J. MOSSINO
J. M. RAKOTOSON

Isoperimetric inequalities in parabolic equations


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Isoperimetric Inequalities in Parabolic Equations.

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0. - Introduction.

Consider the parabolic equation

\[
\begin{aligned}
\frac{\partial u}{\partial t} + H_t(u) + cu &= f & \text{in } Q = (0, T) \times \Omega, \\
u &= 0 & \text{on } \Sigma = (0, T) \times \partial \Omega, \\
u(0, \cdot) &= u_0 & \text{for } t = 0,
\end{aligned}
\]

where \( \Omega \) is a bounded regular domain in \( \mathbb{R}^n \) \( (N \geq 1) \),

\[ H_t(u) = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} a_{ij}(t, x) \frac{\partial u}{\partial x_i}, \]

\( a_{ij} \) satisfy the uniform ellipticity condition (with constant one)

\[ \sum_{i,j=1}^{N} a_{ij}(t, x) \xi_i \xi_j \geq |\xi|^2, \quad \forall \xi \in \mathbb{R}^n; \]

c, \( u_0 \) and \( f \) are non-negative functions; their regularity will be precised later on.

Consider also the equation

\[
\begin{aligned}
\frac{\partial U}{\partial t} - AU &= f & \text{in } \bar{Q} = (0, T) \times \bar{\Omega}, \\
U &= 0 & \text{on } \bar{\Sigma} = (0, T) \times \partial \bar{\Omega}, \\
U(0, \cdot) &= u_0,
\end{aligned}
\]

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where $\Omega$ is the ball of $\mathbb{R}^n$, centered at the origin, which has the same measure as $\Omega$, and $u_a$ (resp. $f(t, \cdot)$) is the rearrangement of $u_0$ (resp. $f(t, \cdot)$) in $\Omega$, which decreases along the radii. This rearrangement is defined as follows.

If $v$ is a real measurable (1) function defined in $\Omega$, the decreasing rearrangement of $v$ is defined in $\Omega^\circ = [0, |\Omega|]$, by

$$v_a(s) = \inf \left\{ \theta \in \mathbb{R}, |v > \theta| \leq s \right\}$$

where $|v > \theta| = \text{meas} \left\{ x \in \Omega, v(x) > \theta \right\}$ (for any measurable set $E$, we denote $|E|$ its measure). The spherical rearrangement of $v$ in $\Omega$, which decreases along the radii is

$$\Omega(x) = v_a(\alpha_n|x|^n), \quad \text{for} \ x \in \Omega,$$

where $\alpha_n$ is the measure of the unit ball of $\mathbb{R}^n$. If $v$ is defined in $(0, T) \times \Omega$, and is measurable with respect to the space variable $x$ of $\Omega$, we consider its rearrangement with respect to $x$:

$$v_a(t, s) = (v(t, \cdot))_a(s) = \inf \left\{ \theta \in \mathbb{R}, |v(t, \cdot) > \theta| \leq s \right\},$$

$$\Omega(t, x) = v_a(t, \alpha_n|x|^n).$$

C. Bandle [2] proved that every strong solution $u$ of problems (1) satisfies

$$\forall t \in [0, T], \forall s \in \Omega^\circ, \quad \int_0^s u_a(t, \sigma) \, d\sigma \leq \int_0^s U_a(t, \sigma) \, d\sigma,$$

which leads to

$$\forall t \in [0, T], \forall \tau \in [1, \infty], \quad \|u(t, \cdot)\|_{L^\tau(\Omega)} \leq \|U(t, \cdot)\|_{L^\tau(\Omega)}.$$  

J. L. Vasquez [9] obtained the same result, if $u$ is a weak solution of a degenerate parabolic equation, the equation of porous media:

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta \varphi(u) & \text{in} \ Q = (0, \infty) \times \mathbb{R}^n, \\
u(0, \cdot) = u_0 & \text{for} \ t = 0,
\end{cases}$$

where $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and continuous, $\varphi(0) = 0$. He used the (1) In the whole paper, we consider the Lebesgue measure.
In this paper, we give a direct proof of (6), (7), valid for every weak solution of problems (1) (see Section 2).

Our method relies on the calculation of the directional derivative of the mapping \( u \to u^* \), that is \( v_{u^*} = \lim_{\lambda \to 0} \frac{(u + \lambda v)_{u^*} - u_{u^*}}{\lambda} \). This calculation was made first by J. Mossino and R. Temam [6], with a direction \( v \) in \( L^p(\Omega) \).

In the first section, we extend their result to functions \( v \) in \( L^p(\Omega) \) \((1 \leq p \leq +\infty)\). Moreover we prove that, if \( u \) belongs to \( H^1(0, T; L^p(\Omega)) \), then \( u^* \) belongs to \( H^1(0, T; L^p(\Omega^*)) \) and

\[
\frac{\partial u^*}{\partial t} = \left( \frac{\partial u}{\partial t} \right)_{u^*}.
\]

Besides, \((\partial u/\partial t)(t, \cdot)\) is shown to be constant on every set where \( u(t, \cdot) \) is constant. The last formula is a crucial point in Section 2.

1. Directional derivative of the rearrangement mapping.

In this Section 1, we assume that \( \Omega \) is a measurable subset of \( \mathbb{R}^N \) \((|\Omega| < \infty, N \geq 1)\). For the sake of completeness, we first recall some properties of rearrangements (see the proofs in [7] for example), and a result of [6].

1.1. Properties of rearrangements.

Let \( u \) be a measurable function: \( \Omega \to \mathbb{R} \) and \( u^* \) be its increasing rearrangement, defined by (2) and

\[
(1.1) \quad u^* = -(-u)^*.
\]

An essential property of rearrangement is that \( u \) and \( u^* \) are equi-measurable:

\[
\forall \theta \in \mathbb{R}, \quad |u < \theta| = \text{meas} \{x \in \Omega, u(x) < \theta\} = |u^* < \theta|,
\]

which implies

\[
(1.2) \quad \int_{\Omega} F(u) \, dx = \int_{\Omega^*} F(u^*) \, ds,
\]

for every Borel measurable \( F: \mathbb{R} \to \mathbb{R}^+ \). Here are some other properties of the increasing rearrangement mapping.
(a) If \( u_1, u_2 \) are two measurable functions such that \( u_1 \leq u_2 \) almost everywhere, then \( u_1^* \leq u_2^* \) everywhere.

(b) For all constants \( C \), \( (u + C)^* = u^* + C \).

(c) More generally, if \( \varphi \) is an increasing function from \( \mathbb{R} \) into \( \mathbb{R} \), then \( \varphi(u)^* = \varphi(u^*) \) almost everywhere.

(d) The mapping \( u \to u^* \) applies \( L^p(\Omega) \) into \( L^{p^*}(\Omega^*) \) (1 \( \leq p \leq \infty \)). It is contracting and norm-preserving.

(e) If \( u \) is in \( L^p(\Omega) \), \( v \) in \( L^{p'}(\Omega)(1/p + 1/p' = 1) \), then

\[
\int_{\Omega} uv \, dx \leq \int_{\Omega} u^* v^* \, dx = \left( \int_{\Omega} u^* v^* \, dx \right).
\]

This inequality is due to Hardy and Littlewood.

We shall use a slight extension of (d):

**Lemma 1.1.** Let \( u: \Omega \to \mathbb{R} \) be measurable, \( v \) in \( L^p(\Omega) \) (1 \( \leq p \leq \infty \)). Then \( (u + v)^* - u^* \) belongs to \( L^{p^*}(\Omega^*) \) and

\[
\|(u + v)^* - u^*\|_{L^{p^*}(\Omega^*)} \leq \|v\|_{L^p(\Omega)}.
\]

This lemma was proved in [7]. For convenience, we reproduce the proof here.

(i) If \( p = \infty \), we have

\[
u - \|v\|_{L^\infty(\Omega)} \leq u + v \leq u + \|v\|_{L^\infty(\Omega)}, \quad \text{a.e.}
\]

By properties (a) and (b) above,

\[
u^* - \|v\|_{L^\infty(\Omega)} \leq (u + v)^* \leq u^* + \|v\|_{L^\infty(\Omega)},
\]

that is

\[
\|(u + v)^* - u^*\|_{L^{\infty}(\Omega^*)} \leq \|v\|_{L^\infty(\Omega)}.
\]

(ii) If \( p < \infty \), we use the truncation

\[
f_n(t) = \begin{cases} 
- n & \text{if } t \leq - n, \\
 t & \text{if } - n \leq t \leq n, \\
 n & \text{if } t \geq n.
\end{cases}
\]
Then $f_n(u)$ and $f_n(u + v)$ are in $L^\infty(\Omega)$. By (c) and (d), $(f_n(u))^* = f_n(u^*)$, $(f_n(u + v))^* = f_n((u + v)^*)$, these functions are in $L^\infty(\Omega^*)$, and

$$
\|f_n((u + v)^*) - f_n(u^*)\|_{L^p(\Omega^*)} = \|(f_n(u + v))^* - (f_n(u))^*\|_{L^p(\Omega^*)} \leq \|f_n(u + v) - f_n(u)\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega)}
$$

(as $f_n$ is contracting). Then, using Fatou lemma,

$$
\|v\|_{L^p(\Omega)} \geq \limsup_{n \to \infty} \int_{\Omega^*} |f_n((u + v)^*) - f_n(u^*)|^p \, ds
$$

and

$$
\int_{\Omega^*} \liminf_{n \to \infty} |f_n((u + v)^*) - f_n(u^*)|^p \, ds
$$

$$
= \int_{\Omega^*} |(u + v)^* - u^*|^p \, ds.
$$

1.2. Directional derivative of the rearrangement mapping. Relative rearrangement.

First, we shall recall a result due to J. Mossino and R. Temam [6].

Consider a couple of functions $(u, v)$, $u : \Omega \to \mathbb{R}$ is measurable, $v$ is in $L^p(\Omega)$ $(1 \leq p \leq \infty)$, and a parameter $\lambda > 0$. By Lemma 1.1, $(u + \lambda v)^* - u^*$ belongs to $L^p(\Omega^*)$, and we can define

$$
(1.4) \quad \varphi_\lambda(s) = \int_0^s \frac{(u + \lambda v)^* - u^*}{\lambda} \, ds.
$$

Thus, $d\varphi_\lambda/ds = ((u + \lambda v)^* - u^*)/\lambda$. By Lemma 1.1,

$$
(1.5) \quad \left\|\frac{d\varphi_\lambda}{ds}\right\|_{L^p(\Omega^*)} \leq \|v\|_{L^p(\Omega)}.
$$

We are going to show that $d\varphi_\lambda/ds$ tends (in the sense of distributions) to $dv/ds$, where

$$
(1.6) \quad \varphi(s) = \begin{cases} 
\int_{u < u^*(s)} v \, dx & \text{if } |u = u^*(s)| = 0, \\
\int_{u < u^*(s)} v \, dx + \int_0^{1 - |u < u^*(s)|} (v|_{\Omega^*}) \, d\sigma, & \text{otherwise},
\end{cases}
$$

-
is the restriction of \( v \) to \( P(s) = \{ u = u^*(s) \} \). The following was proved in [6].

**THEOREM 1.1.** If \( u \) is a measurable function from \( \Omega \) into \( \mathbb{R} \), \( v \) is in \( L^\infty(\Omega) \), then \( w \) is lipschitz,

\[
\left\| \frac{dw}{ds} \right\|_{L^\infty(\Omega)} \leq \| v \|_{L^\infty(\Omega)},
\]

and, when \( \lambda \) decreases to zero,

1. \( w_\lambda \to w \) in \( C^0([0, |\Omega|]) \) that is uniformly;
2. \( \frac{dw_\lambda}{ds} = \frac{(u + \lambda v)^* - u^*}{\lambda} \to \frac{dw}{ds} \) in \( L^\infty(\Omega^*) \) weak *.

\( \square \)

We shall extend Theorem 1.1 to functions \( v \) in \( L^p(\Omega) \) (\( 1 \leq p \leq \infty \)).

**THEOREM 1.1 bis.** Let \( u, v \) be two measurable functions from \( \Omega \) into \( \mathbb{R} \), \( v \) in \( L^p(\Omega) \) (\( 1 \leq p \leq \infty \)). Then \( w \) belongs to \( W^{1,p}(\Omega^*) \),

\[
\left\| \frac{dw}{ds} \right\|_{L^p(\Omega^*)} \leq \| v \|_{L^p(\Omega)},
\]

and, when \( \lambda \) decreases to zero

1. \( w_\lambda \to w \) in \( C^0([0, |\Omega|]) \);
2. \( \frac{dw_\lambda}{ds} = \frac{(u + \lambda v)^* - u^*}{\lambda} \to \frac{dw}{ds} \) in the sense of distributions:

\[
\forall \varphi \in \mathcal{D}(\Omega^*) , \quad \int_{\Omega^*} \frac{dw_\lambda}{ds} \varphi \, ds \to \int_{\Omega^*} \frac{dw}{ds} \varphi \, ds.
\]

(In particular, \( dw_\lambda/ds \to dw/ds \) in \( L^p(\Omega^*) \)-weak if \( 1 < p < \infty \), in \( L^\infty(\Omega^*) \)-weak * if \( p = \infty \)).

\( \square \)

**Proof.** Consider \( v_n \) in \( L^p(\Omega) \); \( w_{\lambda,n}, w_n \) are associated to \( (u, v_n) \) as in (1.4), (1.6). We have

\[
|w_\lambda(s) - w(s)| \leq |w_\lambda(s) - w_{\lambda,n}(s)| + |w_{\lambda,n}(s) - w_n(s)| + |w_n(s) - w(s)|.
\]

By Lemma 1.1,

\[
|w_{\lambda,n}(s) - w_\lambda(s)| = \left| \int_0^s \frac{(u + \lambda v_n)^* - (u + \lambda v)^*}{\lambda} \, d\sigma \right| \leq \| v_n - v \|_{L^p(\Omega)},
\]
and, clearly,
\[ |w_n(s) - w(s)| \leq \|v_n - v\|_{L^p(\Omega)} . \]

Then
\[ \sup_s |w_n(s) - w(s)| \leq \sup_s |w_{\lambda,n}(s) - w_n(s)| + 2\|v_n - v\|_{L^p(\Omega)} . \]

By Theorem 1.1. (i), \(w_{\lambda,n}\) tends to \(w_n\) in \(C^0([0,1],[0,1])\). When \(\lambda\) decreases to zero,
\[ \lim_{\lambda \downarrow 0} \sup_s |w_\lambda(s) - w(s)| \leq 2\|v_n - v\|_{L^p(\Omega)} . \]

We deduce (i). Evidently (ii) follows, as, with \(\varphi\) in \(\mathcal{D}(\Omega^*)\),
\[ \int_{\Omega^*} \frac{dw_\lambda}{ds} \varphi \, ds = -\int_{\Omega^*} \frac{d\varphi}{ds} \, ds \rightarrow -\int_{\Omega^*} \frac{d\varphi}{ds} \, ds \quad \text{(by (i))} \]
\[ = \int_{\Omega^*} \frac{d\varphi}{ds} \varphi \, ds . \]

Now, we shall prove that \(\frac{d\varphi}{ds}\) is in \(L^p(\Omega^*)\), and satisfies (1.7). Taking again \(\varphi\) in \(\mathcal{D}(\Omega^*)\), we have by (1.5)
\[ \left| \int_{\Omega^*} \frac{d\varphi}{ds} \, ds \right| = \left| \int_{\Omega^*} \frac{d\varphi}{ds} \, ds \right| \leq \|\varphi\|_{L^p(\Omega)} \|\varphi\|_{L^p(\Omega^*)} . \]

\((1/p + 1/p' = 1)\). From (i), it follows
\[ \left| \int_{\Omega^*} \frac{d\varphi}{ds} \, ds \right| \leq \|\varphi\|_{L^p(\Omega)} \|\varphi\|_{L^p(\Omega^*)} . \]

If \(p > 1\), \(L^p(\Omega^*)\) is the dual of \(L^p(\Omega^*)\), and we get immediately (1.7). In any case \((p \geq 1)\), we can use the following argument. Let \(v_n\) be a sequence in \(L^p(\Omega)\). As previously, one can prove that
\[ (1.8) \quad \left\| \frac{dw_m}{ds} - \frac{dw_n}{ds} \right\|_{L^p(\Omega^*)} \leq \|v_m - v_n\|_{L^p(\Omega)} , \]

for any \(p > 1\), and, consequently, (passing to the limit) for \(p \geq 1\). Now consider \(v_1, v_2\) in \(L^p(\Omega)\) \((p \geq 1)\), \(v_n\) \((i = 1, 2)\) in \(L^p(\Omega)\), \(v_n \to v_i\) in \(L^p(\Omega)\); \(w_1, w_{\lambda,n}\) are associated to \((u, v_i)\) and \((u, v_{\lambda,n})\) respectively as in (1.6). By (1.8),
\( \frac{dw_n}{ds} \) is a Cauchy sequence in \( L^p(\Omega^*) \). As \( |w_n(s) - w_i(s)| \leq \| v_n - v_i \|_{L^p(\Omega)} \), \( w_n \) tends to \( w \), in \( C^0([0, 1], [\Omega]) \), \( \frac{dw_n}{ds} \to \frac{dw}{ds} \) in \( L^p(\Omega^*) \), and, by passing to the limit in

\[
\left\| \frac{dw_n}{ds} - \frac{dw}{ds} \right\|_{L^p(\Omega^*)} \leq \| v_n - v_i \|_{L^p(\Omega)},
\]

we get

(1.9) \[
\left\| \frac{dw_1}{ds} - \frac{dw_2}{ds} \right\|_{L^p(\Omega^*)} \leq \| v_1 - v_2 \|_{L^p(\Omega)}.
\]

With \( v_1 = v \), \( v_2 = 0 \), we get evidently (1.7). □

Relative rearrangement.

**Definition.** According to J. Mossino and R. Temam [6] the function \( \frac{dw}{ds} \) is called the rearrangement of \( v \) with respect to \( u \), and is denoted by \( v^*_u \).

The usual rearrangement of a function is also the rearrangement of this function with respect to a constant \( (u^*_c = u^*) \) or with respect to itself \( (u^*_u = u^*) \). More generally, if a Borel function \( F: \mathbb{R} \to \mathbb{R} \), and a measurable function \( u: \Omega \to \mathbb{R} \), are such that \( F(u) \) is in \( L^p(\Omega) \), then \( F(u^*) \) is in \( L^p(\Omega^*) \) (by (1.2)) and \( (F(u))^*_u = F(u^*) \). In fact \( (F(u))^*_u = dw/\sqrt{s} \), with

\[
w(s) = \begin{cases} \int_{\alpha} F(u) \, dx & \text{if } |u - u^*(s)| = 0, \\
\int_{\alpha} F(u) \, dx + \int_0^{s-s_\alpha} (F(u)|_{P_s})^* \, d\sigma & \text{otherwise},
\end{cases}
\]

with \( \alpha = u^*(s), P_s = \{u = \alpha\}, |P_s| \neq 0, s_\alpha = |u - \alpha|, \)

\[
= \begin{cases} \int_0^s F(u^*) \, d\sigma & \text{if } |u = u^*(s)| = 0, \\
\int_0^{s_s} F(u^*) \, ds + F(\alpha)(s - s_\alpha) = \int_0^s F(u^*) \, d\sigma & \text{otherwise}
\end{cases}
\]

(by (1.2))

\[
= \int_0^s F(u^*) \, d\sigma.
\]
However, generally, \( v^*_u \) is not an increasing function, the property of equi-
measurability and properties (c), (e) above, for the usual rearrangement, do
not seem to have their analogue for the rearrangement of a function with
respect to another one. But we have, if \( v \) is in \( L^p(\Omega) \) (1 \( \leq p \leq \infty \)), \( u: \Omega \to \mathbb{R} \) is measurable

\[
(a') \quad v_1 \leq v_2 \text{ a.e. implies } (v_1)_u^* \leq (v_2)_u^* \text{ a.e.}
\]

In fact, with \( \varphi \) in \( \mathcal{D}(\Omega^*) \), \( \varphi \geq 0 \),

\[
\int_{\Omega^*} [(v_2)_u^* - (v_1)_u^*] \varphi \, ds = \lim_{\lambda \downarrow 0} \int_{\Omega^*} \varphi \frac{(u + \lambda v_2)_u^* - (u + \lambda v_1)_u^*}{\lambda} \, ds \geq 0
\]

by property (a).

\[(b') \text{ For all constants } C, (v + C)_u^* = v_u^* + C.\]

In fact, with \( \varphi \) in \( \mathcal{D}(\Omega^*) \),

\[
\int_{\Omega^*} (v + C)_u^* \varphi \, ds = \lim_{\lambda \downarrow 0} \int_{\Omega^*} \varphi \frac{(u + \lambda (v + C))_u^* - u_u^*}{\lambda} \, ds
\]

\[
= \lim_{\lambda \downarrow 0} \int_{\Omega^*} \varphi \frac{(u + \lambda v)_u^* - u_u^*}{\lambda} \, ds + \int C \varphi \, ds
\]

(by property (b))

\[
= \int_{\Omega^*} (v_u^* + C) \varphi \, ds.
\]

\[(d') \text{ If } u: \Omega \to \mathbb{R} \text{ is measurable, the mapping } v \to v_u^* \text{ is a contraction from } L^p(\Omega) \text{ into } L^p(\Omega^*) \text{ (1 \( \leq p \leq \infty \)) as we have seen in (1.9).}\]

\[(f') \text{ Besides, the mapping } v \to v_u^* (L^1(\Omega) \to L^1(\Omega^*)) \text{ preserves the integral:}
\]

\[
\int_{\Omega^*} v_u^* \, ds = \int_{\Omega^*} \frac{dw}{ds} \, ds = w(|\Omega|) - w(0) = w(|\Omega|) = \int_{\Omega} v \, dx. \quad \square
\]

One can also define another rearrangement \( v_{*u} \) which is relative to the
directional derivative of the mapping \( u \to u_u^* \) (the decreasing rearrangement
of \( u \)):

\[
v_{*u} = \frac{dw}{ds} = \lim_{\lambda \downarrow 0} \frac{dw_\lambda}{ds}
\]
(the limit is taken in the sense of distributions),
\[
\frac{du_1}{ds} = \frac{(u + \lambda v)_* - u}{\lambda} = -\frac{(-u - \lambda v)_* + (-u)_*}{\lambda}
\]
(by (1.1)). Thus
\[
v_* = (-v)_*.
\]

1.3. Symmetrization of a family of functions.

In this Section, \(u: [0, T] \times \Omega \to \mathbb{R}\) will be a function defined everywhere in \([0, T]\), and almost everywhere in \(\Omega \subset \mathbb{R}^d\). For all \(t\) in \([0, T]\), we denote by \(u(t): \Omega \to \mathbb{R}\), the function \(u(t)(x) = u(t, x)\). (For a fixed \(t\), if no confusion is possible, we shall sometimes write \(u\) instead of \(u(t)\).) We assume that \(u(t)\) is measurable for every \(t\) in \([0, T]\). Then, we can define the function \(u^*: [0, T] \times \Omega^* \to \mathbb{R}\), the increasing rearrangement of \(u\) with respect to the \(x\) variable in \(\Omega\), that is:

\[
\forall t \in [0, T], \forall s \in \Omega^*, \quad u^*(t, s) = (u(t))^*(s) = u^*(t)(s).
\]

We consider now another real function \(v\) defined almost everywhere in \(Q = (0, T) \times \Omega\), such that, for almost every \(t\) in \((0, T)\), \(v(t)\) is in \(L^p(\Omega)\) \((1 \leq p \leq +\infty)\). Then, we can define as in Section 1.2, \((v(t))_{\text{w}(\cdot, t^*)}\), which is in \(L^p(\Omega^*)\)

\[
\|v(t)\|_{L^p(\Omega)} \leq \|v(t)\|_{L^p(\Omega^*)}.
\]

We denote by \(v^*_u\) the function defined almost everywhere in \(Q^* = (0, T) \times \Omega^*\) by

\[
\forall t \in (0, T), \forall s \in \Omega^*, \quad v^*_u(t, s) = (v(t))_{\text{w}(\cdot, t^*)}(s).
\]

The aim of this Section 1.3 is to study the regularity of \(u^*\) with respect to \(t\) (assuming a certain regularity of \(u\) with respect to \(t\)), and to compute \(\partial u^*/\partial t\). We have

**Theorem 1.2.** If \(u\) belongs to \(H^1(0, T; L^p(\Omega))\) \((1 \leq p \leq \infty)\), then \(u^*\) belongs to \(H^1(0, T; L^p(\Omega^*))\), and

\[
\|u^*\|_{H^1(0, T; L^p(\Omega^*))} \leq \|u\|_{H^1(0, T; L^p(\Omega))}.
\]

Moreover

\[
\frac{\partial u^*}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^* = \frac{\partial u}{\partial s} \quad \text{(in the sense of distributions)}
\]
where

\[
(w(t), s) = \begin{cases} 
\int \frac{\partial u}{\partial t} \, dx & \text{if } |u(t) = u(t)^*(s)| = 0, \\
\int_{u(t) < u(t)^*(s)} \frac{\partial u}{\partial t} \, dx + \int_{u(t) < u(t)^*(s)} \left( \frac{\partial u}{\partial t} \bigg|_{u(t) = u(t)^*(s)} \right) \, d\sigma & \text{otherwise.}
\end{cases}
\]

(1.16)

**Proof.** As \( \|u(t)^*\|_{L^p(\Omega^*)} = \|u(t)\|_{L^p(\Omega)} \) (by (1.2)), we have

\[
\|w^*\|_{L^p(0, t; L^p(\Omega^*))} = \|u\|_{L^p(0, t; L^p(\Omega))}.
\]

Besides, by (1.12), (1.13)

\[
\left\| \left( \frac{\partial u}{\partial t} \right)^*(t) \right\|_{L^p(\Omega^*)} \leq \left\| \frac{\partial u}{\partial t} (t) \right\|_{L^p(\Omega)},
\]

\[
\left\| \left( \frac{\partial u}{\partial t} \right)^* \right\|_{L^p(0, t; L^p(\Omega^*))} \leq \left\| \frac{\partial u}{\partial t} \right\|_{L^p(0, t; L^p(\Omega))}.
\]

Thus, we have only to prove (1.15), (1.16). Our proof uses the following lemma (see its proof in the Appendix).

**Lemma 1.2.** Let \( u \) be in \( H^1(0, T; L^p(\Omega)) \) \( (1 \leq p \leq \infty) \),

\[
r_h = \frac{u(t + h) - u(t)}{h} \frac{\partial u}{\partial t}.
\]

Consider a fixed number \( \varepsilon > 0 \). When \( h \) tends to zero, \( r_h \) tends to zero in \( L^p(\Omega_\varepsilon) \), with \( \alpha = \min(p, 2) \), \( Q_\varepsilon = (\varepsilon, T - \varepsilon) \times \Omega \).

Let \( \varphi \) be in \( \mathcal{D}(Q^\varepsilon) \), and let \( \varepsilon > 0 \) be such that the support of \( \varphi \) is included into \( Q^\varepsilon_\varepsilon = (\varepsilon, T - \varepsilon) \times \Omega^\varepsilon \). Consider \( 0 < h < \varepsilon \). We have

\[
\int_{\Omega} \frac{u(t + h) - u(t)}{h} \varphi(t) \, ds \, dt = \int_{\Omega} \frac{(u + h(\partial u/\partial t))^* - u^*}{h} \varphi \, ds \, dt
\]

\[
+ \int_{\Omega} \frac{(u + h(\partial u/\partial t + r_h))^* - (u + h(\partial u/\partial t))^*}{h} \varphi \, ds \, dt.
\]
The first integral in the right hand side is \( \int_0^T A_h(t) \, dt \), where

\[
a.e. \ t, \quad A_h(t) = \int_{\Omega^*} \left( \frac{u(t) + h(\partial u / \partial t)(t))}{h} - u(t) \right) \varphi(t) \, ds \quad \text{as} \quad (h \to 0^+) \int_{\Omega^*} \left( \frac{\partial u}{\partial t} \right)_u^* \varphi(t) \, ds
\]

(by Theorem 1.1 bis), and

\[
|A_h(t)| \leq \left\| \frac{\partial u}{\partial t} \right\|_{L^p(\Omega)} \| \varphi(t) \|_{L^{p'}(\Omega^*)},
\]

with \( 1/p + 1/p' = 1 \) (by property \((d')\) above). Using Lebesgue theorem

\[
\int_0^T A_h(t) \, dt \xrightarrow{(h \to 0^+)} \int_{\Omega^*} \left( \frac{\partial u}{\partial t} \right)_u^* \varphi \, ds \, dt.
\]

The other integral is majorized by

\[
\int_{-\varepsilon}^{T-\varepsilon} \left\| r_h(t) \right\|_{L^p(\Omega)} \| \varphi(t) \|_{L^{p'}(\Omega^*)} \, dt
\]

with \( \alpha = \min \{ p, 2 \} \), \( (1/\alpha + 1/\alpha' = 1) \)

\[
\leq \left\| r_h \right\|_{L^p(\Omega)} \| \varphi \|_{L^{p'}(\Omega^*)},
\]

which tends to zero with \( h \), by Lemma 1.2. Thus

\[
\int_{\Omega^*} \left( \frac{u(t + h)}{h} - u(t) \right) \varphi(t) \, ds \, dt \xrightarrow{(h \to 0^+)} \int_{\Omega^*} \left( \frac{\partial u}{\partial t} \right)_u^* \varphi \, ds \, dt.
\]

But, classically,

\[
\int_{\Omega^*} \left( \frac{u(t + h) - u(t)}{h} \right) \varphi(t) \, ds \, dt \to \int_{\Omega^*} u^* \frac{\partial \varphi}{\partial t} \, ds \, dt.
\]

We conclude that, in the sense of distributions,

\[
\frac{\partial u}{\partial t} \left( \frac{\partial u}{\partial t} \right)_u^* \quad \square
\]
A direct consequence of Theorem 1.2 is the following

**Proposition.** Assume $u$ belongs to $H^1(0, T; L^1(\Omega))$. Then, for almost every $t$ in $(0, T)$, $\partial u/\partial t(t, \cdot)$ is constant (almost everywhere) on any set where $u(t, \cdot)$ is constant (almost everywhere).

**Proof.** If, in the proof of Theorem 1.2, we consider $h < 0$, we get

$$A_h(t) = \int_{\Omega} \left( u(t) + h(\partial u/\partial t)(t) \right)^* u(t)^* \varphi(t) \, ds$$

$$= - \int_{\Omega} \left( u(t) + \frac{(- h)(\partial u/\partial t)}{- h} \right)^* u(t)^* \varphi(t) \, ds,$$

which tends to $- \int_{\Omega} (\partial u/\partial t)^* u(t)^* \varphi(t) \, ds$. Thus, one has, in the sense of distributions,

$$\frac{\partial u^*}{\partial t} = \left( \frac{\partial u^*}{\partial t} \right)_u = - \left( \frac{\partial u^*}{\partial t} \right)_u \quad \left( = \left( \frac{\partial u}{\partial t} \right)^{\ast - u} \text{ by (1.10)} \right),$$

and

$$\left( \frac{\partial u^*}{\partial t} \right)_u = \frac{\partial w}{\partial s} \quad (w \text{ defined in (1.16)}), \quad - \left( \frac{\partial u^*}{\partial t} \right)_u = \frac{\partial w'}{\partial s},$$

with

$$w'(t, s) = \begin{cases} \int_{u(t) < u(t)^*(s)} \frac{\partial u}{\partial t} \, dx & \text{if } |u(t) - u(t)^*(s)| = 0, \\ \int_{u(t) < u(t)^*(s)} \frac{\partial u}{\partial t} \, dx + \int_0^{s-|u(t) - u(t)^*(s)|} \left( - \frac{\partial u}{\partial t} \right)_{u(t) - u(t)^*(s)} \, d\sigma, & \text{otherwise}. \end{cases}$$

The last integral is also

$$\int_0^{s-|u(t) - u(t)^*(s)|} \left( \frac{\partial u}{\partial t} \right)_{u(t) - u(t)^*(s)} \, d\sigma \quad \text{(by (1.1))}.$$ 

Now, fix $t$ in $(0, T)$, such that $\partial w/\partial s = \partial w'/\partial s$ (in $L^1(\Omega^*)$) (this is true for almost every $t$ in $(0, T)$), and consider a flat region of $u(t)\colon P_\theta(t) = \{u(t) = \theta\}$, $|P_\theta(t)| \neq 0$. Set $s_\theta = |u(t) < \theta|$, $s'_\theta = |u(t) \leq \theta|$. As $w(t, 0) = 0 = w'(t, 0)$, one has

$$w(t, s) = \int_0^s \frac{\partial w}{\partial s} (t, \sigma) \, d\sigma = \int_0^s \frac{\partial w'}{\partial s} (t, \sigma) \, d\sigma = w'(t, s),$$
for all $s$ in $Q^*$. Moreover, for all $s$ in $[s_0, s'_0]$, one has, by definition of $w$ and $w'$,

$$\frac{\partial w}{\partial s}(t, s) = \left( \frac{\partial u}{\partial t} \bigg|_{P_0(t)} \right)^* (s - s_0)$$

$$= \frac{\partial w'}{\partial s}(t, s) = \left( \frac{\partial u}{\partial t} \bigg|_{P_0(t)} \right)^* (s - s_0).$$

In particular,

$$\left( \frac{\partial u}{\partial t} \bigg|_{P_0(t)} \right)^* (0) = \sup_{P_0(t)} \inf_{P_0(t)} \frac{\partial u}{\partial t},$$

that is $(\partial u/\partial t)(t, \cdot)$ is constant almost everywhere on $P_0(t)$. $\square$

We shall give now the application to parabolic equations.

2. - Isoperimetric inequalities for linear parabolic equations.

Let us consider first the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}_t(u) + cu = f & \text{in } Q = (0, T) \times \Omega, \\ u = 0 & \text{on } \Sigma = (0, T) \times \partial \Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \end{cases}$$

(2.1)

where $\Omega$ is a bounded regular open set in $\mathbb{R}^n$,

$$\mathcal{A}_t(u) = - \sum_{i,j=1}^N a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$ 

We denote by $A(= A(t, x))$ the matrix $(a_{ij}(t, x))$, as well as the bilinear form on $\mathbb{R}^n$ associated with $A$, and we assume that $A$ satisfies the uniform (with respect to $(t, x)$) ellipticity condition:

$$A(\xi, \xi) = \sum_{i,j=1}^N a_{ij}(t, x) \xi_i \xi_j \geq \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^n.$$
Furthermore, we assume that the data satisfy:

- \( c, f, u_0 \) are non-negative functions; \( c, a_{ij} \) are in \( L^\infty(Q) \);
- \( \partial a_{ij}/\partial t \) are continuous in \( \overline{Q} \), \( f \) is in \( L^1(Q) \), and \( u_0 \) is in \( H^1_0(\Omega) \).

Then, the solution \( u \) is in \( L^\infty(0, T; H^1_0(\Omega)) \), \( \partial u/\partial t \) is in \( L^1(Q) \) (see [5], pp. 113-114, and [4] if \( a_{ij} \neq a_{ji} \)).

Let us introduce the problem

\[
\begin{aligned}
\begin{cases}
\frac{\partial U}{\partial t} - AU = f & \text{in } \overline{Q} = (0, T) \times \Omega, \\
U = 0 & \text{on } \Sigma = (0, T) \times \partial \Omega, \\
U(0, \cdot) = u_0 & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

Where \( \overline{Q}, f, u_0 \) are as in the Introduction.

We are going to compare the solution \( u \) of (2.1) with the solution \( U \) of (2.1). More precisely, we have

**Theorem 2.1.** With the assumptions above,

\[
\forall t \in [0, T], \quad \forall s \in \overline{Q}^*, \quad \int_0^s u_*(t, \sigma) \, d\sigma \leq \int_0^s U_*(t, \sigma) \, d\sigma \leq \int_0^s g(t, \sigma) \, d\sigma
\]

where \( g(t, s) = \int_0^s f_*(\tau, s) \, d\tau + (u_0)_*(s) \). We deduce

\[
\forall t \in [0, T], \forall \tau \in [1, \infty],
\|u(t, \cdot, \tau)\|_{L^2(\Omega)} \leq \|U(t, \cdot, \tau)\|_{L^2(\Omega)} \leq \|g(t, \cdot)\|_{L^2(\Omega)}(\leq \infty),
\]

**Proof.** For a fixed \( t \in [0, T] \), we denote for convenience \( u = u(t) \), \( f = f(t) \). We argue as for the elliptic problem (see [8], [7]). By the maximum principle, we have \( u \geq 0 \). For any \( \theta > 0 \), we get from (2.1),

\[
\int_{\Omega} A(\nabla u, \nabla (u - \theta)_+) \, dx = \int_{\Omega} \left( f - cu \frac{\partial u}{\partial t} \right) (u - \theta)_+ \, dx.
\]

Thus, as in [8], [7], a simple derivation gives:

\[
\frac{d}{d\theta} \int_{u > 0} A(\nabla u, \nabla u) \, dx = \int_{u > \theta} \left( f - cu \frac{\partial u}{\partial t} \right) \, dx.
\]
The uniform ellipticity condition and the Cauchy-Schwartz inequality lead to
\[
\left[- \frac{d}{d\theta} \int_{u > \theta} |\nabla u| \, dx \right]^2 \leq \mu'(\theta) \frac{d}{d\theta} \int_{u > \theta} A(\nabla u, \nabla u) \, dx
\]
where \(\mu(\theta) = |u > \theta|\), and, by (2.5),
\[
(2.6) \quad \left[- \frac{d}{d\theta} \int_{u > \theta} |\nabla u| \, dx \right]^2 \leq - \mu'(\theta) \int_{u > \theta} \left( f - cu - \frac{\partial u}{\partial t} \right) \, dx.
\]
Using a result of Fleming-Rishel, and the isoperimetric inequality for the perimeter in the sense of De Giorgi, we find
\[
(2.7) \quad N^x \sigma^x_{\epsilon} \mu(\theta)^{1-x/|x|} \leq - \frac{d}{d\theta} \int_{u > \theta} |\nabla u| \, dx.
\]
Hence, combining (2.6), (2.7),
\[
(2.8) \quad N^x \sigma^x_{\epsilon} \mu(\theta)^{1-x/|x|} \leq - \mu'(\theta) \int_{u > \theta} \left( f - cu - \frac{\partial u}{\partial t} \right) \, dx.
\]
By the inequality (1.3) of Hardy-Littlewood,
\[
(2.9) \quad \int_{u > \theta} (f - cu) \, dx \leq \int_{u > \theta} f \, dx \leq \int_{u > \theta} f_{\mu(\theta)} \, ds = F(t, \mu(\theta))
\]
if we set
\[
(2.10) \quad F(t, \sigma) = \int_{0}^{\sigma} f_{0}(t, \sigma) \, d\sigma.
\]
For almost every \(\theta\), \(|u = \theta| = 0\), and \(u_{\ast}(\mu(\theta)) = \theta\) because \(u_{\ast}\) is continuous in \([0, |Q|]\) (as \(u\) is in \(H_{0}^{1}(Q)\), \(u\) non-negative, then \(\psi\) is in \(H_{0}^{1}(Q)\), see [7], for example). By Theorem 1.2
\[
\int_{u > \theta} \frac{\partial u}{\partial t} \, dx = \int_{u > u_{\ast}(\mu(\theta))} \frac{\partial u}{\partial t} \, dx = w(t, \mu(\theta)),
\]
with
\[ w(t, s) = \int_{u(0) > u(0,t(\theta))} \frac{\partial w}{\partial t} \, dx \quad \text{if } |u(t) - u(t)_*(\theta)| = 0, \]
\[ \frac{\partial w}{\partial s} = \left( \frac{\partial u}{\partial t} \right)_{u_*} = \frac{\partial u_*}{\partial t}. \]

Thus
\[ \int_{u > 0} \frac{\partial u}{\partial t} \, dx = \int_0^\mu(t) \frac{\partial u_*}{\partial t} \, ds = \frac{\partial k}{\partial t}(t, \mu(\theta)) \]
if we set
\[ (2.11) \quad k(t, s) = \int_0^s u_*(t, \sigma) \, d\sigma \]
\[ \text{(as } (\partial k/\partial t)(t, s) = \int_0^s (\partial u_*/\partial t)(t, \sigma) \, d\sigma). \]

Thus
\[ (2.12) \quad \int_{u > 0} \frac{\partial u}{\partial t} \, dx = \frac{\partial k}{\partial t}(t, \mu(\theta)), \quad \text{a.e. } \theta > 0. \]

From (2.8), (2.9), (2.12), we get
\[ (2.13) \quad 1 \leq -N^{-2} \sigma^2 \mu(\theta) \mu(\theta) \left[ F(t, \mu(\theta)) - \frac{\partial k}{\partial t}(t, \mu(\theta)) \right] \mu'(\theta). \]

As \( F(t, \cdot) - \partial k/\partial t(t, \cdot) \) is continuous in \( \tilde{Q}^* \), then, the function \( H(t, \cdot) \) defined in \( Q^* \) by
\[ H(t, s) = s^{(2/N)-2} \left[ F(t, s) - \frac{\partial k}{\partial t}(t, s) \right] \]
is continuous in \( ]0, |\Omega|] \). By integrating (2.13), we get, for any \( 0 \leq \theta \leq \theta' \),
\[ (2.14) \quad \theta' - \theta \leq -N^{-2} \sigma^2 \mu(\theta) \int_{\mu(\theta)}^{\mu'(\theta)} s^{(2/N)-2} \left[ F(t, s) - \frac{\partial k}{\partial t}(t, s) \right] \, ds. \]

Thus, as in [7], one has for almost every \( s \) in \( \Omega^* \),
\[ (2.15) \quad 0 \leq -\frac{\partial k}{\partial s} = -\frac{d}{ds} \left( u(t)_s \right) \leq N^{-2} \sigma^2 s^{(2/N)-2} \left[ F(t, s) - \frac{\partial k}{\partial t}(t, s) \right]. \]
Hence, \( k \) satisfies
\[
\begin{align*}
\frac{\partial k}{\partial t} - N^2 \alpha_3^{2/3} s^{2-(2/3)} \frac{\partial^2 k}{\partial s^2} &\leq F \quad \text{a.e. in } Q^* = (0, T) \times \Omega^*, \\
k(t, 0) & = 0, \quad \frac{\partial k}{\partial s}(t, |\Omega|) = 0, \quad \forall t \in [0, T], \\
k(0, s) & = \int_0^s (u_0)_* \, d\sigma = k_0(s), \quad \forall s \in \bar{\Omega}^*.
\end{align*}
\]
\( (2.16) \)

Let \( K(t, s) = \int_0^s U_*(t, \sigma) \, d\sigma \), where \( U \) is the solution of \( (2.1) \). We are going to show that the equality is achieved in \( (2.16) \) for \( K \) instead of \( k \).

By the maximum principle, \( U(t, \cdot) \) decreases along the radii in \( \Omega \), and \( (2.1) \) can be written
\[
\frac{\partial U_*}{\partial t} - N^2 \alpha_3^{2/3} s^{2-(2/3)} \frac{\partial U_*}{\partial s} = f_* \quad \text{in } \Omega^*.
\]
By integrating between 0 and \( s \), using the fact that \( s^{2-(2/3)}(\partial U_*/\partial s) = O(s) \) when \( s \) tends to zero (see the remark below) we obtain
\[
\frac{\partial K}{\partial t} - N^2 \alpha_3^{2/3} s^{2-(2/3)} \frac{\partial^2 k}{\partial s^2} = F \quad \text{in } Q^*.
\]

**Remark 2.1.** Using Cauchy-Schwartz inequality in the first line of \( (2.16) \), we get
\[
0 \leq -s^{2-(2/3)} \frac{\partial U_*}{\partial s} \leq N^{-2} \alpha_3^{2/3} s^{1/2} \left[ \|f(t)\|_{L^2(\Omega)} + \|\frac{\partial U_*}{\partial t} (t)\|_{L^2(\Omega)} \right]. \quad \Box
\]

Now, setting \( \chi = k - K \),
\[
\begin{align*}
\frac{\partial \chi}{\partial t} - N^2 \alpha_3^{2/3} s^{2-(2/3)} \frac{\partial^2 \chi}{\partial s^2} &\leq 0 \quad \text{a.e. in } Q^*, \\
\chi(t, 0) & = 0, \quad \frac{\partial \chi}{\partial s}(t, |\Omega|) = 0, \quad \forall t \in [0, T], \\
\chi(0, s) & = 0, \quad \forall s \in \bar{\Omega}^*.
\end{align*}
\]
\( (2.17) \)

The first inequality in \( (2.2) \) will result from a maximum principle for \( \chi \):
LEMMA 2.1. Let \( \chi(t, s) = (k - K)(t, s) = \frac{d}{s} (u_0 - U_0)(t, \sigma) \, d\sigma \). One has \( \chi \leq 0 \) everywhere in \( \bar{Q}^* \).

PROOF OF LEMMA 2.1. Multiplying the inequality in (2.17) by \( s^{(2/N) - 2} \chi_+ \), we get

\[
(2.18) \quad s^{(2/N) - 2} \frac{\partial \chi}{\partial t} \chi_+ \leq N^2 a^{2/N} \frac{\partial \chi}{\partial s} \chi_+ \quad \text{a.e. in } Q^*.
\]

For fixed \( t \), we shall denote \( u, \chi, \ldots \), for simplicity, instead of \( u(t), \chi(t) \ldots \). We shall also denote by \([ \ ] \) a function of \( t \), independent of \( s \). First we prove that \( (\partial^2 \chi/\partial s^2) \chi_+ \) is in \( L^1(Q^*) \). In fact, by Remark 2.1,

\[
\frac{\partial u_0}{\partial s} \leq [ ] s^{(2/N) - (3/2)},
\]

(2.19)

\[
\frac{\partial^2 \chi}{\partial s^2} \leq \frac{\partial u_0}{\partial s} + \left| \frac{\partial U_0}{\partial s} \right| \leq [ ] s^{(2/N) - (3/2)}.
\]

On the other hand

\[
|k| \leq \int_0^1 |u_0| \, d\sigma \leq s^{1/2} \|u\|_{L^1(\Omega)} = [ ] s^{1/2}
\]

(2.20)

\[
|\chi_+| \leq |\chi| \leq |k| + |K|
\]

Thus,

\[
\left| \frac{\partial^2 \chi}{\partial s^2} \chi_+ \right| \leq [ ] s^{(2/N) - 1},
\]

which belongs to \( L^1(Q^*) \). By integrating by parts, we are going to prove that \( \int (\partial^2 \chi/\partial s^2) \chi_+ \, ds \) is non-positive. For \( a > 0 \), as \( \chi \) belongs to \( W^{2,\infty}(a, \partial^* \Omega) \) by (2.19), the following integration by parts is justified

\[
(2.21) \quad \int_a^{a+1} \frac{\partial^2 \chi}{\partial s^2} \chi_+ \, ds = - \int_a^{a+1} \left( \frac{\partial \chi}{\partial s} \right)^2 \, ds - \frac{\partial \chi}{\partial s} (a) \chi_+(a) + \int_a^{a+1} \frac{\partial \chi}{\partial s} \chi_+ \, ds
\]

(we used the fact that \( (\partial \chi/\partial s)(\partial \Omega) = 0 \) by (2.17)). When \( a \) tends to zero, the two integrals tend respectively to \( \int (\partial^2 \chi/\partial s^2) \chi_+ \, ds \) and \( \int (\partial \chi/\partial s)^2 \, ds \) (\( \chi_+ \), as \( \chi \), belongs to \( H^1(Q^*) \)). Now we prove that \( (\partial \chi/\partial s)(a) \chi_+(a) \) tends to zero.
with \(a\). One has

\[
\left| \frac{\partial \chi}{\partial s}(a) \right| = \left| \int_a^b \left( \frac{\partial u_s}{\partial s} - \frac{\partial U_s}{\partial s} \right) ds \right| \leq \left| a^{(2/N)-(1/2)} - |\Omega|^{(2/N)-(1/2)} \right| \quad \text{(by (2.19))}
\]

By (2.20),

\[
\left| \frac{\partial \chi}{\partial s}(a) \chi(a) \right| \leq \left| a^{2/N} - |\Omega|^{(2/N)-(1/2)} a^{1/2} \right|
\]

which tends to zero with \(a\). From (2.21), we get

\[
\int_\Omega \frac{\partial^2 \chi}{\partial s^2} \chi_+ ds = -\int_\Omega \left( \frac{\partial \chi_+}{\partial s} \right)^2 ds \leq 0.
\]

From (2.18),

\[
0 \geq 2 \int_0^t \int_\Omega s^{(1/N)-1} \frac{\partial \chi}{\partial t} \chi_+ ds d\tau = \int_0^t \int_\Omega s^{(1/N)-1} \frac{\partial}{\partial t} (\chi_+^2) ds d\tau = \int_\Omega s^{(2/N)-2} \chi_+^2 ds.
\]

It follows \(\chi_+ \equiv 0\) in \(\bar{Q}^*\). \(\square\)

Now we shall prove the second inequality in (2.2). Let us consider the equation satisfied by \(K\) in \(Q^*\):

\[
F - \frac{\partial K}{\partial t} = -N^2 \alpha_0^{2/N} s^{-(2/N)} \frac{\partial U_s}{\partial s} \geq 0.
\]

Thus,

\[
\int_0^s f_s(t, \sigma) d\sigma \geq \frac{\partial}{\partial t} \int_0^s U_s(t, \sigma) d\sigma.
\]

By integration, we find

\[
\int_0^s U_s(t, \sigma) d\sigma - \int_0^s u_{ss} d\sigma \leq \int_0^s f_s(\tau, \sigma) d\tau. \quad \square
\]

Now, (2.3) is a simple consequence of a lemma in [2] (p. 174), for all \(r\) in \([1, \infty[\), and then for \(r = \infty\).

**Remark 2.2.** If \(f_s(t)\) is absolutely continuous in \([0, |\Omega|]\), for almost every \(t\) in \((0, T)\), then we can obtain an isoperimetric energy inequality:
we get from (2.1)

\[ \int_{\bar{\Omega}} \left( \frac{\partial u}{\partial t} u + A(\nabla u, \nabla u) + cu^2 \right) \, dx = \int_{\bar{\Omega}} fu \, dx \]

\[ \leq \int_{\bar{\Omega}} f_* u_* \, ds \quad \text{(by Hardy-Littlewood inequality)} \]

\[ = -\int_{\bar{\Omega}} \frac{\partial f_*}{\partial s} k \, ds + f_* (|\Omega|) k(\Omega) \]

\[ \leq -\int_{\bar{\Omega}} \frac{\partial f_*}{\partial s} K \, ds + f_* (|\Omega|) K(\Omega) \quad \text{(by Theorem 2.1)} \]

\[ = \int_{\bar{\Omega}} f_* U_* \, ds \]

\[ = \int_{\bar{\Omega}} \left( \frac{\partial U}{\partial t} U + |\nabla U|^2 \right) \, dx . \]

Using the uniform ellipticity condition, we have

\[ \int_{\bar{\Omega}} \left( \frac{\partial u}{\partial t} u + |\nabla u|^2 \right) \, dx \leq \int_{\bar{\Omega}} \left( \frac{\partial U}{\partial t} U + |\nabla U|^2 \right) \, dx , \]

and, by integration

\[ \frac{1}{2} \int_{\bar{\Omega}} u(T)^2 \, dx + \int_{\bar{\Omega}} |\nabla u|^2 \, dx \, dt \leq \frac{1}{2} \int_{\bar{\Omega}} U(T)^2 \, dx + \int_{\bar{\Omega}} |\nabla U|^2 \, dx \, dt . \]

**Appendix.**

In the proof of Lemma 1.2, we shall use the following lemma, whose proof is easy (see [1] for example).

**Lemma A.** Let \( v \) in \( W^{1,\infty}(0, T) \) \((1 \leq \alpha \leq \infty)\). If \( 0 < |h| < \varepsilon \), we have

\[ \int_{\varepsilon}^{T-\varepsilon} \left| \frac{v(t-h) - v(t)}{h} \right| dt \leq \left\| \frac{dv}{dt} \right\|_{L^1(\varepsilon, \varepsilon + |h|, T-\varepsilon + |h|)} . \]
Proof of Lemma 1.2. If \( u \) belongs to \( H^1(0, T; L^p(\Omega)) \), then \( u \) and \( \frac{\partial u}{\partial t} \) belong to \( L^2(0, T; L^p(\Omega)) \subset L^p(\Omega) \);

\[
\text{a.e. } x, \quad u(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x) \in L^p(0, T),
\]

that is

\[
\text{a.e. } x, \quad u(x) \in W^{1,p}(0, T).
\]

We can apply Lemma A, with \( v = u(x) \). For \( 0 < |h| < \varepsilon \), we have, with \( q_n(t, x) = (u(t + h) - u(t))/h \),

\[
\text{a.e. } x, \quad \int_{\varepsilon}^{T-\varepsilon} |q_n(t, x)|^p \, dt \leq \left\| \frac{\partial u}{\partial t} \right\|_{L^p(\varepsilon - |h|, T - \varepsilon + |h|)}^p.
\]

By integrating over \( \Omega \), we get

\[
(A.1) \quad \lim_{h \to 0} \left\| q_n \right\|_{L^p(\Omega)} \leq \left\| \frac{\partial u}{\partial t} \right\|_{L^p(\Omega)}.
\]

1) If \( p > 1 \) (then \( \alpha > 1 \)), it is easy to prove that \( q_n \rightharpoonup \frac{\partial u}{\partial t} \) in \( L^p(\Omega) \), weakly. Then, classically, by (A.1), \( q_n \to \frac{\partial u}{\partial t} \) in \( L^p(\Omega) \) for the strong topology.

2) If \( p = 1 \), then \( \alpha = 1 \). There exists a sequence \( u_n \) in \( H^1(0, T; L^\infty(\Omega)) \) such that \( u_n \rightharpoonup u \) in \( H^1(0, T; L^1(\Omega)) \). Let

\[
r_{hn} = \frac{u_n(t + h) - u_n(t)}{h} - \frac{\partial u_n}{\partial t}.
\]

We have, with the \( L^1(\Omega) \) norms,

\[
\|r_n\| \leq \|r_{hn}\| + \|r_{hn} - r_n\|.
\]

We have just seen (case \( p > 1 \)) that \( r_{hn} \to 0 \) in \( L^1(\Omega) \) (and consequently in \( L^1(\Omega) \)). Besides, by Lemma A,

\[
\left\| \frac{(u_n - u)(t + h) - (u_n - u)(t)}{h} \right\|_{L^p(\Omega)} \leq \left\| \frac{\partial (u_n - u)}{\partial t} \right\|_{L^p(\Omega)}.
\]

Thus

\[
\|r_{hn} - r_n\|_{L^1(\Omega)} \leq 2 \left\| \frac{\partial (u_n - u)}{\partial t} \right\|_{L^p(\Omega)}
\]

which tends to zero with \( n \). It follows that \( r_n \to 0 \) in \( L^1(\Omega) \), and Lemma 1.2 is proved.
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Laboratoire d'Analyse Numérique
C.N.R.S. et Université Paris-Sud
91405 Orsay, France