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Oblique derivative problems and invariant measures


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0. - Introduction.

The present paper is devoted to the study of some second-order oblique derivative problems for an operator $A$ not in the divergence form and for a boundary operator $B$ with H"older continuous coefficients.  

One major purpose is to describe the limiting behaviour of the solutions $u_{\lambda}$ of the unilateral problem:

$$\begin{cases}
\max \left[ u_{\lambda} - \psi; \ A u_{\lambda} + \lambda u_{\lambda} - f \right] = 0 & \text{in } \Omega \\
B u_{\lambda} = 0 & \text{on } \Gamma, 
\end{cases}$$

as the positive parameter $\lambda$ tends to zero. This is motivated by the control theory of stochastic processes. The above boundary value problem actually comprises the Bellman conditions for the optimal stopping of a diffusion process in $\Omega$ reflected at the boundary (see [3]). The case $\lambda > 0$ corresponds to discounted cost functionals while $\lambda = 0$ is related to long run average costs (see [19] and [6] for the case of finite Markov chains).

This problem has been studied by A. Bensoussan and J. L. Lions (see [4]) in the case of regular coefficients. Their main result is that the asymptotic behaviour of $u_{\lambda}$, as $\lambda$ tends to zero, depends on the sign of the average value of $f$ with respect to the measure $m dx$, $m$ being the solution of the problem

$$\begin{cases}
A^* m = 0 & \text{in } \Omega \\
B^* m = 0 & \text{on } \Gamma, \ m > 0 , \ \int_{\overline{\Omega}} m \ dx = 1 , \ m \in C^2(\overline{\Omega}) .
\end{cases}$$

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In the general framework considered in this paper the main difficulties arise from the non-divergence structure of $A$. Actually, the fairly low regularity of the coefficients does not allow a Fredholm alternative approach by integration by parts. Nevertheless, using the Green’s function $G(x, y, t)$ for the initial boundary value problem associated with $A$ and $B$ as constructed in ([13]), we prove the existence of a unique positive function $m \in L^\infty(\Omega)$ such that

$$m = \lambda J^*_\lambda m, \quad \int_\Omega m \, dx = 1.$$ 

Here, $J_\lambda$ is the resolvant of the stationary problem

$$u \in W^2_\lambda(\Omega), \quad Au - \lambda u = f \quad \text{in } \Omega,$$

$$Bu = 0 \quad \text{on } \Gamma,$$

and $J^*_\lambda$ its adjoint operator. Moreover, the identity

$$\int_\Omega \int_\Omega G(x, y, t) f(y) m(x) \, dx \, dy = \int_\Omega f(x) m(x) \, dx$$

holds for every $f \in L^p; \ 1 < p < + \infty$.

The probability measure $d\mu = 1/|\Omega|m \, dx$ can be interpreted then as the invariant measure for a Feller process defined by the transition functions

$$P(x, t, E) = \int_E G(x, y, t) \, dy, \ E \text{ Borel subset of } \Omega.$$ 

This, together with a crucial ergodic property proved in Theorem 3.1, is the key to prove that if the condition $\int f d\mu > 0$ is satisfied, then the $W^2_\lambda$ norms of $u_\lambda$ are uniformly bounded with respect to $\lambda$ and that $u = \lim u_\lambda$ in the weak topology of $W^2_\lambda$ is the unique solution of

$$\max [u - \psi; Au - f] = 0 \quad \text{in } \Omega,$$

$$Bu = 0 \quad \text{on } \partial \Omega.$$ 

We send to Theorem 4.2 for this result and for the limiting behaviour in the case $\int d\mu < 0$.

To conclude this introduction we would like to point out that the analytical approach adopted in this paper to the stochastic control problem
mentioned above yields a complete description of the asymptotic behaviour and of compatibility conditions for the solvability of the limit problem without the restrictions on the regularity of the data required by probabilistic methods (see [19]).

Let us also mention that Theorem 2.1 in Section 2 generalizes various known existence and uniqueness results (see for example [8], [23]) for the unconstrained problem

\[ Au + \lambda u = f \quad \text{in } \Omega, \]
\[ Bu = 0 \quad \text{on } \Gamma. \]

We finally mention that extensions of the results of this paper to integro-differential operators associated to diffusion processes with interior jumps are considered in a forthcoming paper by M. G. Garroni and J. L. Menaldi [12].

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1. Basic assumptions and preliminary results.

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n, \ N > 2 \), with boundary \( \Gamma \) of class \( C^2 \). We shall denote by \( Q_\tau \) the cylinder \( \Omega \times ]0, T[ \), \( 0 < T < +\infty \), and by \( \Sigma_\tau = \Gamma \times [0, T] \) its lateral boundary. Consider the operators

\[
A = -\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial}{\partial x_i},
\]

\[
B = \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i},
\]

whose coefficients are assumed to satisfy

\[
\begin{align*}
& a_{ij}, \ b_i \in C^{\alpha}(\bar{\Omega}, \mathbb{R}) \ \text{for some } 0 < \alpha < 1 \\
& \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2, \quad \mu > 0, \ \forall x \in \Omega, \ \xi \in \mathbb{R}^n \\
& b_i \in C^{\alpha}(\bar{\Gamma}) \\
& \sum_{i=1}^{N} b_i(x) v_i(x) \geq \beta, \quad \beta > 0, \ \forall x \in \Gamma.
\end{align*}
\]

Here \( v = (v_1, ..., v_N) \) is the outward normal vector to \( \Gamma \).
We shall denote by $W^s_p, W^{s,r}_p$ the usual Sobolev spaces for elliptic and parabolic problems with possibly non-integer $s, r$ and by $\| \cdot \|_{s,r}, \| \cdot \|_{s,r,p}$ the corresponding norms (see [11]) and by $\| \cdot \|_p$ the usual $L_p(\Omega)$ norm.

We shall always assume

$$1 < p < \frac{1}{1- \alpha}. \tag{1.5}$$

This condition guarantees that $Bu \in W^{1-\mu}(\Omega)$ for any $u \in W^s_p(\Omega)$.

We shall make use of regularized versions of operators $A, B$.

These are defined by

$$A^n = - \sum_{i,j=1}^{N} a^n_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial}{\partial x_i}, \tag{1.6}$$

$$B^n = \sum_{i=1}^{N} b^n_i(x) \frac{\partial}{\partial x_i}, \tag{1.7}$$

where the coefficients $a^n_{ij}, b^n_i$ are chosen in such a way that

$$\begin{align*}
& a^n_{ij} \in C^\alpha(\bar{\Omega}), \quad a^n_{ij} \rightarrow a_{ij} \quad \text{in } C^{\alpha}(\bar{\Omega}) \quad \text{as } n \rightarrow + \infty, \\
& \sum_{i,j=1}^{N} a^n_{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2, \quad \mu > 0, \ \forall x \in \Omega, \ \xi \in \mathbb{R}^N,
\end{align*} \tag{1.8}$$

$$\begin{align*}
& b^n_i \in C^\alpha(\Gamma), \quad b^n_i \rightarrow b_i \quad \text{in } C^{\alpha}(\Gamma) \quad \text{as } n \rightarrow + \infty, \\
& \sum_{i=1}^{N} b^n_i(x) \nu_i(x) \geq \beta > 0, \quad \forall x \in \Gamma, \tag{1.9}
\end{align*}$$

with constants $\mu, \beta$ as in (1.3), (1.4). In this setting $A^n$ can also be written in divergence form, namely

$$A^n = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a^n_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{N} a_i(x) + \sum_{k=1}^{N} \frac{\partial a^n_k}{\partial x_k}(x) \frac{\partial}{\partial x_i}. \tag{1.10}$$

The symbol $|E|$ stands for the Lebesgue measure of the subset $E \subseteq \Omega$, while $\chi_E$ denotes the characteristic function of $E$.

All equalities and inequalities between $L_p$ functions will be meant to hold almost everywhere. Different constants occurring in various estimates will be denoted by the same letter $C$. 

We are interested in the following oblique derivative problems

\[(I) \quad \begin{cases} 
 u \in \mathcal{W}_2^0(\Omega), \ A u + \lambda u = f & \text{in } \Omega \\
 B u = 0 & \text{on } \Gamma 
\end{cases} \]

and

\[(II) \quad \begin{cases} 
 u \in \mathcal{W}_2^0(\Omega), \ \text{Max} \{ u - \varphi; A u + \lambda u - f \} = 0 & \text{in } \Omega, \\
 B u = 0 & \text{on } \Gamma. 
\end{cases} \]

Here, \( \lambda > 0 \) is a constant, \( f, \varphi \) are given functions.

Our approach to problems (I), (II) relies on the use of the Green’s function \( G = G(x, y, t) \) for the parabolic initial-boundary value problem associated with \( A, B \). The following results is contained, in a more general setting in Garroni-Solonnikov [13]:

**Theorem 1.1.** Under the assumptions (1.3), (1.4), there exists a function \( G(x, y, t) \) such that

\[
\begin{cases} 
 \frac{\partial G}{\partial t} + A x G = \delta_{x \to \delta t} & \text{in } Q, \\
 G(x, y, 0) = 0 & \text{in } \Omega \times \Omega \\
 B x G = 0 & \text{on } \Sigma, 
\end{cases}
\]

and

\[
|G(x, y, t)| \leq C t^{- (N + 1)/2} \exp \left[ -C \frac{||x - y||^2}{t} \right],
\]

\[
\left| \frac{\partial}{\partial x_i} G(x, y, t) \right| \leq C t^{- (N + 1)/2} \exp \left[ -C \frac{||x - y||^2}{t} \right],
\]

\[
\left| \frac{\partial^2 G}{\partial x_i \partial x_j} (x, y, t) - \frac{\partial^2 G}{\partial x_i \partial x_j} (x, y, t) \right| \leq C ||x - y||^{(N + 1 + \alpha)/2} \exp \left[ -C \frac{||x - y||^2}{t} \right],
\]

\[
\left| \frac{\partial G}{\partial t} (x, y, t) - \frac{\partial G}{\partial t} (x, y, t) \right| \leq C ||t - t'||^{\frac{\alpha}{2}} \exp \left[ -C \frac{||x - y||^2}{t} \right].
\]
Here above $s, r$ are non negative integers such that $2s + r = 2$, $d(x)$
denotes the distance of $x$ from $\Gamma$, $a \vee b$ stands for the greatest between the
numbers $a, b$ and $t > t' > 0$.

Let us consider the parabolic mixed problem

\begin{equation}
\begin{cases}
\frac{\partial v}{\partial t} + Av = f & \text{in } Q_T \\
\psi(\cdot, 0) = \varphi & \text{in } \Omega \\
Bv = 0 & \text{on } \Sigma_T,
\end{cases}
\end{equation}

with $\varphi \in W^{2-\frac{1}{p}}(\Omega)$, $f \in L^p(Q_T)$ satisfying the compatibility conditions

\begin{equation}
\begin{cases}
B\varphi = 0 & \text{on } \Gamma (p > 3) \\
\quad = F - \int_0^T \int_{\Omega} \frac{\partial}{\partial t} |B\varphi|^2 \left( \frac{dy}{|x-y|^2 + t} \right)^{p+1} < +\infty, & (p = 3).
\end{cases}
\end{equation}

As a consequence of Theorem 1.1 and a result of Solonnikov ([21]) the
following holds:

**Theorem 1.2.** Let us assume (1.3), (1.4), (1.5). Then, for any $T > 0$,
$f \in L^p(Q_T)$, $\varphi \in W^{2-\frac{1}{p}}(\Omega_x)$ satisfying (1.18), there exists a unique solution
$v \in W^{2,1}(Q_T)$ of (1.17) given by

\begin{equation}
v(x, t) = \int_0^t \int_{\Omega} G(x, y, t - \tau) f(y, \tau) dy d\tau + \int_{\Omega} G(x, y, t) \varphi(y) dy.
\end{equation}

Moreover,

\begin{equation}
\|v\|_{2,1,T} < C[\|f\|_{L^p(Q_T)} + \|\varphi\|_{2-\frac{1}{p}, p} + F],
\end{equation}

for some constant $C = C(T)$ with polynomial growth in $T$.

The next lemma exploits some properties of $G$ that will be useful later.

**Lemma 1.1.** The Green's function $G$ satisfies

\begin{equation}
G(x, y, t) > 0,
\end{equation}

\begin{equation}
\text{for every ball } B \subset \Omega \text{ and } t > 0 \text{ there exists } \gamma > 0 \text{ such that } G(x, y, t) > \gamma \text{ for every } x \in \Omega, y \in B.
\end{equation}

\begin{equation}
\int_B G(x, y, t) dy = 1.
\end{equation}
PROOF. By construction (see [13], Part 2) $G$ is the sum $G_0 + G_1$, where

$$G_0(x, y, t) \simeq Ct^{-N/2} \exp \left\{ - C \frac{|x - y|^2}{t} \right\}, \quad \text{locally},$$

$$G_0(x, y, t) > 0$$

$$G_1(x, y, t) \simeq Ct^{-(N-2)/2} \exp \left\{ - C \frac{|x - y|^2}{t} \right\}.$$ 

Hence,

$$G(x, y, t) \to +\infty \text{ if } t \to 0, \quad \text{with } |x - y|^2 < t.$$

Let us fix an arbitrary $y \in \Omega$. From (1.24) it follows that, for any sufficiently small positive $\varepsilon$,

$$G(x, y, t) > 0$$

for $$(x, t) \in C_\varepsilon(y) = \{(x, t) \in Q_T : |x - y|^2 < t, 0 < t < \varepsilon^2\}.$$ 

Also,

$$\begin{cases}
\frac{\partial G}{\partial t} + A_x G = 0 & \text{in } Q_T - C_\varepsilon(y) \\
G(x, y, 0) = 0 & \text{in } \Omega \times \Omega \\
B_x G = 0 & \text{on } \Sigma_T.
\end{cases}$$

By the estimates (1.15), (1.16), $G$ belongs to $C^{2+\alpha, (2+\alpha)/2}(Q_T - C_\varepsilon(y))$ and is $C^{1+\alpha, (1+\alpha)/2}$ up to the lateral boundary of $Q_T - C_\varepsilon(y)$.

Let us assume that $m = \inf \{G(x, y, t) : (x, t) \in Q_T - C_\varepsilon(y)\}$ is attained at some point $(\bar{x}, \bar{t}) \in Q_T - C_\varepsilon(y)$.

Then the parabolic maximum principle (see [18]) yields $G \equiv m$ on $Q_1 - C_\varepsilon(y)$ and, in particular, on the lateral boundary of $C_\varepsilon(y)$.

Since $\bar{g}$ was arbitrary, this contradicts (1.24).

On the other hand, $m$ cannot be attained on $\Sigma_T$ since this would contradict the oblique derivative condition in (1.26). Therefore, $m$ is attained at some point $(\bar{x}, 0)$. The initial condition in (1.26) then implies

$$G(x, y, t) \geq G(\bar{x}, y, 0) = 0 \quad \text{in } Q_T - C_\varepsilon(y).$$

This, together with (1.25), proves that (1.21) holds true.

Let us choose now $\bar{t}$ such that $0 < \bar{t} < \varepsilon^2$. The same argument as above shows that the infimum $\bar{m}$ of $G$ on $Q_T - Q_{\bar{t}} - C_\varepsilon(y)$ is attained at some
point \((\tilde{x}, y, \tilde{t})\). If it were \(G(\tilde{x}, y, \tilde{t}) = 0\) then, as a consequence of the maximum principle when applied to \(Q - C_\phi(y)\), it would be

\[
G(x, y, t) = 0 \quad \text{in } Q - C_\phi(y).
\]

This is a contradiction, since \(G\) is strictly positive on \(\mathcal{C}_\phi(y)\), (see (1.25)). Therefore, \(\bar{m} > 0\) and (1.22) is proved.

To prove (1.23), it is enough to combine the representation formula (1.19) with the observation that the solution of problem (1.17) with \(f = 0\) and \(\varphi = 1\) is \(v = 1\).

\(\square\)

2. - The case \(\lambda > 0\).

In this section we shall deal with the case \(\lambda > 0\) and prove existence, uniqueness and \(W^2_p\) estimates for the solution of (I) and (II).

For more regular coefficients (that is \(a_{ij}, b_i\) Lipschitz continuous) problem (I) has been extensively studied (see, for example [2]). In the non-variational case (I) has been studied under various type of assumptions by C. Miranda [17], M. Chicco [8] and G. M. Troianiello [23].

The following theorem is related to the results mentioned above. Let us point out, however, that the boundary coefficients \(b_i\) are not required to be Lipschitz continuous and that exponents \(1 < p < 2\) are allowed.

**Theorem 2.1.** Let us assume (1.3), (1.4), (1.5). Then, for any \(\lambda > 0\), \(f \in L_p(\Omega)\), problem (I) has a unique solution given by

\[
(2.1) \quad u_\lambda(x) = \lambda \int_0^\infty \exp\left(-\lambda t\right) \left[\int_\Omega G(x, y, t - \tau)f(y) dy d\tau\right] dt.
\]

Moreover, the estimates

\[
(2.2) \quad \|u_\lambda\|_p \leq \frac{C}{\lambda} \|f\|_p
\]

\[
(2.3) \quad \|u_\lambda\|_{2,p} \leq C[\|f\|_p + \|u_\lambda\|_p]
\]

hold, where \(C\) denotes different positive constants depending on \(p, \mu, \beta\) and the Hölder norms of the coefficients, but independent of \(\lambda > 0\).
PROOF. Let $v = v(x, t)$ be the solution of problem (1.17) with $\varphi = 0$. Then (2.1) becomes

$$u_2(x) = \lambda \int_0^{+\infty} \exp[-\lambda t] v(x, t) \, dt.$$  

Hence,

$$\int_{\Omega} |u_2(x)|^p \, dx \leq C(\lambda) \int_0^{+\infty} \exp[-\lambda t] \|v(x, t)\|_p^p \, dt$$

and

$$= C(\lambda) \frac{d}{dt} \int_0^{+\infty} \exp[-\lambda t] \|v\|^p_{L_p(\Omega)} \, dt,$$

with $C(\lambda) = \lambda^{-\frac{1}{2}p}$.

Taking (1.20) into account, we obtain

$$\|u_2\|_p^p \leq C(\lambda) \frac{d}{dt} \int_0^{+\infty} \exp[-\lambda t] \|v\|^p_{L_p(\Omega)} \, dt = \frac{C}{\lambda^p} \|f\|_p^p$$

and (2.2) is proved.

In a similar way,

$$\int_{\Omega} \left( \frac{\partial}{\partial x_j} u_2 \right)^p \, dx \leq C(\lambda) \lambda \|f\|_p^p \int_0^{+\infty} \exp[-\lambda t] t^{p-1} \, dt = C \|f\|_p^p.$$

By a standard interpolation argument, (2.7) and (2.2) give (2.3).

The next step is to show that $u_2$ is actually a solution of (I). To this purpose observe that

$$Au_2 = \lambda \int_0^{+\infty} \exp[-\lambda t] Av \, dt = \lambda \int_0^{+\infty} \exp[-\lambda t] \left( f - \lambda v \right) \, dt = f - \lambda \int_0^{+\infty} \exp[-\lambda t] \frac{\partial}{\partial t} v \, dt.$$

Hence, integrating by parts and taking the homogeneous initial condition on $v$ into account,

$$Au_2 = f - \lambda u_2 \quad \text{in} \quad \Omega.$$

On the other hand,

$$Bu_2 = \lambda \int_0^{+\infty} \exp[-\lambda t] Bv \, dt = 0 \quad \text{on} \quad \Gamma',$$

since $v$ satisfies the homogeneous boundary condition in (1.17).
Let us prove now that (1) has a unique solution. This amounts to show that if \( z \in W^1_2(\Omega) \) is such that
\[
\begin{align*}
Ax + \lambda z &= 0 \quad \text{in } \Omega, \\
Bz &= 0 \quad \text{on } \Gamma
\end{align*}
\]
then \( z = 0 \).

Let us denote to this purpose by \( v_1 \) and \( v_2 \), the solutions of
\[
\begin{align*}
\frac{\partial v_1}{\partial t} + Av_1 &= 0 \quad \frac{\partial v_2}{\partial t} + Av_2 &= 0 \quad \text{in } Q_T \\
v_1(\cdot, 0) &= z \quad v_2(\cdot, 0) &= -Az = \lambda z \quad \text{in } \Omega \\
Bv_1 &= 0 \quad Bv_2 &= 0 \quad \text{on } \Sigma_T.
\end{align*}
\]

It is easy to check that \( v_1 - z \) and \( \int_0^t v_2(x, s) \, ds \) satisfy the same mixed problem, namely
\[
\begin{align*}
\frac{\partial v}{\partial t} + Av &= \lambda z \quad \text{in } Q_T \\
v(\cdot, 0) &= 0 \quad \text{in } \Omega \\
Bv &= 0 \quad \text{on } \Sigma_T.
\end{align*}
\]

By the uniqueness result in Theorem 1.2 it follows then that
\[
v_1(x, t) - z(x) = \int_0^t v_2(x, s) \, ds,
\]

These yield
\[
(2.8) \quad \frac{\partial v_1}{\partial t} = \lambda v_1,
\]
and, consequently,
\[
\frac{\partial}{\partial t} (\exp [-\lambda t]v_1) = \exp [-\lambda t] \left( \frac{\partial v_1}{\partial t} - \lambda v_1 \right) = 0.
\]

Hence, \( \exp [-\lambda t]v_1 \) equals some function of \( x \) only, say \( C(x) \). The limit relations,
\[
(2.9) \quad \lim_{t \to \infty} \| \exp [-\lambda t]v_1 \|_{L^p(Q)} = 0, \quad \lim_{t \to 0} \| \exp [-\lambda t]v_1 - z \|_{L^p(Q)} = 0,
\]
which follow from (1.20) and Theorem 1.2 imply \( z = 0 \). \( \square \)
REMARK 2.1. The case of an inhomogeneous boundary condition $Bu = Bh \in W^{1,p}_p(I)$ in (I) can be treated in a similar way, via a suitable change of unknown function.

COROLLARY 2.1. The solution $u_\lambda$ of (I) is also given by

$$u_\lambda(x) = \int_0^\infty \exp[-\lambda t] \int_\Omega G(x, y, t)f(y) \, dy \, dt, \quad f \in L_p(\Omega).$$

PROOF. Let us assume temporarily $f \in W^{2-2/p}_{2/p}(\Omega)$ and observe that

$$v(x, t) = \int_\Omega G(x, y, t)f(y) \, dy,$$

is the unique solution of

$$\frac{\partial v}{\partial t} + Av = 0 \quad \text{in} \quad Q_T,$$

$$v(\cdot, 0) = f \quad \text{in} \quad \Omega,$$

$$Bv = 0 \quad \text{on} \quad \Sigma_T.$$

A verification similar to the one in the proof of Theorem 2.1 shows that $u_\lambda$ given by (2.10) is a solution of (I). By the uniqueness part of Theorem 2.1 the representations (2.1), (2.10) must coincide. The validity of (2.10) for general $f \in L_p(\Omega)$ follows by a density argument, taking (2.3) into account.

Let us consider now the unilateral problem (II). The following result holds:

THEOREM 2.2. Let us assume (1.3), (1.4), (1.5), $f \in L_p(\Omega)$ and

$$\psi \in W^{2}_p(\Omega), \quad B\psi > 0 \quad \text{on} \quad \Gamma.$$

Then, for any $\lambda > 0$ there exists a unique solution $u$ of (II). Moreover,

$$u = \sup \{v \in W^{1}_p(\Omega) \mid v \leq \psi, Av + \lambda v \leq f \quad \text{in} \quad \Omega, \quad Bv < 0 \quad \text{on} \quad \Gamma\}$$

and $u$ satisfies the Lewy-Stampacchia inequality

$$f \wedge (A\psi + \lambda \psi) \leq Au + \lambda u \leq f \quad \text{in} \quad \Omega.$$
Proof. Let us consider regularized operators $A^n, B^n(n = 1, 2, \ldots)$ as described in § 1 and the auxiliary problems

\begin{equation}
(2.14) \quad \begin{cases}
    u^n \in W^2_p(\Omega), \quad \max [u^n - \psi^n; A^n u^n + \lambda u^n - f] = 0 \quad \text{in } \Omega \\
    B^n u^n = 0 \quad \text{on } \Gamma,
\end{cases}
\end{equation}

where $\psi^n$ is the unique solution of

\begin{align*}
\begin{cases}
    \psi^n \in W^2_p(\Omega), \\
    A^n \psi^n + \lambda \psi = A \psi + \lambda \psi \\
    B^n \psi^n = B \psi
\end{cases} \quad \text{in } \Omega, \quad \text{on } \Gamma.
\end{align*}

The method of variational inequalities applies to problem (2.14) (see [3], [23]). Hence, (2.14) has a unique solution $u^n$ and the Lewy-Stam-pacchia inequality holds;

\begin{equation}
(2.15) \quad f \wedge (A^n \psi^n + \lambda \psi^n) \ls A^n u^n + \lambda u^n \ls f \quad \text{in } \Omega, \\
0 = 0 \wedge B^n \psi^n \ls B^n u^n \ls 0 \quad \text{on } \Gamma^{(*)}.
\end{equation}

From (2.15) it follows that

\begin{equation*}
A^n u^n + \lambda u^n = g^n \quad \text{in } \Omega,
\end{equation*}

for some $g^n \in L_p(\Omega), f \wedge (A^n \psi^n + \lambda \psi^n) \ls g^n \ls f$.

Hence, taking estimate (2.2), (2.3) into account,

\begin{equation*}
\|u^n\|_{L^p} \ls \frac{C}{\lambda} \|g^n\|_{L^p} \ls C, \quad \text{independent of } n.
\end{equation*}

Therefore at least a subsequence of $u^n$ converges weakly in $W^2_p(\Omega)$ as $n \to +\infty$ to some $u \in W^2_p(\Omega)$.

The passage to the limit in (2.14), (2.15) shows that $u$ is a solution of (II) and (2.13) holds.

Let us prove now that any solution of (II) satisfies (2.12). Of course this will imply uniqueness of the solution.

Let $v$ be an arbitrary function satisfying the inequalities in (2.12). Mimicking an argument due to Troianiello [24], let us define $v^n$ as the solu-

\begin{equation*}
(*) \quad \text{Inequality (2.15) can be proved in the case } 1 \leq p < 2 \text{ using the results of [11].}
\end{equation*}
tion of
\begin{equation}
\begin{cases}
v^n \in W^2_p(\Omega), \\
A^n v^n + \lambda v^n = A v + \lambda v - |(A^n - A)v| - |(A^n - A)u|, \\
B^n v^n = B v \wedge 0 \wedge (B^n - B) u,
\end{cases}
\end{equation}

where \( u \) is a solution of (II).

It is easy to check that
\[
\begin{cases}
A^n(v^n - v) + \lambda(v^n - v) < 0 & \text{in } \Omega \\
B^n(v^n - v) < 0 & \text{on } \Gamma,
\end{cases}
\]

from which it follows that \( v^n < v < \psi \). It is also clear that \( A^n v^n + \lambda v^n < f + (A^n - A)u \) in \( \Omega \), \( B^n v^n < (B^n - B) u \) on \( \Gamma \).

On the other hand any solution \( u \) of (II) satisfies

\[
\begin{cases}
\max[u - v; A^n u + \lambda u - f - (A^n - A)u] = 0 & \text{in } \Omega \\
B^n u^n = (B^n - B) u & \text{on } \Gamma.
\end{cases}
\]

Hence, known results in the variational case (see [23]) yield
\[
v^n < u.
\]

Since \( v^n \rightarrow v \) in \( W^2_p(\Omega) \), the statement (2.12) is proved. \( \square \)

**Remark 2.2.** The same result holds for less regular obstacle of the type
\[
\psi = \bigwedge_{i=1}^k \psi_i, \psi_i \in W^2_p(\Omega),
\]
the inequality (2.12) being substituted by
\[
f \wedge \left( \bigwedge_{i=1}^k (A \psi_i + \lambda \psi_i) \right) < u < f.
\]

Non-homogeneous boundary conditions can be treated by suitable change of variables.

3. – Fredholm alternative and invariant measures.

Let us denote by \( J_\lambda(\lambda > 0) \) the Green operator of problem (I), which associates to any \( f \in L_p(\Omega), 1 < p < 1/(1 - \alpha) \) the unique solution \( u_\lambda = J_\lambda f \) of problem (I).
LEMMA 3.1. $J_\lambda$ is a compact operator acting on $L^p(\Omega)$, $(1 < p < 1/1 - \alpha)$, and $\operatorname{Ker} (I - \lambda J_\lambda) = \mathbb{R}.$

PROOF. The first statement is an immediate consequence of estimate (2.2) and the Rellich compactness theorem.

Since, as it is obvious, $\operatorname{Ker} (I - \lambda J_\lambda)$ is independent of $\lambda$ we can take $\lambda = 1$. Taking (2.10) and Lemma 1.1 into account, we find that

$$J_1 f(x) = \int_0^{+\infty} \exp\left[-t\right] dt \int_{\Omega} G(x, y, t) f(y) dy > 0 \quad \text{in} \quad \Omega,$$

for any $f \in L^p(\Omega)$, $f > 0$, $f \equiv 0$.

Since $J_1 C = C$ for any real constant $C$, the theorem 6.6 of [20] applies, yielding that $\mu = 1$ is a simple eigenvalue of $J_1$. This proves the Lemma.

Now, by the Fredholm alternative, the adjoint equation

$$(3.1) \quad m \in L_p(\Omega), \quad (I - \lambda J_\lambda^*) m = 0 \left( \lambda > 0, \frac{1}{p'} + \frac{1}{p} = 1 \right)$$

has a one dimensional space of solutions and

$$(3.2) \quad u \in L_p(\Omega), \quad (I - \lambda J_\lambda) u = g,$$

is solvable if and only if $g$ satisfies the compatibility condition

$$(3.3) \quad \int_{\Omega} g m dx = 0.$$

The next theorem provides further information on $\operatorname{Ker} (I - \lambda J_\lambda^*)$, which will be of essential use for the result in § 4.

THEOREM 3.1. Let us assume (1.3), (1.4), (1.5).

Then, for every $\lambda > 0$, equation (3.1) has a unique solution $m$ such that

$$(3.4) \quad m > 0, \quad \frac{1}{|\Omega|} \int_{\Omega} m dx = 1, \quad m \in L_p(\Omega).$$
Moreover, for any $t > 0$ and $f \in L_p(\Omega), 1 < p < \infty,$

\begin{equation}
\int \left[ \int_{\Omega} G(x, y, t) f(y) \, dy \right] m(x) \, dx = \int f(x) m(x) \, dx,
\end{equation}

\begin{equation}
\left\| \int_{\Omega} G(x, y, t) f(y) \, dy - \frac{1}{|\Omega|} \int_{\Omega} f(y) m(y) \, dy \right\|_p < K \| f \|_p \exp \left( -\nu t \right), \quad \forall t > 0,
\end{equation}

with $K, \nu$ independent of $f$ (*)

**Proof.** From (3.1) it follows that

\[ \int_{\Omega} m(x) \, dx = \frac{1}{\nu} \int_{\Omega} \exp \left( -\frac{1}{\nu} \int \theta_m \, dx \right) \, dx, \]

\[ m(y) = \int_0^{+\infty} \exp \left( -t \right) \int_{\Omega} G(x, y, t) m(x) \, dx. \]

Then

\[ |m(y)| < \| m \|_{q^*} \int_0^{+\infty} \exp \left( -t \right) \left( \int_{\Omega} \theta_m \, dx \right)^{1/q} dt. \]

Therefore, the choice $q' > N/2$ yields

\[ \| m \|_{q'} < c \| m \|_{q^*}. \]

Let us start by showing that any solution $m$ of (3.1) has a constant sign almost everywhere in $\Omega$.

At this purpose, set

\[ \Omega^+ = \{ x \in \Omega : m(x) > 0 \}, \quad \Omega^- = \Omega - \Omega^+ \]

and assume that $0 < |\Omega^+| < |\Omega|$.

Multiplying (3.1) by $\chi_{\Omega^-}$ one finds that

\begin{equation}
\int_{\Omega^-} m \, dx = \lambda \int_{\Omega^+} m \, dx - \lambda \int_{\Omega^-} m \, dx.
\end{equation}

(*) Further regularity properties of the function $m$ will be the object of another paper (see [12]).
The representation (2.10), together with (1.22), (1.23) of Lemma 1.1, yields

\[ J_\lambda \chi_{\Omega^*}(x) = \int_0^\infty \exp \left[ -\lambda t \right] dt \int_\Omega G(x, y, t) \, dy < \frac{1}{\lambda} \quad \text{in} \, \Omega. \]

Use this in (3.8) to get

\[ \int_\Omega m \, dx > \lambda \int_\Omega mJ_\lambda \chi_{\Omega^*} \, dx > \int_\Omega m \, dx, \]

which is absurd. Hence \(|\Omega^*| = |\Omega|\) and the statement is proved.

Since \(\text{Ker} (I - \lambda J_\lambda^*)\) is one-dimensional, this must contain some \(m > 0\), \(\tilde{m} \neq 0\). Then, (3.4) is satisfied by \(m = \tilde{m}|\Omega|\tilde{m} \, dx\).

Actually, \(m > 0\) in \(\Omega\). Assume by contradiction that \(\Omega^0 = \{x \in \Omega | m = 0\}\) has positive measure and multiply (3.1) by \(\chi_{\Omega^*}\). Then,

\[ 0 = \int_\Omega m\chi_{\Omega^*} \, dx = \lambda \int_\Omega mJ_\lambda \chi_{\Omega^*} \, dx = \lambda \int_\Omega mJ_\lambda \chi_{\Omega^*} \, dx > 0, \]

since \(J_\lambda \chi_{\Omega^*} > 0\) (see the proof of Lemma 3.1).

In order to prove (3.5), let us multiply (3.1) by an arbitrary \(f \in L_p(\Omega)\) to obtain

\[ \frac{1}{\lambda} \int_\Omega mf \, dx = \int_\Omega mJ_\lambda f \, dx = \int_0^\infty \int_\Omega \left[ \int_\Omega G(x, y, t)f(y) \, dy \right] m(x) \, dx. \]

Since \(1/\lambda = \int_0^\infty \exp \left[ -\lambda t \right] dt\), the above yields

\[ \int_0^\infty \exp \left[ -\lambda t \right] dt \int_\Omega \int_\Omega G(x, y, t)f(y) \, dy - f(x) \, m(x) \, dx = 0. \]

The term in square brackets in (3.9) being a continuous function of \(t\), (3.5) is proved.
Let us prove now the estimate (3.6). Following [5], Thm. 3.1 p. 365, we consider, for \( n = 1, 2, \ldots \), and for any Borel subset \( E \) in \( \Omega \), the measures
\[
m_n(E) = \inf_{z \in \partial} \int_{\Omega} G(x, \xi, n) \chi_{E}(\xi) \, d\xi.
\]
\[
M_n(E) = \sup_{y \in \partial} \int_{\Omega} G(y, \xi, n) \chi_{E}(\xi) \, d\xi.
\]
Since
\[
G(x, \xi, n + 1) = \int_{\Omega} G(x, z, n) G(\xi, z, 1) \, dz,
\]
we obtain
\[
M_{n+1}(E) - m_{n+1}(E)
\]
\[
= \sup_{x \in \partial} \int_{\Omega} [G(x, z, 1) - G(y, z, 1)] \, dz \int_{\Omega} G(x, \xi, n) \chi_{E}(\xi) \, d\xi.
\]
If we define now the set \( \Omega^+ = \Omega^{+}_{z,v} \) by
\[
\Omega^+ = \{ z \in \partial | G(x, z, 1) > G(y, z, 1) \},
\]
we have, taking (1.23) of Lemma 1.1, into account,
\[
\int_{\Omega^+} [G(x, z, 1) - G(y, z, 1)] \, dz = 1 - \int_{\Omega^+} G(y, z, 1) \, dz - \int_{\partial^*} G(x, z, 1) \, dz
\]
Hence, by (1.22) of Lemma 1.1,
\[
\int_{\partial^*} [G(x, z, 1) - G(y, z, 1)] \, dz < 1 - \gamma |B|.
\]
Using this in (3.10), we obtain
\[
M_{n+1}(E) - m_{n+1}(E) < \sup_{z,v} \int_{\partial^*} M_n(E) [G(x, z, 1) - G(y, z, 1)] \, dz
\]
\[
+ \int_{\partial^*} m_n(E) [G(x, z, 1) - G(y, z, 1)] \, dz < \sup_{z,v} (M_n(E) - m_n(E)) (1 - \gamma |B|).
\]
and, by iteration,

\begin{equation}
M_{n+1}(E) - m_{n+1}(E) \leq (M_1(E) - m_1(E)) (1 - \gamma|B|)^{n-1}.
\end{equation}

Let us estimate now

\[ \left| \int_{\Omega} G(x, \xi, n + 1) \chi_{\delta}(\xi) \, d\xi - \frac{1}{|\Omega|} \int_{\Omega} \chi_{\delta}(\xi) m(\xi) \, d\xi \right|^p \]

or, which is the same, thanks to (3.5),

\begin{equation}
h(x) = \left| \int_{\Omega} G(x, \xi, n + 1) \chi_{\delta}(\xi) \, d\xi - \frac{1}{|\Omega|} \int \int m(\eta) \left[ \int \int G(\eta, \xi, n + 1) \chi_{\delta}(\xi) \, d\xi \right] \, d\eta \right|^p.
\end{equation}

Now, the obvious inequalities

\begin{align*}
m_{n+1}(E) &< \int_{\Omega} G(x, \xi, n + 1) \chi_{\delta}(\xi) \, d\xi < M_{n+1}(E), \\
m_{n+1}(E) \frac{1}{|\Omega|} \int m(\eta) \, d\eta &< \frac{1}{|\Omega|} \int m(\eta) \left[ \int \int G(\eta, \xi, n + 1) \chi_{\delta}(\xi) \, d\xi \right] \, d\eta \\
&< M_{n+1}(E) \frac{1}{|\Omega|} \int m(\eta) \, d\eta,
\end{align*}

(3.4) and (3.14) yield

\[ h(x) < \left| M_{n+1}(E) - m_{n+1}(E) \right|^p < \left| M_1(E) - m_1(E) \right|^p (1 - \gamma|B|)^{p(n-1)} \]

\[ < \left| M_1(E) \right|^p (1 - \gamma|B|)^{p(n-1)}. \]

At this point from (3.15) it follows that

\[ \left\| \int_{\Omega} h(x) \, dx < K (1 - \gamma|B|)^{p(n-1)} \| \mathcal{X}_E \|_{p'}, \right\|

for some constant \( K \) depending only on \( Q \).

If we choose now \( v = - p \log (1 - \gamma|B|) \), the inequality (3.6) is proved for positive integer \( t = [t] \) and \( f = \chi_E \).
Now for $t > [t]$, we have

$$\left\| \int_\Omega G(x, y, t)f(y) \, dy - \frac{1}{|\Omega|} \int_\Omega f(y)m(y) \, dy \right\| \leq \left\| \int_\Omega G(x, y, t) \left[ \int_\Omega G(y, z, [t]) f(z) \, dz \right] \, dy - \frac{1}{|\Omega|} \int_\Omega f(y)m(y) \, dy \right\| \leq K \|X_t\|_p \exp \left[ -v[t] \right] < K \|X_t\|_p \exp \left[ -v[t] \right].$$

Using a density argument we obtain (3.6). □

We want to give now a probabilistic interpretation of Theorem 3.1. By means of the Green's function one can define a family of probability measures $P(x, t, E)$ on $\Omega$ in the following standard way,

$$P(x, t, E) = \int_E G(x, y, t) \, dy,$$

where $E$ is any Borel subset on $\Omega$. To $P(x, t, E)$ it corresponds a linear operator $\Phi(t)$ defined by

$$\Phi(t) f(x) = \int_\Omega P(x, t, E) f(y) = \int_\Omega G(x, y, t) f(y) \, dy.$$

The easy estimate

$$\left| \Phi(t) f(x) \right| \leq \sup_{y \in \Omega} |f(y)| \int_\Omega G(x, y, t) \, dy,$$

together with the estimates of Theorem 1.1, implies that $\Phi(t)$ is a continuous operator on $C(\Omega)$. Taking the result of § 1 and (3.17) into account it is straightforward to check that

$$\begin{align*}
\Phi(0) &= I, \quad \Phi(t) \circ \Phi(s) = \Phi(t + s), \\
\|\Phi(t)\| &= 1, \\
\Phi(t) f &> 0, \quad \forall f > 0, \quad \Phi(t) C = C, \quad \forall C \in \mathbb{R}, \\
\Phi(t) f &\to f \text{ in } C(\Omega), \quad \text{as } t \to 0^+.
\end{align*}$$

In probabilistic terminology (see [9]) this means that $\Phi(t)$ is a Feller semigroup on $C(\Omega)$. Actually, $\Phi(t)$ is associated with a diffusion process...
in $\Omega$, reflected at its boundary $\Gamma$ according to the vector field $b = (b_1 \ldots b_N)$, (see [22] and for recent results, [15]).

Let us recall for the convenience of the reader the notion of invariant probability measure for such a semigroup. A probability measure $\mu$ on $\Omega$, endowed with the Borel $\sigma$-algebra, is invariant for $\Phi(t)$ if

\begin{equation}
\int_{\Omega} \Phi(t)f \, d\mu = \int_{\Omega} f \, d\mu, \quad \forall f \in C(\Omega),
\end{equation}

(see [9]). From Theorem 3.1 it follows then that

\begin{equation}
d\mu = \frac{1}{|\Omega|} m \, dx
\end{equation}

can be interpreted as an invariant probability measure for the semigroup defined by (3.16). The estimate (3.6) still holds with respect to the norm in $C(\Omega)$ and this implies that $\mu$ is the unique invariant measure of $\Phi(t)$, (see [5]).

When the operator $A$ has Lipschitz continuous coefficients and $b$ is $C^1$, an integration by parts shows that (3.1) becomes the adjoint homogeneous differential problem of (I), namely

\begin{equation}
\begin{aligned}
A^* m &= 0 \quad \text{in } \Omega \\
B^* m &= 0 \quad \text{on } \Gamma,
\end{aligned}
\end{equation}

(see [17], p. 14-15, for an explicit formula for the adjoint operator $B^*$).

We conclude this section by showing that even in the present framework the function $m$ can be constructed purely by PDE methods. At this purpose, let us consider the problems

\begin{equation}
\begin{cases}
m_n \in W^1_2(\Omega), \\
A^*_n m_n = 0 \quad \text{in } \Omega, \\
B^*_n m_n = 0 \quad \text{on } \Gamma, \quad (n = 1, 2, \ldots),
\end{cases}
\end{equation}

where $A^*_n, B^*_n$ are the adjoints of the regularized operators $A_n, B_n$ given by (1.6), (1.7).

It is known (see for example [4]) that (3.22) has a unique solution $m_n$ such that

\begin{align*}
m_n \in C^0(\Omega), \quad m_n > 0, \quad \frac{1}{|\Omega|} \int_{\Omega} m_n \, dx = 1, \quad (n = 1, 2, \ldots).
\end{align*}
Define now probability measures $\mu_n$ on $\Omega$ by

$$
(3.23) \quad d\mu_n = \frac{1}{|\Omega|} m_n \, dx
$$

and denote by $\Phi_n(t)$ the Feller semigroup associated with $A_n$, $B_n$. Then, we have the following:

**Theorem 3.2.** Let us assume (1.3), (1.4), (1.5) and $\alpha > (N + 2)/2$. Then, the measures $\mu_n$ defined by (3.23) are invariant for $\Phi_n(t)$ and converge weakly in the sense of measures, as $n \to +\infty$, to the invariant measure $\mu$ of $\Phi(t)$, defined by (3.20).

**Proof.** Consider the function

$$
\theta_n(t) = \int_\Omega [\Phi_n(t)f(x) - f(x)] m_n(x) \, dx \quad (n = 1, 2, \ldots),
$$

where $f$ is arbitrary in $W^{2-2\alpha}_p(\Omega)$, $p > (N + 2)/2$. Then,

$$
\theta_n'(t) = -\int_\Omega A_n \Phi_n(t)f(x) m_n(x) \, dx, \quad \forall t > 0.
$$

An integration by parts, taking (3.22) into account, shows that $\theta_n'(t) = 0$, $\forall t > 0$ hence $\theta_n(t) = \theta_n(0) = 0$.

A standard density argument completes the proof of the first part of the theorem.

The weak convergence of $\mu_n$ to some probability measure $\tilde{\mu}$ being immediate, the next step is to identify $\tilde{\mu}$ with $\mu$. To this end, let $f \in W^{2-2\alpha}_p(\Omega)$, $p > (N + 2)/2$, and $z_n = (\Phi_n(t) - \Phi(t))f$. Then, $z_n$ satisfies

$$
\begin{cases}
  z_n \in W^{2,1}_p(\Omega_T) \\
  \frac{\partial z_n}{\partial t} + A_n z_n = (A - A_n) \Phi(t)f & \text{in } Q_T \\
  z_n(\cdot, 0) = 0 & \text{in } \Omega \\
  B_n z_n = (B - B_n) \Phi(t)f & \text{on } \Sigma_T,
\end{cases}
$$

for any $T > 0$. By the parabolic estimates (see Theorem 1.2)

$$
(3.24) \quad \|z_n\|_{L^1,\alpha} \leq C [\|(A - A_n) \Phi(t)f\|_p + \|(B - B_n) \Phi(t)f\|_{1-\alpha,\infty}],
$$

with $C$ independent of $n$.  


The right-hand term of (3.24) tends to zero as \( n \to + \infty \) and so
\[
\Phi_n(t)f \to \Phi(t)f, \quad \text{uniformly in } x.
\]
Hence,
\[
\lim_{n \to + \infty} \int_B \Phi_n(t)f \, d\mu_n = \int_B \Phi(t)f \, d\tilde{\mu}, \quad \forall f \in W^{2,2}_{p,0}(\Omega),
\]
and by density the same holds true for \( f \in C(\Omega) \).

Since, trivially,
\[
\lim_{n \to + \infty} \int_B f \, d\mu_n = \int_B f \, d\tilde{\mu},
\]
the measure \( \tilde{\mu} \) is invariant for \( \Phi(t) \). From (3.6) of Theorem 3.1 it follows that
\[
\lim_{k \to + \infty} \Phi(t_k)f(x) = \frac{1}{|\Omega|} \int_B f \, d\mu \quad \text{a.e. in } \Omega,
\]
at least for a subsequence \( t_k \). Therefore, by the Lebesgue dominated convergence theorem,
\[
\lim_{k \to + \infty} \int_B \Phi(t_k)f \, d\tilde{\mu} = \frac{1}{|\Omega|} \int_B \tilde{\mu} \int_B f \, d\mu = \frac{1}{|\Omega|} \int_B f \, d\mu.
\]

Since \( \tilde{\mu} \) is invariant for \( \Phi(t) \), we have also
\[
\int_B \Phi(t)f \, d\tilde{\mu} = \int_B f \, d\tilde{\mu}, \quad \forall t > 0, \quad \forall f \in C(\Omega).
\]

Hence, comparing this with (3.25), \( \tilde{\mu} = \mu \).  \( \Box \)

4. - The limiting case \( \lambda = 0 \).

This section is devoted to the study of problems (I) and (II) in the case \( \lambda = 0 \). Let us observe that
\[
\begin{align*}
\begin{cases}
 u \in W^{2,2}_p(\Omega), & Au = f \quad \text{in } \Omega \\
 Bu = 0 & \text{on } \Gamma
\end{cases}
\end{align*}
\]
is equivalent to
\[
\begin{align*}
\begin{cases}
 u \in L^p(\Omega), & (I - \lambda J_\lambda)u = J_\lambda f \quad \left(\lambda > 0\right) \\
\end{cases}
\end{align*}
\]
Hence, by the results of § 3, (4.1) is solvable if and only if \( f \in L^p(\Omega) \) satisfies

\[
\int_{\Omega} J_\lambda f^m \, dx = 0
\]

for any solution \( m \) of (3.1) or, which is the same,

\[
\int_{\Omega} f^m \, dx = 0.
\]

An immediate consequence of the definition of \( J_\lambda \) and \( m \) which will be useful later on, is that the solution \( u_\lambda \) of (I) satisfies

\[
\int u_\lambda m \, dx = \frac{1}{\lambda} \int_{\Omega} f^m \, dx.
\]

Let us state now the main result concerning problem (4.1) and the behaviour as \( \lambda \to 0 \) of the solutions \( u_\lambda \) of (I).

**Theorem 4.1.** Let us assume (1.3), (1.4), (1.5), \( f \in L^p(\Omega) \). A necessary and sufficient condition for (4.1) to have a solution is that

\[
\int_{\Omega} f^m \, dx = 0.
\]

If (4.6) holds then,

\[
\| u_\lambda \|_{p,\lambda} < C \| f \|_p, \quad C \text{ independent of } \lambda > 0
\]

and (4.1) has a unique solution \( u \) such that

\[
\int_{\Omega} u m \, dx = 0
\]

given by

\[
u = \int_0^\infty \int_{\Omega} G(x, y, t) f(y) \, dy = \lim_{\lambda \to 0} u_\lambda, \quad \text{weakly in } W^p_0(\Omega).
\]
PROOF. If (4.1) or, equivalently, (4.2) has a solution $u$ for a given $j \in L_p(Q)$, then
\[ \int_{\Omega} J_\lambda f m \, dx = \int_{\Omega} (I - \lambda J_\lambda) u m \, dx = \int_{\Omega} u (I - \lambda J_\lambda^*) m \, dx = 0 \]

On the other hand,
\[ \int_{\Omega} J_\lambda f m \, dx = \int_{\Omega} f J_\lambda^* m \, dx = \frac{1}{\lambda} \int_{\Omega} f m \, dx \]

and the necessity of condition (4.6) is established.

Let $u_1, u_2$ be two solutions of (4.1). Then,
\[ u_1 - u_2 \in \text{Ker} (I - \lambda J_\lambda) , \]

and so, by Lemma 3.1, $u_1 - u_2 = C$. If $\int_{\Omega} u_1 m \, dx = 0 = \int_{\Omega} u_2 m \, dx$, then
\[ 0 = C \int_{\Omega} m \, dx , \]
and therefore $C = 0$.

Let us prove now that if (4.6) is satisfied, then
\[ \| u_\lambda \|_{z,p} < C \| f \|_p , \quad C \text{ independent of } \lambda > 0 . \]

Using the H"older inequality in the integral with respect to $t$ in representation (2.10), one obtains
\[ \int_{\Omega} |u_\lambda|^p \, dx \leq \left( \frac{1}{(\lambda + \eta/2)^{2p'}} \int_{\Omega} \exp \left( \frac{\nu f(t)}{2} \right) \right)^{1/p'} \int_{\Omega} G(x, y, t) f(y) \, dy \, dt . \]

Hence, taking (3.6) into account,
\[ \| u_\lambda \|_p \leq \left( \frac{1}{(\lambda + \eta/2)^{2p'}} \right)^{1/p'} \left( \frac{2}{\nu} \right)^{1/p} \| f \|_p \leq C \| f \|_p \]

for some constant $C$ depending on $\nu, p$ not on $\lambda$.

Therefore, the fundamental estimates (2.3) and (4.11) give (4.7). Now standard arguments show that $u_\lambda$ converge weakly in $W^2_p(\Omega)$ as $\lambda \to 0$ to a solution $u$ of (4.1) such that $\int_{\Omega} u m \, dx = 0$. This last fact follows from (4.5).

The estimate (3.6) guarantees that $u$ is given by (4.8). \qed
REMARK 4.1. The uniform convergence of \( u_\lambda \) to \( u \) has been proved by Robin [19] in the case \( f \in C(\bar{\Omega}) \) and for Lipschitz continuous boundary vectorfield \( b \). This result follows, of course, from Theorem 4.1, if \( p > N \).

REMARK 4.2. As a consequence of Theorem 4.1

\[
Av = h - \frac{1}{|\Omega|} \int_B hm \, dx \quad \text{in } \Omega, \quad Bv = 0 \quad \text{on } \Gamma
\]

has a unique solution \( \tilde{v} \in W^1_p(\Omega) \) such that \( \int_B \tilde{v}m \, dx = 0 \), for any given \( h \in L_p(\Omega) \).

From (4.8) it follows that \( \tilde{v} \) satisfies

\[
\tilde{v}(x) = \int_0^T dt \int_B G(x, y, t) \left[ h(y) - \frac{1}{|\Omega|} \int_B h(\xi) m(\xi) \, d\xi \right] \, dy + \int_B G(x, y, T) \tilde{v}(y) \, dy
\]

and the estimate (3.6) shows that

\[
\frac{1}{T} \left\| \int_0^T G(x, y, T) \tilde{v}(y) \, dy - \tilde{v} \right\|_p \leq \frac{1}{T} \left( 1 + K \exp \left[ -vT \right] \right) \to 0 \quad \text{as } T \to +\infty.
\]

Hence, the ergodic formula

\[
\left\| \frac{1}{T} \int_0^T \int_B G(x, y, t) h(y) \, dy \, dt - \frac{1}{|\Omega|} \int_B hm \, dx \right\|_p \to 0 \quad \text{as } T \to +\infty
\]

holds for any \( h \in L_p(\Omega) \).

We are interested now in the behaviour of the solution \( u_\lambda \) of the unilateral problem as \( \lambda \to 0 \).

Let us assume, for simplicity, \( \psi = 0 \). For the case of \( \psi \neq 0 \), see Remark 4.5 below. The next lemma contains two estimates which will be useful later.

**Lemma 4.1.** There exist constants \( C_1, C_2 \) independent of \( \lambda > 0 \) such that

\[
-G \|f\|_p < \lambda \int_B u_\lambda m \, dx,
\]

(4.12)

\[
\left\| u_\lambda - \frac{1}{|\Omega|} \int_B u_\lambda m \, dx \right\|_{L^p} < C_1 \|f\|_p.
\]

(4.13)
PROOF. From the Lewy-Stampacchia estimate (2.13) follows that for some \( g \in L^p_0(\Omega) \), \( 0 \leq g \leq f^+ \),

\[
\begin{aligned}
Au_\lambda + \lambda u_\lambda &= f - g_\lambda & \quad & \text{in } \Omega \\
Bu_\lambda &= 0 & \quad & \text{on } \Gamma.
\end{aligned}
\]

(4.14)

Hence (see 4.5),

\[
0 > \lambda \int_\Omega u_\lambda m \, dx = \int_\Omega (f - g_\lambda) m \, dx > - C_1 \| f \|_p .
\]

Set \( C_2 = 1/|\Omega| \int_\Omega u_\lambda m \, dx \) and \( w_\lambda = u_\lambda - C_1 \). From (4.14) it easily follows that \( w_\lambda \) satisfies

\[
\begin{aligned}
w_\lambda \in W^2_p(\Omega), \ A w_\lambda + \lambda w_\lambda &= f - g_\lambda - \lambda C_1, & \quad & \text{in } \Omega \\
Bu_\lambda &= 0, & \quad & \text{on } \Gamma, \\
\int_\Omega w_\lambda m \, dx &= 0 .
\end{aligned}
\]

(4.15)

Taking into account (4.5) we have necessarily

\[
\int_\Omega (f - g_\lambda - \lambda C_1) m \, dx = 0 .
\]

Hence, estimate (4.7) of Theorem 4.1 yields

\[
\| w_\lambda \|_p \leq C \| f - g_\lambda - \lambda C_1 \|_p \leq C \| f \|_p ,
\]

for some \( C \) independent of \( \lambda \), and (4.13) follows through the basic estimate (2.3). \( \square \)

The next theorem describes the behaviour as \( \lambda \to 0 \) of the solutions \( u_\lambda \) of problem (\( II \)).

**Theorem 4.2.** Let us assume (1.3), (1.4), (1.5), \( f \in L^p(\Omega) \). Then, as \( \lambda \to 0 \),

1) if \( \int_\Omega f m \, dx > 0 \), the solutions \( u_\lambda \) of (\( II \)) converge weakly in \( W^2_p(\Omega) \) to the unique solution \( u \) of

\[
\begin{aligned}
\text{in } \Omega \\
Bu = 0 & \quad & \text{on } \Gamma.
\end{aligned}
\]

(4.16)
2) if $\int_{\Omega} fm \, dx < 0$, then $w_{\lambda} = u_{\lambda} - 1/|\Omega| \int_{\Omega} u_{\lambda} \, m \, dx$ converge weakly in $W_0^1(\Omega)$ to the unique solution $w$ of

\[
\begin{align*}
  w \in W_0^1(\Omega), & \quad Aw = f - \frac{1}{|\Omega|} \int_{\Omega} fm \, dx \quad \text{in } \Omega, \\
  Bu &= 0 \quad \text{on } \Gamma, \\
  \int_{\Omega} wm \, dx &= 0; 
\end{align*}
\]

(4.17)

3) if $\int_{\Omega} fm \, dx = 0$, and $p > N/2$ then $u_{\lambda}$ converge weakly in $W_0^1(\Omega)$ to the maximum solution $u$ of

\[
\begin{align*}
  u \in W_0^1(\Omega), & \quad u < 0, \quad Au = f \quad \text{in } \Omega \\
  Bu &= 0 \quad \text{on } \Gamma. 
\end{align*}
\]

(4.18)

Proof. Let us consider the functions $w_{\lambda} = u_{\lambda} - 1/|\Omega| \int_{\Omega} u_{\lambda} \, m \, dx = u_{\lambda} - C_{\lambda}$. Of course,

\[
\begin{align*}
  w_{\lambda} \in W_0^1(\Omega), & \quad \text{Max} \left[ w_{\lambda} + C_{\lambda}; Aw_{\lambda} + \lambda w_{\lambda} - f + \lambda C_{\lambda} \right] = 0 \quad \text{in } \Omega, \\
  Bw_{\lambda} &= 0 \quad \text{on } \Gamma. 
\end{align*}
\]

(4.19)

By Lemma 4.1, at least a subsequence of $w_{\lambda}$ converges weakly in $W_0^1(\Omega)$ as $\lambda \to 0$ to some $w$ such that $\int_{\Omega} w_{\lambda} \, m \, dx = 0$ and $\lambda C_{\lambda} \to -L < 0$.

Consider the sets $S_{\lambda} = \{ x \in \Omega | w_{\lambda}(x) = -C_{\lambda} \}$. Then,

\[
\int_{\Omega} w_{\lambda} \chi_{S_{\lambda}} \, dx = -C_{\lambda} |S_{\lambda}|. 
\]

(4.20)

Since the left hand term has a finite limit as $\lambda \to 0$, (4.20) shows that if

\[
C_{\lambda} \to -\infty \quad \text{as } \lambda \to 0, 
\]

(4.21)

then $|S_{\lambda}| \to 0$ as $\lambda \to 0$.

From (4.19) it follows that

\[
Aw_{\lambda} + \lambda w_{\lambda} = f - \lambda C_{\lambda} \quad \text{in } \Omega - S_{\lambda}. 
\]
Passing to the limit in the above equations we find that $w$ satisfies

\[
\begin{cases}
    w \in \mathcal{W}_p^2(\Omega), & Aw = f - L \quad \text{in } \Omega \\
    \int_{\partial \Omega} w \mathbf{n} \cdot \mathbf{d}x = 0, & Bw = 0 \quad \text{on } \Gamma,
\end{cases}
\]

(observe, to justify this, that

\[
\int_{S} |Aw_\lambda + \lambda w - f + \lambda C \mathbf{n} \cdot \mathbf{d}x < \|Aw_\lambda + \lambda w\|_{L^p}.
\]

Let us assume now

\[
\int_{\partial \Omega} f \mathbf{n} \cdot \mathbf{d}x > 0.
\]

From (4.22) it follows that $\int_{\partial \Omega} f \mathbf{n} \cdot \mathbf{d}x = \int_{\partial \Omega} f \mathbf{n} \cdot \mathbf{d}x$, which contradicts (4.23), since $L < 0$. Therefore, if (4.23) holds, then $C_\lambda$ is bounded. This yields immediately the boundedness of $u_\lambda$ in $\mathcal{W}_p^2(\Omega)$, (see Lemma 4.1). A passage to the limit in (II) shows that the weak limit $u$ of $u_\lambda$ is a solution of (4.16). Let $\tilde{u}$ be another solution of (4.16). Then $\tilde{u}$ satisfies the inequalities $\tilde{u} < 0$, $A\tilde{u} + \lambda \tilde{u} < f$ in $\Omega$ and $B\tilde{u} = 0$ on $\Gamma$. Hence, by Theorem 2.2, $\tilde{u} < u_\lambda$ for every $\lambda > 0$, so that

\[
\tilde{u} < u.
\]

From (4.16) it follows that

\[
A(u - \tilde{u}) \cdot (u - \tilde{u}) < 0
\]

and, taking (4.24) into account, this gives $A(u - \tilde{u}) < 0$. Hence $u - \tilde{u}$ satisfies

\[
\begin{cases}
    A(u - \tilde{u}) = g \quad \text{in } \Omega \\
    B(u - \tilde{u}) = 0 \quad \text{on } \Gamma
\end{cases}
\]

for some $g < 0$. The necessary condition of Theorem 4.1 implies that $g = 0$, and as a consequence of Lemma 3.1 one obtains $u - \tilde{u} = C$ for some constant $C$. If $C > 0$, then $\tilde{u} = u - C < C < 0$ and therefore (4.16) reduces to

\[
\begin{align*}
    A\tilde{u} &= f \quad \text{in } \Omega, \\
    B\tilde{u} &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

By taking the necessary condition (4.6) into account we obtain $\int_{\partial \Omega} f \mathbf{n} \cdot \mathbf{d}x = 0$ contradicting (4.23). Hence, by virtue of (4.24), $C = 0$ and $u = \tilde{u}$. 
Assume now
\begin{equation}
(4.26) \quad \int_{\Omega} fm \, dx < 0
\end{equation}

and suppose that $C_A$ is bounded below. Then $C_A \to C, \lambda C_A \to 0$ and, passing to the limit in (4.19) as $\lambda \to 0$ one finds that the weak limit $w$ of $w_\lambda$ satisfies
\[
\begin{cases}
\max [w + \overline{C}; Aw - f] = 0 & \text{in } \Omega \\
Bw = 0 & \text{on } \Gamma.
\end{cases}
\]

Hence, for some $g > 0$,
\[
Aw = f - g, \quad Bw = 0.
\]

So that necessarily, $\int_{\partial} (f - g) \, m \, dx = 0$. But this contradicts (4.26) and therefore $C_A \to -\infty$. Hence, (4.22) must hold and the necessary condition (4.6) yields $\int_{\partial} fm \, dx = L[\Omega]$.

This proves the second part of the theorem.

Let us consider finally the case
\begin{equation}
(4.27) \quad \int_{\Omega} fm \, dx = 0.
\end{equation}

By Theorem 4.1 and the assumption $p > N/2$, the solutions $z_\lambda$ of
\begin{equation}
(4.28) \quad \begin{cases}
Az_\lambda + \lambda z_\lambda = f & \text{in } \Omega \\
Bz_\lambda = 0 & \text{on } \Gamma
\end{cases}
\end{equation}
are bounded in $W^{2,p}_r(\Omega)$ and $L_\infty(\Omega)$, uniformly in $\lambda$.

It is easy to check that $z_\lambda - \|z_\lambda\|_\infty$ is a subsolution of (II). Hence, taking (2.12) into account,
\[
u_\lambda > z_\lambda - \|z_\lambda\|_\infty.
\]

From Lemma 4.1 it follows that
\[
\|u_\lambda\|_{L^p} \leq C_2 \|f\|_p - \int_{\Omega} u_\lambda m \, dx < C_2 \|f\|_p + \|z_\lambda - \|z_\lambda\|_\infty \|m\|_\infty,
\]
and therefore $\|u_\lambda\|_{L^p} < C$, for some constant $C$ independent of $\lambda$. By the
Lewy-Stampacchia inequality (2.13), $u_\lambda$ satisfies

\begin{equation}
\begin{cases}
Au_\lambda + \lambda u_\lambda = f - g_\lambda & \text{in } \Omega, \\
Bu_\lambda = 0 & \text{on } \Gamma,
\end{cases}
\end{equation}

for some $g_\lambda > 0$ with $\|g_\lambda\|_p < C$. Let $u, g$ be weak subsequential limits of $u_\lambda$ and $g_\lambda$, respectively. Letting $\lambda \to 0$ in (4.29) one finds that $u$ is a solution of

\begin{equation}
\begin{cases}
Au = f - g & \text{in } \Omega, \\
Bu = 0 & \text{on } \Gamma.
\end{cases}
\end{equation}

Then, necessarily,

$$\int_{\Omega} gm\,dx = \int_{\Omega} fm\,dx = 0,$$

so that $g = 0$. This proves that $u$ satisfies (4.18).

To prove the maximality of $u = \lim_{\lambda \to 0} u_\lambda$, let $\tilde{u}$ be another solution of (4.18). Then, for every $\lambda > 0$,

$$\tilde{u} < 0, \quad A\tilde{u} + \lambda \tilde{u} < f \quad \text{in } \Omega.$$

Hence by (2.12),

$$\tilde{u} < u_\lambda \quad \text{for every } \lambda > 0,$$

so that

$$\tilde{u} \leq \lim_{\lambda \to 0} u_\lambda = u.$$

This shows also that the whole family $u_\lambda$ converges and the proof is complete. \square

**Remark 4.3.** The proof of the theorem shows that also for $\int_B m\,dx = 0$, the functions $u_\lambda = u_\lambda - 1/|\Omega| \int_B m\,dx$ converge weakly in $W^p_0(\Omega)$ to the unique solution of (4.17) without the restriction $p > N/2$.

**Remark 4.4.** The above theorem extends previous results of Bensoussan-Lions [4] and Robin [19].

**Remark 4.5.** The case of an obstacle $\psi \neq 0$ with $B\psi = 0$ in (II) can be dealt with by means of the translation $z_\lambda = u_\lambda - \psi$. Then $z_\lambda$ satisfies

$$\text{Max} [z_\lambda; A z_\lambda + \lambda z_\lambda - f + A \psi + \lambda \psi] = 0.$$
It is easy to check that if \( \int_{\partial \Omega} f m \, dx > 0 \), then
\[
a_\lambda = \int_{\partial \Omega} (f - A \psi - \lambda \psi) m \, dx = \int_{\partial \Omega} (f - \lambda \psi) m \, dx > 0
\]
for sufficiently small \( \lambda \). Hence by theorem 4.2, \( u_\lambda \) converges to the solution \( u \) of \( \max \{u - \psi; Au - f\} = 0 \) in \( \Omega \), \( Bu = 0 \) on \( \Gamma \).

Conversely, if \( \int_{\partial \Omega} f m \, dx < 0 \), then \( a_\lambda < 0 \) for small \( \lambda \) and the result is that \( w_\lambda = u_\lambda - C_\lambda \) converges to the unique solution \( w \) of
\[
w \in W^2_2(\Omega), \quad Aw = f - \frac{1}{|\Omega|} \int_{\partial \Omega} f m \, dx \quad \text{in} \ \Omega, \quad Bw = 0 \quad \text{on} \ \Gamma, \quad \int_{\partial \Omega} \psi m \, dx = 0.
\]

Finally, if \( \int_{\partial \Omega} f m \, dx = 0 = \int_{\partial \Omega} \psi m \, dx \) and \( p > N/2 \), then \( u_\lambda \) converge weakly to the maximum solution \( u \) of
\[
u < \psi, \quad Au = f \quad \text{in} \ \Omega, \quad Bu = 0 \quad \text{on} \ \Gamma.
\]

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