ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

WILHELM KLINGENBERG

Uniform boundary regularity of proper holomorphic maps

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 17, nº 3 (1990), p. 355-364

<http://www.numdam.org/item?id=ASNSP_1990_4_17_3_355_0>

Uniform Boundary Regularity of Proper Holomorphic Maps

WILHELM KLINGENBERG

1. - Introduction

According to recent results in [2] and [10], the family of proper holomorphic maps from $D \subset\subset \mathbb{C}^n$ to $G \subset\subset \mathbb{C}^n$ of multiplicity bounded from above by some $m \in \mathbb{N}$ is normal. That is any sequence f_j of such maps either has a convergent subsequence or is compactly divergent.

If D and G are in addition C^∞ smoothly bounded and pseudoconvex of finite type [6], then by [5], [7] the maps f_j are known to extend smoothly up to the boundary of D. Here we study the behaviour of this extension as $j \to \infty$. We denote by $\text{Prop}(D, G, m)$ the set of proper holomorphic maps from D to G of multiplicity m.

THEOREM 1. *Let $D, G, \subset\subset \mathbb{C}^n$ be C^∞-smoothly bounded pseudoconvex domains of finite type, and $f_j \in \text{Prop}(D, G, m)$, $f_j \to f : D \to \overline{G}$. Then, if $f \in \text{Prop}(D, G, m_0)$, one has*

i) *if $m_0 = m$, then $f_j \to f$ in $C^\infty(\overline{D})$*

ii) *if $m_0 < m$, then $\exists j'$, $\{p_i\}_1^{m-m_0} \subset \partial D$ with $f_{j'} \to f$ in $C^\infty(\overline{D} - \{p_i\})$.*

Otherwise, f is a constant map to some $q \in \partial G$, and ii) holds with $m_0 = 0$ and C^∞ replaced by C^0.

In [1], Bell gave an analogous result for biholomorphic maps. The points $\{p_i\}$ in ii) are limits of $f_{j1}^{-1}(w)$ for $w \in G$. The example of a sequence of m-fold Blanschke products as maps from the unit disc in $\mathbb{C}$ to itself shows that one cannot expect smooth convergence at these points, see also [1], [2] and [10]. The main ingredients of the proof are: the transformation rule for the Bergman kernel function under proper maps [3]; a Proposition of Bell [1] on the density of span $K_2(\cdot, w)$, $w \in G$, in $A^\infty(G) = A(G) \cap C^\infty(\overline{G})$; $C^\infty(\overline{G} \times \overline{G} - \Delta)$-regularity of the Bergman kernel for pseudoconvex domains of finite type, see [4], [9]. Here, Δ is the boundary diagonal of $G \times G$, $A(G)$ the holomorphic functions and

Pervenuto alla Redazione il 19 Luglio 1989.

K_2 the Bergman kernel function of G. Finally a division Theorem in $A^\infty(D)$, which is of independent interest; we write $|\cdot|_{\ell,D}$ for the C^ℓ-sup norm on D.

THEOREM 2: *Let D be a smoothly bounded domain in $\mathbb{C}^n$. Assume that*

a) $u_j \in A^\infty(D)$ *converge in* $C^\infty(\overline{D})$ *to* $u \in A^\infty(D)$.

b) *the order of vanishing of the* u_j, u *in* $\overline{D}$ *is of uniformly bounded order.*

c) $h_j \in A(D)$ *are uniformly bounded:* $|h_j(z)| \leq M$ *for all* $z \in D$, $j \in \mathbb{N}$.

d) *for all* $m \geq 0$, $\{u_j \cdot h_j^m\}_{j\geq 1}$ *is bounded in* $A^\infty(D)$, *that is:* $\forall \ell \geq 1$, $m \geq 0\ \exists c_1(\ell, m) \forall j :\ |u_j \cdot h_j^m|_{\ell,D} \leq c_1(\ell, m)$.

Then $\{h_j\}_{j\geq 1}$ *is bounded in* $A^\infty(D)$: $\forall \ell \exists c_2(\ell, u,$ *finitely many* $c_1)$: $|h_j|_{\ell,D} \leq c_2$.

It is a pleasure to thank my thesis advisor Steven Bell for his advice during this project.

2. - Proof of Theorem 1

If $f \in \mathrm{Prop}(D, G, m)$, then f^{-1} is an m-valued holomorphic map or correspondence from G to D or a holomorphic map $f^{-1} : G \to D^m_{\mathrm{sym}}$, the m-fold symmetric product of D, see [12].

PROPOSITION 3. *Assume* $D, G \subset\subset \mathbb{C}^n$ *and* $f_j \in \mathrm{Prop}(D, G, m)$, $f_j \to f : D \to \overline{G}$. *Then, if* $f \in \mathrm{Prop}(D, G, m_0)$, *one has*

i) *if* $m_0 = m$, *then* $f_j^{-1}(w) \to f^{-1}(w)$ *in* D^m_{sym}.

ii) *if* $1 \leq m_0 < m$, *then there is a subsequence* j' *and an* $(m - m_0)$*-valued holomorphic map* $h : G \to \partial D$ *with* $f_{j'}^{-1} \to (f^{-1} \cup h) : G \to (\overline{D})^m_{\mathrm{sym}}$.

Otherwise, f *is into* ∂G, *and* $\exists j'$, $h : G \to (\partial D)^m_{\mathrm{sym}}$ *with* $f_{j'}^{-1}(w) \to h$. *If in addition D and G are pseudoconvex of finite type, then the maps h above are constant:* $h : G \to \{p_i\}_1^{m-m_0} \subset \partial D$, *and in case* $f(D) \subset \partial G$, f *is constant and* $m_0 = 0$.

PROOF. By [2], [10], either $f \in \mathrm{Prop}(D, G, m_0)$ for some $1 \leq m_0 \leq m$ or f maps D into ∂G. We may pass to a subsequence j' such that $f_{j'}^{-1}$ converges to an m-valued map $F : G \to \overline{D}$. If $f(D) \subset \partial G$, then $F(G) \subset \partial D$, and $h = F$. Otherwise, given $K_1 \subset\subset D$ there exists $K_2 \subset\subset G$ such that $f_j(K_1) \subset K_2$ for all j. Therefore $f_{j'}^{-1} \circ f_{j'} \to F \circ f$ as $j' \to \infty$. Note that we may write $f_j^{-1} \circ f_j = id \cup g_j$, where g_j is an $(m-1)$- valued map from D to D, therefore $F \circ f = id \cup g$. This implies that $F = f^{-1} \cup h$ for some $(m - m_0)$-valued map h. If $m_0 = m$, then f^{-1} is m-valued, so $h = \emptyset$, and $F = f^{-1}$. We see that every subsequence of f_j^{-1} has a subsequence that converges to f^{-1}. This proves i). In case ii) we need to show that $h(G) \subset \partial D$. In this case, if ∂D is pseudoconvex of finite type [6], it does not contain any complex varieties, and h must be constant: $h = \{p_i\}_1^{m-m_0}$ for some $p_i \in \partial D$. Since $f : D \to G$ is proper, given $K_2 \subset\subset G$ there exist $K_1 \subset\subset D$ such that $f^{-1}(K_2) \subset K_1$. Claim: $K_1 \cap F(w) \subset f^{-1}(w)$ for $w \in K_2$. It follows that $h(G) \subset \partial D$. Proof of claim: Let $z_{j'} \in K_1 \cap f_{j'}^{-1}(w)$ and $z_{j'} \to z$. Then $f_{j'}(z_{j'}) = w$, and we may pass to $j' \to \infty$: $f(z) = w$. □

PROPOSITION 4. [1, Fact 1]. *Let $G \subset\subset \mathbb{C}^n$ be a smooth pseudoconvex domain of finite type. Then $\forall r \in \mathbb{N}\ \exists \ell \in \mathbb{N}$, $\{w_k\}_1^\ell \subset G$, $c > 0\ \forall h \in A^\infty(G)$, $p \in G \exists \{c_k\}_1^\ell \in \mathbb{C}$:*

i) $\sum_{k=1}^{\ell} c_k K_2(z, w_k) = h(z) + 0(|z-p|^{r+1})$,

ii) $|c_k| \leq c|h|_{r,G}$

Next consider the transformation formula of the Bergman kernel function under proper maps [3]:

$$u_j(z)K_2(f_j(z), w) = \sum_{i=1}^{m} K_1(z, F_j^{(i)}(w))\overline{U_j^{(i)}(w)}. \tag{1}$$

Here, $\{F_j^{(i)}(w)\}_{i=1}^m = f_j^{-1}(w)$ are the branches of the multi-valued inverses, and $U_j^{(i)} = \det(F_j^{(i)})'$. We follow Bell [1]. Now let $h \in A^\infty(D)$, $q \in D$, $r \in \mathbb{N}$. By Proposition 4 where p is replaced by $f_j(q)$ and by (1) there exist $w_k \in G$, $c_k \in \mathbb{C}$ depending on j with

$$u_j(z)h \circ f_j(z) = \sum_{k=1}^{\ell}\sum_{i=1}^{m} c_k K_1(z, F_j^{(i)}(w_k))U_j^{(i)}(w_k) + 0(|z-q|^{r+1}).$$

In case i) of the Theorem, F_j and U_j converge uniformly on $\{w_k\}$ as $j \to \infty$ by Proposition 3. Then, since $K_1 \in C^\infty(\overline{D} \times \overline{D} \backslash \Delta)$ (see [4]), and since the c_k are bounded independently of q and j, we conclude that $\{u_j \cdot h \circ f_j\}$ is bounded in $C^\infty(\overline{D})$. Letting $h = 1$, we conclude that $\{u_j\}$ is bounded in $C^\infty(\overline{D})$ and therefore converges in $C^\infty(\overline{D})$ to $u = \det f'$. By [5], u and u_j vanish at most of order $m \cdot n$ at any point in $\overline{D}$. Letting $h(w) = w_i^m$, $h_j = h \circ f_j$ for $i = 1, \ldots, n$, $m \geq 0$, we finally verify the assumptions c) and d) of Theorem 2. We may then conclude that $\{f_j\}$ is bounded in $C^\infty(\overline{D})$. This proves part i).

In case ii) by Proposition 3 we may pass to a subsequence j' such that $\{u_{j'} \cdot h \circ f_{j'}\}$ is bounded in $C^\infty(\overline{D}\backslash\{p_i\})$. Here again the regularity of K_1 is used. A local version of Theorem 2 allows to conclude that convergence of f_j takes place in $C^\infty(\overline{D}\backslash\{p_i\})$. As to the case of f being a constant map, the same reasoning as above shows that for some subsequence j', u_j, converges to $u \equiv 0$ in $C^\infty(\overline{D}\backslash\{p_i\}_1^m)$. Now the proof of Theorem 1, part B in [1] yields the conclusion that convergence takes place in $C^0(\overline{D}\backslash\{p_i\})$.

3. - A Division Theorem with Estimates

Assuming that $h \in A(D)$ is bounded and that $uh^m \in A^\infty(D)$ for all $m \geq 0$, we wish to show that h is in $A^\infty(D)$ and give estimates for h. Certainly, this cannot hold if u vanishes to infinite order at some point in $\overline{D}$. One is reduced to studying the question in the neighbourhood of a point p in ∂D at which u vanishes of finite order k. We restrict the considered functions u, h to a complex line L_p at p with this property. The division of uh^m by u will be carried out on such lines L_x for $x \in \partial D \cap U$, U a neighbourhood of p, and we will prove that for every ℓ, the function $h|_{\overline{D} \cap L_x \cap U}$ is in $C^\ell(D \cap L_x \cap U)$ with uniform estimates in x. The point is to keep track of the C^ℓ-sup norm estimate of h during the division process which proceeds by dividing the zeroes of u out of uh^m one at a time. To facilitate this procedure we introduce a normalizing transformation of $D \cap U$ which preserves analyticity on the complex lines $D \cap L_x \cap U$. We may choose holomorphic coordinates $(z_1, \dots, z_n)$ such that L_p is the z_1-axis and p is the origin. Let $x = (z_2, \dots, z_n) \in \mathbb{R}^{2n-2}$ and G be a smooth domain in D with $\partial D \cap U = \partial G \cap U$ for a neighbourhood U of the origin such that for some $r > 0$ and $|x| < r$ the slices $G_x = \{z \in \mathbb{C} : (z, x) \in G\} \subset \mathbb{C}$ are simply connected. Let $a \in \mathbb{C}$ be a fixed point which lies in all G_x and let Φ_x be the Riemann mapping function from G_x onto the unit disc Δ, with $\Phi_x(a) = 0$ and $\phi_x'(a) > 0$. Note that $0 \in \partial G_0$. Next let Ψ denote a conformal map of the unit disc onto $\Delta_- = \Delta \cap \{\operatorname{Im} z < 0\}$ which takes $\Phi_0(0) \in \partial A$ to $0 \in \partial \Delta_-$. The coordinate change given by $(z, x) \to (\Psi \circ \Psi_x(z), x/r)$ transforms $\bigcup_{|x|<r} G_x \subset D$ to $\Delta_- \times V \subset \mathbb{C} \times \mathbb{R}^{2n-2}$, where $V = \{|x| < 1\}$. One knows from the classical theory of conformal mappings that this change is C^∞-smooth up to $\partial G \cap U_1$ and maps this set onto $\{(-1, +1) \times V\}$ (by normalization) for some neighbourhood U_1 of the origin in $\mathbb{C}^n$. The function $u \in A(G)$ is transformed to a smooth function $u(z, x)$ on $\Delta_- \times V$ which is holomorphic in z for fixed x. For smooth functions u on $\Delta_- \times V$ we define the norm

$$|u(\cdot, x)|_{\ell, \Delta_-} := \sup_{\substack{z \in \Delta_- \\ i+j \leq \ell}} \left| \frac{\partial^{i+j}}{\partial z^i \partial \overline{z}^j} u(z, x) \right|.$$

Next we define the class of functions we will work in.

DEFINITION: a) $\Gamma^-(\ell, c)$ is the set of complex valued functions u on $\Delta_- \times V$ with $u(\cdot, x) \in A(\Delta_-)) \cap C^\ell(\overline{\Delta}_-)$, and $|u(\cdot, x)|_{\ell, \Delta_-} \leq c$ for each $x \in V$.

b) $\Gamma(\ell, c)$ are the functions on $\Delta \times V$ with $u(\cdot, x) \in A(A_-) \cap C^\ell(\overline{\Delta})$ and $|u(\cdot, x)|_{\ell, \Delta} \leq c$ for each $x \in V$.

Let $u^{(i)}(z, x) = \frac{1}{i!} \frac{\partial^i}{\partial z^i} u(z, x)$ and $u^{(i,j)} = \frac{1}{i! j!} \frac{\partial^{i+j}}{\partial z^i \partial \overline{z}^j} u$. The objective of this section is to prove the following.

THEOREM 5. *Let $h(\cdot, x) \in A(\Delta_-)$ and $|h(\cdot, x)| \leq c_1$ for $x \in V$, and assume that for all $m \geq 0$ there exists $c_1(m)$ with $u \cdot h^m \in \Gamma^-(\ell_1, c_1(m))$. Assume* *furthermore that* u *vanishes of order* k *at* $(0,0)$ *and that for some* $c_2 > 0$*:*

i) $|u^{(k,0)}(0,0)| \geq c_2^{-1}$

ii) $u(z,\cdot) \in C^1(V)$ *and* $\left|\frac{d}{dx}u(z,\cdot)\right| \leq c_2$ *on* $\Delta_- \times V$.

Then $h \in \Gamma^-(\ell, c)$*, where* $\ell = \left(\frac{1}{2}\right)^k \ell_1 - 2k - 2$*, and* c *depends only on* ℓ_1*,* k*, finitely many* c_1*, and* c_2*.*

Theorem 5 implies Theorem 2: Note that the assumption i) of Theorem 5 is verified uniformly for all u_i since they converge in $C^\infty(\overline{D})$ to u which we assume to vanish of at most finite order in $\overline{D}$. Assumption ii) follows from $|u_i \cdot h_i^0|_{1,D} = |u_i|_{1,D} \leq c_1(0,1)$. Now the conclusion of Theorem 9 gives for all ℓ uniform C^ℓ-estimates for $h_i|_{L\cap D\cap U}$ for complex lines L transversal to the boundary of D and some neighbourhood U of any boundary point of D.

Note that by Cauchy estimates, the uniform boundedness of h_i in D gives uniform boundedness of $h_i|_K$ in $C^\infty(K)$ for compact subsets K of D. Pick any point p in $D\backslash K$. Any ℓ-th order derivarive of u at p can be expressed as a finite linear combination of derivatives of u in the direction of complex lines L transversal to the boundary. Since we can choose the L from an open cone of directions at each boundary point, we conclude that the sequence h_i is bounded in $C^\infty(\overline{D})$.

The proof of Theorem 5 proceeds by four propositions. We closely follow Diederich-Fornaess [7]. Here is a well-known fact on bounded extension [8, p. 277].

PROPOSITION 6. *Let* $u \in \Gamma^-(\ell, c)$*. Then there exists a* $v \in \Gamma(\ell, c)$ *with* $v(\cdot, x)|\Delta_- = u(\cdot, x)$.

LEMMA 7. *Let* $u \in \Gamma(\ell, c_1)$ *and* $\varsigma_1, V \to \Delta$ *be any map. Then there exists* $\tilde{u} \in \Gamma(\ell, c), c$ *depending only on* ℓ *and* c_1*, with*

a) $u(\cdot, x) = \tilde{u}(\cdot, x)$ *in* Δ_-

b) $\tilde{u}(z,x) = \sum_0^{\ell-1} u^{(i)}(\varsigma_1(x), x) \cdot (z - \varsigma_1(x))^i + \sigma_\ell(z,x)$*, where* σ_ℓ *vanishes of order* ℓ *at* $z = \varsigma_1(x)$ *for all* $x \in V$.

PROOF. Conclusion b) says that the anti-holomorphic derivatives of $\tilde{u}$ up to order $\ell - 1$ vanish at ς_1.

The Taylor expansion for u at ς_1 is given by

$$u(z,x) = \sum_{i+j=0}^{\ell-1} u^{(i,j)}(\varsigma_1, x)(z - \varsigma_1)^i(\overline{z} - \overline{\varsigma}_1)^j$$

$$+\frac{1}{(\ell-1)!}(z - \varsigma_1)^{-\ell} \sum_{i+j=\ell} \int_{\varsigma_1}^{z} (z-w)^{\ell-1} u^{(i,j)}(w,x)dw \cdot (z - \varsigma_1)^i(\overline{z} - \overline{\varsigma}_1)^j$$

Clearly b) holds for $\tilde{u} \equiv u$ if $\varsigma_1(x) \in \overline{\Delta}_-$, and if $\varsigma_1(x) \notin \overline{\Delta}_-$, we set

$$\tilde{u} = u - \sum_{\substack{j \geq 1 \\ i+j \leq \ell-1}} \varphi\left(\frac{z - \varsigma_1}{\operatorname{Im} \varsigma_1}\right) \cdot u^{(i,j)}(\varsigma_1, x) \cdot (z - \varsigma_1)^i (\overline{z} - \overline{\varsigma}_1)^j.$$

Here, $\varphi \in C_0^\infty\left(\frac{1}{2}\Delta\right)$, $\varphi \equiv 1$ for $|z| < \frac{1}{4}$. We see that a) and b) hold. Note that since $u \in A(\Delta_-) \cap C^\ell(\overline{\Delta}_-)$, $u^{(0,1)}$ vanishes of order $\ell - 1$ on $\operatorname{Im} z = 0$. We may estimate

$$|u^{(i,j)}(z,x)| \leq c_2 \cdot |u|_\ell \cdot |\operatorname{Im} z|^{\ell - i - j}, \; i + j \leq \ell, \; j > 1.$$

Here c_2 depends only on ℓ, c_1. Denote by A_{ij} the entries of the above sum. For $z \in \operatorname{supp} \varphi\left(\frac{z - \varsigma_1}{\operatorname{Im} \varsigma_1}\right)$, we have $|z - \varsigma_1| \leq |\operatorname{Im} z|$, and for $z \notin \operatorname{supp}$, A_{ij} and all its derivatives vanish. Therefore

$$|A_{ij}(z,x)| \leq c_3 |u|_\ell \cdot |\operatorname{Im} z|^\ell.$$

Now every derivative up to order ℓ of A_{ij} with respect to z or $\overline{z}$ will take away one power of $|\operatorname{Im} z|$ in this estimate and change the constant c_3, making it dependent on the first ℓ derivatives of φ.

We conclude that $|\tilde{u}|_\ell \leq c_4 \cdot |u|_\ell \leq c_4 c_1$. □

LEMMA 8. *Let $u \in \Gamma(\ell, c_1)$, $\varsigma_1 : V \to \Delta$ satisfy the conclusion b) of Lemma 7 and $u(\varsigma_1, x) = 0$. Then there exists $u_1 \in \Gamma(\ell - 1, c)$, c depending only on ℓ and c_1, with*

$$u = (z - \varsigma_1) \cdot u_1 \qquad \text{on } \Delta \times V.$$

PROOF. Let σ_ℓ denote the ℓ-th order Taylor remainder term in the development of $u(z,x)$ around $(\varsigma_1(x), x)$. Define

$$u_1 = \sum_1^{\ell-1} u^{(i)}(\varsigma_1, x) \cdot (z - \varsigma_1)^{i-1} + \frac{\sigma_\ell(z,x)}{z - \varsigma_1}.$$

Then

$$\left| \sum_1^{\ell-1} u^{(i)} \cdot (z - \varsigma_1)^{i-1} \right|_{\ell-1} \leq c_2 \cdot |u|_\ell \text{ in } \Delta \times V.$$

The expression $\left(\frac{\sigma_\ell}{z - \varsigma_1}\right)^{(i,j)}$ for $i + j \leq \ell - 1$ is a sum of terms of the form $\sigma_\ell^{(p,q)} \cdot (z - \varsigma_1)^{-r}$ with $p + q + r \leq \ell, r \geq 1$. From the integral formula for σ_ℓ it follows that $|\sigma_\ell^{(p,q)}(z,x)| \leq c_3 |u|_\ell \cdot |z - \varsigma_1|^{\ell - p - q}$. This implies $|u_1|_{\ell-1} \leq c_4 \cdot |u_\ell| \leq c_4 c_1$. □

PROPOSITION 9. *Let* $u(\cdot, x) \in \Gamma(\ell, c_1)$ *vanish of order* $k \leq \ell - 1$ *at* 0, *and*

i) $|u^{(k,0)}(0,0)| \geq c_1^{-1}$

ii) $u(z,\cdot) \in C^1(V)$ *and* $\left|\frac{d}{dx}u(z,\cdot)\right| \leq c_1$ *on* $\Delta \times V$.

Then, after shrinking Δ, V *to* Δ_ϵ, V_r, *where* ϵ, r *depend only on* k, ℓ, c_1, *there exist* $u_k \in \Gamma(\ell - k, c)$, c *depending only on* k, ℓ, c_1 *and maps* $\varsigma_j : V \to \Delta$, $j = 1, \dots, k$ *with*

a) $$u = u_k \prod_1^k (z - \varsigma_j) \text{ on } \Delta_- \times V$$

b) $$|u_k(z,x)| \geq 2^{-k-3}c_1^{-1} \text{ on } \Delta \times V.$$

PROOF. By the k-th order vanishing of u at 0,

$$u(z,0) = \sum_{i+j=k} u^{(i,j)}(0,0)z^i\bar{z}^j + \sigma_{k+1}(z,0).$$

Since $u \in A(\Delta_-) \cap C^\ell(\overline{\Delta}_-)$, we have $u^{(i,j)}(0) = 0$ for $j \geq 1$. Therefore

$$\begin{aligned} u(z,0) &= u^{(k,0)}(0)z^k + \sigma_{k+1}(z) \\ &= z^k \cdot \left(u^{(k,0)}(0) + \frac{\sigma_{k+1}}{z^k}\right) \\ &= z^k v(z). \end{aligned}$$

Since $|\sigma_{k+1}(z,0)| \leq c_2|u|_{k+1} \cdot |z|^{k+1} \leq c_2 c_1 |z|^{k+1}$, we see that $|v(z)| \geq \frac{1}{2}c_1^{-1}$ for $|z| < \frac{1}{2}\frac{1}{c_1^2 c_2} =: \epsilon_0$. Therefore $|u(z,0)| \geq \frac{\epsilon^k}{2} \cdot c_1^{-1}$ for $\epsilon \leq |z| < \epsilon_0$. By assumption ii), there exists $r_0(\epsilon, c_1)$ with

(2) $$|u(z,x)| \geq \frac{\epsilon^k}{4}c_1^{-1} \text{ for } \epsilon \leq |z| < \epsilon_0, \ |x| < r_0.$$

We now see that

(3) $\log\ u(\cdot, x)$ increases its value by $2\pi ik$ around $|z| = \epsilon_0$ for $|x| \leq r_0$.

Therefore, there exists a map $\varsigma_1 : V_{r_1} \to \Delta_{\epsilon_0/2}$ with $u(\varsigma_1(x), x) = 0$.
Applying Lemma 7 to u, ς_1 gives a $\tilde{u} \in \Gamma(\ell, c_3)$ with properties a) and b).
Next apply Lemma 8 to $\tilde{u}, \varsigma_1$ and get $u_1 \in \Gamma(\ell - 1, c_4)$ with

(4) $$\tilde{u} = u_1 \cdot (z - \varsigma_1) \text{ on } \Delta_{\epsilon_0} \times V_{r_1}.$$

Since $\tilde{u} = u$ on $|z| = \epsilon_0$, we conclude from (4) that $u_1 \neq 0$ on $|z| = \epsilon_0$ and (3) holds for u_1 and $k - 1$.

We may repeat this argument k times and conclude that

$$\tilde{u}_j = u_{j+1} \cdot (z - \varsigma_{j+1}) \text{ on } \Delta_{\epsilon_0} \times V_{r_1} \tag{5}$$

with $u_{j+1} \epsilon \Gamma(\ell - k, c_5)$, $j = 1, \dots, k-1$. Therefore

$$\tilde{\tilde{u}} = u_k \cdot \prod_1^k (z - \varsigma_1) \text{ on } \Delta_{\epsilon_0} \times V_{r_1}.$$

Since $u = \tilde{u} = \tilde{\tilde{u}}$ on $\Delta_{\epsilon_0,-} \times V_{r_1}$, this proves a). To prove b), we make the following claim:

Given $\epsilon > 0$, $\exists r(\epsilon, c_1, k)$ such that $|\varsigma_j(x)| < \epsilon$ for $|x| < r$, $j = 1, \dots, k$.

PROOF. Let $\epsilon < \epsilon_0$ be given. By (4), $\tilde{u}(\varsigma_1(x), x) = 0$ for $|x| < r_1$. Since $\tilde{u}(\varsigma_1, x) = 0 = u(\varsigma_1, x)$, (2) implies that $|\varsigma_1(x)| \leq \epsilon$. Continuing inductively, assume that $\varsigma_1, \dots, \varsigma_j$ have modulus smaller that ϵ. By (5), we have $\tilde{u}_j(\varsigma_{j+1}, x) = 0$ for $|x| < r_1$. Now $\tilde{\tilde{u}} = u_j \cdot \prod_1^j (z - \varsigma_i)$. Outside Δ_-, this $\tilde{\tilde{u}}$ does not have to coincide with the $\tilde{\tilde{u}}$ above. Since in Lemma 7 we have $\tilde{u}_j(\varsigma_{j+1}) = u_j(\varsigma_{j+1})$, we conclude that $\tilde{\tilde{u}}(\varsigma_{j+1}) = 0$. If now also $u(\varsigma_{j+1}) = 0$, then (2) implies that $|\varsigma_{j+1}(x)| < \epsilon$. Otherwise one has $\tilde{\tilde{u}}(\varsigma_{j+1}) \neq u(\varsigma_{j+1})$, and since by construction $\tilde{\tilde{u}}$ differs from u only in ϵ-neighbourhoods of $\varsigma_1, \dots, \varsigma_j$, we have for some q

$$|\varsigma_{j+1}(x)| \leq |\varsigma_{j+1} - \varsigma_q| + |\varsigma_q| < 2\epsilon.$$

This proves the claim.

To conclude the proof, note that given $\epsilon > 0$, we have by (2) for $|z| = \epsilon$

$$|u_k(z, x)| = \frac{|u(z, x)|}{\prod_1^k |z - \varsigma_j|} \geq \frac{\epsilon^k}{4} \frac{c_1^{-1}}{(2\epsilon)^k} = 2^{-k-2} c_1^{-1}$$

for $|x| < r$, r chosen as in the claim. Since $u_k \epsilon \Gamma(\ell - k, c_5)$, we may choose ϵ small enough, depending on k, c_1, c_5 such that this implies $|u_k(z, x)| \geq 2^{-k-3} c_1^{-1}$ for $|z| < \epsilon$. □

Proof of Theorem 5: First we apply Proposition 9 to u, which gives $u = u_k \prod_1^k (z - \varsigma_j)$ on $\Delta_- \times V$.

We will successively divide the $(z - \varsigma_j)$ out of uh. To retain estimates on the way, we need to take into account those for $uh^m \in \Gamma^-(\ell_1, c_1(m))$ which by Proposition 6 we may assume to lie in $\Gamma(\ell_1, c_1(m))$.

a) Note that $\frac{(uh)^2}{z - \varsigma_1} = (uh^2) \cdot u_k \prod_2^k (z - \varsigma_j)$ on $\Delta_- \times V$. By the assumption concerning uh^2, and Proposition 9 concerning u_k, we may conclude that

$$\frac{(uh)^2}{z - \varsigma_1} \in \Gamma^-(\ell_1 - k, c_2). \tag{6}$$

We wish to show that $\frac{uh}{z-\varsigma_1} \in \Gamma^-(\ell, c)$ for some ℓ, c. If $\varsigma_1 \in \overline{\Delta}_1$, then (6) implies that $(hu)(\varsigma_1, x) = 0$, and we are done by Lemma 8 with $\ell = \ell_1 - 1$.

If $\varsigma_1 \notin \overline{\Delta}_-$, we proceed as follows. Since $uh \in \Gamma(\ell_1, c_1)$, we may apply Lemma 7 to uh, ς_1 :

$$(\widetilde{uh}) = \sum_0^{\ell-1}(uh)^{(i)}(\varsigma_1, x)(z-\varsigma_1)^i + \sigma_\ell(z,x).$$

Now

$$\frac{(\widetilde{uh})^2}{z-\varsigma_1} = \frac{(uh^{(0)})^2}{z-\varsigma_1} + 2uh^{(0)} \cdot \left(\sum_1^{\ell-1} uh^{(i)}(z-\varsigma_1)^{i-1} + \frac{\sigma_\ell}{z-\varsigma_1}\right) + \frac{1}{z-\varsigma_1}\left(\sum_1^{\ell-1} uh^{(i)}(z-\varsigma_1)^i + \sigma_\ell\right)^2.$$

Since $\widetilde{uh} = uh$ on $\Delta_- \times V$, the left hand side is in $\Gamma^-(\ell_1 - k, c_2)$. As in the proof of Lemma 8 we may estimate the second and third terms on the right hand side to see that they are in $\Gamma(\ell_1 - 1, c_3)$. Therefore $\frac{(uh^{(0)})^2}{z-\varsigma_1} \in \Gamma^-(\ell_1 - k, c_4)$. By differentiating, this implies $\frac{(uh^{(0)})^2}{(z-\varsigma_1)^{\ell_1-k+1}} \in \Gamma^-(0, c_5)$, and $\frac{uh^{(0)}}{(z-\varsigma_1)^p} \in \Gamma^-(0, c_6)$. Here, $p = \left[\frac{\ell_1 - k + 1}{2}\right]$. Assume $p \geq 2$, and consider the integral

$$\int_{\frac{-i}{2}}^{z} \frac{uh^{(0)}}{(w-\varsigma_1)^p} dw = \frac{1}{1-p}\left(\frac{uh^{(0)}}{(z-\varsigma_1)^{p-1}} - \frac{uh^{(0)}}{\left(\frac{-i}{2}-\varsigma_1\right)^{p-1}}\right),$$

where we integrate along a straight line. The left hand side is in $\Gamma^-(1, c_7)$, and since $\varsigma_1 \notin \overline{\Delta}_-$, the second term on the right hand side, independent of z, is bounded by $\frac{1}{p-1}2^{p-1}c_1$. We conclude that $\frac{uh^{(0)}}{(z-\varsigma_1)^{p-1}} \in \Gamma^-(1, c_8)$. Repeating this gives $\frac{uh^{(0)}}{z-\varsigma_1} \in \Gamma^-(p-1, c_9)$. Now we have on $\Delta_- \times V$:

$$\frac{uh}{z-\varsigma_1} = \frac{uh^{(0)}}{z-\varsigma_1} + \sum_1^{\ell-1} uh^{(i)} \cdot (z-\varsigma_1)^{i-1} + \frac{\sigma_\ell}{z-\varsigma_1} \in \Gamma^-(p-1, c_{10}).$$

b) We prove by induction the following statement:

$$\text{(7i)} \qquad u_k \prod_i^k (z-\varsigma_j)h^m \in \Gamma^-(\ell, c) \quad \forall m \geq 0.$$

($i = 1$) is the assumption of the Theorem. ($i = 2$) was proved in part a) for $m = 1$. We will show that (7i) implies (7i+1). Let $g_m = u_k \prod_i^k (z-\varsigma_j)h^m \in \Gamma^-(\ell, c)$, and note that

$$\frac{g_m^2}{z-\varsigma_i} = g_{2m} \cdot u_k \cdot \prod_{i+1}^{k} (z-\varsigma_i) \text{ on } \Delta_- \times V.$$

The right hand side is in $\Gamma^-(\ell - k, c_1)$. We may now conclude as in a) that this implies

$$\frac{g_m}{z-\varsigma_1} \in \Gamma^-(\ell_1, c_2) \text{ where } \ell_1 = \left[\frac{\ell - k + 1}{2}\right] - 1.$$

c) Letting $m = 1$ in $(i = k)$, we get $u_k \cdot h \in \Gamma^-(\ell, c)$. Now the conclusions of Proposition 9 allow us to infer that $h = \frac{(u_k \cdot h)}{u_k} \in \Gamma^-(\ell - k, c_1)$. The expression given for ℓ_1 in Theorem 5 now follows from the one in part b) above. □

REFERENCES

[1] S. BELL, *Compactness of families of holomorphic mappings up to the boundary*, Springer LNM 1268, 29-42.

[2] S. BELL, *A generalization of Cartan's Theorem to proper holomorphic mappings*, J. Math. Pures Appl., **67**, 1988, 85-92.

[3] S. BELL, *Proper holomorphic mappings and the Bergman kernel function*, Duke Math. J. **48** (1981), 167-175.

[4] S. BELL, *Differentiability of the Bergman kernel and pseudo-local estimates*, to appear in Math. Zeit.

[5] S. BELL - D. CATLIN, *Boundary regularity of proper holomorphic mappings*, Duke Math. J. **49** (1982), 385-396.

[6] J. D'ANGELO, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1982), 615-637.

[7] K. DIEDERICH - J.E. FORNAESS, *Boundary regularity of proper holomorphic mappings*, Invent. Math. **67** (1982), 363-384.

[8] G. FOLLAND, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton 1976.

[9] N. KERZMAN, *The Bergman kernel function: Differentiability at the boundary*, Math. Ann. **195** (1972), 149-158.

[10] W. KLINGENBERG - S. PINCHUK, *Normal families of proper holomorphic correspondences*, University of Minnesota Mathematics Report #88-113.

[11] R. NARASIMHAN, *Several complex variables*, Chicago Lectures in Mathematics, University of Chicago Press (1971).

[12] H. WHITNEY, *Complex Analytic Varieties*, Addison Wesley, Reading, Mass., 1972.

School of Mathematics
University of Minnesota
Minneapolis, MN 55455