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Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the $n$-laplacian

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Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the \( n \)-Laplacian

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1. - Introduction

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) with smooth boundary. We are looking for a solution of the following problem:

Let \( 1 < p \leq n \), find \( u \in W^{1,p}_0(\Omega) \backslash \{0\} \) such that

\[
\Delta_p u = f(x, u)|u|^{p-2} \quad \text{in } \Omega
\]

\[
u \geq 0,
\]

where \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplacian and \( f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a \( C^1 \)-function with \( f(x, 0) = 0 \), \( f(x, t) \geq 0 \) for \( t \geq 0 \) and of critical growth.

For \( p = 2 \) and \( n \geq 3 \), Brézis-Nirenberg [4] have studied the existence and non-existence of solution of (1.1) when \( f \) has critical growth of the form \( u^{(n+2)/(n-2)} + \lambda u \). A generalization of this result, on the same lines, for the \( p \)-Laplacian with \( p \leq n \) and \( p^2 \leq n \), has been studied by Garcia Azorero-Peral Alonso [7]. When \( p = n \), in view of the Trudinger [13] imbedding, a critical growth function \( f(x, u) \) behaves like \( \exp \left( b|u|^{n/(n-1)} \right) \) for some \( b > 0 \). In this context, when \( p = n = 2 \) and \( \Omega \) is a ball in \( \mathbb{R}^2 \), existence of a solution of (1.1) has been studied by Adimurthi [1], Atkinson-Peletier [2]. The method used by Atkinson-Peletier is a shooting method and hence cannot be generalized to solve (1.1) in an arbitrary domain. Whereas in Adimurthi [1], (1.1) is solved via variational method which is in the spirit of Brézis-Nirenberg [4] and, based on this method, we prove the following main result in this paper.

Let \( f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)}) \) be a function of critical growth and \( F(x, t) \) be its primitive (see definition (2.1)). For \( u \in W^{1,n}_0(\Omega) \), let

\[
J(u) = \frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \int_{\Omega} F(x, u) \, dx
\]
\( \lambda_1(u) = \inf \left\{ \int_\Omega |\nabla u|^n \, dx; \ u \in W_0^{1,n}(\Omega), \ \int_\Omega |u|^n \, dx = 1 \right\} \)

\( \alpha_n = n\omega_n^{1/(n-1)}, \) where \( \omega_n = \text{Volume of } S^{n-1}. \)

**THEOREM** Let \( f(x,t) = h(x,t)\exp(b|x|^{n/(n-1)}) \) be a function of critical growth on \( \Omega. \) Then

1) \( J : W_0^{1,n}(\Omega) \to \mathbb{R} \) satisfies the Palais-Smale Condition on the interval \( (-\infty, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \).

2) Let \( f'(x,t) = \frac{\partial}{\partial t} f(x,t) \) and further assume that

\( \sup_{x \in \Omega} f'(x,0) < \lambda_1(\Omega) \)

\( \lim_{t \to \infty} \inf_{x \in \Omega} h(x,t)^n = \infty, \)

then there exists some \( u_0 \in W_0^{1,n}(\Omega) \setminus \{0\} \) such that

\( \Delta_n u_0 = f(x,u_0)u_0^{n-2} \quad \text{in } \Omega \)

\( u_0 \geq 0 \)

\( u_0 = 0 \quad \text{on } \partial \Omega. \)

The method adopted to solve (1.7) in Brézis-Nirenberg [4] does not work because of the critical growth is of exponential type. Here we adopt the method of artificial constraint due to Nehari [11]. The main idea of the proof is as follows:

Define

\( a(\Omega, f)^n = \inf \left\{ J(u); \ \int_\Omega |\nabla u|^n \, dx = \int_\Omega f(x,u)u^{n-1} \, dx, \ u \neq 0 \right\}, \)

then the minimizer of (1.8) is a solution of (1.7).

It has to be noted that \( \alpha_n \) is the best constant appearing in Moser’s [10] result about the Trudinger’s imbedding of \( W_0^{1,n}(\Omega) \). In view of this, one expects that \( J \) should satisfy the Palais-Smale Condition on \( (-\infty, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \). Therefore, in order to get a minimizer of (1.8), the question remains to show that

\( a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \)
and this has been achieved by showing the following relation

\[(1.10) \quad \sup_{\Omega} \int_{\Omega} f(x, a(\Omega, f)w)w^{n-1} dx \leq a(\Omega, f). \]

In the forthcoming paper (jointly with Yadava), we discuss the bifurcation and multiplicity results for (1.7) when \(n = 2\).

2. - Preliminaries

Let \(\Omega\) be a bounded domain with smooth boundary. In view of the Trudinger-Moser \([13,10]\) imbedding, we have the following definition of functions of critical growth.

**DEFINITION 2.1.** Let \(h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}\) be a \(C^1\)-function and \(b > 0\). Let \(f(x, t) = h(x, t) \exp \left( bt^{n/(n-1)} \right)\). We say that \(f\) is a function of critical growth on \(\Omega\) if the following holds:

- There exist constants \(M > 0\), \(\sigma \in [0, 1)\) such that, for every \(c > 0\), and for every \((x, t) \in \Omega \times (0, \infty)\),
  - (H1) \(f(x, 0) = 0\), \(f(x, t) > 0\), \(f(x, t)t^{n-1} = f(x, -t)(-t)^{n-1}\);
  - (H2) \(f'(x, t) > \frac{f(x, t)}{t}\), where \(f'(x, t) = \frac{\partial f}{\partial t}(x, t)\);
  - (H3) \(F(x, t) \leq M(1 + f(x, t)t^{n-2+\sigma})\), where
    \[F(x, t) = \int_0^t f(x, s)s^{n-2} ds\]
    is the primitive of \(f\);
  - (H4) \(\lim_{t \to \infty} \sup_{x \in \Omega} h(x, t) \exp \left( -ct^{n/(n-1)} \right) = 0\),
    \(\liminf_{t \to \infty} \inf_{x \in \Omega} h(x, t) \exp \left( ct^{n/(n-1)} \right) = \infty\).

Let \(A(\Omega)\) denote the set of all functions of critical growth on \(\Omega\).

**EXAMPLES.** In view of \((H_1)\), it is enough to define \(f\) on \(\overline{\Omega} \times (0, \infty)\).

1) For \(m \geq 1\), \(b > 0\), \(\beta \geq 0\) and \(0 \leq \alpha < \frac{n}{n-1}\), \(f(x, t) = t^m \exp(\beta t^\alpha) \exp \left( bt^{n/(n-1)} \right)\) is in \(A(\Omega)\).
2) \(f(x, t) = t^2e^{-t} \exp \left( t^{n/(n-1)} \right)\) is in \(A(\Omega)\).
3) Let \(f(x, t) = h(x, t) \exp \left( bt^{n/(n-1)} \right)\), satisfying \((H_1)\) and \((H_4)\).
Further assume that $h'(x, t) \geq \frac{h(x, t)}{t}$ for $(x, t) \in \bar{\Omega} \times (0, \infty)$. Then $f$ is in $A(\Omega)$.

For

$$\frac{f'(x, t)}{f(x, t)} = \frac{h'(x, t)}{h(x, t)} + \frac{nb}{n-1} t^{1/(n-1)} > \frac{1}{t}$$

and hence $f$ satisfy $(H_2)$.

Let $\epsilon > 0$, and $\sigma = \frac{1}{n-1}$

$$F(x, t) - F(x, \epsilon) = \frac{n-1}{nb} \int_\epsilon^t h(x, s)s^{n-1-\sigma} \frac{d}{ds} \exp \left( bs^{n/(n-1)} \right) ds$$

$$\leq \frac{n-1}{nb} \left[ f(x, t)t^{n-2-\sigma} - f(x, \epsilon)t^{n-2-\sigma} \right].$$

This implies that there exists a constant $M > 0$ such that $F(x, t) \leq M[1 + f(x, t)t^{n-2-\epsilon}]$ for $(x, t) \in \bar{\Omega} \times (0, \infty)$. This shows that $f$ satisfy $(H_3)$ and hence $f \in A(\Omega)$.

Let $W_0^{1,n}(\Omega)$ be the usual Sobolev space and $f(x, t) = h(x, t) \exp(bt^{n/(n-1)})$ be in $A(\Omega)$. For $u \in W_0^{1,n}(\Omega)$, define

(2.1) \[ \|u\|^n = \int_\Omega |\nabla u|^n \, dx \]

(2.2) \[ J(u) = \frac{1}{n} \|u\|^n - \int_\Omega F(x, u) \, dx \]

(2.3) \[ I(u) = \frac{1}{n} \int_\Omega f(x, u)u^{n-1} \, dx - \int_\Omega F(x, u) \, dx \]

(2.4) \[ \partial B(\Omega, f) = \left\{ u \in W_0^{1,n}(\Omega) \setminus \{0\}; \|u\|^n = \int_\Omega f(x, u)u^{n-1} \, dx \right\} \]

(2.5) \[ \frac{a(\Omega, f)^n}{n} = \inf \{ J(u); \ u \in \partial B(\Omega, f) \} \]

(2.6) \[ \lambda_1(\Omega) = \inf \left\{ \|u\|^n; \int_\Omega |u|^n \, dx = 1 \right\} \]

$\alpha_n = n\omega_n^{1/(n-1)}$, where $\omega_n = \text{Volume of } S^{n-1}$.\]
DEFINITION OF MOSER FUNCTIONS. Let \( x_0 \in \Omega \) and \( R \leq d(x_0, \partial \Omega) \), where \( d \) denotes the distance from \( x_0 \) to \( \partial \Omega \). For \( 0 < \ell < R \), define

\[
m_{\ell, R}(x, x_0) = \begin{cases} 
\frac{1}{\omega_n} \left( \frac{R}{\ell} \right)^{1 - \frac{1}{n}} & \text{if } 0 \leq |x - x_0| \leq \ell \\
\frac{\log \frac{r}{\ell}}{(\log \frac{R}{\ell})^{1/n}} & \text{if } \ell \leq r = |x - x_0| \leq R \\
0 & \text{if } |x - x_0| \geq R.
\end{cases}
\]

Then it is easy to see that \( m_{\ell, R} \in W^{1,n}_0(\Omega) \) and \( \|m_{\ell, R}\| = 1 \).

For the proof of our theorem, we need the following two results whose proof is found in Moser [10] and P.L. Lions [9] respectively.

**THEOREM 2.1 (Moser).**
1) Let \( u \in W^{1,n}_0(\Omega) \), and \( p < \infty \), then

\[
\exp \left( |u|^{n/(n-1)} \right) \in L^p(\Omega).
\]

2) \( \left( \frac{\alpha_n}{b} \right)^{n-1} = \max \left\{ \frac{c^n}{\|u\|^{n/(n-1)}}, \sup_{\|w\| \leq 1} \int_{\Omega} \exp \left( b c^{n/(n-1)} |w|^{n/(n-1)} \right) \, dx < \infty \right\}.
\]

**THEOREM 2.2 (P.L. Lions).** Let \( \{u_k; \|u_k\| = 1\} \) be a sequence in \( W^{1,n}_0(\Omega) \) converging weakly to a non-zero function \( u \). Then, for every \( p < \frac{1}{(1 - \|u\|^{n-1})^{-1/(n-1)}} \),

\[
\sup_k \int_{\Omega} \exp \left( p \alpha_n |u_k|^{n/(n-1)} \right) \, dx < \infty.
\]

3. - Proof of the Theorem

We need a few lemmas to prove the theorem. The proof of the following lemma is given in the appendix.

**LEMMA 3.1.** Let \( f \in A(\Omega) \). Then we have

1) If \( u \in W^{1,n}_0(\Omega) \), then \( f(x, u) \in L^p(\Omega) \) for all \( p \geq 0 \).

2) \( \left( \frac{\alpha_n}{b} \right)^{n-1} = \max \left\{ \frac{c^n}{\|u\|^{n/(n-1)}}, \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw) w^{n-1} \, dx < \infty \right\}.
\]

3) Let \( \{u_k\} \) and \( \{v_k\} \) be bounded sequences in \( W^{1,n}_0(\Omega) \) converging weakly and for almost every \( x \) in \( \Omega \) to \( u \) and \( v \) respectively. Further assume that

\[
\lim_{k \to \infty} \|u_k\|^{n} < \left( \frac{\alpha_n}{b} \right)^{-1}.
\]

Then, for every integer \( \ell \geq 0 \),

\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k) v_k^\ell \, dx = \int_{\Omega} f(x, u) v^\ell \, dx.
\]
4) Let \( \{u_k\} \) be a sequence in \( W_0^{1,n}(\Omega) \) converging weakly and for almost every \( x \) in \( \Omega \) to \( u \), such that

\[
\sup_k \int_\Omega f(x, u_k) u_k^{n-1} \, dx < \infty.
\]

Then, for any \( 0 \leq r < 1 \),

\[
\lim_{k \to \infty} \int_\Omega f(x, |u_k|) \, |u_k|^{n-2+r} \, dx = \int_\Omega f(x, |u|) \, |u|^{n-2+r} \, dx.
\]

\[
\lim_{k \to \infty} \int_\Omega F(x, u_k) \, dx = \int_\Omega F(x, u) \, dx.
\]

5) \( I(u) \geq 0 \) for all \( u \) and \( I(u) = 0 \) iff \( u \equiv 0 \). Further, there exists a constant \( M_1 > 0 \) such that, for all \( u \in W_0^{1,n}(\Omega) \),

\[
\int_\Omega f(x, u) u^{n-1} \, dx \leq M_1 (1 + I(u)).
\]

**Lemma 3.2.** Let \( f = h \exp \left( b\frac{t^{n/(n-1)}}{t} \right) \in A(\Omega) \) and define

\[
h_0(t) = \inf_{x \in \Omega} h(x, t), \quad M_0 = \sup_{t \geq 0} h_0(t) t^{n-1}, \quad R_0 = \sup_{x \in \Omega} d(x, \partial \Omega),
\]

and

\[
k_0 = \begin{cases} \left( \frac{n}{R_0} \right)^{n/(n-1)} M_0^{-1/(n-1)} & \text{if } M_0 < \infty \\ 0 & \text{if } M_0 = \infty. \end{cases}
\]

Let \( a \geq 0 \) be such that

\[
\sup_{\|w\| \leq 1} \int_\Omega f(x, aw) w^{n-1} \, dx \leq a.
\]

If \( \frac{k_0}{b} < 1 \), then \( a^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \).

**Proof.** From 2) of lemma 3.1, we have \( a^n \leq \left( \frac{\alpha_n}{b} \right)^{n-1} \). Suppose \( a^n = \left( \frac{\alpha_n}{b} \right)^{n-1} \). Let \( x_0 \in \Omega \) such that \( d(x_0, \partial \Omega) = R_0 \) and \( 0 < \ell < R_0 \). Let

\[
m_\ell(x) = m_{\ell, R_0}(x, x_0).
\]
be the Moser functions and
\[ t = a \omega_n^{-1/n} \left( \log \frac{R_0}{\ell} \right)^{(n-1)/n}, \]
then from (3.1) we have
\[
\begin{align*}
& a \geq \int_{\Omega} f(x, am\ell)m_{m\ell}^{n-1} \, dx \\
& \geq \int_{B(x_0, \ell)} h_0(am\ell)m_{m\ell}^{n-1} \exp \left( b a^{n/(n-1)} m_{m\ell}^{n/(n-1)} \right) \, dx \\
& = \frac{h_0(t)t^{n-1}\omega_n R_0^n}{na^{n-1}}. \\
\end{align*}
\]
This implies that
\[
\left( \frac{\alpha_n}{b} \right)^{n-1} = a^n \geq \frac{h_0(t)t^{n-1}\omega_n R_0^n}{n}.
\]
That is, for all \( t \in (0, \infty) \),
\[
b \leq \left( \frac{n}{R_0} \right)^{n/(n-1)} \left( h_0(t)t^{n-1} \right)^{-1/(n-1)}
\]
and hence
\[
b \leq \left( \frac{n}{R_0} \right)^{n/(n-1)} \inf_{t \geq 0} \left( h_0(t)t^{n-1} \right)^{-1/(n-1)} \leq k_0
\]
which contradicts the hypothesis \( b > k_0 \). Hence \( a^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \) and this proves the lemma.

Lemma 3.3. (Compactness Lemma). Let \( f \) be in \( A(\Omega) \) and \( \{u_k\} \) be a sequence in \( W_0^{1,n}(\Omega) \) converging weakly and for almost every \( x \) in \( \Omega \) to a non-zero function \( u \). Further, assume that

(i) There exists \( C \in \left( 0, 1/n \left( \frac{\alpha_n}{b} \right)^{n-1} \right] \) such that \( \lim_{k \to \infty} J(u_k) = C \);
(ii) \( \|u\|^n \geq \int_{\Omega} f(x, u)u^{n-1} \, dx \);
(iii) \( \sup_{k} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx < \infty \);
then
\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx = \int_{\Omega} f(x, u)u^{n-1} \, dx.
\]
PROOF. From 5) of lemma 3.1, \( I(u) > 0 \). Therefore, from (ii) we have
\[ J(u) \geq I(u) > 0 \] and
\[ J(u) \leq \lim_{k \to \infty} J(u_k) = C. \] Hence we can choose an \( \epsilon > 0 \) such that
\[(C - J(u)) (1 + \epsilon)^{n-1} < \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1}. \]

Let \( \beta = \int_{\Omega} F(x, u) \, dx \). Then, from (iii) and 4) of lemma 3.1, we have
\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\}
= n(C + \beta).
\]

From (3.2) and (3.3) we can choose a \( k_0 > 0 \) such that, for all \( k \geq k_0 \),
\[(1 + \epsilon)^{n-1} \left( \frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n < \frac{C + \beta}{C - J(u)} = \left( 1 - \frac{\|u\|^n}{n(C + \beta)} \right)^{-1}.
\]

Now choose \( p \) such that
\[(1 + \epsilon)^{n-1} \left( \frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n \leq p^{n-1} < \frac{C + \beta}{C - J(u)}. \]

Applying theorem 2.2 to the sequence \( \frac{u_k}{\|u_k\|} \) and using (3.3) and (3.5), we have
\[
\sup_k \int_{\Omega} \exp \left[ p\alpha_n \left( \frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] \, dx < \infty.
\]

From (3.5) and (3.6), we have
\[
\sup_k \int_{\Omega} \exp \left( (1 + \epsilon)^{n-1} b |u_k|^{n/(n-1)} \right) \, dx
\leq \sup_k \int_{\Omega} \exp \left[ p\alpha_n \left( \frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] \, dx < \infty.
\]

Let
\[ M_1 = \sup_{(x, t) \in \bar{\Omega} \times \mathbb{R}} |h(x, t)t^{n-1}| \exp \left( -\frac{b}{2} |t|^{n/(n-1)} \right). \]
and $N > 0$. Then from (3.7) we have

$$\int_{|u_k| \geq N} f(x, u_k)u_k^{n-1} dx = \int_{|u_k| \geq N} h(x, u_k)u_k^{n-1} \exp \left( b|u_k|^{n/(n-1)} \right) dx \leq M_1 \int_{|u_k| \geq N} \exp \left( -\frac{b}{2}|u_k|^{n/(n-1)} \right) \exp \left[ (1 + \epsilon)b|u_k|^{n/(n-1)} \right] dx = O \left( \exp \left( -\frac{b}{2}N^{n/(n-1)} \right) \right).$$

Hence

$$\int_{\Omega} f(x, u_k)u_k^{n-1} dx = \int_{|u_k| \leq N} f(x, u_k)u_k^{n-1} dx + O \left( \exp \left( -\frac{b}{2}N^{n/(n-1)} \right) \right).$$

Now letting $k \to \infty$, and $N \to \infty$ in the above equation, we obtain

$$\lim_{k \to \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} dx = \int_{\Omega} f(x, u)u^{n-1} dx.$$

This proves the lemma.

**LEMMA 3.4.** Let $f \in A(\Omega)$ and assume that

(i) $\lim_{t \to \infty} h_0(t)t^{n-1} = \infty$,

where $h_0(t) = \inf_{x \in \Omega} h(x, t)$;

(ii) $\sup_{x \in \Omega} f'(x, 0) < \lambda_1(\Omega)$;

then

$$0 < a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1}.$$

**PROOF.** The lemma is proved in several steps.

**STEP 1.** $a(\Omega, f) > 0$.

Suppose $a(\Omega, f) = 0$. Then there exists a sequence $\{u_k\}$ in $\partial B(\Omega, f)$ such that $J(u_k) \to 0$ as $k \to \infty$. Since $J(u_k) = I(u_k)$, hence from 5) of lemma 3.1

$$\sup_{k} \int_{\Omega} f(x, u_k)u_k^{n-1} dx < \infty$$

(3.9)

$$\sup_{k} ||u_k||^n < \infty.$$
Then, by extracting a subsequence, we can assume that \( \{u_k\} \) converges weakly and for almost every \( x \) in \( \Omega \) to a function \( u \). Now by Fatou's lemma,

\[
0 \leq I(u) \leq \lim_{k \to \infty} I(u_k) = \lim_{k \to \infty} J(u_k) = 0.
\]

Hence \( u \equiv 0 \). From (3.9) and 4) of lemma 3.1, we have

\[
(3.12) \quad \lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.
\]

Let \( v_k = \frac{u_k}{\|u_k\|} \) and converging weakly to \( v \). Using \( u_k \in \partial B(\Omega, f) \), (3.12), 3) of lemma 3.1 and (ii), we have

\[
1 = \lim_{k \to \infty} \int_{\Omega} \frac{f(x, u_k)}{u_k} v_k^n \, dx
\]

\[
= \int_{\Omega} f'(x, 0)v^n \, dx < \lambda_1(\Omega) \int_{\Omega} v^n \, dx \leq 1,
\]

which is a contradiction. This prove step 1.

**STEP 2.** For every \( u \in W^{1,n}_0(\Omega) \setminus \{0\} \), there exists a constant \( \gamma > 0 \) such that \( \gamma u \in \partial B(\Omega, f) \). Moreover, if

\[
(3.13) \quad \|u\|^n \leq \int_{\Omega} f(x, u)u^{n-1} \, dx,
\]

then \( \gamma \leq 1 \) and \( \gamma = 1 \) iff \( u \in \partial B(\Omega, f) \).

For \( \gamma > 0 \), define

\[
\psi(\gamma) = \frac{1}{\gamma} \int_{\Omega} f(x, \gamma u)u^{n-1} \, dx.
\]

Then, from 3) of lemma 3.1 and (ii), we have

\[
\lim_{\gamma \to 0} \psi(\gamma) = \int_{\Omega} f'(x, 0)u^n \, dx < \|u\|^n,
\]

\[
\lim_{\gamma \to \infty} \psi(\gamma) = \infty.
\]

Hence there exists \( \gamma > 0 \) such that \( \psi(\gamma) = \|u\|^n \); this implies that \( \gamma u \in \partial B(\Omega, f) \). From \((H_1)\) and \((H_2)\), it follows that \( \frac{f(x, tu)}{t} u^{n-1} \) is an
increasing function for $t > 0$. Hence, if $u$ satisfies (3.13), it follows that $\gamma \leq 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$. This proves step 2.

STEP 3. $a(\Omega, f)^n < \left( \frac{a_n}{b} \right)^{n-1}$.

Let $w \in W_0^{1,n}(\Omega)$ such that $\|w\| = 1$. From step 2, we can choose a $\gamma > 0$ such that $\gamma w \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \leq J(\gamma w) \leq \frac{\gamma^n}{n} \|w\|^n = \frac{\gamma^n}{n};$$

this implies that $a(\Omega, f) \leq \gamma$. Using again the fact that $\frac{f(x, tw)}{t} w^{n-1}$ is an increasing function of $t$ in $(0, \infty)$ and $\gamma w \in \partial B(\Omega, f)$, we have

$$\int_{\Omega} \frac{f(x, a(\Omega, f)w)}{a(\Omega, f)} w^{n-1} dx \leq \int_{\Omega} \frac{f(x, \gamma w)}{\gamma} w^{n-1} dx = 1.$$

This implies that

\begin{equation}
(3.14) \sup_{\|w\| \leq 1} \int_{\Omega} f(x, a(\Omega, f)w) w^{n-1} dx \leq a(\Omega, f). \tag{3.14}
\end{equation}

Now from (i), (3.14) and lemma 3.2 we have $a(\Omega, f)^n < \left( \frac{a_n}{b} \right)^{n-1}$. This proves the lemma.

**Lemma 3.5.** Let $f \in A(\Omega)$ and $u_0 \in \partial B(\Omega, f)$ such that $J'(u_0) \neq 0$ ($J'(u)$ denote the derivative of $J$ at $u$). Then

$$J(u_0) > \inf\{J(u); \ u \in \partial B(\Omega, f)\}.$$

**Proof.** Choose $h_0 \in W_0^{1,n}(\Omega)$ such that $\langle J'(u_0), h_0 \rangle = 1$ and, for $\alpha, t \in \mathbb{R}$, define $\sigma_t(\alpha) = \alpha u_0 - th_0$. Then

$$\lim_{t \to 0} \frac{d}{dt} J(\sigma_t(\alpha)) = -\langle J'(u_0), h_0 \rangle = -1$$

and hence we can choose $\epsilon > 0$, $\delta > 0$ such that, for all $\alpha \in [1 - \epsilon, 1 + \epsilon]$ and $0 < t \leq \delta$,

\begin{equation}
(3.15) \quad J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u_0). \tag{3.15}
\end{equation}

Let

$$\rho_t(\alpha) = \|\sigma_t(\alpha)\|^n - \int_{\Omega} f(x, \sigma_t(\alpha)) \sigma_t(\alpha)^{n-1} dx.$$
Since \( f(x, \alpha u_0) u_0^{-n-1} \) is an increasing function of \( \alpha \) and using \( u_0 \in \partial B(\Omega, f) \), by shrinking \( \epsilon \) and \( \delta \) if necessary, we have, for \( 0 < t \leq \delta \), \( \rho_t(1 - \epsilon) > 0 \) and \( \rho_t(1 + \epsilon) < 0 \). Hence there exists \( \alpha_t \) such that \( \rho_t(\alpha_t) = 0 \). Therefore \( \rho_t(\alpha_t) \) is in \( \partial B(\Omega, f) \). Hence from (3.15) we have

\[
\inf \{ J(u); u \in \partial B(\Omega, f) \} \leq J(\rho_t(\alpha_t)) < J(\alpha_t u_0) \leq \sup_{t \in \mathbb{R}} J(tu_0) = J(u_0).
\]

This proves the lemma.

**PROOF OF THE THEOREM.**

1) *Palais-Smale Condition.* Let \( C \in \left( -\infty, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \right) \) and \( \{ u_k \} \) be a sequence such that

\[
\lim_{k \to \infty} J(u_k) = C
\]

(3.16)

\[
\lim_{k \to \infty} J'(u_k) = 0.
\]

Let \( h \in W_0^{1,n}(\Omega) \), then we have

\[
\langle J'(u_k), h \rangle = \int_\Omega |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla h \, dx - \int_\Omega f(x, u_k) u_k^{n-2} h \, dx.
\]

Hence we have

(3.19)

\[
J(u_k) - \frac{1}{n} \langle J'(u_k), u_k \rangle = I(u_k).
\]

**CLAIM 1.**

(3.20)

\[
\sup_k \| u_k \| + \sup_k \int_\Omega f(x, u_k) u_k^{n-1} \, dx < \infty.
\]

Since \( \{ J(u_k) \} \) and \( \{ J'(u_k) \} \) are bounded and hence from (3.19), \( I(u_k) = O(\| u_k \|) \). Now from 5) of lemma 3.1, we have \( \int_\Omega f(x, u_k) u_k^{n-1} \, dx = O(\| u_k \|) \).

Now from (H3) it follows that

\[
\int_\Omega F(x, u_k) \, dx = O(\| u_k \|)
\]

and, by using the boundedness of \( J(u_k) \), we obtain \( \| u_k \|^n = O(\| u_k \|) \). This implies (3.20) and hence the claim.
By extracting a subsequence, we can assume that

\[(3.21) \quad u_k \rightharpoonup u_0 \text{ weakly and for almost all } x \text{ in } \Omega. \]

**CASE (I).** \( C \leq 0. \)

From Fatou's lemma and 5) of lemma 3.1, we have

\[
0 \leq I(u_0) \leq \lim_{k \to \infty} I(u_k)
\]

\[= \lim_{k \to \infty} \left\{ J(u_k) - \frac{1}{n} \langle J'(u_k), u_k \rangle \right\}
\]

\[= C. \]

Hence \( u_0 \equiv 0. \) If \( C < 0, \) no Palais-Smale sequence exists. If \( C = 0, \) then from (3.20) and 4) of lemma 3.1 we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.
\]

This proves that \( u_k \to 0 \) strongly.

**CASE (II).** \( C \in \left( 0, \frac{1}{n} \left( \frac{a_n}{b} \right)^{n-1} \right). \)

**CLAIM 2.** \( u_0 \not\equiv 0 \) and \( u_0 \in \partial B(\Omega, f). \)

Suppose \( u_0 \equiv 0. \) Then, from (3.20) and 4) of lemma 3.1, we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\}
\]

\[= nC < \left( \frac{a_n}{b} \right)^{n-1}. \]

Hence, from 3) of lemma 3.1 and (3.22), we have

\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx = 0.
\]

This implies that \( \lim_{k \to \infty} I(u_k) = 0 \) and hence from (3.19)

\[
0 < C = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle \right\} = 0
\]
which is a contradiction. Hence \( u_0 \neq 0 \). From (3.20) and 4) of lemma 3.1, taking \( h \in C_0^\infty(\Omega) \) and letting \( k \to \infty \) in (3.19), we obtain

\[
\int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \cdot \nabla h \, dx = \int_\Omega f(x, u_0) u_0^{n-2} h \, dx.
\]

By density, the above equation holds for all \( h \in W_0^{1,n}(\Omega) \). Hence, by taking \( h = u_0 \), we obtain

\[
(3.23) \quad \|u_0\|^n = \int_\Omega f(x, u_0) u_0^{n-1} \, dx.
\]

Hence \( u_0 \in \partial B(\Omega, f) \) and this proves the claim.

Now from (3.20) and claim 2, \( \{u_k, u_0\} \) satisfy all the hypotheses of the compactness lemma 3.3. Hence we have

\[
\|u_0\|^n \leq \lim_{k \to \infty} \|u_k\|^n
\]

\[
= n \lim_{k \to \infty} \left\{ J(u_k) + \int_\Omega F(x, u_k) \, dx \right\}
\]

\[
= n \lim_{k \to \infty} \left\{ I(u_k) + \frac{1}{n} (J'(u_k), u_k) + \int_\Omega F(x, u_k) \, dx \right\}
\]

\[
= \lim_{k \to \infty} \left\{ \int_\Omega f(x, u_k) u_k^{n-1} \, dx + (J'(u_k), u_k) \right\}
\]

\[
= \int_\Omega f(x, u_0) u_0^{n-1} \, dx = \|u_0\|^n.
\]

This implies that \( u_k \) converges to \( u_0 \) strongly. This proves the Palais-Smale condition.

2) Existence of Positive Solution. Since the critical points of \( J \) are the solutions of the equation (1.7) and \( J(u) = J(|u|) \) for all \( u \) in \( \partial B(\Omega, f) \) and hence in view of lemma 3.5, it is enough to prove that there exists \( u_0 \neq 0 \) such that

\[
(3.24) \quad \frac{a(\Omega, f)^n}{n} = J(u_0).
\]

Let \( u_k \in \partial B(\Omega, f) \) such that

\[
\lim_{k \to \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.
\]
Since $J(u_k) = I(u_k)$, and hence by 5) of lemma 3.1

\begin{equation}
\sup_k \int_\Omega f(x, u_k) u_n^{n-1} \, dx < \infty,
\end{equation}

(3.25)

\begin{equation}
\sup_k \|u_k\| < \infty.
\end{equation}

(3.26)

Hence we can extract a subsequence such that

$u_k \to u_0$ weakly and for almost all $x$ in $\Omega$.

CLAIM 3. $u_0 \neq 0$ and

\begin{equation}
\|u_0\|^n \leq \int_\Omega f(x, u_0) u_0^{n-1} \, dx.
\end{equation}

(3.28)

Suppose $u_0 \equiv 0$, then from (3.25) and 4) of lemma 3.1

\begin{equation}
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_\Omega F(x, u_k) \, dx \right\}
\end{equation}

\begin{equation}
= a(\Omega, f)^n.
\end{equation}

(3.29)

From lemma 3.4, we have $0 < a(\Omega, f)^n < (\frac{\alpha_n}{b})^{n-1}$. Hence, from (3.29) and 3) of lemma 3.1, we have

\begin{equation}
\lim_{k \to \infty} \int_\Omega f(x, u_k) u_k^{n-1} \, dx = 0.
\end{equation}

This implies that

\begin{equation}
0 < \frac{a(\Omega, f)^n}{n} = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} I(u_k) = 0,
\end{equation}

which is a contradiction. This proves $u_0 \neq 0$. Suppose (3.28) is false, then

\begin{equation}
\|u_0\|^n > \int_\Omega f(x, u_0) u_0^{n-1} \, dx.
\end{equation}

(3.30)

Now from (3.25), (3.30) and $0 < a(\Omega, f)^n < (\frac{\alpha_n}{b})^{n-1}$, \{u_k, u_0\} satisfy all the hypotheses of lemma 3.3. Hence

\begin{equation}
\lim_{k \to \infty} \int_\Omega f(x, u_k) u_k^{n-1} \, dx = \int_\Omega f(x, u_0) u_0^{n-1} \, dx.
\end{equation}
This implies that
\[ \|u_0\|^n \leq \lim_{k \to \infty} \|u_k\|^n = \lim_{k \to \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx \]
\[ = \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx. \]
contradicting (3.30). This proves the claim.

Now from (3.28) and step 2 of lemma 3.4, there exists \( 0 < \gamma \leq 1 \) such that \( \gamma u_0 \in \partial B(\Omega, f) \). Hence
\[ \frac{a(\Omega, f)_n}{n} \leq J(\gamma u_0) = I(\gamma u_0) \]
\[ \leq I(u_0) \leq \lim_{k \to \infty} I(u_k) \]
\[ = \lim_{k \to \infty} J(u_k) = \frac{a(\Omega, f)_n}{n}. \]

This implies that \( \gamma = 1 \) and \( u_0 \in \partial B(\Omega, f) \). Hence \( J(u_0) = \frac{a(\Omega, f)_n}{n} \) and this proves the Theorem.

4. Concluding Remarks

REMARK 4.1. (Regularity). From Di-Benedetto [6], Tolksdorf [12] and Gilbarg-Trudinger [8], any solution of (1.7) is in \( C^{1,\alpha}(\Omega) \) if \( n > 3 \) and in \( C^{2,\alpha}(\overline{\Omega}) \) if \( n = 2 \).

REMARK 4.2. Let \( f \in A(\Omega) \) and \( f'(x, 0) < \lambda_1(\Omega) \) for all \( x \in \overline{\Omega} \). We prove the existence of a solution for (1.7) under the assumption that
\[ \lim_{t \to \infty} \inf_{x \in \Omega} h(x, t)t^{n-1} = \infty. \]  

The only place where it is used is to show that \( a(\Omega, f)^n < \left( \frac{\alpha a}{b} \right)^{n-1} \). But, from lemma 3.2, this inequality holds if
\[ \frac{k_0}{b} < 1. \]

Hence the theorem is true under the less restrictive condition (4.2).
Now the question is what happens if \( \frac{b_0}{b} \geq 1 \) or the condition (4.1) is not satisfied. In this regard, we have (jointly with Srikanth - Yadava) obtained a partial result, which states that there are functions \( f \in A(\Omega) \) such that

\[
\liminf_{t \to \infty} \inf_{x \in \Omega} h(x, t)t^{n-1} = 0
\]

for which no solution to problem (1.7) exists if \( \Omega \) is a ball of sufficiently small radius. In this context, we raise the following question:

**Open Problem.** Let \( \Omega \) be a ball and \( f \in A(\Omega) \) such that \( \sup_{x \in \Omega} f'(x, 0) \leq 1 \). Is (4.2) also a necessary condition to obtain a solution to the problem (1.7).

In the case \( n = 2 \), this question is related to the question of Brézis [3]: “where is the border line between the existence and non-existence of a solution of (1.7)?”

**REMARK 4.3.** Let \( \beta \geq 0 \), then by using the norm

\[
\left( \int_{\Omega} |\nabla u|^n \, dx + \beta \int_{\Omega} |u|^n \, dx \right)^{1/n}
\]

in \( W^{1,n}_0(\Omega) \), the Theorem still holds if we replace \(-\Delta u\) by \(-\Delta u + \beta|u|^{n-2}u\) in the equations (1.7).

Due to this and using a result of Cherrier [5], it is possible to extend the Theorem to compact Riemann surfaces.

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5. - Appendix

**PROOF OF THE LEMMA 3.1.**

1) Let \( f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)}) \in A(\Omega) \). From (H4), for every \( \epsilon > 0 \), there exists a \( C(\epsilon) > 0 \) such that

\[
|f(x, t)| \leq C(\epsilon) \exp\left( (b + \epsilon)|t|^{n/(n-1)} \right)
\]

and hence, from theorem 2.1, \( f(x, u) \in L^p(\Omega) \) for every \( p < \infty \).

2) From (H4), for every \( \epsilon > 0 \), there exist positive constants \( C_1(\epsilon) \) and \( C_2(\epsilon) \) such that

\[
|f(x, t)t^{n-1}| \leq C_1(\epsilon) \exp\left( b(1 + \epsilon)|t|^{n/(n-1)} \right)
\]
Hence, if $c > 0$ such that
\[ \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} \, dx < \infty, \]
it implies that, for every $\epsilon > 0$,
\[ \sup_{\|w\| \leq 1} \int_{\Omega} \exp \left( b(1 - \epsilon)c^{n/(n-1)}|w|^{n/(n-1)} \right) \, dx < \infty. \]

Therefore, from Theorem 2.1, we have
\[ (1 - \epsilon)^{n-1}c^n \leq \left( \frac{\alpha_n}{b} \right)^{n-1}. \]

This implies that
\[ \sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} \, dx < \infty \right\} \leq \left( \frac{\alpha_n}{b} \right)^{n-1}. \]

On the other hand, if $c^n < \left( \frac{\alpha_n}{b} \right)^{n-1}$, then by choosing $\epsilon > 0$ such that
\[ (1 + \epsilon)^{2n-1}c^n < \left( \frac{\alpha_n}{b} \right)^{n-1}, \]
from Theorem 2.1 and from (5.1), we have
\[ \sup_{\|w\| \leq 1} \int_{\Omega} f(x, (1 + \epsilon)cw)w^{n-1} \, dx \]
\[ \leq C_1(\epsilon) \sup_{\|w\| \leq 1} \int_{\Omega} \exp \left[ b \left( (1 + \epsilon)c|w| \right)^{n/(n-1)} \right] \, dx < \infty \]
this proves
\[ \sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1} \, dx < \infty \right\} = \left( \frac{\alpha_n}{b} \right)^{n-1}. \]

3) Since $\lim_{k \to \infty} \|u_k\|^n < \left( \frac{\alpha_n}{b} \right)^{n-1}$, from 2) we can choose a $p > 1$ such that
\[ c_1^p = \sup_k \int_{\Omega} |f(x, u_k)|^p \, dx < \infty. \]
Let $\frac{1}{p} + \frac{1}{q} = 1$ and

$$c_2^\ell = \sup_k \int_\Omega |v_k|^{\ell q} \, dx.$$ 

Then, for any $N > 0$ and by Holder’s inequality,

$$\left| \int_{|u_k| > N} \frac{f(x, u_k)}{u_k} v_k^\ell \, dx \right| \leq \frac{1}{N} \int_\Omega |f(x, u_k)| |v_k^\ell| \, dx \leq \frac{c_1 c_2}{N}.$$

Hence

$$\int_\Omega \frac{f(x, u_k)}{u_k} v_k^\ell \, dx = \int_{|u_k| \leq N} \frac{f(x, u_k)}{u_k} v_k^\ell \, dx + O(1/N).$$

By dominated convergence theorem, letting $k \to \infty$ and then $N \to \infty$ in the above equation, it implies that

$$\lim_{k \to \infty} \int_\Omega \frac{f(x, u_k)}{u_k} v_k^\ell \, dx = \int_\Omega \frac{f(x, u)}{u} v^\ell \, dx.$$ 

4) Let $N > 0$, then

$$\int_{|u_k| > N} f(x, |u_k|)|u_k|^{n-2+\tau} \, dx \leq \frac{1}{N^{1-\tau}} \int_\Omega f(x, |u_k|)|u_k|^{n-1} \, dx$$

$$= \frac{1}{N^{1-\tau}} \int_\Omega f(x, u_k)u_k^{n-1} \, dx = O \left( \frac{1}{N^{1-\tau}} \right).$$

Hence

$$\int_\Omega f(x, |u_k|)|u_k|^{n-2+\tau} \, dx = \int_{|u_k| \leq N} f(x, |u_k|)|u_k|^{n-2+\tau} \, dx + O \left( \frac{1}{N^{1-\tau}} \right).$$

By dominated convergence theorem, letting $k \to \infty$ and $N \to \infty$ in the above equation, we obtain

$$\lim_{k \to \infty} \int_\Omega f(x, |u_k|)|u_k|^{n-2+\tau} \, dx = \int_\Omega f(x, |u|)|u|^{n-2+\tau} \, dx. \quad (5.3)$$

Now from $(H_3)$,

$$|F(x, t)| \leq M(1 + |f(x, t)| |t|^{n-2+\tau}).$$
for some $\sigma \in [0, 1)$. Hence, from (5.3) and the dominated convergence theorem,

$$\lim_{k \to \infty} \int_{\Omega} F(x, u_k) \, dx = \int_{\Omega} F(x, u) \, dx.$$  

5) From (H2) we have, for $t > 0$,

$$\frac{\partial}{\partial t} \left[ f(x, t)t^{n-1} - nF(x, t) \right] = \left[ f'(x, t) - \frac{f(x, t)}{t} \right] t^{n-1} > 0. \quad (5.4)$$

Therefore from (H1) and (5.4), $f(x, t)t^{n-1} - nF(x, t)$ is an even positive function and increasing for $t > 0$. This implies that $I(u) \geq 0$ and $I(u) = 0$ iff $u \equiv 0$. From (H3) we have

$$nI(u) = \int_{\Omega} \left[ f(x, u)u^{n-1} - nF(x, u) \right] \, dx$$

$$\geq \int_{\Omega} \left[ f(x, u)u^{n-1} - nM(1 + |f(x, u)| |u|^{n-2+\sigma}) \right] \, dx$$

$$\geq C_1 + \frac{1}{2} \int_{|u| \geq C_2} f(x, u)u^{n-1} \, dx$$

for some constants $C_1$ and $C_2 > 0$. This implies that there exists a constant $M_1 > 0$ such that

$$\int_{\Omega} f(x, u)u^{n-1} \, dx \leq M(1 + I(u)).$$

This proves the lemma 3.1.

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