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with applications to measure theory


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A Linear Radon-Nikodym Type Theorem for $C^*$-Algebras
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0. - Introduction

In a previous paper [10] (see also [11]) the second author defined the notion of absolute continuity for (non-normal) bounded linear forms on $C^*$-algebras and proved a non-commutative Radon-Nykodym type theorem which generalized the quadratic version of S. Sakai [13]. Here in section 1 we give an extension of Sakai’s linear version [13] in the context of $C^*$-algebras. As in [10] the normality of the functionals in question need not be assumed and Sakai’s condition of strong domination is here replaced by absolute continuity. In contrast to our linear version, the quadratic version of [10] is valid only for positive functionals. In commutative $C^*$-algebras both linear and quadratic versions (essentially) coincide.

Section 2 is devoted to applications of our abstract results to measure theory. We show that the classical Lebesgue-Radon-Nikodym theorem as well as its generalization to finitely additive measures due to S. Bochner [1] and C. Fefferman [7] can be obtained as direct consequences of our results applied to a certain commutative $C^*$-algebra $B(\Omega, \Sigma)$.

1. - The linear Radon-Nikodym type theorem for $C^*$-algebras

Let $A$ be a $C^*$-algebra with positive part $A_+$ and unit ball $S$. Let $f$ be a positive bounded linear functional and $g$ an arbitrary bounded linear functional on $A$. $g$ is said to be absolutely continuous with respect to $f$, if one of the following equivalent conditions is fulfilled (see [10]):

(i) For every $\varepsilon > 0$ there exits $\delta > 0$ such that $|g(x)| < \varepsilon$ whenever $x \in A_+ \cap S$ and $f(x) < \delta$.

(ii) For every sequence $\{x_n\}$ in $A_+ \cap S$ with $\lim_{n \to \infty} f(x_n) = 0$, it follows that $\lim_{n \to \infty} g(x_n) = 0$.

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For \( y \in A \), the linear functional \( x \rightarrow f(yz + xy)/2 \) (\( z \in A \)) is denoted by \( f_y \). Since \( f \) is continuous, \( f_y \) is continuous and \( \|f_y\| \leq \|f\| \|y\| \). If \( y \) is self-adjoint, \( f_y \) is self-adjoint (i.e. \( f_y(x^*) = f_y(x^2) \); \( x \in A \)); but \( f_y \) need not be positive, if \( y \) is positive.

**LEMMA (1.1)** For \( y \in A \) and \( x \in A_+ \) we have the following inequality:

\[
|f_y(x)| \leq \|f\|^{1/2}\|y\|^{1/2}\|x\|^{1/2}f(x)^{1/2}.
\]

**PROOF.** From the Cauchy-Schwarz inequality for positive functionals it follows that

\[
|f_y(x)| = \frac{1}{2}|f(yz + xy)| \leq \frac{1}{2}(|f(yz)| + |f(xy)|)
= \frac{1}{2}(|f((yz^{1/2}z^{1/2})| + |f(x^{1/2}(x^{1/2}y))|)
\leq \frac{1}{2}[f(yz^{1/2}f(x)^{1/2} + f(x)^{1/2}f(y^*zy)^{1/2}]
\leq \|f\|^{1/2}\|y\|^{1/2}\|x\|^{1/2}f(x)^{1/2}.
\]

From Lemma (1.1) we immediately obtain the following:

**LEMMA (1.2)** For \( y \in A \), \( f_y \) is absolutely continuous with respect to \( f \).

Since the set of all bounded linear functionals on \( A \) which are absolutely continuous with respect to \( f \) is a closed linear subspace of the topological dual space \( A^* \), each element of the closure of the set \( \{f_y : y \in A\} \) is absolutely continuous with respect to \( f \). Now we will show that the converse is also valid.

**THEOREM (1.3)** Let \( f \) be a positive bounded linear functional and \( g \) an arbitrary bounded linear functional on the \( C^*\)-algebra \( A \).

(i) \( g \) is absolutely continuous with respect to \( f \), if and only if there exists a sequence \( \{y_n\} \) in \( A \) such that

\[
\lim_{n \to \infty} \|g - f_{y_n}\| = 0.
\]

(ii) \( g \) is self-adjoint and absolutely continuous with respect to \( f \), the \( y_n \) in (1) can be chosen self-adjoint.

(iii) \( g \) is positive and absolutely continuous with respect to \( f \), the \( y_n \) in (1) can be chosen positive.

(iv) \( 0 \leq g \leq f \), the \( y_n \) in (1) can be chosen such that \( y_n \in A_+ \cap S \).

Before proceeding to the proof we need to recall some pertinent facts. The second dual \( A^{**} \) of the \( C^*\)-algebra \( A \) is an (abstract) \( W^*\)-algebra in a natural manner (with the Arens multiplication). Moreover, \( A \) is a \( \sigma(A^{**}, A^*)\)-dense \( C^*\)-subalgebra of \( A^{**} \), when it is canonically embedded into \( A^{**} \), and the continuous linear functionals (positive linear functionals) on \( A \) coincide precisely with the
restrictions of the normal linear functionals (positive normal functionals) on $A^\ast$ to $A$. The image of $g, f \in A^\ast$ under the canonical embedding $A^\ast \to A^{***}$ will again be denoted by $g$ resp. $f$. (See [5], [8] and in particular [13]).

In [10], (Lemma (2.2)) it is shown that $g$ is absolutely continuous with respect to $f$ if and only if the image of $g$ under the canonical embedding $A^\ast \to A^{***}$ is absolutely continuous with respect to the canonical image of $f$. For this reason we need not distinguish between $g, f \in A^\ast$ and their canonical images in $A^{***}$. These facts are very important for the following proof of the theorem.

**PROOF of (iv).** Let $0 \leq g \leq f$. We consider the set

$$K := \{f_y : y \in A_+ \cap S\}.$$  

$K$ is a non-empty convex subset of the dual space $A^\ast$. Let $\overline{K}$ be its closure in the norm topology on $A^\ast$, and suppose that $g \not\in \overline{K}$.

From the Hahn-Banach theorem it follows that there exist $a \in A^{**}$ and $\gamma \in \mathbb{R}, \gamma < 1$, such that

$$\text{Re} \, g(a) = 1, \text{Re} \, f_y(a) \leq \gamma \text{ for all } y \in A_+ \cap S.$$  

Choose $b := (a + a^*)/2 \in A^{**}$. Then, since $g$ and $f_y(y \in A_+)$ are self-adjoint:

$$g(b) = \text{Re} \, g(a) = 1,$$

$$f_y(y) = f_y(b) = \text{Re} \, f_y(a) \leq \gamma \text{ for all } y \in A_+ \cap S.$$  

Since $f$ and the mappings $x \rightarrow bx$, and $x \rightarrow xb$ are $\sigma(A^{**}, A^\ast)$-continuous on $A^\ast$ (see [13]), $f_y$ is $\sigma(A^{**}, A^\ast)$-continuous on $A^{**}$. From Kaplansky’s density theorem it follows that $A_+ \cap S$ is $\sigma(A^{**}, A^\ast)$-dense in the positive part of the unit ball of $A^{**}$ (see [10] Lemma (2.1)). Therefore

$$f_y(y) \leq \gamma \text{ for all } y \in A^{**} \text{ with } y \geq 0 \text{ and } \|y\| \leq 1.$$  

The self-adjoint element $b$ has an orthogonal decomposition $b = b^+ - b^-$, where $b^+, b^- \in A^{**}; b^+, b^- \geq 0$ and $b^+b^- = 0 = b^-b^+$.

Let $q \in A^{**}$ be the support of $b^+$. Then

$$1 > \gamma \geq f_y(q) = f(bq + qb)/2 = f(b^+) \geq g(b^+) \geq g(b) = 1.$$  

This is the desired contradiction.

**PROOF of (iii).** Let $g \geq 0$ be absolutely continuous with respect to $f$ and consider the set

$$M := \{f_y : y \in A_+\}.$$  

$M$ is a non-empty convex cone in the dual space $A^\ast$. Let $\overline{M}$ be its closure in the norm topology and suppose that $g \not\in \overline{M}$. 

As in the above proof of (iv) there exists a self-adjoint element \( b \in A^{**} \) such that
\[
g(b) = 1, \\
f_b(y) = f_b(b) \leq 0 \text{ for all } y \in A_+.
\]
Since \( A_+ \) is \( \sigma(A^{**}, A^*) \)-dense in the positive part of the \( W^* \)-algebra \( A^{**} \) and since \( f_b \) is \( \sigma(A^{**}, A^*) \)-continuous, we obtain
\[
f_b(y) \leq 0 \text{ for all } y \geq 0, y \in A^{**}.
\]
Let \( b^+, b^- \in A^{**} \) be the positive and the negative part of \( b \) and let \( q \in A^{**} \) be the support of \( b^+ \). Then
\[
0 \geq f_b(q) = f(bq + qb)/2 = f(b^+) \geq 0, \text{ thus } f(b^+) = 0.
\]
From the absolute continuity we conclude that \( g(b^+) = 0 \).
Finally we obtain the following contradiction:
\[
1 = g(b) = g(b^+) - g(b^-) = -g(b^-) \leq 0.
\]

\textbf{PROOF} of (ii). Let \( g \) be self-adjoint and absolutely continuous with respect to \( f \) and consider the set
\[
L := \{ f_y : y \in A_h \},
\]
where \( A_h \) denotes the self-adjoint (= hermitian) part of \( A \).

\( L \) is a real-linear subspace of \( A^* \). Let \( \overline{L} \) be its norm closure and suppose that \( g \not\in \overline{L} \). Again, as above, there exists a self-adjoint element \( b \in A^{**} \) such that
\[
g(b) = 1; f_y(b) = 0 \text{ for all } y \in A_h.
\]
Since \( A_h \) is \( \sigma(A^{**}, A^*) \)-dense in the self adjoint part of \( A^{**} \) and since \( f_b \) is \( \sigma(A^{**}, A^*) \)-continuous, we have
\[
f_b(y) = 0 \text{ for all self-adjoint } y \in A^{**}.
\]
Let \( b = b^+ - b^- \) be the orthogonal decomposition of \( b \) in \( A^{**} \), and let \( q, p \) be the supports of \( b^+, b^- \) in \( A^{**} \). Then
\[
0 = f_b(q) = f(bq + qb)/2 = f(b^+);
0 = f_b(p) = f(bp + pb)/2 = -f(b^-).
\]
From the absolute continuity it follows that
\[
g(b^+) = g(b^-) = 0.
\]
Thus \( g(b) = g(b^+) - g(b^-) = 0 \). This contradicts the fact that \( g(b) = 1 \).

**Proof** of (i). The fact that condition (1) implies the absolute continuity of \( g \) with respect to \( f \) follows from Lemma (1.2). The converse is obtained by applying part (ii) to the real and imaginary parts of \( g \).

In the sequel let \( A \) be a \( W^* \)-algebra with predual \( A_* \). The linear version of S. Sakai’s Radon-Nikodym theorem is an immediate consequence of our Theorem (1.3).

**Corollary (1.4)** (S. Sakai) Let \( g, f \) be positive linear functionals on the \( W^* \)-algebra \( A \), where \( f \) is normal and \( g \leq f \). Then there exists \( y_0 \in A \), \( 0 \leq y_0 \leq 1 \), such that

\[
g(x) = \frac{1}{2} f(y_0 x + xy_0) \quad (x \in A).
\]

**Proof.** From Theorem (1.3) (iv) it follows that there is a sequence \( \{y_n\} \) in \( A_* \cap S \), such that

\[
\lim_{n \to \infty} \|g - f_{y_n}\| = 0.
\]

Since \( A_* \cap S \) is \( \sigma(A, A_*) \)-compact and since the mapping \( y \to f_y \) from \( A \) with the \( \sigma(A, A_*) \)-topology to \( A^* \) with the \( \sigma(A^*, A) \)-topology is continuous, the set

\[
K := \{f_y : y \in A_* \cap S\}
\]

is a \( \sigma(A^*, A) \)-compact subset of \( A^* \). Therefore \( K \) is closed in the \( \sigma(A^*, A) \)-topology and hence in the norm topology on \( A^* \).

From formula (1) of Theorem (1.3) we conclude that \( g \in K \); i.e., there is \( y_0 \in A_* \cap S \) such that

\[
g(x) = f_{y_0}(x) = \frac{1}{2} f(y_0 x + xy_0) \quad (x \in A).
\]

**Remark.** In Corollary (1.4) the element \( y_0 \) can be chosen such that \( 0 \leq y_0 \leq s(f) \), where \( s(f) \) is the support of the positive normal functional \( f \). (If need be one can replace the \( y_0 \) of Corollary (1.4) by \( s(f)y_0s(f) \).) With this additional restraint \( y_0 \) is uniquely determined as we shall prove below. In particular if \( f \) is faithful (i.e., \( s(f) = 1 \)), then the \( y_0 \) of Corollary (1.4) is uniquely determined.

To show the uniqueness of \( y_0 \), let \( y_0, y_1 \in A \) be such that \( f_{y_0} = f_{y_1} = g \) and \( 0 \leq y_0 \leq s(f), 0 \leq y_1 \leq s(f) \). Then

\[
0 = f_{y_0} - f_{y_1} = f_{y_0 - y_1};
\]

\[
0 = f_{y_0 - y_1}(y_0 - y_1) = f((y_0 - y_1)^2).
\]

Let \( q \) be the support of \((y_0 - y_1)^2\); then \( f(q) = 0 \) (see [14] 5.15), and therefore

\[
q \leq 1 - s(f).
\]
On the other hand, since $0 \leq y_0, y_1 \leq s(f)$, it follows for $i = 0, 1$ that:

$$0 \leq (1 - s(f))y_i(1 - s(f)) \leq (1 - s(f))s(f)(1 - s(f)) = 0.$$ 

$$\Rightarrow 0 = (1 - s(f))y_i(1 - s(f))$$

$$= [y_i^{1/2}(1 - s(f))]^*[y_i^{1/2}(1 - s(f))].$$

$$\Rightarrow 0 = y_i^{1/2}(1 - s(f)).$$

$$\Rightarrow 0 = y_i(1 - s(f)).$$

Then

$$0 = (y_0 - y_1)^2(1 - s(f))$$

$$\Rightarrow q \leq s(f).$$

Thus

$$q = 0, \text{ and hence } (y_0 - y_1)^2 = 0; \text{ i.e., } y_0 = y_1.$$ 

In Corollary (1.4) we have not required that $g$ be normal. This follows automatically from $0 \leq g \leq f$, when $f$ is normal. It is, in fact, the case that a positive linear functional $g$ is normal if it is absolutely continuous with respect to a positive normal functional $f$.

The above Theorem (1.3) should be compared with Theorem (2.6) from [10]; in the commutative case they coincide for the most part. But the linear version (1.3) has two advantages: it provides an equivalent characterization of absolute continuity, and the “smaller” functional $g$ need not be positive. It is for this reason that we prefer the linear version for the measure theoretical applications in the next section. However, the quadratic version (2.6) from [10] seems to be more suitable for applications to operator algebras (see section 3 of [10] for a variety of such applications including new proofs of two classical results in the theory of von Neumann algebras due to J. von Neumann and R. Pallu de la Barrière).

2. - Applications to additive set functions

Let $\Omega$ be an arbitrary set and let $B(\Omega)$ be the algebra (pointwise operations) of all bounded complex-valued functions on $\Omega$. $B(\Omega)$ is a commutative $C^*$-algebra for the sup norm $\|\|_\infty$.

Now let $\Sigma$ be a field of subsets of $\Omega$. The linear combinations of characteristic functions of sets in $\Omega$ are called primitive functions. The set of all primitive functions is a subalgebra of $B(\Omega)$; it is denoted by $P(\Omega, \Sigma)$. The closure of $P(\Omega, \Sigma)$ in $B(\Omega)$ is a $C^*$-subalgebra of $B(\Omega)$ and will be denoted by $B(\Omega, \Sigma)$. If $\Sigma$ is a $\sigma$-field, $B(\Omega, \Sigma)$ consists of all bounded measurable complex-valued functions on $(\Omega, \Sigma)$. $B(\Omega) = B(\Omega, \Sigma_0)$, where $\Sigma_0$ is the family of all subsets of $\Omega$. 

The dual space of $B(\Omega, \Sigma)$ is isometrically isomorphic to the Banach space $ba(\Omega, \Sigma)$ which consists of all bounded (finitely) additive complex set functions on $\Sigma$; the norm $\| \cdot \|_v$ on $ba(\Omega, \Sigma)$ is given by the total variation. The isomorphism is defined as follows: every $f \in B(\Omega, \Sigma)^*$ is mapped onto $\mu_f \in ba(\Omega, \Sigma)$ such that the following equation is fulfilled:

$$f(x) = \int x \, d\mu_f \quad (x \in B(\Omega, \Sigma)).$$

This isomorphism preserves order; and $f$ is self-adjoint if and only if $\mu_f$ is real-valued.

On the linear space $ba(\Omega, \Sigma)$ a second norm $\| \cdot \|_\infty$ can be introduced:

$$\|\mu\|_\infty := \sup_{E \in \Sigma} |\mu(E)|.$$

These norms are equivalent: $\| \|_\infty \leq \| \|_v \leq 4\| \|_\infty$.

The notion of absolute continuity for measures (= countably additive set functions) is extended to (finitely) additive set functions in the following way (see [1], [2], [6], [7]):

**DEFINITION (2.1)** Let $\nu, \mu \in ba(\Omega, \Sigma), \mu \geq 0$. Then $\nu$ is said to be absolutely continuous with respect to $\mu$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|\nu(E)| < \delta$ for $E \in \Sigma$ implies that $\mu(E) < \delta$.

**REMARKS 2.2** Let $\nu, \mu \in ba(\Omega, \Sigma), 0$. (i) $\nu$ is absolutely continuous with respect to $\nu$, iff for every sequence $\{E_n\}$ in $\Sigma$, $\lim \mu(E_n) = 0$ implies $\lim \nu(E_n) = 0$. (ii) $\nu$ is absolutely continuous with respect to $\mu$, iff the variation, $|\nu|$, is absolutely continuous with respect to $\mu$. (iii) Let $\Sigma$ be a $\sigma$-field and let $\nu, \mu$ be countably additive; then $\nu$ is absolutely continuous with respect to $\mu$, iff $\mu(E) = 0$ for $E \in \Sigma$ implies that $\nu(E) = 0$. (For the proofs see [6] chap.III.)

The following proposition illustrates the relationship between absolutely continuous functionals on a $C^*$-algebra and absolutely continuous set functions.

**PROPOSITION (2.3)** Let $g, f$ be bounded linear functionals on the $C^*$-algebra $B(\Omega, \Sigma)$ and suppose that $f \geq 0$. Then $g$ is absolutely continuous with respect to $f$, iff $\mu_g$ is absolutely continuous with respect to $\mu_f$.

**PROOF:** The necessity of the condition is obvious, since the characteristic functions are positive elements of $B(\Omega, \Sigma)$ of norm 1. To prove the sufficiency let $\mu_g$ be absolutely continuous with respect to $\mu_f$; then the variation $|\mu_g|$ is absolutely continuous with respect to $\mu_f$ as well.

Since $P(\Omega, \Sigma)$ is dense in $B(\Omega, \Sigma)$, it is sufficient to consider only primitive functions. Let $\{x_n\}$ be a sequence in $P(\Omega, \Sigma)$ with $0 \leq x_n \leq 1$ and $\lim f(x_n) = 0$. We will show: $\lim g(x_n) = 0$. Let $\varepsilon > 0$. Since $x_n$ is primitive, the sets $E_n := \{t \in \Omega : x_n(t) \geq \varepsilon\}$ are elements of $\Sigma$. From $f \geq 0$ it follows
that for every \( n \in \mathbb{N} \):

\[
f(x_n) \geq f(\epsilon \chi_{E_n}) = \epsilon \mu_f(E_n) \geq 0,
\]

where \( \chi_{E_n} \) denotes the characteristic function of \( E_n \).

Therefore

\[
\lim_{n \to \infty} \mu_f(E_n) = 0.
\]

Since \( |\mu_f| \) is absolutely continuous with respect to \( \mu_f \), it follows that

\[
\lim_{n \to \infty} |\mu_f|(E_n) = 0.
\]

Thus there is an \( n_0 \in \mathbb{N} \) such that \( |\mu_f|(E_n) < \epsilon \) for all \( n \geq n_0 \), and since \( 0 \leq x_n \leq 1 \), we get for all \( n \geq n_0 \):

\[
|g(x_n)| = |\int \chi_{E_n} \, d\mu_f| \leq \int \chi_{E_n} \, d|\mu_f| = \int \chi_{E_n} \, d|\mu_f| + \int \chi_{E_n^c} \, d|\mu_f|
\]

\[
\leq |\mu_f|(E_n) + \epsilon |\mu_f|(E_n^c) < \epsilon + \epsilon \|\mu_f\|_v = \epsilon (1 + \|\mu_f\|_v),
\]

where \( E_n^c \) denotes the complement. Hence \( \lim g(x_n) = 0 \).

Next we apply our Theorem (1.3) to the \( C^* \)-algebra \( B(\Omega, \Sigma) \) and obtain a generalization of the classical Lebesgue-Radon-Nikodym theorem for (finitely) additive set functions due to S. Bochner [1].

**THEOREM (2.4)** Let \( \Omega \) be a set, and let \( \Sigma \) be a field of subsets of \( \Omega \). Let \( \nu, \mu \in \text{ba}(\Omega, \Sigma) \) be such that \( \mu \) is positive and \( \nu \) is absolutely continuous with respect to \( \mu \).

(i) Then there is a sequence \( \{y_n\} \) of primitive functions on \( \Omega \) such that:

1. \( \nu(E) = \lim_{n \to \infty} \int_E y_n \, d\mu \) uniformly for \( E \in \Sigma \)
2. \( \lim_{n,m \to \infty} \int_{\Omega} |y_n - y_m| \, d\mu = 0 \).

(ii) If \( \nu \) is real-valued (positive), the \( y_n \) in (i) can be chosen as real-valued (non-negative) primitive functions.

**PROOF.** We consider the commutative \( C^* \)-algebra \( B(\Omega, \Sigma) \) and the linear functionals \( g := \int d\nu, f := \int d\mu \). By Proposition (2.3) \( g \) is absolutely continuous with respect to \( f \), and we can therefore apply our Theorem (1.3). Thus there exists a sequence \( \{y'_n\} \) in \( B(\Omega, \Sigma) \) such that

\[
\lim_{n \to \infty} \|g - f y'_n\| = 0.
\]
Since $P(\Omega, \Sigma)$ is dense in $B(\Omega, \Sigma)$, we can find $y_n \in P(\Omega, \Sigma)$ such that
\[ ||y_n - y'_n||_\infty < \frac{1}{n} \quad (n \in \mathbb{N}). \]
Then
\[
\begin{align*}
||g - f_{y_n}|| &\leq ||g - f_{y'_n}|| + ||f_{y'_n} - f_{y_n}|| \\
&\leq ||g - f_{y'_n}|| + ||f|| \cdot ||y'_n - y_n||_\infty \\
&\leq ||g - f_{y'_n}|| + \frac{||f||}{n}
\end{align*}
\]
and hence we conclude that
\[
(3) \quad \lim_{n \to \infty} ||g - f_{y_n}|| = 0.
\]
For $E \in \Sigma$ we have
\[
(1) \quad \nu(E) = g(\chi_E) = \lim_{n \to \infty} f_{y_n}(\chi_E) = \lim_{n \to \infty} \int_E y_n \, d\mu.
\]
Moreover from (3) we have
\[
(2) \quad 0 = \lim_{n,m \to \infty} ||f_{y_n} - f_{y_m}|| = \lim_{n,m \to \infty} ||(y_n - y_m)\mu||_v
\]
\[= \lim_{n,m \to \infty} \int_\Omega |y_n - y_m| \, d\mu.
\]
(2) implies uniform convergence for $E \in \Sigma$ in (1).

(ii) follows in the same way from Theorem (1.3) parts (ii) and (iii).

Finally let $\Sigma$ be a $\sigma$-field and let $\nu, \mu \in \text{ba}(\Omega, \Sigma)$ be countably additive, where $\mu$ is positive and $\nu$ is absolutely continuous with respect to $\mu$. Then the space $L^1(\mu)$ is complete and from (2) it follows that there is an $h \in L^1(\mu)$ such that:
\[0 = \lim_{n \to \infty} \int_\Omega |h - y_n| \, d\mu.
\]
Hence $\nu(E) = \lim_{n \to \infty} \int_E y_n \, d\mu = \int_E h \, d\mu$ for all $E \in \Sigma$, which is the classical Lebesgue-Radon-Nikodym theorem for finite measures.

REMARKS. (a) Different proofs of Theorem (2.4) may be found in [1], [6], or [7]. C. Fefferman generalizes this theorem in [7] for an arbitrary (not necessarily positive) $\mu \in \text{ba}(\Omega, \Sigma)$.

(b) In this section we have applied our results to the $C^*$-algebra $B(\Omega, \Sigma)$. Similarly we could consider the commutative $C^*$-algebra $C_0(T)$ consisting of all continuous complex-valued functions on some locally compact Hausdorff space $T$ which vanish at infinity; but since the continuous functionals on $C_0(T)$ correspond precisely to the regular complex Borel measures on $T$, we would obtain the Lebesgue-Radon-Nikodym theorem only for regular measures, whereas the example $B(\Omega, \Sigma)$ leads us to much more general results.
REFERENCES