1 - p^e_2)(jh - r)
\]

whenever \( jh - r < (j + 1)h - \tau \), where \( j = 0, \ldots, j_h \) with \( t_h - h < t < t_h \), \( t_h := j_h h \), and given \( t \) near \( T \). In this definition \( p^e_1(t) := p^e_1(t) \) for \( t < 0 \) and the initial value \( p^e_1(t) := p^e_1(t) \) for \( t > 0 \) is chosen in \( H^1(\Omega) \) such that \( \frac{\partial}{\partial t} p^e_1 - \frac{\partial}{\partial x} p^e_2 < p_{\max} \) and

\[
\int_\Omega \left( \Psi(s^e_0) - \int_0^1 (s^e_0 - s_0(\xi)) d\xi \right) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

By construction \( v^e_1 \) are of class \( K \) and \( \partial_x v^e_1 \) are in \( L^1(\Omega \times [0, T]) \). Furthermore by (2.10)

\[
v^e_1(t) = w_1(t) \pm \zeta_1(t) \quad \text{for} \quad t = jh - r.
\]

Then the \( \zeta_1 \) terms in (2.1) give the assertion provided we can show that for \( \varepsilon = 0 \) the right side in (2.1) does not exceed the left in the limit \( \varepsilon \to 0, h \to 0, \) and \( \delta \to 0. \) First let us consider the parabolic terms. For almost all \( \tau \) almost everywhere in \( \Omega \) writing \( s_1(t) \) for \( s_1(x, (p_1 - p_2)(x, t)) \)

\[
\int_{jh - r}^{(j + 1)h - r} s_1 \partial_x (v^e_1 - v^e_2)
\]

\[
= \int_{jh - r}^{(j + 1)h - r} \chi \left( \{ p_{\min} < w^e_1 - w^e_2 < p_{\max} \} \right) \left( s_1 - s_1((j + 1)h - \tau) \right) \partial_x (w^e_1 - w^e_2)
\]

\[
+ s_1((j + 1)h - \tau) \left( (w^e_1 - w^e_2)(j + 1)h - \tau) - (w^e_1 - w^e_2)(jh - r) \right)
\]
The second term tends to zero as $\varepsilon \to 0$, hence summing over $j$ and integrating over $\Omega$ we obtain

\begin{equation}
\left(2.13\right) \quad \lim_{\varepsilon \to 0} \int_0^{t_\varepsilon} \int_\Omega \partial_t (v_1^{\text{res}} - v_2^{\text{res}}) \nonumber
\end{equation}

\[
\begin{align*}
&\sum_{j=0}^{t_\varepsilon-1} \int_\Omega \left| s_1 - s_1((j + 1)\tau - \tau) \right| \left| \partial_i (p_1^p - p_2^p) \right| \\
&- \sum_{j=0}^{t_\varepsilon-1} \int_\Omega \left| s_1 - s_1((j + 1)\tau - \tau) \right| \left| \partial_i (p_1^{pe} - p_2^{pe}) \right| \\
&+ \sum_{j=0}^{t_\varepsilon-1} \int_\Omega \left( (p_1^p - p_2^p)((j + 1)\tau - \tau) - (p_1^{pe} - p_2^{pe})(j\tau - \tau) \right) \\
&+ \sum_{j=0}^{t_\varepsilon-1} \int_\Omega \left( (p_1^p - p_2^p)((j + 1)\tau - \tau) - (p_1^{pe} - p_2^{pe})(j\tau - \tau) \right).
\end{align*}
\]

For the first term on the right of (2.1) we have

\begin{equation}
\left(2.14\right) \quad \int_\Omega \left( (s_1(t_\varepsilon - \tau)(v_1^{\text{res}} - v_2^{\text{res}})(t_\varepsilon - \tau) - s_1^0(v_1^{\text{res}} - v_2^{\text{res}})(0)) \right) \nonumber
\end{equation}

\[
\begin{align*}
&\leq C \int_\Omega \left| (p_1^p - p_2^p)(t_\varepsilon - \tau) - (p_1^{pe} - p_2^{pe})(t_\varepsilon - \tau) \right| \\
&+ C \int_\Omega \left| (p_1^p - p_2^p)(0) - (p_1^{pe} - p_2^{pe})(0) \right| \\
&+ \sum_{j=0}^{t_\varepsilon-1} \int_\Omega \left( s_1(t_\varepsilon - \tau)(p_1^p - p_2^p)(t_\varepsilon - \tau) - s_1^0(p_1^p - p_2^p)(0) \right).
\end{align*}
\]
Subtracting (2.13) from (2.14) we obtain for the last terms on the right

\[ \int \sum_{j=0}^{h-1} \left( s_1(j+1) - s_1(j) \right) \left( p_1^\tau - p_2^\tau \right) (j \tau - r) \]

\[ = \int \sum_{j=0}^{h-1} \left( s_1(j+1) - s_1(j) \right) \left( \frac{s_1^0 - s_1^0}{(j \tau - r)} \right) \]

Integrating over \( \tau \) from 0 to \( h \) and dividing by \( h \) this converges to

\[ \int (\Psi(s_1(t_0)) - \Psi(s_1^0)) \]

for almost all \( t_0 \), where (2.12) is used. This is the parabolic term on the left of (2.1). Thus we have to verify that the remaining terms in (2.13) and (2.14) are small. Since it was assumed that \( \partial_t p_1^D \) are in \( L^r(\Omega \times [0, T]) \) for some \( r > 1 \) the first term on the right of (2.13) is small for small \( h \) after performing the mean over \( \tau \). The second term is estimated uniformly in \( \tau \) by

\[ C \int \max \left( |\partial_t \max (p_1^D - p_2^D, 0)| \right) \]

which tends to zero with \( \rho \). The first term on the right of (2.14) converges for almost all \( t_0 \) in the mean over \( \tau \) to

\[ C \int |(p_1^D - p_2^D)(t_0) - (p_1^D - p_2^D)(t_0)| \]

\[ < C \int \max \left( (p_1^D - p_2^D)(t_0) - p_\max, 0 \right) + \max \left( p_\min - (p_1^D - p_2^D)(t_0), 0 \right) \]

which tends to zero as \( \rho \to 0 \). The same holds for the second term.

Now let us consider the elliptic term on the right of (2.1). First we note that for almost all \( \tau \) as \( \varepsilon \to 0 \) the functions \( \tau^\varepsilon \) converge in \( L^2(0, T; H^2(\Omega)) \)
to \( v^*_i \), which is defined as in (2.11) with
\[
\omega^*_i := p^D_i(t) + (p^q_i - p^D_i)(jh - \tau) \quad \text{for } jh - \tau < t < (j + 1)h - \tau.
\]

Therefore as \( \epsilon \rightarrow 0 \),

\[
\text{we see that}
\]

Similarly we argue in \( \phi \rightarrow 0 \). Consequently, we have
\[
\text{In we have}
\]

we see that
\[
|\nabla (p^D_1 - p^D_2)| = \frac{1}{2} \psi'(p^D_1 - p^D_2)|\nabla (p^D_1 - p^D_2)|,
\]
\[
|\nabla (p^D_2 - p^D_2)| < (1 + \frac{1}{2} \psi'(p^D_1 - p^D_2))|\nabla (p^D_1 - p^D_2)|.
\]

Therefore as \( \phi \rightarrow 0 \)
\[
\int_0^{t_f} \int \chi(\{p^D_1 - p^D_2 > 0\})|\nabla (p^D_1 - p^D_2)|^2 \leq \int_0^{t_f} \int \chi(\{p^D_1 - p^D_2 > p^q_1\})|\nabla (p^D_1 - p^D_2)|^2
\]
\[
= \int_0^{t_f} \int \chi(\{p^D_1 - p^D_2 = p^q_1\})|\nabla (p^D_1 - p^D_2)|^2 = 0.
\]

Similarly we argue in \( \{p^D_1 - p^D_2 < 0\} \). Hence it remains to show that for \( \phi \rightarrow 0 \)
\[
\int_0^{t_f} \int \nabla p^q_i \left( \sum_{j} k_{ij}(s_i) \nabla u_j + k_i(s_i) e_i \right)
\]
\[
\text{does not exceed the second integral on the left of (2.1). In } \{w^q_{\min} < u_1 - u_2 < w^q_{\max}\} \text{ we have } p^q_i = p_i \text{ and therefore}
\]
\[
\nabla p^q_i = \frac{1}{k_i(s_i)} \sum_{j} k_{ij}(s_i) \nabla u_j.
\]

In \( \{u_1 - u_2 > w^q_{\max}\} \) we have
\[
p^q_i = u^q_i = u_1 - \frac{1}{2}(u_1 - u_2 - w^q_{\max}) \quad \text{and} \quad p^q_2 = p^q_1 - p^q_{\max}.
\]
hence
\[ \nabla p_t^e - \frac{1}{k_i(s_i)} \sum_j k_{ij}(s_i) \nabla u_j = \nabla (p_t^e - u_t) = -\frac{1}{2} \nabla (u_t - u_2). \]

Since \( \nabla (u_t - u_2) \) and \( \sum_j k_{ij}(s_i) \nabla u_j \) are in \( L^2(\Omega \times [0, T]) \) we conclude
\[ \int_0^T \int_\Omega \chi(\{u_1 - u_2 < u_{\text{max}}\}) (\nabla p_t^e - \frac{1}{k_i(s_i)} \sum_j k_{ij}(s_i) \nabla u_j) \left( \sum_i k_{ij}(s_i) \nabla u_j + k_i(s_i) e_i \right) \]
\[ \leq C \left( \int_0^T \int_\Omega \chi(\{u_1 - u_2 < u_{\text{max}}\}) |\nabla (u_1 - u_2)|^2 \right)^{1/2} \]
\[ \rightarrow C \left( \int_0^T \int_\Omega \chi(\{u_1 - u_2 < u_{\text{max}}\}) |\nabla (u_1 - u_2)|^2 \right)^{1/2} = 0. \]

Finally let us consider the integral with \( \nabla p_t^e \). In \( \{u_1 - u_2 = u_{\text{max}}\} \) we have \( k_2(s_2) = 0 \), hence the integral gives no contribution. In \( \{u_{\text{max}} < u_1 - u_2 < u_{\text{max}}\} \) we compute
\[ \nabla p_t^e - \frac{1}{k_2(s_2)} \sum_j k_{2j}(s_2) \nabla u_j = \nabla p_t^e - ((1 - \kappa) \nabla u_1 + \kappa \nabla u_2) = \left( -\frac{1}{2} + \kappa \right) \nabla (u_1 - u_2), \]
where
\[ \kappa := \sqrt{\frac{k_2^e(s_2^e(0))}{k_2^e(s_2^e(p_1 - p_2))}}. \]

Since \( \kappa \) is unbounded we have to argue more carefully. From (2.1) we know that
\[ \frac{1}{k_2(s_2)} \left| \sum_j k_{2j}(s_2) \nabla u_j \right| \in L^1(\Omega \times [0, T]), \]
that is,
\[ \sqrt{k_2(s_2)} ((1 - \kappa) \nabla u_1 + \kappa \nabla u_2) \in L^1(\Omega \times [0, T]), \]
and therefore also
\[ \sqrt{k_2(s_2)} \kappa \nabla (u_1 - u_2) \in L^1(\Omega \times [0, T]). \]
We conclude

\[
\int_0^T \int_\Omega \chi(\{u_{\text{max}}^0 < u_1 - u_2 < u_{\text{max}}\}) \left( \nabla p_i^e - \frac{1}{k_2(s_2)} \sum_I k_2(s_2) \nabla u_i \right) \left( \sum_I k_2(s_2) \nabla u_i + k_2(s_2) c_2 \right) d\Omega dt
\]

\[
= \int_0^T \int_\Omega \chi(\{u_{\text{max}}^0 < u_1 - u_2 < u_{\text{max}}\}) \left( -\frac{1}{2} + \varepsilon \right) \nabla(u_1 - u_2) k_2(s_2) \left( (1 - \varepsilon) \nabla u_1 + \varepsilon \nabla u_2 + c_2 \right) d\Omega dt
\]

\[
< C \left( \int_0^T \int_\Omega \chi(\{u_{\text{max}}^0 < u_1 - u_2 < u_{\text{max}}\}) k_2(s_2) \left( -\frac{1}{2} + \varepsilon \right) \nabla(u_1 - u_2) \right)^2 \rightarrow 0
\]

as \( \varepsilon \rightarrow 0 \). This completes the proof.

2.8. REMARK. For a weak solution \( p_1, p_2 \) define

\[
P_i := \frac{1}{k_i(s_i(p_1 - p_2))} \sum_I k_i(s_i(p_1 - p_2)) \nabla u_i
\]

in \( \{ k_i(s_i(p_1 - p_2)) > 0 \} \) and \( P_1 = 0 \) in \( \{ p_1 - p_2 = p_{\text{min}} \} \), \( P_2 = 0 \) in \( \{ p_1 - p_2 = p_{\text{max}} \} \). Then

\[
k_i(s_i(p_1 - p_2)) P_i \in L^1(\Omega \times [0, T]),
\]

and \( P_i \) is the limit of gradients in the following sense. If \( p_i^e \) are defined as in the proof of Lemma 2.7 then as shown above

\[
P_i = \nabla p_i^e \quad \text{in} \quad \{ p_{\text{min}} < p_1 - p_2 < p_{\text{max}}^e \}
\]

and

\[
\int_0^T \int_\Omega k_i(s_i(p_1 - p_2)) |P_i - \nabla p_i^e|^2 \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

Moreover the variational inequality (2.1) reads

\[
\int_\Omega (\mathcal{W}(s_i(p_1 - p_2)(t)) - \mathcal{W}(s_i^e)) - \int_\Omega ((s_i(p_1 - p_2)(t) (v_1 - v_2) (t) - s_i^e(v_1 - v_2)(0)))
\]

\[
+ \int_0^t \int_\Omega (p_1 - p_2) \partial_t (v_1 - v_2) + \sum_I \int_0^T \int_\Omega k_i(s_i(p_1 - p_2)) (P_i + e_i) (P_i - \nabla v_i) < 0
\]

for all \((v_1, v_2) \in \mathcal{K}\) with \( \partial_t v_i \in L^1(\Omega \times [0, T]) \) and for almost all \( t \).
3. Continuity of the saturation.

We shall prove that the saturation \( s = s_i(p_i - p_s) \) is continuous in space and time (see Theorem 3.5). For this we introduce the mean pressure \( u \) defined by (1.15) and consider the transformed system (1.16), (1.17), which consists of a parabolic equation for \( v \) and an elliptic equation for \( u \). Therefore \( u \) and \( v \) are the natural variables of the problem if one wants to separate the parabolic and elliptic nature of the system (1.1).

As assumption we need that the diffusion coefficient for \( v \) degenerates at most at one side, and that the mean pressure \( u \) is bounded. Sufficient conditions for \( u \) being bounded are given in 3.9. To perform the estimates we also need various conditions on the coefficients, in particular they should be smooth enough as functions of the space variable (see 3.1).

The proof of the Regularity theorem 3.5 consists of two parts, an estimate from above (see 3.7) and from below (see 3.8). The proofs of these estimates (see section 4 and 5) follow the lines of the De Giorgi techniques, where the special features here are the degeneracy of the coefficient \( a \) in the parabolic equation for \( v \) and the coupling to the elliptic equation for \( u \).

3.1. Assumption on the data. Let \( D \) an open subset of \( \Omega \). We assume that \( s_i \) is continuous differentiable with respect to the second variable in \( D \times \{ p_{\min} < z < p_{\max} \} \), and we assume the following qualitative behavior of the coefficients

\[
\partial_z s_i(x, z) > 0 ,
\]

\[
\frac{\partial_i k_i(x, s_i(x, z))}{\partial_z s_i(x, z)} > c(\delta) > 0 \quad \text{for } x \in D, \quad z < p_{\max} + p_{\max} ,
\]

\[
\left| \frac{k_i(x, s_i(x, z))}{\partial_z s_i(x, z)} \right| < C \quad \text{for } x \in D \text{ and } z < 0 \ (> 0) \text{ if } i = 1 \ (2) ,
\]

\[
|k_i(x, s_i(x, z))(\partial_z k_i)(x, s_i(x, z)) + k_i(x, s_i(x, z))(\partial_z^2 k_i)(x, s_i(x, z))| < C .
\]

For the dependence on \( x \) we assume

\[
s_i(x, z) \to 0 \quad \text{as } \quad z \to p_{\min} \quad \text{uniformly for } x \in D ,
\]

\[
|\nabla s_i| < C ,
\]

\[
\left| \frac{k_i(x, s_i(x, z))}{\partial_z s_i(x, z)} \cdot \nabla s_i(x, z) \right| < C \quad \text{for } x \in D \text{ and } z < 0 \ (> 0) \text{ if } i = 1 \ (2) ,
\]
3.2. REMARK. The coefficients defined in (1.19) satisfy

\[ c < k < C, \quad |d| < C. \]

From (3.3), (3.7), and (3.9) it follows that

\[ |a| + |b| + |e| < C, \]

and (3.2) implies that for every \( \delta > 0 \)

\[ \inf_{x \in D, |x| < 1-\delta} a(x, z) > 0, \]

that is, the diffusion coefficient \( a \) for \( v \) is coercive near 0 and degenerate at 1, but we impose no restriction on the nature of this degeneracy.

By (3.4), (3.8), and (3.6)

\[ |\partial_z d| + |\nabla_z d| < C. \]

Moreover if

\[ A(x, z) := \int_0^z a(x, \xi) \, d\xi, \]

then (3.7) and (3.10) imply that

\[ |\nabla A| < C. \]

In the proof of the regularity theorem we will use these properties of the coefficients only (besides (3.6)).

First let us prove that the transformed functions \( u \) and \( v \) satisfy the differential equations (1.16) and (1.17).

3.3. LEMMA. Assume the data satisfy 2.1, 2.2, and 3.1. For any weak solution \( p_1, p_2 \) define \( u, v \) as in (1.14) and (1.15). Then \( \partial_t (p_1 v) \in L^2(0, T; \)
\( \hat{H}^{1,\gamma}(\Omega) \) with initial values \( s^\omega_0 \) (in the sense of (2.6)) and in this space the differential equations

\[
\begin{align*}
(3.11) \quad & \nabla \cdot \left( k(v) \lim_{\delta \to 0} \nabla u^\omega + e(v) \right) = 0 \quad \text{(define } v := - \left( k(v) \lim_{\delta \to 0} \nabla u^\omega + e(v) \right) ), \\
(3.12) \quad & \partial_t (s^\omega v) = \nabla \cdot \left( \lim_{v \to 0} a(v) \nabla \min(v, 1 - v) + b(v) + d(v) v \right)
\end{align*}
\]

are satisfied with coefficients \( k, a, b, d \) given by (1.19). The limits in (3.11) and (3.12) exist in \( L^1(\Omega \times [0, T]) \), where \( u^\omega \) is defined in (3.13) below (see Remark 3.4). In particular \( u^\omega \) and \( \min(v, 1 - v) \) belong to \( L^1(0, T; H^{1,\gamma}(\Omega)) \).

**Proof.** By (2.7) we have to consider

\[
(3.13) \quad \sum_i k_{ii}(s_i(p_i - p_0)) \nabla u_i + k_i(s_i(p_i - p_0)) e_i, \quad i = 1, 2.
\]

The sum of both expression in (3.13) equals

\[
k(v) \left( \frac{1}{k(v)} \sum_i k_{ii}(s_i(p_i - p_0)) \nabla u_i \right) + \sum_i k_i(s_i(p_i - p_0)) e_i.
\]

Since \( \partial_t (s_i(p_i - p_0) + s_i(p_i - p_0)) = 0 \), the only thing to show for \( u \) is that \( \nabla u^\omega \) has the desired limit. Here

\[
(3.14) \quad u^\omega(x, t) := p_i^\omega(x, t) + \int_0^{x^\omega(s_0)} \frac{k_1(x, s_1(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))} d\xi,
\]

where \( p_i^\omega \) and \( p_i^\omega \) are defined as in (2.8) and (2.9). Then

\[
\nabla u^\omega = \frac{1}{k(v)} \sum_i k_{ii}(s_i(p_i - p_0)) \nabla u_i^\omega + \int_0^{x^\omega(s_0)} \nabla_s \left( \frac{k_1(s_1(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} \right) d\xi.
\]

In \( \{ u_{\min}^\omega < u_1 - u_2 < u_{\max}^\omega \} \) we have \( u_i^\omega = u_i \) and therefore in this region (3.13) equals

\[
k(v) \nabla u^\omega + e(v).
\]

In \( \{ u_1 - u_2 > u_{\max}^\omega \} \)

\[
u^\omega = 1/2 \left( u_1 + u_2 + u_{\max}^\omega \right) - \int_0^{u_{\max}^\omega} \frac{k_2(s_2(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))} d\xi.
\]
hence in the \(L^2(D \times ]0, T[)\) norm
\[
\chi\{u_1 - u_2 > u_{\text{max}}\} \nabla u^q
= \chi\{u_1 - u_2 > u_{\text{max}}\} \left(\frac{\nabla u_1 + \nabla u_2}{2} + \int_0^{p_{\text{max}}} \nabla_s \left(\frac{k_1(s_1(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))}\right) d\xi\right)
\]
\[-\chi\{u_1 - u_2 = u_{\text{max}}\} \left(\nabla u_1 + \int_0^{p_{\text{max}}} \nabla_s \left(\frac{k_1(s_1(\xi))}{k_1(s_1(\xi)) + k_2(s_2(\xi))}\right) d\xi\right),
\]
but also
\[
\frac{1}{k(v)} \sum_{i=1}^{d} k_i s_i(p_1 - p_2) \nabla u_j = \nabla u_1 \quad \text{in} \quad \{u_1 - u_2 = u_{\text{max}}\}.
\]

Now let us look at the equation for, e.g., \(p_2\). An easy computation shows that (writing \(k_i\) for \(k_i(s_i(p_1 - p_2))\))
\[
-\left(\sum_j k_{ij} \nabla u_j + k_2 e_2\right)
= \frac{k_1 k_2}{k_1 + k_2} \left(\frac{1}{k_1} \sum_j k_{ij} \nabla u_j - \frac{1}{k_2} \sum_j k_{ij} \nabla u_j\right) + \frac{k_1 k_2}{k_1 + k_2} (e_2 - e_2) + \frac{k_2}{k_1 + k_2} v.
\]
In \(\{u_1 - u_2 = u_{\text{min}}\text{ or } u_{\text{max}}\}\) the first term vanishes. Therefore as \(q \to 0\) this term is the limit of
\[
\chi\{u_{\text{min}} < u_1 - u_2 < u_{\text{max}}\}) \left(\frac{k_1 k_2}{k_1 + k_2} \left(\frac{1}{k_1} \sum_j k_{ij} \nabla u_j - \frac{1}{k_2} \sum_j k_{ij} \nabla u_j\right)\right)
= \frac{k_1 k_2}{(k_1 + k_2)} s(t) \left(\nabla s(t) - \nabla (s(t)/s_0)\right),
\]
if we define
\[
v^q(x, t) := \frac{s_1(x, p_1^q - p_2^q)(x, t)}{s_0(x)}.
\]
This shows that
\[
\partial_t s^q = \nabla \cdot \left(\lim_{\sigma \to 0} a(v) \nabla v^q + b(v) + d(v) u^q\right).
\]
The same is true for
\[
v^{\sigma q}(x, t) := \frac{s_1(x, \max (p^q_{\text{min}}, \min (p^q_{\text{max}}, (p_1 - p_2)(x, t))))}{s^q(x)}
\]
as \(\sigma \to 0\) and \(q \to 0\) independently. It was assumed that \(a(v) > c_\sigma > 0\) in
\[ \{0 < s \varphi < s_1(p_{\text{max}}^0)\}. \] Since \( \varphi_{\sigma}^{\varphi} = 0 \) in \( \{s \varphi < s_1(p_{\text{min}}^0)\} \) and in \( \{s \varphi > s_1(p_{\text{max}}^0)\} \) we infer that \( \varphi_{\sigma}^{\varphi} \) has a limit in \( L^2(D \times ]0, T[) \) as \( \sigma \to 0 \). Hence

\[ \varphi_{\sigma}(x, t) := s_1 \left( x, \min \left( p_{\text{max}}^0, (p_1 - p_0)(x, t) \right) \right) / s^\sigma(x) \]

is of class \( L^2(0, T; H^{1,2}(D)) \). Moreover, \( a(\varphi) \nabla \varphi_{\sigma} \) is estimated by a function in \( L^2(D \times ]0, T[) \), hence for small \( \delta > 0 \) the function \( a(\varphi) \min (\varphi, 1 - \delta) \) is near \( a(\varphi) \nabla \varphi \) in \( L^2(\Omega \times ]0, T[) \) uniformly in \( \varphi \). (Note that \( a(0) \) needs not to be defined, since \( \varphi_{\sigma} = 0 \) in \( \{\varphi = 0\} \). Since

\[ \min (\varphi, 1 - \delta) = \min (\varphi, 1 - \delta) \]

for small \( \varphi \) and fixed \( \delta \) (by (3.5)) we infer that

\[ \lim_{\varphi \to 0} a(\varphi) \nabla \varphi_{\sigma} = \lim_{\delta \to 0} a(\varphi) \nabla \min (\varphi, 1 - \delta), \]

which proves the assertion.

3.4. REMARKS.

1) If \( u_0 \) is bounded in \( L^2(D \times ]0, T[) \) then it has a limit \( u \) in \( L^2(0, T; H^{1,2}(D)) \) and (3.11) means

\[ \nabla \cdot (k(\varphi) \nabla u + e(\varphi)) = 0. \]

Moreover \( u \) is given by (1.15) (see prof. of Lemma 3.9). In the following theorem we will assume that \( u_0 \) are uniformly bounded functions.

2) It is known that using test functions of the form

\[ \varphi(\varphi) \eta^2 \]

in (3.12) for bounded functions

\[ \eta \in L^2(0, T; H^{1,2}(D)) \quad \text{with} \quad \partial_t \eta \in L^2(D \times ]0, T[), \]

one gets for almost all \( 0 < t_1 < t_2 < T \)

\[ \int_{t_1}^{t_2} \langle \partial_t (s^\sigma \varphi), \varphi(\varphi) \eta^2 \rangle = \int_{s^\sigma \varphi(t)}^{s^\sigma \varphi(t)} \left( \int_{s}^{\sigma(t)} q(\xi) \, d\xi \right)_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \int_{s}^{s^\sigma \varphi(t)} \left( \int_{z}^{s^\sigma \varphi(t)} q(\xi) \, d\xi \right). \]
Here \( \varphi \) is any Lipschitz function with \( \varphi'(z) = 0 \) for \( z \) near 1, and \( z_0 \) any real number.

3) It also is a standard calculation that for test functions (\( u \) as in 1)

\[ u \eta^2 \]

in (3.11) with \( \eta \in H^{1,2}(D) \) one gets for almost all \( t \)

\[ \int_D |\nabla u(t)|^2 \eta^2 < C \int_D (u(t)^2 |\nabla \eta|^2 + \eta^2) \]

3.5. REGULARITY THEOREM. Assume the data satisfy 2.1, 2.2 and 3.1, and suppose \( p_1, p_2 \) is a weak solution with \( u \in L^p(D \times ]0, T[) \) (satisfying (3.15)). Then

\( s, (p_1 - p_2) \in C^0(D \times ]0, T[) \).

The modulus of continuity of \( s \), can be estimated by the estimates on the coefficients made in 2.1 and 3.1, the distance to the boundary of \( D \times ]0, T[, \) and the supremum of \( |\eta| \).

For the proof we introduce

3.6. NOTATION. Let \( (x_0, t_0) \in D \times ]0, T[ \). For \( R > 0 \) the parabolic cylinders are denoted by

\[ Q_R := B_R(x_0) \times ]t_0 - R^2, t_0[. \]

Furthermore

\[ Q^\alpha_R := B_R(x_0) \times ]t_0 - \alpha R^2, t_0[ \quad \text{for } \alpha > 0 \]

and

\[ Q^0_{\sigma_1, \alpha_2} := B_{(1-\sigma_1)R}(x_0) \times ]t_0 - (1-\sigma_2) \alpha R^2, t_0[ \quad \text{for } \sigma_1, \sigma_2 \in ]0, 1[. \]

We define

\[ \|w\|_{Q_R}^2 := \text{ess sup}_{t_0 - R^2 < t < t_0} \int_{B_R(x_0)} |w|^2 + \int_{Q_R} |\nabla w|^2, \]

and similarly for the cylinders \( Q^0_{\sigma_1, \alpha_2} \). In the following \( 0 < R < R_0 \) with \( Q_0 \subset D \times ]0, T[, \) and \( \mu^+, \mu^- \) are any numbers with

\[ (3.17) \]

\[ \begin{cases} \text{ess sup}_{Q_0} v < \mu^+ < 1, \\ \text{ess inf}_{Q_0} v > \mu^- > 0, \end{cases} \]
hence

\[ \text{ess osc } v < \mu^+ - \mu^- . \]

Furthermore, \( \omega \) is any positive number satisfying

\[ \mu^+ - \mu^- < \omega < 2(\mu^+ - \mu^-) . \]

By Remark 3.2

\[ q_\phi(\omega) := \inf_{x \in D, 0 \leq z \leq 1 - \omega/4} a(x, z) \]

is positive.

3.7. Proposition. With the notation of 3.6 there exists a small constant \( c_0 > 0 \) independent of \((x_0, t_0), R, \) and \( \omega, \) such that if

\[ \frac{\text{meas } (Q_R \cap \{ v > \mu^+ - \omega/2 \})}{\text{meas } Q_R} < e^q q_1(\omega) \]

then

\[ \text{ess osc } v < \frac{\delta}{5} \omega . \]

Here

\[ q_1(\omega) := (c_0 q_\phi(\omega))^{q_1} . \]

Proof. See section 4.

3.8. Proposition. Suppose that

\[ \frac{\text{meas } (Q_R \cap \{ v < \mu^- + \omega/4 \})}{\text{meas } Q_R} < 1 - c_0 q_1(\omega) \]

with \( c_0 \) and \( q_1 \) as in Proposition 3.7. Moreover suppose that \( R \) is small enough, precisely,

\[ R < \frac{\omega}{2 q(\omega)} \]

and

\[ \frac{\text{osc } s}{s} < \frac{1}{8} (c_0 q_1(\omega))^2 . \]

Then

\[ \text{ess osc } v < (1 - 2^{q(\omega)-1}) , \]

where

\[ R^* = c_1 R^{q_1/s} . \]
Here \( c_1 \) is a small constant independent of \((x_0, t_0), R, \) and \( \omega, \) and \( q(\omega) \) is a decreasing function of \( \omega \) independent of \((x_0, t_0)\) and \( R \) (see (5.14)).

**Proof.** See section 5.

**Proof of Theorem 3.5.** We apply Proposition 3.7 and Proposition 3.8 inductively in order to prove continuity at \((x_0, t_0) \in D \times ]0, T[.\) First, the largest oscillation of \( v \) is 1, therefore we start with selecting \( R_0 \) to be so small that the closure of \( Q_{2R_0} \) is contained in \( D \times ]0, T[ \) and that

\[
R_0 < \frac{1}{2q(\delta)} \quad \text{and} \quad \frac{\text{osc}}{2R_0} < \frac{1}{8}(c_0q_1(1))^2.
\]

Define two sequences of real numbers \( R_n \) and \( \omega_n \) as follows

\[
1 = \omega_1, \quad \omega_{n+1} = \omega_n \left( 1 - 2^{-q(\omega_n-1)} \right),
\]

\[
R_1 = R_0, \quad R_{n+1} = \min \left\{ \frac{1}{2} c_1 R_n^{2/3}, \omega_{n+1} \frac{\omega_n^{2-q(\omega_n+1)}}{2}, \sigma_0 \left( \frac{1}{8}(c_0q_1(\omega_{n+1})) \right) \right\}.
\]

Here \( c \) and \( q(\omega) \) are the quantities in Proposition 3.8, and \( c_0 \) and \( q_1(\omega) \) are as in Proposition 3.7. The function \( \sigma_0 \) describes the continuity of \( s_0, \) that is, \( \sigma_0 \) is continuous with \( \sigma_0(0) = 0 \) and

\[
\frac{\text{osc}}{R_n} < \delta \quad \text{if} \quad R < \sigma_0(\delta).
\]

(By (3.6) we can choose \( \sigma_0(\delta) = c\delta, \) but this strong condition on \( s_0 \) is needed only to control the coefficient \( d, \) see Remark 3.2.) Obviously \( \omega_n \to 0 \) and \( R_n \to 0 \) as \( n \to \infty, \) for \( R_{n+1} < R_n/2, \) provided \( c_1 \) is small enough.

By construction we have

\[
R_n < \frac{\omega_n}{2q(\omega_n)} \quad \text{and} \quad \frac{\text{osc}}{R_n} < \frac{1}{8}(c_0q_1(\omega_n))^2.
\]

Let us assume that

\[
\text{ess \ osc} \ v < \omega_n,
\]

which is true for \( n = 1. \) Then we can choose \( \mu_n^+ \) and \( \mu_n^- \) such that (3.17) and (3.18) is satisfied. Obviously we must have either

\[
\text{meas} \ \left( Q_{R_n} \cap \left\{ v > \frac{\mu_n^+ - \omega_n}{2} \right\} \right) < c_0q_1(\omega_n) \ \text{meas} (Q_{R_n})
\]

(3.21)
or

\[ (3.22) \quad \text{meas} \left( Q_{E_n} \cap \left\{ \psi > \mu^+_n - \frac{\omega_n}{2} \right\} \right) > c_2 \phi_1(\omega_n) \text{meas} (Q_{E_n}). \]

If \((3.21)\) occurs, by Proposition 3.7 we have

\[ \text{ess osc } \psi \leq \text{ess osc } \psi \leq \frac{3}{4} \omega_n < \omega_{n+1}, \]

since \(q(\omega_n) > 1\). If \((3.22)\) holds then either

\[ \mu^+_n + \frac{\omega_n}{4} < \mu^+_n - \frac{\omega_n}{2} \]

or

\[ \mu^+_n + \frac{\omega_n}{4} > \mu^+_n - \frac{\omega_n}{2}. \]

In the second case

\[ \text{ess osc } \psi \leq \mu^+_n - \mu^-_n < \frac{3}{4} \omega_n < \omega_{n+1}, \]

and in the first case

\[ \text{meas} \left( Q_{E_n} \cap \left\{ \psi < \mu^-_n + \frac{\omega_n}{4} \right\} \right) < (1 - c_2 \phi_1(\omega_n)) \text{meas} (Q_{E_n}). \]

Therefore by Proposition 3.8 in view of \((3.21)\) we must have

\[ \text{ess osc } \psi \leq \text{ess osc } \psi < \omega_n (1 - 2^{-q(\omega_n)}) = \omega_{n+1}. \]

We obtain inductively that \((3.20)\) holds for all \(n\). This proves the continuity of \(\psi\) and supplies a modulus of continuity implicitly.

Finally let us verify the condition that \(u\) is bounded, which was needed for the proof of the regularity theorem.

3.9. \textsc{Lemma}. \textit{Suppose that in addition to the assumption in the existence theorem 2.6 the condition (3.9) is satisfied in the entire domain \(\Omega\). Furthermore assume that if \(\mathcal{K}^{n-1}(I^\gamma_1 \cap I^\gamma_2) > 0\) then \(p_{\min} > -\infty\) and}

\[ \int_0^{p_{\max}} \frac{k_1(x, s_1(x, \xi))}{k_1(x, s_1(x, \xi)) + k_2(x, s_2(x, \xi))} d\xi < C \quad \text{for } x \in I^\gamma_1 \cap I^\gamma_2. \]
If $\mathcal{K}^{n-1}(I_2^P \cap I_1^Q) > 0$ then the corresponding properties are assumed. The conclusion is that $u$ is locally bounded in $\Omega \times [0, T]$.

**Proof.** Define $w^e$ as in (3.14). Then $w^e \in L^2(0, T; H^{1,\omega}(\Omega))$ as shown in Lemma 3.3. On $I_1^P \cap I_2^P$ we have

$$\|w^e\|_{L^2(0, T; H^{1,\omega}(\Omega))} < C$$

since $p_1^P$ are bounded. Note that $C$ is independent of $\varrho$. Since

$$w^e = \int_0^{p_1 - p_2} \frac{k_{4}(s_{1}(\xi))}{k_{1}(s_{1}(\xi)) + k_{2}(s_{2}(\xi))} d\xi < C$$

we see that on $I_1^P \cap I_2^P$

$$w^e < p_1^P - \int_0^{p_{\min}} \frac{k_{2}(s_{2}(\xi))}{k_{1}(s_{1}(\xi)) + k_{2}(s_{2}(\xi))} d\xi < C$$

and

$$w^e > p_1^P - \int_0^{p_{\max}} \frac{k_{2}(s_{2}(\xi))}{k_{1}(s_{1}(\xi)) + k_{2}(s_{2}(\xi))} d\xi > C.$$ 

Let

$$\varphi(z) := \min (z + C, \max (z - C, 0)).$$

Then $\varphi(w^e) \in L^2(0, T; H^{1,\omega}(\Omega))$ and $\psi(w^e) = 0$ on $(I_1^P \cup I_2^P) \times [0, T]$. In the proof of Lemma 2.7 it suffices to assume that

$$\zeta_1 = 0 \quad \text{on } I_1^P \times [0, T],$$

$$\zeta_1 = 0 \quad \text{on } (I_1^P \cup I_2^P) \times [0, T].$$

Therefore setting $\zeta_1 = \zeta$ where $\zeta \in L^2(0, T; H^{1,\omega}(\Omega))$ with

$$\zeta = 0 \quad \text{on } (I_1^P \cup I_2^P) \times [0, T],$$

and

(3.23) $\zeta \in L^1(\Omega \times [0, T])$, and $\zeta(T) = 0$,
we obtain almost everywhere in time

\[ 0 = \int_{\Omega} \left( \sum_{i} \left( \sum_{j} k_{ij}(s_{i}(p_{1} - p_{2})) \nabla u_{j} + k_{ij}(s_{i}(p_{1} - p_{2})) \right) \nabla \zeta \right) \]

\[ = \int_{\Omega} (k(v) \lim_{\varepsilon \to 0} \nabla u_{\varepsilon} + e(v)) \nabla \zeta \]

where the last identity was proved in Lemma 3.3.

By approximation we see that we can ignore the condition (3.23), hence we are able to set \( \zeta = \varphi(u^\varepsilon) \), which yields

\[ \int_{\Omega} |\nabla \varphi(u^\varepsilon(t))|^2 \leq C \left( 1 + \int_{\Omega} \chi(\{|u^\varepsilon(t)| > C\}) |\lim_{\varepsilon \to 0} \nabla u_{\varepsilon}(t) - \nabla u^\varepsilon(t)|^2 \right) \]

where the last integral integrated over time tends to zero with \( \varepsilon \). Since \( \varphi(u^\varepsilon(t)) \) vanishes on \( \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \), we conclude that

\[ \int_{\Omega} |u^\varepsilon(t)|^2 \leq C \left( 1 + \int_{\Omega} |\lim_{\varepsilon \to 0} \nabla u_{\varepsilon}(t) - \nabla u^\varepsilon(t)|^2 \right) \]

Then multiplying (3.11) by \( \eta^2 u^\varepsilon(t) \) with \( \eta \in C^0_c(\Omega) \) we obtain that also

\[ \int_{\Omega} \eta^2 |\nabla u^\varepsilon(t)|^2 \leq C \left( 1 + \int_{\Omega} |\lim_{\varepsilon \to 0} \nabla u_{\varepsilon}(t) - \nabla u^\varepsilon(t)|^2 \right) \]

In particular \( u^\varepsilon \) is bounded in \( L^2(0, T; H^{1/2}_{\text{loc}}(\Omega)) \), hence it has a weak limit \( u \). Since

\[ \int_{t_1}^{t_2} \int_{\Omega} \eta^2(|u|^2 + |\nabla u|^2) \leq \liminf_{\varepsilon \to 0} \int_{t_1}^{t_2} \int_{\Omega} \eta^2(|u^\varepsilon|^2 + |\nabla u^\varepsilon|^2) \leq C(t_2 - t_1) \]

we see that \( u \in L^\infty(0, T; H^{1/2}_{\text{loc}}(\Omega)) \). Moreover \( u \) satisfies (3.15). Then the De Giorgi estimate (see e.g. [11]) says that \( u \in L^\infty(0, T; L^\infty(\Omega)) \).

Since \( u^i = u_i \) in \( \{u^i_{\text{min}} \leq u_i - u_2 < u^i_{\text{max}} \} \) we conclude that \( u \) is given by (1.15) in \( \{u_{\text{min}} < u_1 - u_2 < u_{\text{max}} \} \). If \( u_{\text{max}} < \infty \) then in \( \{u_1 - u_2 = u_{\text{max}} \} \)

\[ u^\varepsilon = \frac{1}{2} (u_1 + u_2)^{\frac{u^i_{\text{max}}}{u^i_{\text{max}}}} - \int_{0}^{u^i_{\text{max}}} \frac{k_i(s_i(\xi))}{k_i(s_i(\xi)) + k_i(s_i(\xi))} d\xi \].
The first term converges uniformly to $u_1 = p_1$. By (3.9) also the second term converges uniformly, and the limit must be finite since $w$ has a weak limit (we assume that $\{u_1 - u_2 = u_{\text{max}}\}$ has positive measure). Thus in $\{u_1 - u_2 = u_{\text{max}}\}$ the second formula for $u$ in (1.15) holds. Similarly in $\{u_1 - u_2 = u_{\text{min}}\}$ the first formula of (1.15) is relevant for $u$.

The statement of Lemma 3.9 in connection with the assumption in Theorem 3.5 is not quite satisfactory, since in case $k_1(z) < C_2$ condition (3.2) implies that $p_{\text{min}} = -\infty$. But then Lemma 3.9 does not cover the case that $I^0_1 \cap I^0_2$ is non-empty. Note also that $u$ is bounded for the second example following (1.12).

4. - Proof of Proposition 3.7.

Let $v_0 := \min(v, \mu - \omega/4)$ and let $k$ be any number satisfying

\[
(4.1) \quad \mu - \frac{\omega}{2} < k < \mu - \frac{\omega}{4}.
\]

First we will establish the following estimate for $(v_0 - k)^+ = \max(v_0 - k, 0)$ for any numbers $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$

\[
(4.2) \quad \| (v_0 - k)^+ \|^2_{q_0(\sigma_1, \sigma_2)} \leq \frac{C}{q_0(\omega)} (\sigma_1 R^{-2} + (\sigma_2 R^2)^{-1}) \text{ meas} (Q_R \cap \{v > k\}).
\]

Then we apply this estimate inductively for a sequence of values $R$ and $k$ in order to obtain Proposition 3.7.

To prove (4.2) we select the test function

$(v_0 - k)^+ \eta$

in (3.12) in the time interval $[t_0 - R^2, t_1]$ with $t < t_0$. Here $\eta$ is a cut-off function in $O(t_0)$ with $0 < \eta < 1$ and

\[
(4.3) \quad \eta = 1 \quad \text{in} \quad Q_R(\sigma_1, \sigma_2), \quad \eta = 0 \quad \text{on the parabolic boundary of} \quad Q_R,
\]

\[
|\nabla \eta| < C(\sigma_1 R)^{-1}, \quad |\Delta \eta| < C(\sigma_1 R)^{-2}, \quad 0 < \partial_t \eta < C(\sigma_2 R^2)^{-1}.
\]

Since

\[
\Phi(v) := \int_0^1 \left( \min \left( \xi, \mu - \frac{\omega}{4} - k \right)^+ \right) d\xi
\]

\[
= \frac{1}{2} |(v_0 - k)^+|^2 + \left( \mu - \frac{\omega}{4} - k \right) \left( v - \left( \mu - \frac{\omega}{4} \right)^+ \right)
\]
we obtain using 3.4.2

\[ \int_{\mathbb{R}^n} s_\omega \eta(t)^2 \Phi(v(t)) + \int_{t_\omega - R^2}^t \int_{\mathbb{R}^n} a(v) \eta^2 |\nabla(v_\omega - k)^+|^2 \]

\[ = \int_{t_\omega - R^2}^t \int_{\mathbb{R}^n} \left( s_\omega \Phi(v) \partial_t \eta^2 - a(v)(v_\omega - k)^+ \nabla v \nabla \eta^2 - (b(v) + d(v) v) \nabla ((v_\omega - k)^+ \eta^2) \right). \]

The first term on the left is

\[ \geq c \int_{\mathbb{R}^n} \eta(t)^2 |(v_\omega(t) - k)^+|^2 \]

and for the second integral we have

\[ \geq q_\omega(\omega) \int_{t_\omega - R^2}^t \int_{\mathbb{R}^n} \eta^2 |\nabla(v_\omega - k)^+|^2 \]

since the integrand vanishes in \( \{v > \mu^+ - \omega/4 \} \). The function \( q_\omega \) is defined in 3.6. The first term on the right is

\[ \leq C(\sigma_2 R^2)^{-1} \int_{\mathbb{R}^n} \chi(\{v > k\}) \]

and the following term can be treated in the following way using the properties of the function \( A(x, z) \) in 3.2.

\[ = \int_{t_\omega - R^2}^t \int_{\mathbb{R}^n} \int_{t_\omega - R^2}^t \left( A(v)(\nabla(v_\omega - k)^+ \nabla \eta^2 + (v_\omega - k)^+ \Delta \eta^2) + (v_\omega - k)^+ (\nabla A)(v) \nabla \eta^2 \right) \]

\[ \leq \delta \int_{t_\omega - R^2}^t \int_{\mathbb{R}^n} \eta^2 |\nabla(v_\omega - k)^+|^2 + C(q_\omega)^{-1} \int_{t_\omega - R^2}^t \int_{\mathbb{R}^n} \chi(\{v > k\}) \]

for every \( \delta > 0 \). The \( b \)-term easily can be estimated by the same expres-
Collecting these estimates we obtain choosing $\delta = \frac{1}{4} \varphi_4(\omega)$

\[
(4.4) \quad c \int_{B_n} \eta(t) \left( |v_n(t) - k|^2 + \frac{1}{2} \varphi_4(\omega) \right) \int_{t_n}^t \int_{\tilde{B}_n} \eta^2 |\nabla (v_n - k)^+|^2 \, d\tilde{x} \, d\tilde{t}
\]

\[
< C \left( \frac{1}{\varphi^2(\omega)} (\sigma_n \tilde{R})^{-1} + (\sigma_n^2 \tilde{R}^2)^{-1} \right) \int_{Q_n} \chi(\{v > k\}) - \int_{t_n}^t \int_{\tilde{B}_n} v \delta (v) \nabla ((v_n - k)^+ \eta^2) \, d\tilde{x} \, d\tilde{t}.
\]

The last term we transform as follows

\[
= - \int_{t_n}^t \int_{\tilde{B}_n} \nabla \eta^2 \, d\tilde{x} \, d\tilde{t}
\]

\[
- \int_{t_n}^t \int_{\tilde{B}_n} \left( \int_{k} \delta (\tilde{v}) \, d\tilde{t} \right) \left( \int_{\tilde{k}} (\nabla \delta)(\tilde{x}) \, d\tilde{t} \right) \eta^2.
\]

Using the fact that $v$ is divergence free this equals

\[
- \int_{t_n}^t \int_{\tilde{B}_n} \left( \int_{k} (\nabla \eta^2) \, d\tilde{t} \right) \int_{\tilde{k}} (\nabla \delta)(\tilde{x}) \, d\tilde{t} \eta^2.
\]

By the assumption on the coefficient $\delta$ this is estimated by

\[
< C \int_{t_n}^t \int_{B_n} (|\nabla u| + 1)(v_n - k)^+ (\eta |\nabla \eta| + \eta^2)
\]

\[
< \delta \int_{t_n}^t \int_{B_n} |\nabla u| \eta^2 (v_n - k)^+ \eta^2 + \frac{C}{\delta} \int_{t_n}^t \int_{B_n} \chi(\{v > k\})(|\nabla \eta|^2 + \eta^2).
\]

Using the estimate in 3.4.3 and the assumption that $u$ is bounded, the integral involving $|\nabla u|^2$ is bounded by

\[
\int_{t_n}^t \int_{B_n} (|\nabla (v_n - k)^\eta|^2 + |(v_n - k)^+ |^2 \eta^2)
\]

\[
< C \int_{t_n}^t \int_{B_n} (\eta^2 |\nabla (v_n - k)^\eta|^2 + \chi(\{v > k\})(|\nabla \eta|^2 + \eta^2)).
\]
We substitute this estimate in (4.4) and obtain for a suitable choice of $\delta$

$$
\varepsilon \int_{\mathbb{R}^n} \eta(t)^2 \left[ (v_\omega(t) - k)^+ \right]^2 + \frac{1}{3} q_\phi(\omega) \int_{t_0 - R^2}^{t_0} \int_{\mathbb{R}^n} \eta^2 \left| \nabla (v_\omega - k)^+ \right|^2 \\
< C \left( \frac{1}{q_\phi(\omega)} (\sigma_1 R^{-2})^{-2} + (\sigma_2 R^2)^{-1} \right) \int_{Q_R} \mathcal{Z} \{ \{ \{ v > k \} \} \}.
$$

Since $\eta = 1$ in $Q_R(\sigma_1, \sigma_2)$ this implies (4.2).

Now we will use (4.2) over a sequence of shrinking cylinders $Q_{k_n}$ and increasing levels $k_n$ given by

$$R_n := \frac{R}{2} + \frac{R}{2^{n+1}} \quad \text{and} \quad k_n = \mu^+ - \frac{\omega}{2} + \frac{\omega}{8} - \frac{\omega}{2^{n+3}} \quad \text{for } n \geq 0.
$$

Then $k_0 = \mu^+ - \frac{\omega}{2}$ and $k_n$ is increasing in $n$ with

$$\mu^+ - \frac{\omega}{2} < k_n < \mu^+ - \frac{\omega}{4}.
$$

Consequently $k_n$ can be used as level in the inequality (4.2) and with

$$
\begin{align*}
\sigma_1 := 1 - \frac{R_{n+1}}{R_n}, \quad \sigma_1 R_n = \frac{R}{2^{n+2}}, \\
\sigma_2 := 1 - \frac{R_{n+1}^2}{R_n^2}, \quad \sigma_1 R_n^2 = \frac{R^2}{2^{n+2}},
\end{align*}

we obtain

$$\left\| (v_\omega - k_n)^+ \right\|_{Q_{k_n+1}}^2 \leq C \frac{2^{2n}}{q_\phi(\omega)^2 R^2} \text{meas } (Q_R \cap \{ v > k_n \}).
$$

But the left side controls $\text{meas } (Q_{k_n+1} \cap \{ v > k_{n+1} \})$ from above as follows. By an embedding Lemma for functions in $L^2(t_0 - R_{n+1}^2, t_0; H^1(B_{R_{n+1}}))$ (see [12; II (3.9)]) we have

$$\int_{Q_{k_n+1}} \left| (v_\omega - k_n)^+ \right|^2 < C \text{meas } (Q_{R_{n+1}} \cap \{ v > k_{n+1} \})^{2/(N+2)} \| (v_\omega - k_n)^+ \|_{Q_{R_{n+1}}}^2.
$$

For the integral on the left we have

$$\int_{Q_{k_n+1}} \left| v_\omega - k_n \right|^2 > \int_{Q_{k_n+1} \setminus \{ v > k_{n+1} \}} \left| (v_\omega - k_n)^+ \right|^2 > (k_{n+1} - k_n)^2 \text{meas } (Q_{R_{n+1}} \cap \{ v > k_{n+1} \}).
$$
Since \( k_{n+1} - k_n = \alpha / 2^{n+4} \) we obtain the recursive inequality

\[
\text{meas}(Q_{R^n} \cap \{ v > k_{n+1} \}) < \frac{C 2^{4n}}{\omega^2 \sigma_\omega} \text{meas}(Q_{R^n} \cap \{ v > k_n \})^{1/2(\omega^2 + 4)}.
\]

Dividing by \( R^{n+2} \) and setting

\[
y_n := \frac{1}{R^{n+2}} \text{meas}(Q_{R^n} \cap \{ v > k_n \})
\]

we have the dimensionless form

\[
y_{n+1} < \frac{C 2^{4n}}{\omega^2 \sigma_\omega} y_n^{1/2(\omega^2 + 4)}.
\]

It follows (see [11; 2 Lemma 4.7] or [12; II Lemma 5.6]) that \( y_n \to 0 \) as \( n \to \infty \) if

\[
y_n < \left( \frac{\omega^2 \sigma_\omega}{C} \right)^{(\omega^2 + 4)/2} 2^{-(\omega^2 + 4)^2}.
\]

But this condition for \( y_0 \) is the assumption in Proposition 3.7 for a suitable choice of \( \sigma_\omega \). Consequently

\[
\text{meas}(Q_{R^n} \cap \{ v > \mu^+ - \frac{3}{8} \omega \}) = 0,
\]

hence

\[
\text{ess osc}_{Q_{R^n}} v < \mu^+ - \frac{3}{8} \omega - \frac{5}{8} \omega.
\]

The proposition is proved.

5. – Proof of Proposition 3.8.

We divide the proof into several steps. First we prove a logarithmic estimate (Lemma 5.2), which implies that for large \( p \) the set \( \{ v > \mu^- + 2^{-\rho} \omega \} \) covers a certain portion of \( B_R \) at all times (Lemma 5.3). From this we conclude that for large \( p \) the set \( \{ v < \mu^- + 2^{-\rho} \omega \} \) is a small portion in \( Q_R \), provided \( R \) is small enough (Lemma 5.5). Then if we decrease \( R \) further this set has measure zero in \( Q_R \) (Lemma 5.6).

5.1. Definition. Let \( p, p_0 \) be positive numbers. Then for \( z < (2^{-p}

where is defined provided
We also set
which co and qi as in Proposition 3.7. It is assumed that co is small enough to provide that $\alpha < \frac{1}{4}$.

5.2. **Lemma.** Let $k < \mu^{-} + 2^{-p} \omega$ and $2 < p < p_{0} - 1$. Then for $t_{0} - R^{2} < t_{1} < t < t_{0}$

$$
\int_{B_{(1-\sigma)}R} s_{n} \eta^{2}((v(t) - k)^{-}) < \int_{B_{n}} s_{n} \eta^{2}((v(t_{1}) - k)^{-}) + C \frac{p^{2} - p}{\eta_{\alpha}(\omega)} \left( \frac{1}{\sigma_{1}^{4}} + \left( \frac{2^{\mu_{1}}R^{1}}{\omega} \right) \right) R^{N},
$$

where $C$ is a constant independent of $\omega, k, p, p_{0}$.

**Proof.** Since $\psi^{2}$ is a $C^{1,1}$ function and since $(v - k)^{-} \in L^{2}(t_{0} - R^{2}, t_{0} ; H^{1,2}(B_{n}))$ with $(v - k)^{-} < k - \mu^{-}$ we can use

$$
- (\psi^{2})'((v - k)^{-}) \eta^{2}
$$
as test function in (3.12) in the time interval $[t_{1}, t]$, where $\eta$ is a cut off function in $C^{0}(B_{n})$ with $0 < \eta < 1$ and

$$
\eta = 1 \text{ in } B_{(1-\sigma)R}, \quad \eta = 0 \text{ on } \partial B_{n}, \quad |\nabla \eta| < C(\sigma_{1}R)^{-1}.
$$

We obtain (using 3.4.2), where $\psi^{2\nu}$ means $(\psi^{2})^{\nu}$

$$
\int_{B_{n}} s_{n} \eta^{2} \psi^{2}\left((v(t) - k)^{-}\right) + \int_{t_{1}}^{t} \int_{B_{n}} a(v) \eta^{2} \psi^{2\nu}\left((v - k)^{-}\right)|\nabla (v - k)^{-}|^{2}
$$

$$
= \int_{B_{n}} s_{n} \eta^{2} \psi^{2}\left((v(t_{1}) - k)^{-}\right)
$$

$$
+ \int_{t_{1}}^{t} \int_{B_{n}} \left( a(v) \psi^{2\nu}\left((v - k)^{-}\right) \nabla v \nabla \eta^{2} + (b(v) + \delta(v)(v) \nabla \psi^{2\nu}\left((v - k)^{-}\right) \eta^{2} \right),
$$

In the second integral on the left we can estimate $a(v) \geq \eta_{\alpha}(\omega)$, since $\psi^{2\nu}\left((v - k)^{-}\right) = 0$ in $\{v > k\}$ and $k < \mu^{-} + \omega/4 < \mu^{+} < \omega/4 < 1 - \omega/4$. Further-
more, since \( \psi'^2(0) = 0 \) and
\[
(5.3) \quad \psi'^2 = 2(1 + \psi)\psi'^2, \quad \text{hence } \frac{(\psi'^2)^2}{\psi'^2} = \frac{2\psi^2}{1 + \psi} < 2\psi,
\]
the a-term on the right is estimated by
\[
< C \int_{t_1}^{t} \int_{B_n} \psi'^2((v - k)^-)|\nabla(v - k)^-|\eta|\nabla\eta|
\]
\[
< \delta \int_{t_1}^{t} \int_{B_n} \eta^2 \psi'^2((v - k)^-)|\nabla(v - k)^-|^2 + \frac{C}{\delta} \int_{t_1}^{t} \int_{B_n} \psi((v - k)^-)|\nabla\eta|^2.
\]
Similarly the b-term is estimated by
\[
< C \int_{t_1}^{t} \int_{B_n} \left(\psi'^2((v - k)^-)|\nabla(v - k)^-|\eta^2 + \psi'^2((v - k)^-)\eta|\nabla\eta|\right)
\]
\[
< \delta \int_{t_1}^{t} \int_{B_n} \eta^2 \psi'^2((v - k)^-)|\nabla(v - k)^-|^2
\]
\[
+ \frac{C}{\delta} \int_{t_1}^{t} \int_{B_n} \left(\frac{(\psi'^2)^2}{\psi}((v - k)^-)\eta^2 + \psi((v - k)^-)|\nabla\eta|^2\right)
\]
\[
< \delta \int_{t_1}^{t} \int_{B_n} \eta^2 \psi'^2((v - k)^-)|\nabla(v - k)^-|^2
\]
\[
+ \frac{C}{\delta} \int_{t_1}^{t} \int_{B_n} \left(1 + \psi((v - k)^-)(\psi'((v - k)^-)^2)\eta^2 + \psi((v - k)^-)|\nabla\eta|^2\right).
\]
Collecting these estimates and choosing \( \delta = c\eta\phi(\omega) \) we find
\[
(5.4) \quad \int_{B_n} d\vartheta \eta^2 \psi'^2((v(t_i) - k)^-) + c\eta\phi(\omega) \int_{t_1}^{t} \int_{B_n} \eta^2 \psi'^2((v - k)^-)|\nabla(v - k)^-|^2
\]
\[
< \int_{B_n} d\vartheta \eta^2 \psi'^2((v(t_i) - k)^-)
\]
\[
+ \frac{C}{\theta^2(\omega)} \int_{t_1}^{t} \int_{B_n} \left(1 + \psi((v - k)^-)(\psi'((v - k)^-)^2)\eta^2 + \psi((v - k)^-)|\nabla\eta|^2\right)
\]
\[
+ \int_{t_1}^{t} d(v) \nabla(\psi'^2((v - k)^-)\eta^2).
\]
We transform the last integral as follows using the fact that \( v \) is divergence-free.

\[
= -\int_{t_1}^{t} \int_{B_R} v \nabla (d(w)) \psi (v - k)^{-} \eta^2
\]

\[
=-\int_{t_1}^{t} \int_{B_R} v (\partial_v d(w)) \nabla (v - k)^{-} + (\nabla \partial_v d(w)) \psi (v - k)^{-} \eta^2.
\]

Therefore using (5.3) again

\[
\left| \int_{t_1}^{t} \int_{B_R} d(v) \psi (v - k)^{-} \eta^2 \right| \leq C \int_{t_1}^{t} \int_{B_R} (|\nabla u| + 1)(|\nabla (v - k)^{-} | + 1) \psi (v - k)^{-} \eta^2
\]

\[
\leq \delta \int_{t_1}^{t} \int_{B_R} \eta^2 \psi (v - k)^{-} (|\nabla (v - k)^{-}|^2 + 1) + C \delta \int_{t_1}^{t} \int_{B_R} \eta^2(|\nabla u|^2 + 1) \psi.
\]

Now observe that

\[
(5.5) \quad \begin{cases} 
\psi(v - k)^{-} \leq \log \frac{2 - p \omega}{2 - p \omega} < \log 2(p^p - p), \\
\psi'(v - k)^{-} \leq \frac{1}{2 - p \omega}.
\end{cases}
\]

Hence with an appropriate choice of \( \delta \) the estimate (5.4) becomes

\[
\int_{B_R} s \eta^2 \psi^2 (v(t) - k)^{-} \leq \int_{B_R} s \eta^2 \psi^2 (v(t) - k)^{-} + \frac{C(p - p^p)}{\varphi_0(\omega)} \int_{t_1}^{t} \int_{B_R} \left( \frac{\eta^2}{(2 - p \omega)^2} + |\nabla \eta|^2 + \eta^2(|\nabla u|^2 + 1) \right).
\]

The \( \nabla u \) term can be estimated as in 3.4.3. Then using the properties of the function \( \eta \) the assertion follows immediately.

As a consequence we obtain

5.3. LEMMA. There exists a large number \( p(\omega) \) independent of \( R \), such
that

\[ R < \frac{\omega}{3^{2p} q} \]

implies

\[ \frac{\text{meas} \left( B_R \cap \{ v(t) < \mu + 2^{-p} \omega \} \right)}{\text{meas} \left( B_R \right)} < 1 - \alpha^2 + \text{osc}_{B_R} s_0 \]

for all \( t \) with \( t_0 - \alpha R^2 < t < t_0 \). Here \( \alpha \) is the number defined in (5.2).

**Proof.** Consider the logarithmic estimate established in the previous lemma with

\[ p = 2, \quad k = \mu + \frac{\omega}{4}, \quad \text{and} \quad p_0 > 3, \]

where \( p_0 \) has to be chosen. If \( R < \omega/2^p \) then

\[ \int_{B_{t_0 - \alpha R^2}} s_0 \psi^2 \left( (v(t) - k)^- \right) \leq \int_{B_{t_0}} s_0 \psi^2 \left( (v(t_1) - k)^- \right) + \frac{C(p_0 - 2)}{q(\omega) \sigma_1^2} R^p. \]

Now the assumption in Proposition 3.8 is

\[ \frac{\text{meas} \left( Q_R \cap \{ v < k \} \right)}{\text{meas} \left( Q_R \right)} < 1 - c_0 \psi_1(\omega) = 1 - 2\alpha. \]

Since the left side can be written as

\[ \frac{1}{R^2} \int_{t_0 - R^2}^{t_0} \frac{\text{meas} \left( B_R \cap \{ v(t) < k \} \right)}{\text{meas} \left( B_R \right)} \, dt, \]

we find a time \( t_1 \in [t_0 - R^2, t_0 - \alpha R^2] \) with

\[ \frac{\text{meas} \left( B_R \cap \{ v(t_1) < k \} \right)}{\text{meas} \left( B_R \right)} < \frac{1 - 2\alpha}{1 - \alpha}. \]

Hence using (5.5) we obtain for the integral on the right of (5.6)

\[ \int_{B_{t_0}} s_0 \psi^2 \left( (v(t_1) - k)^- \right) \leq \left( \sup_{Q_R} s_0 \right) \left( \log 2 \right)^2 (p_0 - 2)^2 \text{meas} \left( B_R \cap \{ v(t_1) < k \} \right) \]

\[ < \left( \sup_{Q_R} s_0 \right) \left( \log 2 \right)^2 (p_0 - 2)^2 \frac{1 - 2\alpha}{1 - \alpha} \text{meas} \left( B_R \right). \]
The left hand side of (5.6) is estimated as follows:

\[
\int_{B(t-\sigma_1,\varepsilon)} s_\varepsilon \psi^2 \left( (v(t) - k)^- \right) > \int_{B(t-\sigma_1,\varepsilon) \cap \{ v(t) < \mu^--2^{-p_\varepsilon} \}} s_\varepsilon \psi^2 \left( (v(t) - k)^- \right)
\]

\[
= \int_{B(t-\sigma_1,\varepsilon) \cap \{ v(t) < \mu^--2^{-p_\varepsilon} \}} s_\varepsilon \max \left( 0, \log \left( \frac{\omega/4}{\omega/4 - (v(t) - k)^- + 2^{-p_\varepsilon} \omega} \right) \right)^2
\]

\[
> \left( \inf_{\varepsilon \sigma_\varepsilon} \max \left( 0, \log \left( \frac{\omega/4}{\omega/4 + (2^{-p_\varepsilon} \omega - \omega/4) + 2^{-p_\varepsilon} \omega} \right) \right)^2 \right)
\]

\[
\cdot \text{meas} \left( B(t-\sigma_1,\varepsilon) \cap \{ v(t) < \mu^--2^{-p_\varepsilon} \} \right)
\]

\[
> \left( \inf_{\varepsilon \sigma_\varepsilon} \log(2)^2 \left( \frac{p_\varepsilon - 3}{p_\varepsilon} \right)^2 \right) \text{meas} \left( B(t-\sigma_1,\varepsilon) \cap \{ v(t) < \mu^--2^{-p_\varepsilon} \} \right)
\]

\[
> \left( \inf_{\varepsilon \sigma_\varepsilon} \log(2)^2 \left( \frac{p_\varepsilon - 3}{p_\varepsilon} \right)^2 \left( \text{meas} \left( B \cap \{ v(t) < \mu^--2^{-p_\varepsilon} \} \right) - \sigma_1 N \text{meas} \left( B_R \right) \right) \right).
\]

Substituting these estimates in (5.6) we get

\[
\text{meas} \left( B \cap \{ v(t) < \mu^--2^{-p_\varepsilon} \omega \} \right)
\]

\[
\leq \left( 1 + C \log s_\varepsilon \right) \left( \frac{p_\varepsilon - 2}{p_\varepsilon - 3} \right)^2 \frac{1 - 2\varepsilon}{1 - \varepsilon} + C \frac{p_\varepsilon - 2}{\varphi_\varepsilon(\omega) \sigma_\varepsilon^2 (p_\varepsilon - 3)^2} + \sigma_1 N.
\]

This inequality holds for almost all \( t \in [t_1, t_\varepsilon[, \) all \( \sigma_1 \in ]0, 1[, \) and all \( p_\varepsilon > 3. \)

Furthermore it is essential that the first term tends to 1 as \( \varepsilon \to 0, \) \( p_\varepsilon \to \infty, \) and \( R \to \infty, \) and that the remainder is small if \( \sigma_1 \to 0 \) and \( p_\varepsilon \to \infty \) in a suitable manner. To be precise we choose

\[
\sigma_1 = \frac{2 \varepsilon^2}{3 N^2}.
\]

and then \( p_\varepsilon \) large enough such that

\[
\frac{C}{\varphi_\varepsilon(\omega) \sigma_\varepsilon^2 (p_\varepsilon - 3)^3} \leq \frac{3}{2} \varepsilon^2
\]

and

\[
\left( \frac{p_\varepsilon - 2}{p_\varepsilon - 3} \right)^2 \leq (1 - \varepsilon)(1 + 2\varepsilon).
\]

The lemma is proved.
Recalling the definition of $\alpha$ and $\varphi_1(\omega)$ it is readily seen that a suitable choice of $p_0 = p(\omega)$ would be

$$(5.7) \quad p(\omega) := 3 + \frac{C}{(\omega \varphi_0(\omega))^{\frac{m_0+1}{2}}}$$

for a constant $C$ independent of $\omega$.

Next we will show that the relative measure of $\{v < \mu^+ + 2^{-\omega} \}$ in $Q^+_R$ is small provided $R$ is small enough. For this we need

5.4. LEMMA. There is a constant $C$ such that if $0 < k < \mu^- + \omega/4$ and $0 < \beta < 1$, then

$$\| (v - k)^- \|_{Q^+_R} \leq \frac{C}{\varphi_0(\omega)^2} \left( (\sigma_1 R)^{-\beta} + (\sigma_2 \beta R^2)^{-1} \right) \int_0^t \| (v - k)^- \|^2_{Q^+_R} + \frac{C}{\varphi_0(\omega)^2} \text{meas} (Q^+_R \cap \{v < k\}).$$

PROOF. As in section 4 we select the test function

$$-(v - k)^- \eta$$

in the time interval $[t_0 - \beta R^2, t]$ with $t < t_0$. Here $\eta$ is a suitable cut off function in $C^0(\overline{Q^+_R})$ with $\eta = 1$ in $Q^+_R(\sigma_1, \sigma_2)$ (see (4.3)). From (3.12) and 3.4.2 we get

$$(5.8) \quad \frac{1}{2} \int_{B_R} s \varphi_1(t)^2 |(v(t) - k)^-|^2 + \int_{t_0 - \beta R^2}^t \int_{B_R} s \varphi_1(v) \eta^2 |\nabla (v - k)^-|^2$$

$$= \int_{t_0 - \beta R^2}^t \int_{B_R} (s \varphi_1(v - k)^- \tilde{\eta} \eta^2 + a(v)(v - k)^- \nabla v \nabla \eta^2$$

$$+ (b(v) + d(v) \nabla v) \nabla ((v - k)^- \eta^2)).$$

Since

$$k < \mu^- + \frac{\omega}{4} < \mu^+ - \frac{\omega}{4} < 1 - \frac{\omega}{4}$$

we have $a(v) > \varphi_0(\omega)$ in $\{v < k\}$, which estimates the second term on the left. The first term on the right is

$$< C(\sigma_1 \beta R^2)^{-1} \int_{Q^+_R} |(v - k)^-|^2$$
and the second term (in contrast to section 4) we are able to estimate by

$$< \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} \eta^2 |(v - k)^-| \eta_\nabla $$

$$< \delta \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} \eta^2 |\nabla (v - k)^-|^2 + \frac{C}{\delta} (\sigma_\gamma R)^{-2} \int_{\mathbb{R}^n} \eta^2 |(v - k)^-|^2 .$$

Similarly for the $b$-term

$$< C \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} \eta^2 |(\nabla (v - k)^-|^2 + \frac{C}{\delta} \int_{\mathbb{R}^n} \eta^2 \chi\{v < k\} .$$

Note that the term with the characteristic function has no $R^{-2}$ factor in front. Combining these estimates and choosing $\delta = o(p_0(\omega))$ the identity (5.8) becomes

$$(5.9) \quad c \int_{\mathbb{R}^n} \eta(t) \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} \eta^2 |(v(t) - k)^-|^2 + c o(p_0(\omega)) \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} \eta^2 |\nabla (v - k)^-|^2$$

$$< C \left( \frac{1}{p_0(\omega)} (\sigma_\gamma R)^{-2} + (\sigma_\delta R)^{-1} \right) \int_{\mathbb{R}^n} \eta^2 |(v - k)^-|^2$$

$$+ \frac{C}{p_0(\omega)} \int_{\mathbb{R}^n} \eta^2 \chi\{v < k\} + \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} b(v) \nabla ((v - k)^- \eta^2) .$$

The last term we transform as in section 4

$$= \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} b(v) \left( \nabla (v - k)^- \eta^2 - \eta^2 \nabla \int_k \nabla (v - k)^- d(\xi) \nabla (v - k)^- \nabla (v - k)^- \right)$$

$$= \int_{t_s - \beta R^n}^t \int_{\mathbb{R}^n} b(v) \left( \nabla (v - k)^- d(\xi) - d(v) \right) \nabla (v - k)^- \eta^2 + \eta^2 \int_k \nabla (v - k)^- d(\xi) \nabla (v - k)^- \eta^2 .$$
Since $d$ is Lipschitz continuous, this is

$$
\leq C \int_{t \leq R^2} \int_0^1 (|\nabla u| + 1) (|v(k) - k|^2 |\nabla \eta| + \eta^2 (v(k) - k))
$$

$$
\leq \delta \int_{t \leq R^2} \int_0^1 (|\nabla u|^2 |v(k) - k|^2 + \frac{C}{\delta} \int (|v(k) - k|^2 |\nabla \eta|^2 + \chi(\{v < k\}) \eta^2).
$$

The integral involving $|\nabla u|^2$ can be estimated by 3.4.3. Using the assumption that $u$ is bounded this integral is

$$
\leq \int_{t \leq R^2} \int_0^1 (|\nabla (v(k) - \eta)|^2 + |v(k) - k|^2 \eta^2)
$$

$$
\leq 2 \int_{t \leq R^2} \int_0^1 \eta^2 |\nabla (v(k) - k)|^2 + 2 |v(k) - k|^2 (|\nabla \eta|^2 + \eta^2).
$$

We substitute these estimates in (5.9). Choosing $\delta = c\varrho_0(\omega)$ we obtain

$$
\int_{B_\varrho} \eta(t)^2 |v(t) - k|^2 + c\varrho_0(\omega) \int_{t \leq R^2} \int_0^1 \eta^2 |\nabla (v(k) - k)|^2
$$

$$
\leq C \left( \frac{1}{\varrho_0(\omega)} (\sigma_1 R)^{-s} + (\sigma_2 \beta R)^{-1} \right) \int (|v(k) - k|^2 + \frac{C}{\varrho_0(\omega)} \int \chi(\{v < k\}),
$$

and the lemma is proved.

### 5.5. Lemma.
Consider the cylinder $Q^c$ with $\omega$ as in (5.2). For every $\theta > 0$ there exists a number $q = q(\omega, \theta) > p(\omega)$ such that if

$$
R \leq \frac{\omega}{\sqrt{q}} \quad \text{and} \quad \text{osc} \varrho_0 \leq \frac{\alpha^2}{\sqrt{q}},
$$

then

$$
\frac{\text{meas} (Q^c \cap \{v < \mu - 2^{-q} \omega\})}{\text{meas} (Q^c)} < \theta.
$$

**Proof.** Let $q > p(\omega)$ and

$$
l = \mu - 2^{-q} \omega, \quad k = \mu - 2^{-q-1} \omega.
$$
Then for \( t_0 - \alpha R^2 < t < t_0 \) by [11; Lemma 3.5]

\[
(l - k) \operatorname{meas} \left( B_n \cap \{ v(t) < k \} \right) < \frac{CR^{n+1}}{\operatorname{meas} \left( B_n \cap \{ v(t) > l \} \right)} \int_{B_n \cap \{ t < v(t) < l \}} |\nabla v(t)|.
\]

By virtue of Lemma 5.3 and the assumption we have

\[
\operatorname{meas} \left( B_n \cap \{ v(t) > l \} \right) > c \left( \alpha^2 - \text{osc} s \right) R^n > c \alpha^2 R^n,
\]

therefore

\[
\frac{\omega}{2^{q+1}} \operatorname{meas} \left( B_n \cap \{ v(t) < k \} \right) < \frac{CR}{\alpha^2} \int_{B_n \cap \{ t < v(t) < l \}} |\nabla v(t) - l|^{-1}.
\]

Now integrate over \( t \), square both sides, and use Hölder's inequality to obtain

\[
\left( \frac{\omega}{2^{q+1}} \right)^2 \operatorname{meas} \left( Q_n^2 \cap \{ v < k \} \right)^2 < \frac{CR^2}{\alpha^4} \operatorname{meas} \left( Q_n^2 \cap \{ k < v < l \} \right) \int_{Q_n^2} |\nabla v - l|^{-2}.
\]

To estimate the integral on the right side we apply Lemma 5.4 over the cylinders \( Q_n^2 \) and \( Q_n^{2h} \), where \( Q_n^2 = Q_n(\frac{1}{2}, 1 - \alpha/4) \), and to the level \( l \). We obtain

\[
\int_{Q_n^2} |\nabla v - l|^{-2} < \frac{C}{q^2(\alpha)^2} \left( \text{ess sup}_{Q_n^2} |v - l|^{-1} \right)^2 + R^2)^{-\frac{q}{2}}.
\]

Since

\[
\text{ess sup}_{Q_n^2} |v - l|^{-1} < l - \mu = 2^{-q} \omega
\]

and \( R < 2^{-q} \omega \) by assumption, inequality (5.10) becomes

\[
\operatorname{meas} \left( Q_n^2 \cap \{ v < k \} \right)^2 < \frac{R^{n+2}}{\alpha^4 q^2(\alpha)^2} \operatorname{meas} \left( Q_n^2 \cap \{ k < v < l \} \right).
\]

Adding this inequality for \( q = p(\omega), \ldots, q_0 - 1 \) yields

\[
(q_0 - p(\omega) - 1) \operatorname{meas} \left( Q_n^2 \cap \{ v < \mu + 2^{-q_0} \omega \} \right)^2 < \frac{C}{\alpha^4 q^2(\alpha)^2} R^{2(n+2)}.
\]

To prove the lemma we have only to choose \( q_0 = q(\omega, \theta) \) large enough,
that is,

\begin{equation}
q(\omega, \theta) = p(\omega) + 1 + \frac{C}{\varphi_1(\omega)^q \varphi_q(\omega)^q \theta^q}.
\end{equation}

5.6. Lemma. Let \( \alpha = \alpha(\omega) \) as in (5.2). There is a large number \( q = q(\omega) \) such that if

\[
R^\frac{\omega}{2^q} \quad \text{and} \quad \frac{\omega}{q} \leq \frac{\omega}{2^q}.
\]

then

\[
\text{meas} \left( Q_{n+2}^* \cap \{ v < \mu - 2^{-q-1} \omega \} \right) = 0.
\]

Proof. Consider the cylinders \( Q_{n+2}^* \) and the levels \( k_n \) defined by

\[
R_n := \frac{R}{2} + \frac{R}{2^{n+1}} \quad \text{and} \quad k_n := \mu + \frac{\omega}{2^{q+2}} + \frac{\omega}{2^{q+n+1}}
\]

for \( n > 0 \) with \( q := q(\omega, \theta) \) from the previous lemma, where \( \theta > 0 \) has to be chosen. Then \( R_n = R, \) \( k_n = \mu + 2^{-q-1} \omega \) and \( k_n \) is decreasing in \( n. \) By the embedding lemma [12; (3.9)] we have

\[
\int_{Q_{n+1}^*} |(v - k_n)|^2 \leq C \text{meas} \left( Q_{n+1}^* \cap \{ v < k_n \} \right)^{2/(N+2)} \left[ (v - k_n)^- \right]_{Q_{n+1}^*}.
\]

For the integral on the left side we have

\[
\int_{Q_{n+1}^*} |(v - k_n)|^2 \geq \int_{Q_{n+1}^* \cap \{ v < k_n \}} |(v - k_n)|^2 > (k_n - k_{n+1})^2 \text{meas} \left( Q_{n+1}^* \cap \{ v < k_{n+1} \} \right),
\]

and \( k_n - k_{n+1} = 2^{-q-n-1} \omega. \) The norm on the right hand side we estimate by Lemma 5.4. Since \( Q_{n+1}^* = Q_n^*(\sigma_1, \sigma_2) \) with \( \sigma_1, \sigma_2 \) as in (4.5) this gives

\[
\| (v - k_n)^- \|^2_{Q_{n+1}^*} \leq \frac{C}{\varphi_{q}(\omega)^2} \left( C_{R}^{2n} \text{ess sup} \left( v - k_n \right)^- \right)_{Q_n^*} \text{meas} \left( Q_n^* \cap \{ v < k_n \} \right).
\]

But

\[
\text{ess sup} \left( v - k_n \right)^- \leq k_n - \mu \leq \frac{\omega}{2^q}.
\]
Therefore we get the recursive estimate
\[ \text{meas} \left( Q_{x_{n+1}}^a \cap \{ v < k_{n+1} \} \right) \leq \frac{C_2 4^n}{q(\omega)^3 R^2} \left( 1 + \left( \frac{2^\epsilon R}{\omega} \right)^n \right) \text{meas} \left( Q_{x_n}^a \cap \{ v < k_n \} \right)^{1+\epsilon/(n+2)}. \]

Thus assuming \( R < 2^{-\epsilon} \omega \) and setting
\[ y_n := \frac{\text{meas} \left( Q_{x_n}^a \cap \{ v < k_n \} \right)}{\text{meas} \left( Q_{x_n}^a \right)}, \]
we obtain
\[ y_{n+1} \leq \frac{C \omega^{2/(n+2)}}{q(\omega)^3} 2^{4\epsilon} y_{n+1}^{1+\epsilon/(n+2)}. \]

From [12: II Lemma 5.6] it follows that \( y_n \to 0 \) as \( n \to \infty \) if
\[ y_n < \left( \frac{q(\omega)}{C \omega^{2/(n+2)}} \right)^{1+\epsilon/(n+2)} = \frac{q(\omega)^{n+2}}{\alpha}. \]

Thus the assumption follows if (5.12) is satisfied. In fact if we use
\[ \theta = \theta(\omega) := \frac{q(\omega)^{n+2}}{\alpha}, \]
then (5.12) is just the statement of Lemma 5.5. Therefore the lemma holds if \( q(\omega) := q(\omega, \theta(\omega)) \). In a precise way, combining (5.2), (5.7), (5.11), and (5.13) we get
\[ q(\omega) = 4 + C q(\omega)^{-\epsilon} (q(\omega)^{-\epsilon} + q(\omega)^{-n}) \cdot \]

5.7. END or THE PROOF OF PROPOSITION 3.8. Lemma 5.6 shows that \( v > p_0 + 2^{-\epsilon(\omega)} \omega \) almost everywhere in \( Q_{x_n}^a \). We have to choose \( R^* \) such that \( Q_{x_n}^a \) contains \( Q_{R^*} \). By (5.14)
\[ R < \frac{\omega}{C q(\omega)^{n+2}}, \]
hence \( \sqrt{\omega} > c \sqrt{\omega} R^{1/6} \). Therefore \( R^* = c \sqrt{\omega} R^{1/6} \) is an appropriate choice.

6. – Some generalizations.

We want to show in this section that the local continuity of \( v \) still holds if \( a(v) \) is degenerate also at \( v = 0 \), but the degeneracy is mild. In a precise
way instead of (3.2) we assume

\[
\begin{align*}
\begin{cases}
    a(x, z) > c(\delta) > 0 & \text{for } x \in D, \delta < z < 1 - \delta, \\
    a(x, z) > c|\log z|^{-\sigma} & \text{for } x \in D, 0 < z < \frac{1}{2},
\end{cases}
\end{align*}
\]

where \(0 < \sigma < 1/(N + 2)\). Then Theorem 3.5 remains valid under the stronger assumption that

\[
u \in L^\infty(0, T; H_{\log}^{1, 2}(\Omega)),
\]

which was established in the proof of Lemma 3.9. The proof of the regularity theorem follows from Propositions 3.7 and 3.8 which are stated exactly as before. Some modifications occur in the proof of such propositions, and we limit ourselves here to indicate such changes.

Since Proposition 3.7 essentially involves only values of \(v\) near 1, the proof remains unchanged. We only have to choose \(\omega < \frac{3}{4}(\mu^+ - \mu^-)\) (see (3.18)). Then \(\mu^+ - \omega/4 > \frac{1}{2}\omega\) and therefore the estimates are unchanged if

\[
\phi_k(\omega) := \inf \left\{ a(x, z); x \in D, \frac{\omega}{6} < z < 1 - \frac{\omega}{4} \right\}.
\]

To prove 3.8 we have to note that now \(\nabla v\) is not defined near \(\{v = 0\}\), that is, (3.12) now reads

\[
\partial_t(s_k v) = \nabla \cdot \left( \lim_{\epsilon \to 0} \lim_{h \to 0} a(v) \nabla \min (v_h, 1 - q) + b(v) + d(v)v \right)
\]

where \(v_h := \max (v, h)\). In the proof of Lemma 5.2 we now use \((-v^2)' ((v_h - k)^-)\eta^2\) as test function. Then in the elliptic term \(a(v) > \phi_k(\omega)\), where

\[
\phi_k(\omega) := \min \left( \phi_k(\omega), c|\log h|^{-\sigma} \right).
\]

The integral involving \(v\) is now estimated by

\[
\left| \int_{t_1}^t \int_{B_R} d(v_h) v \nabla (\psi^2 ((v_h - k)^-)\eta^2) \right|
\]

\[
+ C \int_{t_1}^t \int_{B_R} |v| \left| \nabla (\psi^2 ((v_h - k)^-)\eta^2) \right|
\]

\[
\leq \delta \int_{t_1}^t \int_{B_R} \psi^2 ((v_h - k)^-) \left( \eta^2 |\nabla (v_h - k)^-|^2 + 1 \right) + h^2 |\nabla \eta|^2
\]

\[
+ \frac{C}{\delta} \int_{t_1}^t \int_{B_R} |v|^2 \eta^2 (\psi((v_h - k)^-) + h^2 \psi^2 ((v_h - k)^-))
\].
Therefore using the notation

$$
\psi_k^2(x) := \begin{cases} 
\psi^2(x) & \text{for } x_1 > k - h , \\
\psi^2(k - h) + \psi^2(k - h)(x - (k - h)) & \text{for } x_1 < k - h ,
\end{cases}
$$

the statement of Lemma 5.2 becomes

$$
\int_{B_k - \omega} s_0 \psi_k^2 \left( (v(t) - k)^- \right) < \int_{B_k} s_0 \psi_k^2 \left( (v(t_1) - k)^- \right) + C \frac{p - p}{p_\omega} \left( \frac{1}{\sigma_1^2} + \left( \frac{2pR}{\omega} \right)^2 \left( 1 + \frac{\varphi_\omega(\omega)}{\sigma_1} \right)^2 \left( \frac{2pR}{\omega} \right)^4 \right) R^{2\gamma} .
$$

Then in the proof of Lemma 5.3 we use the fact that

$$
\psi^2((v_h - k)^-) < \psi_k^2((v - k)^-) < \psi^2((v - k)^-) .
$$

We also let

$$
h = 2^{-p_\omega} .
$$

Hence the statement of the lemma remains exactly the same if we choose $p(\omega)$ so that

$$
p(\omega) \varphi_\omega(\omega) > C \quad (h \text{ as above}) ,
$$

which is possible provided $\sigma < 1$. Similarly in the proof of Lemma 5.4 we use $-(v_h - k)^- \eta^2$ as test function (now $h$ is again any sufficient small positive number). The additional $\nu$ term now is

$$
\frac{C h}{\sigma_1^2} \int_{t_1 - \beta R} \int_{B_k} |v|^2 (v_h - k)^- |\eta| |\nabla \eta| 
\leq \frac{C}{\delta} \sup_{t_1 - \beta R < t < t_1} \int_{B_k} |v|^2.
$$

Since it was assumed that $u$ is in $L^\infty(0, T; H^{1, \omega}_{loc}(D))$ it follows from elliptic estimates that locally in $D$

$$
\int_{B_k} |v(t)|^2 \leq OR^{2\gamma + \gamma}
$$

uniformly in $t$ for some $\gamma > 0$. Therefore Lemma 5.4 is now stated with $v$
replaced by \( v_h \) and \( \varphi_h(\omega) \) replaced by \( \varphi_h(\omega) \) and with the additional term

\[
\frac{\beta h^2}{\alpha^4} R^{n+\gamma}
\]

on the right side of the estimate. Moreover the term \( \int_{\partial\Omega} |\nabla (v_h - k)^-|^2 \) is now replaced by \( \varphi_h(\omega) \int_{\partial\Omega} |\nabla (v_h - k)^-|^2 \) because of assumption (6.1). Proceeding in the proof with Lemma 5.5 we see that if \( R < 2^{-s} \omega \) and \( h < k = \mu + 2^{-s-1} \omega \)

\[q = p(\omega), \ldots, q_0 - 1,\]

where \( q_0 \) has to be chosen. Therefore we have to use the value \( h = 2^{-s} \omega \), hence

\[
C_4(h, \omega) \leq \frac{C}{\alpha^4} \left( \frac{1}{\varphi_h(\omega)} \right)^2 + h^s R^\gamma.
\]

We wish to add (6.2) for \( q = p(\omega), \ldots, q_0 - 1, \) where \( q_0 \) has to be chosen. Therefore we have to use the value \( h = 2^{-s} \omega \), hence

\[
C_4(h, \omega) \leq \frac{C}{\alpha^4} \left( \frac{1}{\varphi_h(\omega)} + R^\gamma \right) \leq \frac{C}{\alpha^4 \varphi_h(\omega)}.
\]

Repeating the iteration process described in Lemma 5.6 we see that we have to choose \( q_0 \) so that

\[
\frac{C_1}{\sqrt{\alpha^4 \varphi_h(\omega)}} \leq C_4 \varphi_h(\omega)^{N+1},
\]

and this is possible if \( 0 < \sigma < 1/(N + 2) \).

REFERENCES


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