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About a theorem of Paolo Codecà’s and omega estimates for arithmetical convolutions. Second part

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1. - Introduction

Consider the real valued functions $h$ defined on $[1, \infty)$

\begin{equation}
    h(x) = \sum_{n \leq x} \alpha(n)n^af\left(\frac{x}{n}\right),
\end{equation}

where $\alpha(n)$ is a sequence of real numbers satisfying

\begin{equation}
    \sum_{n \leq x} |\alpha(n)| = O(x),
\end{equation}

$-1 \leq \alpha < 0$, $f$ is a periodic function of period 1, of bounded variation on $[0, 1]$ and such that

\begin{equation}
    \int_{0}^{1} f(u)du = 0,
\end{equation}

and $z = z(x) \leq x$ is a positive, strictly increasing, continuous and unbounded function ($z$ will always be assumed to satisfy these properties in the sequel).

We say that $h$ is $C_\alpha(\alpha, \alpha, f)$.

In the first part of this work [8], inspired by an article of Codecà's [1], I considered functions $g$ nearly $C_\alpha(-1, \alpha, f)$ (i.e. short of a $o(1)$), where $\alpha$ possesses an asymptotic mean $K$, and such that

\[ g - K \int_{1}^{\infty} u^{-1}f(u)du \]
is nearly $C_x(-1, \alpha, f)$ for some $z = z(x) = o(x)$, as $x \to \infty$.

In the case where $z$ may be taken small enough, I obtained a general expression for the mean of $g$ on an arithmetical progression $An + B$, $n \leq x$, (Theorem 1), from which I could then, for particular functions $g$, prove omega estimates by suitably choosing the parameters $A, B,$ and $x$.

Among the functions for which I found this process to be successful are the classical error terms $H$ and $E$ related to the Euler function $\phi$ and to the sum-of-divisors function $\sigma$, an error term of Landau (see e.g. [14]) related to the function $n/\phi(n)$, and the “Chowla-Walum functions”

\[
G_{a,k}(x) := \sum_{n \leq x} n^a \psi_k \left( \frac{x}{n} \right)
\]

(where $\psi_k(y) = B_k([y])$) denotes the $k$-th Bernoulli polynomial of argument the fractional part $\{y\}$ of $y$ when $a \leq -1$. This brings us to the main purpose of this sequel.

The $G_{a,k}$ are related to various divisor problems (see e.g. [5], [6], [9], [10], [11]). Conjectures were proposed as to the “best” $O$ and $\Omega$ estimates satisfied by the functions $G_{a,k}$, originating with the Piltz-Hardy-Landau conjecture on the famous Dirichlet divisor problem, generalized in 1963 by Chowla and Walum [2]. If we gather the conjectures that appear reasonable so far in view of the various investigations made by a number of authors, we can state them in the compact form described below.

Let $a_k(a)$ and $b_k^*(a)$, where $*$ is allowed to denote $+$, $-$, $\pm$, or nothing at all (i.e. not even a blank), be the smallest $a$, respectively the largest $b$, for which $G_{a,k}(x) = O(x^{a+\varepsilon})$, resp. $G_{a,k}(x) = \Omega(x^{b-\varepsilon})$, for every $\varepsilon > 0$. Set

\[
g(a) = \begin{cases} 
\frac{9}{2} + \frac{1}{4} & \text{if } a \geq -\frac{1}{2}, \\
0 & \text{if } a \leq -\frac{1}{2}.
\end{cases}
\]

**CONJECTURE.** For every real number $a$, every positive integer $k$, and for $*$ denoting $+$ and $-$, we have

\[
(O.k.a.) \quad a_k(a) \leq g(a)
\]

and

\[
(\Omega.k.a) \quad b_k^*(a) \geq g(a).
\]

**REMARKS.**

(i) The Piltz-Hardy-Landau conjecture is in this notation $(O.1.0)$; the Chowla-Walum conjecture is: $(O.k.a)$ for all $a \geq 0$ and all positive integers $k$.

(ii) The assertions “$(O.k.a)$ and $(\Omega.k.a)$” and “$a_k(a) = g(a)$” are equivalent.
For a brief review of the results known to date towards these conjectures see [11].

In the first part of this paper [8] (see the Addendum), the truth of \((\Omega_{\pm,k,a})\), \(k = 1, 2, \ldots\), is proved for \(a \leq -1\). Here, through an extension of the main result of [8] to \(G_{a}(x, \alpha, f)\), \(-1 < a < 0\), for suitable \(x\) and \(\alpha\) (Theorem 1 in Section 2 below), we obtain \(\Omega\)-estimates for the \(G_{a,k}(x)\) (Theorem 2 just below), and as a corollary the truth of \(\Omega_{\pm,k,a}\), \(k = 1, 2, \ldots\), for \(a \leq -\frac{1}{2}\).

**THEOREM 2.** For \(-1 < a < 0\) and every positive integer \(k\), we have

\[
G_{a,k}(x) = \Omega_{\pm} \left( \exp \left\{ (1 + o(1)) \xi_{a,k} \frac{(-a/2)^{1+a}}{1 + a} \frac{(\log x)^{1+a}}{\log \log x} \right\} \right),
\]

where

\[
\xi_{a,k} = \begin{cases} 
1 & \text{if } k = 1 \text{ or } k \text{ is even}, \\
2^a & \text{if } k > 1 \text{ is odd}.
\end{cases}
\]

As another corollary of Theorem 2, we obtain in Section 3:

**THEOREM 3.** Let \(E_{a}(x)\) be the error term

\[
E_{a}(x) := \sum_{n \leq x} \sigma_{a}(n) \left( \frac{\zeta(1 + a)}{1 + a} x^{1+a} + \zeta(1 + a) x - \zeta(-a) \frac{x^a}{2} \right),
\]

for \(a \neq -1, 0\).

Then, for \(\frac{25}{38} < |a| < 1\),

\[
E_{a}(x) = \Omega_{\pm} \left( x^{\frac{\min}{2}} \exp \left\{ (1 + o(1)) \frac{|a|}{1 - |a|} \frac{(\log x)^{1-|a|}}{\log \log x} \right\} \right).
\]

It appears that no nontrivial \(\Omega\)-estimate for \(E_{a}(x)\), with \(-1 < a < -\frac{1}{2}\), was known so far; and, when \(a\) is positive, (1.7) improves in the indicated range both results

\[
E_{a}(x) = \Omega_{\pm} \left( (x \log x)^{\frac{\alpha}{2}} \right)
\]

and

\[
E_{a}(x) = \Omega(x^{a})
\]

of Hafner’s [3], and should be compared, on the one hand with MacLeod’s [7]

\[
\lim_{x \to \infty} E_{a}(x) x^{a} = \pm \frac{\zeta(a)}{2}, \quad a > 1,
\]
and our \[12\]

\[(1.12) \quad \lim_{x \to \infty} \frac{E_1(x)}{x \log \log x} \geq \pm \frac{e^7}{2},\]

and, on the other hand, with the following consequence of conjecture (D.l.a)

\[(1.1) \quad E_a(x) = O(x^{a+\varepsilon}), \quad a \geq \frac{1}{2},\]

(see (3.10)).

Finally in Section 4 we give another application of Theorem 1. We define the functions

\[(1.13) \quad P_a(x) = \sum_{n \leq \sqrt{x}} n^a \cos \left( \frac{x}{n} \right)\]

and

\[(1.14) \quad Q_a(x) = \sum_{n \leq \sqrt{x}} n^a \sin \left( \frac{x}{n} \right).\]

In [8] we prove (see [4])

\[(1.15) \quad P_{-1}(x) = \Omega_\pm (\log \log x)\]

and

\[(1.16) \quad Q_{-1}(x) = \Omega_\pm \left( (\log \log x)^\frac{3}{2} \right).\]

Here we obtain

**THEOREM 4.** For \(-1 < a < 0\), we have

\[(1.17) \quad P_a(x) = \Omega_\pm \left( \exp \left\{ (1 + o(1)) \left( \frac{(-a/2)^{1+a}}{1 + a} \frac{(\log x)^{1+a}}{\log \log x} \right) \right\} \right)\]

and

\[(1.18) \quad Q_a(x) = \Omega_\pm \left( \exp \left\{ (1 + o(1))2^a \left( \frac{(-a/2)^{1+a}}{1 + a} \frac{(\log x)^{1+a}}{\log \log x} \right) \right\} \right).\]

2. - The main result

Let the notation be that of Section 1 and consider a function \(h\) being some \(C_\varepsilon(a, \alpha, f)\), where \(-1 < a < 0\) and, in addition to (1.2), the arithmetical function \(\alpha\) satisfies the submultiplicative property

\[(2.1) \quad |\alpha(nm)| = O(|\alpha(n)\alpha(m)|).\]
for all positive integers $n$ and $m$. Let $A = A(x) > 0$ be an integer valued function, and $B = B(x) \geq 0$ (we do not require that $B$ be an integer: see [8, Addendum]). Then we have

**Theorem 1.** Set

$$u = u(x) := x(Ax + B),$$

$$v_i := x \left( \frac{Ax}{2^i} + B \right), \quad i \geq 0,$$

$$\alpha_0(A) := \sum_{d|A} |\alpha(d)|d^\beta,$$

and suppose that there exists a function $\eta = \eta(x)$ decreasing to 0 as $x \to \infty$ and such that

$$(2.2) \quad A \leq v_N,$$

where $N := \left\lceil -\frac{\log \eta}{\log 2} \right\rceil$. Then

$$\frac{1}{x} \sum_{n \leq x} h(An + B)$$

$$(2.3) \quad = \sum_{k \leq \eta} \alpha(k)k^\alpha \left( \frac{1}{k^*} \sum_{n \leq k^*} f \left( \frac{n}{k^*} + \frac{B(k)}{k} \right) \right)$$

$$+ o \left( \frac{u^{\alpha+2}}{x} + v_\eta^\alpha \alpha_0(A) + \eta \alpha_\eta(A) \right),$$

where $k^*$ denotes $\frac{k}{\lambda(A,k)}$.

**Proof.** The proof goes along the same line as that of Theorem 1 in [8]. We let $w(k)$ be the inverse function of $v(y) := x(Ay + B)$, $1 \leq y \leq x$ if $k \geq u(1)$, and $w(k) = 1$ otherwise, and we obtain as in [8]

$$(2.4) \quad \frac{1}{x} \sum_{n \leq x} h(An + B) = \beta + \delta + \varepsilon,$$

where

$$(2.5) \quad \beta := \sum_{k \leq \eta} k^\alpha \alpha(k) \left( \frac{1}{k^*} \sum_{n \leq k^*} f \left( \frac{n}{k^*} + \frac{B(k)}{k} \right) \right),$$
where by (1.2),

\[(2.6) \quad \varepsilon = O \left( \frac{1}{x} \sum_{k \leq u} k^a k^* |\alpha(k)| \right) = O \left( \frac{u^{a+2}}{x} \right),\]

and where

\[(2.7) \quad \delta = O \left( \frac{1}{x} \sum_{k \leq u} k^{a-1}(A, k) |\alpha(k)| w(k) \right).\]

In order to estimate \(\delta\) we define, as in [8],

\[R_i := \begin{cases} 
\{ k \in \mathbb{N} | \max(v_1, v_i) < k \leq v_{i-1} \} & \text{if } i = 1, 2, \ldots, M := \left\lceil \frac{\log x}{\log 2} \right\rceil + 1, \\
\{ k \in \mathbb{N} | k \leq v := v(1) \} & \text{if } i = M + 1,
\end{cases}\]

and may thus rewrite the sum in (2.7) as

\[(2.8) \quad \sum_{i=1}^{M+1} \sum_{k \in R_i} (A, k) k^{a-1}|\alpha(k)| w(k) \leq \sum_1 + \sum_2 + \sum_3,
\]
say, where \(\sum_3 := \sum_{k \leq v} k^a |\alpha(k)|\), and

\[(2.9) \quad \sum_{1,2} := \sum_i \frac{x}{2i-1} \sum_{k \in R_i} (A, k) k^{a-1} |\alpha(k)|,
\]
the ranges of summation in (2.9) being respectively

\[1 \leq i \leq N \text{ and } N + 1 \leq i \leq M\]

(\(\sum_3\) corresponding to \(i = M + 1\)). By (1.2)

\[(2.10) \quad \sum_3 = O(u^{a+1}) = O(u^{a+2}).\]

By (2.1) the inside sum on the right side of (2.9) is a \(O\) of

\[(2.11) \quad \sum_{d|A} d^a |\alpha(d)| \sum_{d^* \geq \max(1, \frac{x}{2})} d^{a-1} |\alpha(d^*)|,
\]
which in turn is, by (1.2), a \(O\) of

\[(2.12) \quad \sum_{d|A} d^a |\alpha(d)| = \alpha_a(A),\]
and, in the case of $\sum_1$, is by (2.2) a $O$ of

$$\sum_{d|A} |\alpha(d)|v^a_d \leq v^a_N \alpha_0(A).$$

Thus

$$\sum_2 = O(x\eta \alpha_n(A))$$

and

$$\sum_1 = O(xv^a_N \alpha_0(A)).$$

The theorem now follows from (2.6), (2.7), (2.8), (2.10), (2.14) and (2.15).

\[\square\]

3. - Proofs of Theorems 2 and 3

In this section we let $h$ be a $G_{a,\ell}$ as in (1.4). Thus $z = \sqrt{x}$, $f = \psi_\ell$ and $\alpha(n) = 1$ for all $n$. A well known identify for Bernoulli polynomials $[13, (6.1)]$ implies that

$$\frac{1}{k^*} \sum_{n \leq k^*} \psi_\ell \left( \frac{n}{k^*} + \frac{B}{k} \right) = \frac{1}{k^*} \psi_\ell \left( \frac{B}{A, k} \right),$$

whence, if $B = O(A)$ and $A = o(x)$, an application of Theorem 1 yields

$$\frac{1}{x} \sum_{n \leq x} G_{a,\ell}(An + B)$$

$$= \sum_{k \leq u} (A, k)^{a-\ell} \psi_\ell \left( \frac{B}{(A, k)} \right) + O[A(Ax)^{a-\ell} + A^a \sigma_0(A)] + o(\sigma_0(A))$$

and, with a special choice of the parameters $A$, $B$ and $x$, the

**Lemma 1.** If $-1 < a < 0$ and

$$A = \prod_{p \leq y} p x^{-\frac{a}{\ell}},$$

where $P_\ell = \{p \equiv 1(2), p \text{ prime}\}$ if $\ell$ is either 1 or even, and $P_\ell = \{p \equiv 1(3), p \text{ prime}\}$ if $\ell > 1$ is odd, then there are non negative numbers $B_i < A$, $i =$
1, 2, such that as $y \to \infty$,

$$
\frac{1}{x} \sum_{n \leq x} G_{a, \ell}(An + B) = \begin{cases} 
\Omega_+(\sigma_a(A)), & \text{if } B = B_1 \\
\Omega_-(\sigma_a(A)), & \text{if } B = B_2
\end{cases}
$$

PROOF. We shall make use of the following elementary properties of the Bernoulli polynomials (see [13], Chapter I).

(i) With the choice of $A$ and with $B = 0$ the right side of (3.2) becomes, for a certain set $D$ of integers containing 1,

$$
\psi_\ell(0) \sum_{d \mid A} d^\alpha \sum_{d' \leq \frac{1}{2}} d'^{a-\ell} + o(\sigma_a(A)),
$$

and we thus obtain the $\Omega_-$-estimate for $\ell = 1$, and one of the $\Omega$-estimates for $\ell$ even.

(ii) Let $B = A - 1$ and $\ell = 1$. The right side of (3.2) becomes

$$
\frac{1}{2} \sum_{k \leq \sqrt{\tau}} (A, k) k^{a-1} - \zeta(1 - a) + O(u^a) + o(\sigma_a(A)),
$$

since

$$
\psi_1 \left( \frac{A - 1}{(A, k)} \right) = \frac{1}{2} - \frac{1}{(A, k)}.
$$

Whence the $\Omega_-$-estimate in the case where $\ell = 1$.

(iii) With $B = A/2$ we obtain, by virtue of (3.4), the other $\Omega$-estimate for $\ell$ even.

(iv) Finally, when $\ell > 1$ is odd, each one of the choices $B = A/3, B = 2A/3$ yields one of the $\Omega$-estimates (again we use (3.4)).

Now we need an estimate for $\sigma_a(A)$.

**Lemmas.** For $-1 < a < 0$ and

$$
A = \prod_{p \leq \sqrt{\tau} \quad (p \mid p \not\equiv k(n))} p,
$$

where $(k, n) = 1$, we have

$$
\sigma_a(A) = \exp \left( (1 + o(1)) \left\{ \frac{\phi(n)^a}{a + 1} \left( \log A \right)^{a+1} \right\} \right).
$$
PROOF. For this choice of \( A \) we have

\[
\log \sigma_a(A) = \sum_{p \leq y, p \equiv k(n)} \log(1 + p^a)
\]

(3.8) \[
= \sum_{p \leq y, p \equiv k(n)} p^a + \begin{cases} 
O(1), & -1 < a < -\frac{1}{2}, \\
O(\log \log y), & a = -\frac{1}{2}, \\
O(y^{a+1}), & -\frac{1}{2} < a < 0.
\end{cases}
\]

Now the Euler summation formula and the prime number theorem for arithmetical progressions yield

(3.9) \[
\sum_{p \leq y} p^a = (1 + o(1)) \frac{\phi(n)^a}{a + 1} \frac{y^{a+1}}{\log y},
\]

and the lemma follows, after another application of the prime number theorem.

\[\square\]

Theorem 2 is now a direct consequence of Lemmata 1 and 2. \[\square\]

As for theorem 3, it easily follows from Theorem 2, from [9, (1.3)]

(3.10) \[
E_a(x) = -x^a G_{-a,1}(x) - G_{a,1}(x) + O(x^{\frac{3}{4}}), \quad |a| < 2,
\]

and from [10, (5.4)]

(3.11) \[
\alpha_1(b) \leq \frac{b}{2} + \frac{25}{76}, \quad b > 0.
\]

\[\square\]

4. - Proof of Theorem 4

We shall need

LEMMA 3. With the notation of Theorem 1, we have

(4.1) \[
\sum_{n \leq k^*} \sin \cos \left( 2\pi \left( \frac{n}{k^*} + \frac{B}{k} \right) \right) = \begin{cases} 
\sin \cos \left( \frac{2\pi B}{k} \right), & \text{if } k^* = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

The proof of which is quite straightforward.

To prove (1.17) we set

(4.2) \[
A = \prod_{2 < p \leq y} p = x^{-\frac{1}{2}}, \quad B = 0, \quad z = \sqrt{x},
\]
and we obtain, with Theorem 1 and Lemma 3,
\begin{equation}
\frac{1}{x} \sum_{n \leq x} P_{a}(2\pi(A_n + B)) = \sum_{k|A} k^{a} + o(\sigma_{a}(A)) = \sigma_{a}(A)(1 + o(1))
\end{equation}
which, with Lemma 2, implies the $\Omega_{+}$-estimate; as for the $\Omega_{-}$-estimate, it is obtained similarly with $B = A/2$ instead of $B = 0$ in (4.2).

To prove the $\Omega_{-}$-estimate in (1.18), we set
\begin{equation}
A = 4B = x^{- \frac{A}{4}}, \quad B = \prod_{\substack{p \leq x \atop p \equiv 1(4)}} p, \quad z = \sqrt{x}.
\end{equation}
Theorem 1 and Lemma 3 yield this time
\begin{equation}
\frac{1}{x} \sum_{n \leq x} Q_{a}(2\pi B(4n + 1)) = 4^{a} \sum_{k|B} k^{a} + o(\sigma_{a}(A)) = 4^{a} \sigma_{a}(B)(1 + o(1)),
\end{equation}
and we conclude again with Lemma 2. The $\Omega_{-}$-estimate is similarly obtained with
\begin{equation}
A = \frac{4B}{3} = x^{- \frac{A}{6}} = 4D, \quad D = \prod_{\substack{p \leq x \atop p \equiv 1(4)}} p, \quad z = \sqrt{x}.
\end{equation}

REFERENCES


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