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Biholomorphic equivalence of bounded Reinhardt domains


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Biholomorphic Equivalence of Bounded Reinhardt Domains.

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Introduction.

A domain $D$ in a complex Banach space $E$ with basis $(e_n)_{n \in \mathbb{N}}$ is said to be Reinhardt (w.r.t. $(e_n)$) if it contains the origin and is invariant under the transformations

$$\sum_n x_n e_n \to \sum_n \lambda_n x_n e_n \quad \forall |\lambda_n| = 1.$$ \hspace{1cm} (1)

It is known that $E$ contains a bounded Reinhardt domain precisely when $(e_n)$ is unconditional. In this case, an appropriate diagonal linear isomorphism $T: E \to E$ normalizes $D$, i.e.,

$$e_n \in \partial \bar{D}, \quad \text{and} \quad \lambda e_n \in \partial \bar{D} \Rightarrow |\lambda| < 1 \quad \forall n \in \mathbb{N},$$

where $\bar{D} = TD$. $E$ may then be given an equivalent norm $\| \cdot \|$ for which $co(D)$ is the unit ball and $(e_n)$ is 1-unconditional.

Let $D$ and $\bar{D}$ be bounded normalized Reinhardt domains in $E$ and $\bar{E}$ w.r.t. the 1-unconditional bases $(e_n)$ and $(\bar{e}_n)$. $D$ and $\bar{D}$ are said to be biholomorphically equivalent if there is a biholomorphic map $\psi: D \to \bar{D}$. In [10], Sunada has shown that for finite dimensional $E$ and $\bar{E}$, $D$ and $\bar{D}$ are biholomorphically equivalent iff there is a surjective linear isomorphism $T: D \to D$ which is basic, i.e., there is a permutation $\sigma$ so that $T(e_n) = \bar{e}_{\psi(n)} \forall n$. We extend this result to infinitely many coordinates.

The methods used in [10] are Lie algebraic and peculiar to finite dimensions. We'll deduce the theorem instead by using the $D$ skew-hermitian operators on $E$ to examine the $\| \cdot \|$ isometric structure induced by $D$ on

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the coordinate subspaces of \( E \). Specifically, we'll find a maximal partition \( \mathcal{C} \) of \( N \) such that \( E_c = \{ e_i : i \in c \} \) is isometrically Hilbert space and \( D_c = D \cap E_c \) is the Hilbert unit ball \( \forall e \in \mathcal{C} \), and such that \( x \in D \) is determined by the sequence \( (\| x_e \|_1)_{c \in \mathcal{C}} \), where \( \| \cdot \|_1 \) is the Hilbertian norm on \( E_c \) and \( x_e \) is the canonical projection of \( x \) onto \( E_c \).

For \( D = B_x \) this structure was studied by Schneider and Turner [8] in finite dimensions and by Fleming and Jamison [4, 5] and Kalton and Wood [6] in infinite dimensions, where \( \{ E_c : c \in \mathcal{C} \} \) are referred to as the Hilbert components of \( E \). Stachó, using different methods, re-discovered Hilbert components in [9]. The papers of Vigué [12] and Barton et al. [2] using Jordan theoretic techniques developed principally by Vigué [12] and Braun et al. [3], may be viewed in part as uncovering certain Hilbert components of \( E \) induced by arbitrary bounded Reinhardt domains admitting of a non-linear biholomorphic automorphism.

In § 1 we'll use an elementary argument to establish the Hilbert components induced by a bounded normalized Reinhardt domain \( D \). The argument is motivated in part by Auerbach [1], in which it is shown that a bounded group of linear transformations is a subgroup of a group of unitary transformations. We'll see in particular that the Hilbert components induced by \( D \) are generally proper subspaces of those induced by \( \sigma(D) \). In § 2 we'll prove Sunada's theorem. The argument here is standard; half of it is similar to that of [5]. The normal form of a bounded Reinhardt domain (cf. [2]) is described in § 3. This form shows the very special geometric structure required of a bounded Reinhardt domain to support a nonlinear biholomorphic automorphism. A subset of the Hilbertian components induced by \( D \) can be computed from the parameters of the normal form, and these parameters furnish a set of biholomorphic invariants of \( D \).

§ 1. We first recall some background (cf. [11]). Let \( D \) be a bounded domain containing the origin, and let \( G_0(D) \) be the linear and continuous automorphisms of \( D \). The infinitesimal transformations of \( G_0(D) \) are a Banach Lie sub-algebra \( g(D)^+ \) of the linear operators on \( E \), called the \( D \) skew-hermitian operators on \( E \). \( g(D)^+ \) is closed under the Lie bracket

\[
[f, g] = g \circ f - f \circ g, \quad f, g \in g(D)^+.
\]

Represent \( f \in g(D)^+ \) as a matrix \( (f_{ij})_{i,j \in \mathbb{N}} \) where

\[
f_{ij} = \langle f(e_i), e_j^* \rangle
\]

and \( e_j^* \) is the coefficient functional associated with \( e_j \). When \( D \) is Reinhardt,
\( i a \in g(D)^+ \) whenever \( a \) is a real diagonal matrix \( (a_j) \) since

\[
\exp (i a)(x) = \sum_j \exp (i a_j) x_j e_j .
\]

Let \( h \) be a linear operator on \( E \) such that \( \iota h \in g(D)^+ \). Then for any real diagonal matrices \( a \) and \( b \),

\[
[\iota b, [i a, \iota h]] \in g(D)^+ .
\]

In particular, choosing \( a_k = 1 \) and \( a_i = 0 \) \( \forall i \neq k \), and \( b_1 = 1, b_i = 0 \) \( \forall i \neq 1 \), for some \( k \neq l \), we find that \( \iota (h e k + h e k) \in g(D)^+ \) where \( e_{ij} \) is the elementary matrix with a 1 in the \((i, j)^{th}\) place and zeros elsewhere.

Let \( D \) be a bounded normalized Reinhardt domain.

**Lemma 1:** (compare with [9, Lemma 3.6] and [6, Proposition 4.2]).

Let \( k \neq l \). Then the following are equivalent:

(i) \( \exists h \in g(D)^+ \) such that \( h e k \neq 0 \).

(ii) \( |x|^2 + |y|^2 = |x|^2 + |y|^2 \) and \( |x| = |y| \) \( \forall i \neq k, l \) imply \( x \in D \) iff \( y \in D \).

**Proof.** Suppose (i) holds. We may assume \( \iota h = \iota (x e k + \beta e k) \in g(D)^+ \) for some \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 0 \). Let \( g = \exp (i \gamma h) \) for arbitrary \( t \in \mathbb{R} \) and write \( D_0 = D \cap [e_k, e_k] \). Then \( g|_{D_0} \cap [e_k, e_k] \) is the identity and \( g|_{D_0} \in g(D_0) \), i.e., we may assume that \( D = D_0 \) and \( \iota h \in g(D_0)^+ \). By calculating \( \exp (i \gamma h) \), \( t \in \mathbb{R} \), one easily finds that \( \alpha \beta > 0 \) and \( (i/\gamma) h \in g(D_0) \), where \( \gamma^2 = \alpha \beta \).

Since \( D_0 \) is normalized,

\[
\lambda \frac{i \beta}{\gamma} e_k \in D_0 \Rightarrow |\lambda| < 1 , \quad \text{and}
\]

\[
\frac{i \beta}{\gamma} e_k \in D_0 \Rightarrow \left| \frac{\lambda i \beta}{\gamma^2} \right| < 1 .
\]

Since \( e_k, e_k \in D_0 \), we may choose \( |\lambda| \) as near to 1 as we please, and so conclude that \( |\alpha \beta / \gamma^2| = 1 \). Similarly, \( |\alpha / \gamma| = 1 \). It follows that

\[
\alpha = \beta .
\]

The argument of [6, Proposition 4.2] can be easily adapted to complete the proof of (i) \( \Rightarrow \) (ii).
Now assume (ii) holds. Let \(|x| = 1\), fix \(x \in D_0\), and let
\[
y = (x_k \cos t + iax_1 \sin t)e_k + (x_1 \cos t + i\tilde{a}x_1 \sin t)e_1
\]
for arbitrary \(t \in \mathbb{R}\). Then \(|y_k|^2 + |y_1|^2 = |x_k|^2 + |x_1|^2\), so \(y \in D_0\): This shows that \(i(x_k e_k + \tilde{a}e_1) \in g(D_0)^+.\)

REMARK. If \(k \neq l\) satisfy (ii), the above arguments shows that any \(g\) in \(G_0(B_U)\) is naturally in \(G_0(G_0)\), and \(g \oplus id_{[e_i: 0 \neq e_k, 0]} \in G_0(D)\). In particular, for each \(x \in E\) there exists \(g \in G_0(D)\) with
\[
g(x) = (|x_k|^2 + |x_1|^2)e_k + \sum_{i \neq k, l} x_i e_i.
\]

Define a relation on \(N\) by \(k \sim l\) if \(k = l\) or \(\exists f \in g(D)^+\) with \(f_{kl} \neq 0\). The proof of \((i) \Rightarrow (ii)\) shows that \(f_{kl} = -f_{lk}\), so \(\sim\) is symmetric. If \(i(xe_k + \tilde{a}e_k)\) and \(i(\beta e_k + \beta e_k)\) are in \(g(D)^+\), then so is their Lie bracket, which is just \(i(\alpha e_k + \alpha e_k)\). So \(\sim\) is an equivalence relation. Let \(E\) denote the induced equivalence classes. For \(x \in E\), let \(x_c\) be the canonical projection of \(x\) onto \(E_c = [e_i: i \in c], c \in G\). Write \(D_c = D \cap E_c\), and for each \(c \in G\) choose a distinguished \(i_c \in c\).

If, for some \(c \in G\) and \(x \in E\), \(x_c\) has only finitely many nonzero coordinates, then the remark following Lemma 1 may be repetitively applied to find a \(g \in G_0(D)\) such that
\[
g(x) = \|x\|_2^2 e_{i_c} + \sum_{i \neq c} x_i e_i,
\]
where \(\|x\|_2 = \left(\sum_{i} |x_i|^2\right)^{\frac{1}{2}}\).

Let \(y^n = \sum_{i=1}^{n} y_i e_i\), and let \(c \in G\) such that \(c \cap \{1, \ldots, n\} \neq \emptyset\). Then \(\exists g_c \in G_0(D)\) such that
\[
g_c(y^n) = \|y^n\|_2^2 e_{i_c} + \sum_{i \notin c} y_i^n e_i.
\]

By composing the automorphisms \(g_c\) we get a \(g_n \in G_0(D)\) such that
\[
g_n(y^n) = \sum \|y^n\|_2^2 e_{i_c},
\]
where the summation is over all \(c \in G\) with \(c \cap \{1, \ldots, n\} \neq \emptyset\). Since \(G_0(D)\) is naturally embedded in \(G_0(co(D))\), every \(g \in G_0(D)\) is a \(\|\cdot\|\)-isometry. Hence,
if $m > n$, then
\[
\|g_m(y^n) - g_n(y^n)\| = \|g_m(y^n) - y^n\| = \|y^n - y^n\|.
\]
So $(\sum \|y^n_e\| \cdot e_e)_n$ converges in $E$. Also,
\[
\|x - g_n^{-1}(\sum \|y^n_e\| \cdot e_e)\| = \|g_n x - \sum \|y^n_e\| \cdot e_e\| = \|\sum x_i e_i\|
\]
so $g_n^{-1}(\sum \|y^n_e\| \cdot e_e)$ and consequently $\sum \|y^n_e\| \cdot e_e$ both converge to an element in $D$ if and only if $x \in D$.

Should $x \in E$ for some $e \in \mathcal{C}$, then $\|y^n\| = \|g_n x^n\| = \|y^n\| \forall n$. Hence $\|x\| = \|x\|_2$. Summarizing the above discussion, we've shown (compare with [4, Lemma 4.2]):

**Lemma 2.** Let $x \in E$. Then $\|x\| = \|x\|_2 \forall e \in \mathcal{C}$. Furthermore,
\[
x \in D \quad \text{iff} \quad \sum_{e} \|x\|_2 e_e \in D.
\]
As a consequence we obtain

**Lemma 3.** $D_c \cong B_{t_1} \forall e \in \mathcal{C}$.

**Proof.** By the normalization of $D$ and Lemma 2 we may consider $D_c$ to be a subset of $B_{t_1}$. Since $D_c$ contains a relatively open neighborhood of the origin, $\exists \theta < \zeta < 1$ such that $|t| < \zeta$ implies $te_e \in D_c$. By Lemma 2, $\zeta B_{t_1} \subseteq D_c$. Hence, the $\|\cdot\|$ and $\|\cdot\|_2$ topologies on $E_c$ coincide. Thus, $D$ is an open connected subset of $B_{t_1}$. Since each of the sets
\[
D_c \cap \{x \in B_{t_1} : \|x\|_2 < t\}, \quad \text{and}
\]
\[
D_c \cap \{x \in B_{t_1} : t < \|x\|_2 < 1\}, \quad 0 < t < 1
\]
is open and nonempty, their union cannot exhaust $D_c$. Therefore, $\forall 0 < t < 1 \exists x \in D_c$ with $\|x\|_2 = t$. Another application of Lemma 2 completes the proof.

It's now a small step to

**Proposition 4.** $\bigoplus_{e \in \mathcal{C}} G_e(D_c) = \bigoplus_{e \in \mathcal{C}} G_e(B_{t_1}) \subseteq G_e(D)$.

**Proof.** Let $g_e \in G_e(D_c) \forall e \in \mathcal{C}$ and let $g = \bigoplus g_e$. Let $x \in E$. By Lemma 3,
We conclude this section by noting that the embedding of \( G_\phi(D) \) into \( G_\phi(co(D)) \) induces an embedding of \( g(D)^+ \) into \( g(co(D))^+ \), and so \( \mathcal{C} \) refines the equivalence classes induced by \( co(D) \). The refinement is strict in general, as is apparent from the example

\[
D = B_1 \setminus \{ x \in B_1 : |x_1| = |a_1|, |x_2| = |a_2| \},
\]

where \( 0 < |a_1|^2 + |a_2|^2 < 1 \). Choosing \( |a_1| \neq |a_2| \) ensures, in particular, that the map \( g \in G_\phi(B_1^1) \) given by \( g(e_1) = e_2 \) and \( g(e_2) = e_1 \) is not in \( G_\phi(D) \).

\textbf{§ 2.} In this section we prove our main result.

\textbf{Theorem.} Let \( D \) and \( \tilde{D} \) be bounded normalized Reinhardt domains in \( E \) and \( \tilde{E} \) with respect to the bases \( (e_n) \) and \( (\tilde{e}_n) \). Then \( D \) is biholomorphically equivalent to \( \tilde{D} \) if and only if there is a surjective linear isomorphism \( \psi : E \to \tilde{E} \) taking \( D \) onto \( \tilde{D} \) such that \( \psi(\tilde{e}_1) = \tilde{e}_\sigma(1) \) for some bijection \( \sigma : \mathbb{N} \to \mathbb{N} \).

\textbf{Proof.} Sufficiency is trivial.

Suppose that \( D \) is biholomorphically equivalent to \( \tilde{D} \) and let \( \psi : D \to \tilde{D} \) be a biholomorphic mapping. We first show that there is a surjective linear isomorphism \( T : E \to \tilde{E} \) taking \( D \) onto \( \tilde{D} \). The argument is standard. For \( x \in D \) denote the orbit of \( x \) under \( G(D) \), the biholomorphic automorphisms of \( D \), by

\[
G(D) \cdot x = \{ g(x) : g \in G(D) \}.
\]

If \( f \in G(D) \), then \( \psi f \psi^{-1} \in G(\tilde{D}) \), so

\[
\psi f(x) = \psi f \psi^{-1}(\psi(x)) \in G(\tilde{D}) \cdot \psi(x) \quad \forall f \in G(D).
\]

Hence,

\[
\psi(G(D) \cdot x) \subseteq G(\tilde{D}) \cdot \psi(x).
\]

Likewise,

\[
\psi^{-1}(G(\tilde{D}) \cdot \psi(x)) \subseteq G(D) \cdot x.
\]
Thus,
\[ \varphi(G(D)\cdot x) = G(\bar{D})\cdot \varphi(x). \]

Kaup and Upmeier [7] have shown that
\[ G(D)\cdot 0 = \{ x \in D : G(D)\cdot x \text{ is a closed complex submanifold of } D \}. \]

Since \( \varphi \) preserves these properties, \( G(\bar{D})\cdot \varphi(0) \) is a closed complex submanifold of \( \bar{D} \). Hence, \( \varphi(0) \in G(\bar{D})\cdot 0 \), and so \( \exists g \in G(\bar{D}) \) with \( \varphi(0) = 0 \). By H. Cartan’s theorem, \( \varphi \) is linear. \( T = g \varphi \) is the desired mapping. Observe that by Proposition 4 and Lemma 1
\[ i h \in g(D)^+ \text{ if and only if } h = \bigoplus_{c} h_c, \]
where \( i h_c \in g(D_c)^+ \) for all \( c \in \mathcal{C} \).

List the elements of \( \mathcal{C} = \{ c_1, c_2, \ldots \} \), and \( \mathcal{F} = \{ \bar{c}_1, \bar{c}_2, \ldots \} \), where each set is finite or infinite according to the cardinality of \( \mathcal{C} \) and \( \mathcal{F} \). Write
\[ T = (T_{i,j})_{i,j} \quad \text{where } T_{i,j} : E_{c_i} \to \overline{E_{\bar{c}_j}}, \]
and
\[ T^{-1} = (S_{i,j})_{i,j} \quad \text{where } S_{i,j} : \overline{E_{\bar{c}_j}} \to E_{c_i}. \]

Fix \( k \) and \( i h \in g(D_k)^+ \). Since \( T \circ \exp (g(D)^+) \circ T^{-1} = \exp (T \circ g(D)^+ \circ T^{-1}) \) it follows that \( T \circ g(D)^+ \circ T^{-1} \subseteq g(\bar{D})^+ \). Hence,
\[ T \circ (i h \otimes 0) \circ T^{-1} = (T_{i,k} \circ i h \circ S_{k,i})_{i,j} = \bigoplus_j i h_{\bar{c}_j} \]
for some \( i h_{\bar{c}_j} \in g(\bar{D}_{\bar{c}_j})^+ \), where \( 0 \) is the zero map on \( E_{k-i} \). Consequently
\[ (2.1) \quad T_{i,k} \circ i h \circ S_{k,i} = 0 \quad \text{for all } i \neq j \text{ and } i h \in g(D_k)^+. \]

Since \( T \) is invertible there is a \( j \) with \( S_{k,j} \neq 0 \). Because \( D_k \cong B_k \) every \( g \in G(D_k) \) may be written \( g = \partial h_1 - h_2 \), where \( \partial h_1, h_2 \in g(D_k)^+ \). Hence (2.1) implies \( T_{i,k} = 0 \) for all \( i \neq j \). Since \( T \) is invertible, \( T_{i,k} \neq 0 \). Thus for each \( k \) there is a unique \( j \) with \( T_{i,k} \neq 0 \). This defines a map
\[ \tau : \mathcal{C} \to \mathcal{F} \]
by
\[ \tau(c_k) = \bar{c}_{\tau(k)}. \]
$T$'s invertibility ensures that $\tau$ is a bijection. Consequently,

$$T|_{E_0}^* E_0 \to \mathcal{E}_{\tau(t)}.$$

Hence, there is an appropriate choice of $\tilde{g}_x \in G_0(\tilde{D}_x)$ so that $(\oplus \tilde{g}_x) \circ T$ is the desired mapping.

§3. We first recall some notation and background (see [2] and [12]). Suppose that $D$ supports a nonlinear biholomorphic automorphism, so that $G(D) \cdot 0 \not\subset 0$. Then $\exists I \subset \mathbb{N}$ with $E_i = [e_i]$, $i \in I$ $= [G(D) \cdot 0]$, a partition $\mathcal{P}$ of $I$ with $E_x = [e_i]$, $i \in p \approx I$ and $D \cap E_j \cong B(\mathbb{C})$, and nonnegative constants $r_{n,j}$ with $\sup_{\mathcal{P}} \sum r_{n,j} < \infty$ so that

$$D = \left\{ \sum_{\mathcal{P}} x_p + \sum_x x_j e_j : (\|x_n\|_2^2) \in B_2, \sum_{e_i} x_j e_j \in D_1 \right\},$$

where $J = \mathbb{N} \setminus I$, $x_{\mathcal{P}}$ denotes $\sum_{p \in \mathcal{P}} x_p$, $D_1 = D \cap E_1$, and

$$q_j(x_{\mathcal{P}}) = \prod_{p \in \mathcal{P}} (1 - \|x_p\|_2^2)^{r_{n,j}}.$$

We'll abbreviate these notations by writing

$$D = B_1[+]D_1,$$

where $B_1 = B_{1(\mathbb{C})}$. For each $k \in J$, define

$$S_k = \{ j \in J : r_{n,j} = r_{n,k} \ \forall p \in \mathcal{P} \}.$$

Then $S_k \cap S_i \neq \emptyset$ $\Rightarrow S_k = S_i$ and $\bigcup_k S_k = J$. Thus, the distinct members $\mathcal{P}$ of $\{ S_k : k \in J \}$ form a partition of $J$. For $j, k \in \mathcal{P}$, we write $r_{n,s}$ for the common value of $r_{n,j}$ and $r_{n,k}$, and we write $q_j$ for the function

$$q_j(x_{\mathcal{P}}) = \prod_{p \in \mathcal{P}} (1 - \|x_p\|_2^2)^{r_{n,s}}, \quad s \in \mathcal{P}.$$

With these notations,

$$x \in D \iff x_{\mathcal{P}} \in B_1 \quad \text{and} \quad \sum_{p \in \mathcal{P}} \frac{x_p}{q(p_{\mathcal{P}})} \in D_1,$$
where \( x_s \) is the canonical projection of \( x \) onto \( E_s = [e_i : i \in s] \). Observe that, since \( q_s(x) \) depends only on \( \|x^p\|_{p < s} \),

\[
\bigoplus_{p < s} G_q(D \cap E_p) \oplus G_q(D_1) \subseteq G_q(D).
\]

Associated with \( D \) is its triple product \( \{ , , \} : E \times E \times E \to E \). We'll not study the triple product here, but simply use several properties it possesses, namely that it is symmetric bilinear in the outer two variables, conjugate linear in the middle variable, and satisfies

\[
\{ x \xi x \} = 0 \quad \forall \xi \in E, \quad x \in E,
\]

\[
\{ x \xi y \} = - (x|\xi) \sum_{p} r_{s, y} \quad \forall \xi, x \in E, \quad \forall y \in E
\]

\[
\{ x \xi y \} = - \frac{1}{2} \sum_{p} (y_{s, p} x_{s, p} + y_{s, p} x_{s, p}) \quad \forall \xi, x, y \in E.
\]

where \( (\cdot | \cdot) \) is the inner product on \( E_p \). In [3] and [12] it's been shown that \( \mathfrak{h} \in g(D^+) \) implies

\[
\mathfrak{h}(x \xi y) = - \{ x, \mathfrak{h}(\xi), y \} + \{ \mathfrak{h}(x), \xi, y \} + \{ x, \xi, \mathfrak{h}(y) \} \quad \forall \xi \in E, \quad \forall x, y \in E.
\]

It may occur that \( (x \xi y) \cdot 0 \not\subset \{ 0 \} \), in which case \( D_1 \) may be decomposed

\[
D_1 = B_s[+D_2].
\]

Assume this process continues at least \( n \) times, so

\[
D = D_0 = B_1[+D_1] = B_1[+B_2[+D_3]] = \cdots = B_1[+B_2[+B_3[+\cdots [B_{n-1}[+]D_n] \cdots]]].
\]

Write \( P_s, \mathcal{P}_s, p^k_s \), etc., for the quantities and objects associated with \( D_{s-1} \), \( 1 < k < n \). Define

\[
C_1 = \mathcal{P}_1,
\]

\[
C_k = \{ s_1 \cap \cdots \cap s_{k-1} \cap p_k : s_i \in \mathcal{P}_i, p_k \in \mathcal{P}_k \}, \quad 1 < k < n.
\]
Theorem. \( \bigcup_{k=1}^{n} \mathcal{C}_k \subseteq \mathcal{C} \), where \( \mathcal{C} \) determines the Hilbertian components induced by \( D \).

The theorem is established by the following two lemmas. Note that if \( c \in \mathcal{C}_k \), then \( \exists p \in \mathcal{P}_k \) with \( c \subseteq p \), so \( D_c = D \cap E_c \cong B_1 \).

Lemma 6. If \( h \in g(D)^+ \), then \( h(E_c) \subseteq E_c \) \( \forall c \in \bigcup_{k=1}^{n} \mathcal{C}_k \).

Proof. Choose \( p \in \mathcal{P}_1 \) and \( i \in p \). Then

\[
- h(e_i) = h(\{e_i, e_i, e_i\}) = - \{e_i, h(e_i), e_i\} + 2\{e_i, e_i, h(e_i)\} = (e_i|\langle h(e_i) \rangle_p)e_i - (e_i|e_i)(h(e_i))_p - (\langle h(e_i) \rangle_p|e_i)e_i \in E_p.
\]

So \( h(E_p) \subseteq E_p \) \( \forall p \in \mathcal{P}_1 \).

Now choose \( s_0 \in \mathcal{S}_1 \), and let \( p \) and \( i \) be as above. Then for \( y \in E_{s_0} \),

\[
r_{p,s_0}h(y) = - h(\{e_i, e_i, y\}) = \{e_i, h(e_i), y\} - \{h(e_i), e_i, y\} - \{e_i, e_i, h(y)\} = - (e_i|h(e_i))_{r_{p,s_0}}y + (\langle h(e_i) \rangle_p|e_i)r_{p,s_0}y + (e_i|e_i) \sum_{s \in \mathcal{S}_1} r_{p,s}(h(y)).
\]

Since \( (e_i|e_i) = 1 \), this implies that

\[
r_{p,s_0}(h(y)) = r_{p,s}(h(y)) \quad \forall p \in \mathcal{P}_1,
\]

for all \( s \neq s_0 \) for which \( (h(y))_s \neq 0 \). It follows from the definition of \( \mathcal{S}_1 \) that \( (h(y))_s = 0 \) \( \forall s \neq s_0 \). Thus \( h(E_s) \subseteq E_s \) \( \forall s \in \mathcal{S}_1 \). In particular, \( h \) decomposes so that \( \langle h \rangle_{I_{D_1}} \in g(D_1)^+ \). Hence, the above argument can be repeated for \( D_1, ..., D_{n-1} \) to yield

\[
h(E_p) \subseteq E_p \quad \forall p \in \bigcup_{k=1}^{n} \mathcal{P}_k
\]

\[
h(E_s) \subseteq E_s \quad \forall s \in \bigcup_{k=1}^{n} \mathcal{P}_k.
\]

Taking intersections completes the proof.

In view of Lemma 1, Lemma 6 establishes that for each \( c \in \bigcup_{k=1}^{n} \mathcal{C}_k \), there is \( c' \in \mathcal{C} \) with \( c' \subseteq c \). The reverse inclusion follows from
Lemma 7. Let $c_0 \in \mathcal{C}_k$ for some $1 \leq k \leq n$. Suppose $y_i = x_i \forall i \notin c_0$ and that $\|y_c\|_2 = \|x_c\|_2$. Then $x \in D \iff y \in D$.

Proof. It suffices to show that if $g_0 \in G_0(D_{c_0}) = G_0(B_{c_0})$, then $g = g_0 \oplus \text{id}_{\mathcal{E}_N \backslash c_0} \in G_0(D)$. Write $c_0 = s_1 \cap \ldots \cap s_{k-1} \cap p_k$ for some $s_i \in \mathcal{S}_i$ and $p_k \in \mathcal{P}_k$.

Fix $1 \leq i \leq k-1$ and $s \in \mathcal{S}_i$. If $s \neq s_i$, then $g|_{s_i} = \text{id}$. If $s = s_i$, then $g|_{s_i} = g_0 \oplus \text{id}_{\mathcal{E}_N \backslash s_i}$, so $g(E_i) \subseteq E_i$. Hence,

$$g(E_i) \subseteq E_i \quad \forall s \in \bigcup_{i=1}^{k-1} \mathcal{S}_i.$$ 

It follows that

$$g \in G_0(D) \iff \forall x \in D, \quad g(x) \in D$$

$$\iff \forall x \in D, \quad x_{s_i} \in B_1 \quad \text{and} \quad \sum_{\mathcal{P}_i} \frac{(g(x))_{\mathcal{P}_i}}{q_{\mathcal{P}_i}(x_{\mathcal{P}_i})} \in D_1$$

$$\iff \forall x \in D, \quad x_{s_i} \in B_1 \quad \text{and} \quad g \left( \sum_{\mathcal{P}_i} \frac{x_{\mathcal{P}_i}}{q_{\mathcal{P}_i}(x_{\mathcal{P}_i})} \right) \in D_1,$$

which is implied by $g|_{s_i} \in G_0(D_i)$. Repeating this argument $k-1$ times we see that

$$g|_{s_{k-1}} \in G_0(D_{k-1}) \Rightarrow g|_{s_{k-2}} \in G_0(D_{k-2}) \Rightarrow \ldots \Rightarrow g \in G_0(D).$$

Since $g|_{s_{k-1}} \in \bigoplus_{s \in \mathcal{P}_k} G_0(D \cap E_s) \oplus G_0(D_k) \subseteq G_0(D_{k-1})$ the proof is complete. \Box

We conclude this section with the following:

Proposition 8. Let $D$ and $\tilde{D}$ be biholomorphically equivalent bounded normalized Reinhardt domains in $E$ and $\tilde{E}$. Then the matrices $(r_{s,x})_{s,x}$ and $(\tilde{r}_{s,x})_{s,x}$ agree up to a permutation.

Proof. In light of the Theorem of § 2, we may assume there is a basic isomorphism $T : D \to \tilde{D}$ and a bijection $\tau : \mathcal{C} \to \tilde{E}$ such that $T(E_i) = E_i(\xi)$, $\forall \xi \in \mathcal{E}$. By [12, Theorem 2.2], $T|_{s_i} = \tilde{B}_i$ and $T|_{s_{k-1}} = \tilde{B}_1$, and if $\xi \in E_i$, $x \in E$, then

$$T(\{x \xi x\}_{s}) = \{T(x), T(\xi), T(x)\}_{\tilde{s}}$$

where all quantities with a $\sim \circ$ above them refer to $\tilde{D}$. In particular,
and Then
Since we have
Using the definition of \( \tilde{\mathcal{F}} \), it follows that there is a unique \( \tilde{s} \), depending on \( s \), with \( (T(x))_s \neq 0 \), and that \( r_{p,s} = \tilde{r}_{p,\tilde{s}} \) \( \forall p \in \mathcal{P} \). The map \( s \to \tilde{s}(s) \) determines a bijection \( \sigma: \mathcal{G} \to \tilde{\mathcal{F}} \) since \( T \) is invertible. Evidently,

\[
(r_{p,s})_{\mathcal{G},\mathcal{P}} = (\tilde{r}_{p,\tilde{s}})_{\tilde{\mathcal{F}},\tilde{\mathcal{P}}}. \quad \blacksquare
\]

**Remark.** If \( D \) can be decomposed \( n \) times, iteration of the above argument shows that \( \tilde{D} \) can be decomposed \( n \) times, and that \( (r_{p,s})_{\mathcal{G},\mathcal{P}} \) and \( (\tilde{r}_{p,s})_{\tilde{\mathcal{G}},\tilde{\mathcal{P}}} \) agree up to a permutation \( \forall 1 < k < n \).

**BIBLIOGRAPHY**


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