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1. - Introduction

It is well-known that the full regularity of the elliptic systems

$$D_\alpha A^\alpha(Du) = 0$$

in two dimensions can (under standard assumptions) be proved by using $W^{1,2+d}_-$ estimates for linear elliptic systems with $L^\infty$ coefficients. (See, for example, M. Giaquinta [2]). The purpose of this paper is to show that a similar method can be used when dealing with nonlinear parabolic systems

$$\frac{\partial u_i}{\partial t} = D_\alpha A^\alpha(Du).$$

The idea is to show that $\frac{\partial u_i}{\partial t}$ is bounded in $L^\infty(-T, 0; L^{2d}(\Omega))$ and then apply the theory of elliptic systems. The required estimate is obtained by using estimates for solutions of linear parabolic systems with $L^\infty$ coefficients. (See Lemma 1). In the two-dimensional case we get full regularity.

2. - Preliminaries

Let $n \geq 2$, $N \geq 1$. We shall be dealing with open sets $Q = \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $T > 0$. A typical point of $\mathbb{R}^{n+1}$ is denoted by $z = (x, t), x \in \mathbb{R}^n, t \in \mathbb{R}$.

For $\delta > 0$ we let

$$\Omega_\delta = \{x \in \Omega, \text{ dist } (x, \partial \Omega) > \delta \}$$

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and
\[ Q_\delta = \Omega_\delta \times (-T + \delta, 0). \]
For \( x \in \mathbb{R}^n \) and \( \rho > 0 \) we define
\[ B_{x,\rho} = \{ y \in \mathbb{R}^n, |x - y| < \rho \}. \]
If \( a, b \in \mathbb{R} \), we denote by \( a \land b \) the minimum of the two numbers.

The Sobolev spaces \( W^k_p \), \( W_0^k \) are defined in the standard way.

The space \( L^2(\Omega) \) is denoted by \( W^{1,0}_2(\Omega) \). The norm \( \| \cdot \|_{2,\Omega} \) on \( W^{1,0}_2(Q) \) is defined by
\[ [u]_{2,\Omega} = \left\| \int_Q \left( |u|^2 + \sum_{i=1}^n |D_i u|^2 \right) \right\|^{\frac{1}{2}}. \]

The spaces \( L^\infty(-T, 0; L^p(\Omega)) \), \( p \geq 1 \) will be denoted by \( L^{p,\infty}(Q) \) and the corresponding norm is denoted by \( \| \cdot \|_{p,\infty,\Omega} \).

The usual \( L^p \)-norm is denoted by \( \| \cdot \|_{p,\Omega} \).

Let us consider the nonlinear parabolic system

(1) \[ \frac{\partial u_i}{\partial t} - D_\alpha A_\alpha^i(Du) = 0 \quad (i = 1, \ldots, N) \]

where \( u = (u_1, \ldots, u_N) \), \( Du = (D_\alpha u_i)_{1 \leq i \leq N, 1 \leq \alpha \leq n} = (\frac{\partial u_i}{\partial \xi_\beta})_{1 \leq i \leq N, 1 \leq \alpha \leq n} \) is the gradient matrix of \( u \) and the summation over repeated indexes is understood.

We shall suppose that the functions \( A_\alpha^i \) have continuous derivatives satisfying

(2) \[ \left\{ \sum_{i,j} \sum_{\alpha,\beta} \left| \frac{\partial A_\alpha^i}{\partial \xi_\beta^j} (\xi) \right|^2 \right\} \leq M \]

and
\[ \frac{\partial A_\alpha^i}{\partial \xi_\beta^j} (\xi) \pi_\alpha^i \pi_\beta^j \geq \nu |\pi|^2, \quad \nu > 0, \]

for every \( \xi, \pi \in \mathbb{R}^n \).

(Of course, for higher regularity results we have to assume higher smoothness of \( A_\alpha^i \)).

By a weak solution of (1) we mean a function \( u \in W^{1,0}_2(Q) \) satisfying
\[ \int_Q \left( u_i \frac{\partial \varphi_i}{\partial t} - A_\alpha^i(Du) D_\alpha \varphi_i \right) \, dz = 0 \]

for every \( \varphi \in W^{1,0}_2(Q) \).
We shall also be dealing with linear strongly parabolic systems

\[ \frac{\partial u_i}{\partial t} - D_\alpha a_{ij}^\alpha D_\beta u_j = 0 \quad (i = 1, \ldots, N) \]

where \( a_{ij}^\alpha = a_{ij}^\alpha(z) \) are \( L^\infty \)-functions in \( Q \) satisfying for almost every \( z \in \Omega \) the conditions

\[ \left\{ \sum_{i,j} \sum_{\alpha,\beta} |a_{ij}^\alpha|^2 \right\}^{1/2} \leq M \]

and

\[ a_{ij}^\alpha \xi_j^\alpha \xi_i \geq \nu |\xi|^2 \]

for every \( \xi \in \mathbb{R}^{nN} \). By a weak solution of (4) we mean a function \( u \in W^{1,0}_2(Q) \) satisfying

\[ \int_Q \left( u_i \frac{\partial \varphi_i}{\partial t} - a_{ij}^\alpha D_\beta u_j D_\alpha \varphi_i \right) dz = 0 \]

for every \( \varphi \in W^1_2(Q) \).

We shall use the following well-known results.

(i) If \( u \) is a weak solution of (1) or (4), then \( u \) is continuous in time with respect to the \( L^2 \)-norm. More precisely, if \( \Omega' \subset \subset \Omega \), then the map \( t \rightarrow u(\cdot, t) \) from \((-T, 0) \) into \( L^2(\Omega') \) is continuous. (See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva [4], Chap. 3, Lemma 4.3).

(ii) We have the imbedding

\[ L^{2,\infty}(Q) \cap W^{1,0}_2(Q) \hookrightarrow L^\Phi(Q) \]

where

\[ q_0 = \begin{cases} \frac{2(n+2)}{n}, & \text{if } n > 2 \\ \text{is any number } \in [1,4) & \text{if } n = 2 \end{cases} \]

(See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva [4], Chap. 2).

We denote by \( c_i \) various constants. The value of these constants can depend on \( \nu, M, \Omega, T, n \) and \( N \). The dependence on additional parameters will be indicated.
3. - $L^p$-estimates

The first statement of the following Lemma is well known (see, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3). The second statement will be used for the $L^\infty(-T,0; L^{2^*}(\Omega))$-estimate mentioned in the introduction.

**LEMMA 1.** Let $u$ be a weak solution of the linear system (4). Then, for any $\delta > 0$,

- (i) $u \in L^{2,\infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$

and

- (ii) For every $p \in [2, (2 + \frac{1}{N^*}) \wedge q_0)$ the function $u$ belongs to $L^{p,\infty}(Q_\delta)$ and

$$
\|u\|_{p,\infty,Q_\delta} \leq c_2(\delta, p)\|u\|_{2,Q}.
$$

**PROOF.** Let $\gamma \geq 1$ and let $k > 0$ be such that $\text{meas}\{z \in Q, |u(z)| = k\} = 0$. Define $g_k : [0, \infty) \to \mathbb{R}$ by

$$
g_k(t) = \begin{cases} 
t^\gamma, & \text{if } 0 \leq t \leq k, \\
k^\gamma + \gamma k^{\gamma-1}(t - k), & \text{if } t \geq k.
\end{cases}
$$

Clearly $g'_k(t) = \gamma(t \wedge k)^{\gamma-1}$ and

$$
g''_k(t) = \begin{cases} 
\gamma(\gamma - 1)t^{\gamma-2}, & \text{if } 0 < t < k, \\
0, & \text{if } t > k.
\end{cases}
$$

Define also the function $\eta^k : \mathbb{R}^N \to \mathbb{R}$ by

$$
\eta^k(u) = g_k(|u|^2) = \begin{cases} 
t^\gamma, & \text{if } |u| < k, \\
k^\gamma + \gamma k^{\gamma-1}(|u| - k), & \text{if } |u| > k.
\end{cases}
$$

We have

$$
\eta^k_{u_i}(u) = \frac{\partial \eta^k}{\partial u_i}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}u_i
$$

and

$$
\eta^k_{u_i u_j}(u) = \frac{\partial^2 \eta^k}{\partial u_i \partial u_j}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}(\delta_{ij} + 2(\gamma - 1)d_{ij}(u)).
$$

In the second formula we assume $|u|^2 \neq k$ and

$$
d_{ij}(u) = \begin{cases} 
0, & \text{if } |u|^2 > k, \\
\frac{u_i u_j}{|u|^2}, & \text{if } |u|^2 < k.
\end{cases}
$$
Let $\omega_\epsilon$ be a family of symmetric mollifying functions satisfying

$\omega_\epsilon \in D(\mathbb{R})$, $\omega_\epsilon(t) \geq 0$,
$\omega_\epsilon(t) = \omega_\epsilon(-t)$,
support $\omega_\epsilon \subset (\epsilon, -\epsilon)$,
$\int_{\mathbb{R}} \omega_\epsilon = 1$.

For $f \in L^1(Q)$ let us denote by $(f)_\epsilon$ the function defined a.e. in $Q$ by

$$(f)_\epsilon(x,t) = \int_{\mathbb{R}} f(x,t-s)ds.$$  

(We extend $f$ by zero outside $Q$). Let $\psi \in W^{1,1}_0(\Omega \times (-T + \epsilon, -\epsilon))$. Following E. Giusti, M. Giaquinta [3] we set $\varphi = (\psi)_\epsilon$ in ($*$) and we see that

$$
\int_Q \frac{\partial (u_\epsilon)}{\partial t} \psi_i dz = - \int_Q (a_{ij}^{\alpha \beta} D_\beta u_j)_\epsilon D_\alpha \psi_i dz.
$$

(5)

Let $\theta \in D(\Omega)$ with $0 \leq \theta \leq 1$ and $\theta = 1$ on $\Omega_\delta$ and let $\rho \in D(-T, 0)$, $\rho \geq 0$.
For $\epsilon$ sufficiently small we can use (5) with

$$
\psi_i = \eta^k_{u_i}(u_\epsilon) \theta^2 \rho^2
$$

to get

$$
\int_Q \left( \frac{\partial}{\partial t} \eta^k_{u_i}(u_\epsilon) \right) \theta^2 \rho^2 dz = - \int_Q (a_{ij}^{\alpha \beta} D_\beta u_j)_\epsilon D_\alpha (\eta^k_{u_i}(u_\epsilon)) \theta^2 \rho^2 dz
- \int_Q (a_{ij}^{\alpha \beta} D_\beta u_j)_\epsilon \eta^k_{u_i}(u_\epsilon) 2\theta D_\alpha \theta \rho^2 dz.
$$

Integrating by parts on the left-hand side, letting $\epsilon \to 0$ and then using the chain rule for the derivative $D_\alpha (\eta^k_{u_i}(u_\epsilon))$ (which is legal) we see that

$$
\int_Q \eta^k(u) \theta^2 (\rho^2)' dz = - \int_Q a_{ij}^{\alpha \beta} D_\beta u_j \eta^k_{u_i}(u) D_\alpha u_i \theta^2 \rho^2 dz
- \int_Q a_{ij}^{\alpha \beta} D_\beta u_j \eta^k_{u_i}(u) 2\theta D_\alpha \theta \rho^2 dz.
$$

(6)

Since $|d_{ij}| \leq 1$ we see that if $0 \leq 2(\gamma - 1) < \frac{K}{NM}$ then the matrix

$a_{ij}^{\alpha \beta} = a_{ij}^{\alpha \beta}(\theta_i + 2(\gamma - 1)d_{ii}(u))$ satisfies the condition $(2')$ with $\nu$ replaced by $\nu_1 = \nu - 2(\gamma - 1)NM$. 


We can estimate the right-hand side of (6) by

\[ \nu_1 \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 dz \]

\[ + 2M \left\{ \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \right\}^{\frac{1}{2}} \]

\[ \times \left\{ \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz \right\}^{\frac{1}{2}} \]

\[ \leq -\frac{\nu_1}{2} \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 dz \]

\[ + \frac{4M^2}{\nu_1} \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz. \]

Let \( t_1 \in (-T + \delta, 0) \). As we have remarked in Section 2, under our assumptions the function \( t \rightarrow u(\cdot, t) \) is continuous mapping of \((-T, 0)\) into \(L^2(\Omega_\delta)\). Hence we can use (6) with \( \rho \) defined by

\[ \rho^2(t) = \begin{cases} 
0 & \text{if } t \in (-T, -T + \frac{\delta}{2}) \\
\frac{2}{\delta} (T + t - \frac{\delta}{2}) & \text{if } t \in (-T + \frac{\delta}{2}, -T + \delta) \\
1 & \text{if } t \in (-T + \delta, t_1) \\
0 & \text{if } t \in (t_1, 0).
\end{cases} \]

We get

\[ \int_Q \eta^h(u(x, t_1)) \theta(x) dx + \frac{\nu_1}{2} \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \]

\[ \leq c_3 \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz + \int_{\Omega \times (-T, t_1)} (\rho^2 \gamma^k(u) \theta^2) dz. \]

Letting \( \gamma = 1 \) we get (i).

We can use (i) and the imbedding

\[ L^{2,\infty}(Q) \cap W^{1,0}_2(Q) \hookrightarrow L^p(Q) \]

to infer that \( \|u\|_{\infty, Q_1} \leq c_4(\delta) \|u\|_{2, Q} \). Using this and letting \( k \to \infty \) in (7) we get (ii) with \( p = 2\gamma \).
LEMMA 2. Let $u$ be a weak solution of the nonlinear system (1). Then $u \in W_2^1(Q_\delta)$, the derivatives $D_i u$, $i = 1, \ldots, n$ and $D_{n+1} u = \frac{\partial u}{\partial t}$ belong to the space $L^{p, \infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$ and for each $i = 1, \ldots, n, n + 1$

$$
\|D_i u\|_{p, \infty, Q_\delta} + \|D_{n+1} u\|_{2, Q_\delta} \leq [u]_{2, Q}.
$$

PROOF. As above, we denote by $Du$ the vector $(D_1 u, \ldots, D_n u) \in \mathbb{R}^{nN}$. Let us fix an index $r$, $1 \leq r \leq n + 1$ and let $\varepsilon_r \in \mathbb{R}^n \times \mathbb{R}$ be the $r$-th vector of the canonical basis. Let $\delta' > 0$. For $0 < h < \delta'$ let

$$
u_h(z) = h^{-1}[u(z) - u(z - \varepsilon_r)].$$

Define the functions $a_{hij}^{\alpha\beta} \in L^\infty(Q_{\delta'})$ for a.e. $z \in Q_{\delta'}$ by

$$a_{hij}^{\alpha\beta}(z) = \int_0^1 A_{i,\alpha}^{\alpha'}(Du(z) - hDu_h(z) + \tau hDu_h(z))d\tau.$$ 

It is not difficult to see that $u_h$ is the weak solution of the linear system

$$\frac{\partial u_h}{\partial t} - D_\alpha a_{hij}^{\alpha\beta} D_\beta u_{hij} = 0$$

in $Q_{\delta'}$. The functions $a_{hij}^{\alpha\beta}$ clearly satisfy the conditions (2') and (3'). Hence, by Lemma 1

$$
\|u_h\|_{p, \infty, Q_{\delta'}} \leq c_6(\delta', p)\|u_h\|_{2, Q_{\delta'}}
$$

$$
\|Du_h\|_{2, Q_{\delta'}} \leq c_7(\delta')\|u_h\|_{2, Q_{\delta'}}.
$$

Suppose first $1 \leq r \leq n$. In this case the difference is taken in the direction of the space variables. Since $u \in W_2^{1,0}(Q)$, we have

$$
\|u_h\|_{2, Q_{\delta'}} \leq \|D_r u\|_{2, Q}. 
$$

Using Nirenberg’s Lemma we see from (8) that $Du \in W_2^{1,0}(Q_{\delta'})$ and

$$
\|D_r u\|_{p, \infty, Q_{\delta'}} \leq c_6(\delta', p)\|D_r u\|_{2, Q_{\delta'}}
$$

$$
\|D^2 u\|_{2, Q_{\delta'}} \leq c_7(\delta')\|D_r u\|_{2, Q_{\delta'}}.
$$

for every $1 \leq r \leq n$. Now let $r = n + 1$. Following S. Campanato [1] we notice that we can use equation (1) and the $L^2$-estimate of $D_\alpha D_\beta u$ obtained above to infer that $\frac{\partial u}{\partial t} \in L^2(Q_{2\delta'})$ and

$$
\frac{\partial u}{\partial t} \leq c_8(\delta')\|Du\|_{2, Q}. 
$$
Now we can use (8) with $Q$ replaced by $Q_{2\delta'}$ and using (11) we get by the same argument as above

$$\| \frac{\partial u}{\partial t} \|_{p, \infty, Q_{2\delta'}} \leq c_9(\delta', p)\| Du \|_{2, Q}$$

$$\| D_\delta \frac{\partial u}{\partial t} \|_{2, Q_{2\delta'}} \leq c_9(\delta')\| Du \|_{2, Q}.$$ 

The proof is finished.

**THEOREM 1.** Let $u$ be a weak solution of the system (1) and let $p$ be the exponent from Lemma 1. Then for each $\delta > 0$

$$\frac{\partial u}{\partial t} \in L^{p, \infty}(Q_{\delta})$$

and

$$u \in L^\infty(-T + \delta, 0; W^2_2(\Omega_{\delta}))$$

for some $q = q(\nu, M, p, \delta)$ with $2 < q < p$. Moreover

$$\| u \|_{L^\infty(-T + \delta, 0; W^2_2(\Omega_{\delta}))} + \| \frac{\partial u}{\partial t} \|_{p, \infty, Q_{\delta}} \leq c_{10}(\delta, p, q)\| u \|_{2, Q}.$$ 

**PROOF.** Let $\delta' > 0$. We notice that $u$ can be considered as a weak solution of the linear system (4) with

$$\alpha^\beta_{ij}(\tau) = \int_0^1 A^\alpha_{ij}(r Du(z)) d\tau.$$ 

(See, for example S. Campanato [1]). Using this and Lemma 1 we get estimates for the norms $\| u \|_{p, \infty, Q_{\delta'}}$ and $\| u \|_{2, Q_{\delta'}}$. Now we can use Lemma 2 to get estimates of the norms $\| Du \|_{p, \infty, Q_{2\delta'}}$, $\| \frac{\partial u}{\partial t} \|_{p, \infty, Q_{2\delta'}}$. Lemma 2 also implies $D_\alpha D^\beta u \in L^2(Q_{2\delta'})$. We see that equation (1) is satisfied pointwise almost everywhere in $Q_{2\delta'}$ and that for almost every $t \in (-T + 2\delta', 0)$ the function $u(\cdot, t)$ belongs to $W^2_2(\Omega_{2\delta'})$ and is the weak solution of the elliptic system

$$D_\alpha A^\alpha_{ij}(Du) = \frac{\partial u_i}{\partial t}$$

in $\Omega_{2\delta'}$. We can now use well-known $L^p$-estimates for elliptic systems (see Lemma 3 below). The proof is finished.

**LEMMA 3.** Let $p > 2$ and let $g \in L^p(\Omega)$. Let $u \in W^1_2(\Omega)$ be a weak solution of the elliptic system

$$(12) \quad D_\alpha A^\alpha_{ij}(Du) = g_i \quad i = 1, \ldots, n$$
Then there exists \( q = q(v, M, p) > 2 \) such that \( u \in W^{2,q}_{q,\text{loc}}(\Omega) \). Moreover, for every \( \delta > 0 \)
\[
\|u\|_{W^{2,q}_{q}(\Omega)} \leq c_{11}(\nu, M, p, q, \delta)(\|u\|_{W^{2,q}_{q}(\Omega)} + \|g\|_{p,\Omega}).
\]

**Proof.** Using the standard difference quotient technique, it is not difficult to verify that the following computations are legal.

Let \( 1 \leq s \leq n \). We let \( v = D_{s}u \) and take the \( s \)-th derivative of (12). We get
\[
D_{\alpha}a_{ij}^{\alpha\beta}D_{\beta}v_{j} = D_{s}g_{i}
\]
where \( a_{ij}^{\alpha\beta}(z) = A_{i,j}^{\alpha\beta}(Du(z)) \).

This implies
\[
\frac{\nu}{2}\int_{\Omega} \xi^{2}|Dv|^{2}dx \leq c_{12}(\nu, M)\int_{\Omega}(|v|^{2}|D\xi|^{2} + |g|^{2})dx
\]
for every \( \xi \in D(\Omega) \) (Cacciopoli's inequality). The required estimate can now be obtained by using the technique of reverse Hölder inequalities. (See, for example, M. Giaquinta [2], Chap. 5, Theorem 2.2). The proof is finished.

**Corollary.** Let the assumptions of Theorem 1 be satisfied.

(i) If \( n \leq 4 \), then \( u \) is Hölder continuous in \( Q \).

(ii) If \( n \leq 2 \), then \( Du \) is Hölder continuous in \( Q \).

(iii) If \( n \leq 2 \) and the functions \( A_{ij}^{\alpha\beta} \) are smooth, then the solution \( u \) is smooth.

**Remark.** If \( n > 3 \), then \( Du \) may not be continuous. Examples are provided by nonregular solutions of elliptic systems. These can be found in J. Nečas [5].

**Proof of the Corollary.** Let \( \delta > 0 \).

(i) Since \( W^{2,4}(\Omega_{\delta/2}) \hookrightarrow C^{0,\alpha}(\Omega_{\delta}) \) with \( \alpha = (2 - \frac{n}{4}) \wedge 1 \), we have \( u \in L^{\infty}(-T + \delta, 0; C^{0,\alpha}(\Omega_{\delta})) \).

Since we have also \( \frac{\partial u}{\partial t} \in L^{2}(Q_{\delta}) \), \( u \) is Hölder continuous by Lemma 4 below.

(ii) In this case we have \( W^{2,4}(\Omega_{\delta/2}) \hookrightarrow C^{1,\beta}(\Omega_{\delta}) \) \( \beta = 1 - \frac{n}{4} \).

Hence \( Du \in L^{\infty}(-T + \delta, 0; C^{0,\beta}(\Omega_{\delta})) \). Using the Hölder continuity of \( u \) it is easy to see that in fact \( Du(-, t) \in C^{0,\beta}(\Omega_{\delta}) \) for every \( t \in (-T + \delta, 0) \), the \( C^{0,\beta} \)-norm being bounded independently of \( t \).

Now we can use Lemma 3.1, Chap. 2 from O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva [4] to infer that \( Du \) is Hölder continuous in \( Q_{\delta} \).

(iii) The higher regularity follows in the standard way from the theory of linear equations.
LEMMA 4. Let $\alpha > 0$, $q > 1$, $\delta > 0$ and suppose
\[ u \in L^\infty(-T,0;C^{0,\alpha}(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^q(Q). \]

Denote $K_1 = \|u\|_{L^\infty(-T,0;C^{0,\alpha}(\Omega))}$, $K_2 = \|\frac{\partial u}{\partial t}\|_{L^q(Q)}$. Then there exists $K = K(K_1, K_2, \delta)$ such that
\[ |\tilde{u}(x,t_1) - \tilde{u}(x,t_2)| \leq K|t_1 - t_2|^\beta \]
for every $x \in \Omega_\delta$ and every $t_1$, $t_2 \in (-T,0)$, where $\beta = \frac{\alpha/q'}{\alpha + n/q}$, $q' = \frac{q}{q-1}$ and $\tilde{u}$ is a suitable representative of $u$.

PROOF. Suppose first that $u$ is continuous. Let $x \in \Omega_\delta$ and let $0 < \rho < \delta$. Define
\[ w_\rho(t) = \frac{1}{|B_{x,\rho}|} \int_{B_{x,\rho}} u(y,t)dy. \]

It is easy to see that $w_\rho$ is bounded in $L^q(-T,0)$ by $c_1\rho^{-\frac{n}{2}}K_2$.

Let $t_1, t_2 \in (-T,0)$. We can write
\[ |u(x,t_1) - u(x,t_2)| \leq |u(x,t_1) - w_\rho(t_1) + w_\rho(t_1) - w_\rho(t_2) + w_\rho(t_2) - u(x,t_2)| \]
\[ \leq 2K_1\rho^n + c_1 K_2 \rho^{-\frac{n}{2}}|t_1 - t_2|^{\frac{1}{2}}. \]

The proof is easily finished by using this inequality with $\rho = |t_1 - t_2|^\frac{1}{2}$.

REMARK. It is not difficult to see that if the boundary of $\Omega$ is sufficiently regular (say, lipshitzian), then $K$ can be chosen independent of $\delta$.

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