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Rearrangement and continuity properties of $BMO(\phi)$ functions on spaces of homogeneous type


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Rearrangement and Continuity Properties of
*BMO*(ϕ) Functions on Spaces of Homogeneous Type

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In this note we study the behaviour of the non-increasing rearrangement of functions satisfying conditions on their mean oscillation over balls on spaces of homogeneous type. We extend a result of S. Spanne [S] and as a corollary we get extensions of the results of Campanato [C], John and Nirenberg [J-N] and Meyers [M]. The central tool is an extension of A.P. Calderón’s proof of John-Nirenberg Lemma [N]. Related results can be found in [M-S].

Let X be a set. A symmetric function {0} is a quasi-distance if \( d(x, y) = 0 \) iff \( x = y \) and there is a constant \( K \) such that \( d(x, z) \leq K[d(x, y) + d(y, z)] \) for every \( x, y \) and \( z \) in X. The ball with center \( x \in X \) and radius \( r > 0 \) is the set \( B(x, r) = \{ y \in X : d(x, y) < r \} \). We shall say that a measure \( \mu \) defined on a \( \sigma \)-algebra containing the balls satisfies a doubling condition if and only if there is a positive constant \( A \) such that

\[
0 < \mu(B(x, 2r)) \leq A_\mu(B(x, r)) < \infty,
\]

for every \( x \in X \) and every \( r > 0 \). If \( d \) is a quasi-distance on \( X \) and \( \mu \) satisfies a doubling condition, then we say, following [C-W], that \( (X, d, \mu) \) is a space of homogeneous type.

Let \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be an increasing function satisfying the \( \Delta_2 \) Orlicz’s condition \( \phi(2r) \leq C\phi(r) \) for some positive constant \( C \) and every \( r > 0 \) (see (K-R)). We say that a locally integrable function \( f : X \rightarrow \mathbb{R} \) belongs to the class \( BMO(\phi) \) if and only if there exists a positive constant \( D \) such that the inequality

\[
\frac{1}{\mu(B)} \int_B |f - m_B(f)| \, d\mu \leq D\phi(\tau(B)),
\]

holds for every ball \( B \) in \( X \), where \( \tau(B) \) is the radius of the ball \( B \) and \( m_B(f) = \mu(B)^{-1} \int_B f \, d\mu \).

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Given a ball $B$ and a non-negative measurable function $g$ on $B$, we write $\eta_B$ for the distribution function of $g$ on $B$, i.e.

$$\eta_B(s) = \mu(\{x \in B : g(x) > s\}), \quad s \geq 0;$$

$$\psi_B(t) = \sup\{\sigma : \eta_B(\sigma) > t\}.$$ 

The functions $\psi_B$ and $g$ have the same distribution function and, consequently, $\psi_B$ contains the integral properties of $g$. On the other hand, while $g$ is a function defined on the abstract space $X$, $\psi_B$ is a function of a real variable. For the basic properties of the rearrangement see (Z).

Given a ball $B = B(x, r)$ we write $\breve{B}$ for $B(x, 2Kr)$ and $\psi_B$ for the non-increasing rearrangement of the function $f(x) - I$ on $B$.

The main result in this note is the following theorem.

**THEOREM.** Let $(X, d, \mu)$ be a space of homogeneous type such that continuous functions are dense in $L^1(X, d, \mu)$. Then a function $f$ belongs to $BMO(\phi)$ if and only if there exist positive constants $\alpha$, $\beta$ and $\gamma$ such that for every ball $B = B(x, r)$ the inequality

$$\psi_B(t) \leq \beta \left( \int_{\frac{r}{\gamma K^2}}^{r} \frac{\phi(\xi)}{\xi d\xi} \right)^{\frac{1}{\alpha}}$$

holds for every $t \in (0, \gamma \mu(B))$.

**COROLLARY.** Let $(X, d, \mu)$ be as in the Theorem.

(a) A locally integrable function $f$ is of bounded mean oscillation ($\phi \equiv 1$) if and only if there exist positive constants $\beta$ and $\gamma$ such that for every ball $B = B(x, r)$ the inequality

$$\psi_B(t) \leq \beta \log \left( \frac{\gamma 5 K^2 \mu(B)}{t} \right)$$

holds for every $t \in (0, \gamma \mu(B))$.

(b) If $\int_{0}^{1} \frac{\phi(\xi)}{\xi} d\xi < \infty$, then a function $f$ in $BMO(\phi)$ is continuous and

$$|f(x) - f(y)| \leq C \int_{0}^{d(x,y)} \frac{\phi(\xi)}{\xi} d\xi,$$

which in the special case of $\phi(\xi) = \xi^\alpha$ is equivalent to $|f(x) - f(y)| \leq C(d(x,y))^\alpha$.

The following lemma contains three simple but useful properties of $BMO(\phi)$ functions.
LEMMA.
(1). If \( f \in BMO(\phi) \), then \( |f| \in BMO(\phi) \).

(2). Let \( f \) be a locally integrable function on \( X \). If there is a constant \( D \) such that for every ball \( B \) there exists a constant \( m_B \) satisfying
\[
\frac{1}{\mu(B)} \int_B |f - m_B| \, d\mu \leq D\phi(r(B)),
\]
then \( f \in BMO(\phi) \).

(3). Let \( \bar{B} \) denote the ball with the same center as \( B \) and twice its radius. If \( f \in BMO(\phi) \), then there exists a constant \( \bar{D} \) such that the inequality
\[
\frac{1}{\mu(B)} \int_B |f - m_B(f)| \, d\mu \leq \bar{D}\phi(r(B)),
\]
holds for every ball \( B \) in \( X \).

The next covering Lemma is a slight modification of that in [C-W].

(4). COVERING LEMMA. Let \( (X, d, \mu) \) be a space of homogeneous type. Let \( \mathcal{B} = \{ B_\alpha = B(x_\alpha, r_\alpha) : \alpha \in \Gamma \} \) be a family of balls in \( X \) such that \( \bigcup_{\alpha \in \Gamma} B_\alpha \) is bounded. Then there exists a sequence of disjoint balls \( \{ B_i \} \subset \mathcal{B} \) such that for every \( \alpha \in \Gamma \) there exists \( i \) satisfying \( r_\alpha \leq 2r_i \) and \( B_\alpha \subset B(x_i, 5K^2r_i) \).

From now on \( B_0 = B(x_0, r_0) \) is a given ball in \( (X, d, \mu) \) and \( f \) a function in \( BMO(\phi) \) such that \( m_{B_0}(2K_0) = 0 \). Set \( M = 5K^2 \) and \( \lambda_j = \sum_{k=0}^{j-1} \phi \left( \frac{r_0}{M^k} \right) \).

LEMMA. There is a constant \( C_1 \) depending only on \( K, A, C \) and \( D \) such that the inequality
\[
m_{B(z, r)}(|f|) < C_1 \lambda_i
\]
holds for every \( z \in B_0 \) and every \( r \in \left[ \frac{r_0}{M^{i+1}}, \frac{r_0}{M^i} \right] \).

PROOF. Since, for \( r \in \left[ \frac{r_0}{M^{i+1}}, \frac{r_0}{M^i} \right] \), we have
\[
m_{B(z, r)}(|f|) \leq \frac{\mu \left( B \left( x, \frac{r_0}{M^i} \right) \right)}{\mu \left( B \left( x, \frac{r_0}{M^{i+1}} \right) \right)} m_{B \left( x, \frac{r_0}{M^i} \right)} \left( \frac{r_0}{M^i} \right) |f|,
\]
inequality (5) will follow if we prove that there is a constant \( C_2 \) depending
only on $K, A, C$ and $D$ such that the inequality

\[(6) \quad m_{B\left(x, \frac{r_0}{M^i}\right)}(\|f\|) \leq C_2 \lambda_i \]

holds for every $X \in B_0$ and every $i \in \mathbb{N}$. In order to prove (6), let us first observe that for $x \in B_0$ we have

\[B\left(x, \frac{r_0}{M^i}\right) \subset B\left(x, \frac{r_0}{M^{i-1}}\right) \subset \cdots \subset B\left(x, \frac{r_0}{M}\right) \subset B(x_0) \subset B(x_0, 2Kr_0).\]

From (1), it follows that

\[
\left| m_{B\left(x, \frac{r_0}{M^i}\right)}(\|f\|) - m_{B\left(x_0, 2Kr_0\right)}(\|f\|) \right|
\leq \sum_{h=1}^{i} \left| m_{B\left(x, \frac{r_0}{M^h}\right)}(\|f\|) - m_{B\left(x_0, 2Kr_0\right)}(\|f\|) \right|
+ \left| m_{B(x,r_0)}(\|f\|) - m_{B\left(x_0, 2Kr_0\right)}(\|f\|) \right|
\]

\[(7) \quad \leq \sum_{h=1}^{i} \frac{\mu(B(x, \frac{r_0}{M^h}))}{\mu(B(x_0, 2Kr_0))} \int_{B\left(x_0, \frac{r_0}{M^h}\right)} \left| f \right| - m_{B\left(x_0, 2Kr_0\right)}(\|f\|) \, d\mu
+ \frac{1}{\mu(B(x,r_0))} \int_{B(x,r_0)} \left| f \right| - m_{B\left(x_0, 2Kr_0\right)}(\|f\|) \, d\mu\]

\[
\leq 2D \left\{ \sum_{h=1}^{i} \frac{\mu(B(x, \frac{r_0}{M^h}))}{\mu(B(x_0, 2Kr_0))} \phi\left(\frac{r_0}{M^{h-1}}\right) + \frac{\mu(B(x_0, 2Kr_0))}{\mu(B(x,r_0))} \phi(2Kr_0) \right\}
\]

\[
\leq C_3 \sum_{h=0}^{i-1} \phi\left(\frac{r_0}{M^h}\right) = C_3 \lambda_i.
\]

Since $m_{B(x_0, 2Kr_0)}(f) = 0$, we have $m_{B\left(x_0, 2Kr_0\right)}(\|f\|) \leq D\phi(2Kr_0)$. Now (6) follows from (7) and the last inequality.
Let $t$ be a positive real number. Let us consider the set

$$\Omega^j_t = \{ x \in B_0 : \text{there exists } r \in \left(0, \frac{r_0}{M}\right) \text{ such that } m_{B(x,r)}(|f|) > t \lambda_j \}$$

and, given $x \in \Omega^j_t$,

$$R^j_t(x) = \left\{ r \in \left(0, \frac{r_0}{M}\right) : m_{B(x,r)}(|f|) > t \lambda_j \right\}.$$

(8) LEMMA. $R^j_t(x) \subset \left(0, \frac{r_0}{M^{j+1}}\right)$ provided that $t > C_1$.

PROOF. Given $r \in \left[\frac{r_0}{M^{j+1}}, \frac{r_0}{M^j}\right]$ there is an $h \leq j$ such that

$$\frac{r_0}{M^{j+1}} \leq \frac{r_0}{M^{j+1}} \leq r < \frac{r_0}{M^h} \leq \frac{r_0}{M}.$$

From (5) we have

$$m_{B(x,r)}(|f|) < C_1 \lambda_h \leq t \lambda_h \leq t \lambda_j,$$

so that $r \notin R^j_t(x)$.

(9) LEMMA. Let $n$ be a given positive integer. For $k = 1, 2, \ldots, n$ there is a function $r^k$ defined on $\Omega^k_t$ such that

(10) $r^k(x) \in R^k_t(x)$;

(11) $0 < r^k(x) < \frac{r_0}{M^{k+1}}$;

(12) $m_{B(x,r^k(x))}(|f|) > t \lambda_k \geq m_{B(x,M^{k+1}(x))}(|f|)$;

(13) $r^{k-1}(x) \geq r^k(x), \ x \in \Omega^k_t$.

PROOF. Given $x \in \Omega^n_t$ pick $r^n(x) \in R^n_t(x)$ in such a way that

$$Mr^n(x) \notin R^n_t(x).$$

The second inequality in (11) for $k = n$ follows from Lemma (8). The second inequality in (12) holds since $Mr^n(x) \notin R^n_t(x)$ and $Mr^n \in \left(0, \frac{r_0}{M}\right)$. Let us now define $r^{n-1}$. Observe that $\Omega^n_t \subset \Omega^{n-1}_t$. If $x \in \Omega^{n-1}_t - \Omega^n_t$ then we get $r^{n-1}(x)$ in the same way as we have got $r^n$. If $x \in \Omega^n_t$, then pick $r^{n-1}(x) \in R^{n-1}_t(x)$ in such a way that $r^{n-1}(x) \geq r^n(x)$ and $Mr^n(x) \notin R^{n-1}_t(x)$. 

Given $k = 1, 2, \ldots, n$, set
\[ B^k = \{B(x, r^k(x)) : x \in \Omega^k \}. \]

(14) LEMMA. For each $k = 1, \ldots, n$, there exists a sequence $\{x_i^k : i \in \mathbb{N}\}$ of points in $\Omega^k$ such that the following properties hold:

15) $B(x_i^k, r^k(x_i^k)) \cap B(x_j^k, r^k(x_j^k)) = \emptyset$, $i \neq j$;

16) for every $x \in \Omega^k$ there exists $i \in \mathbb{N}$ such that $r^k(x) \leq 2r^k(x_i^k)$ and
\[ B(x, r^k(x)) \subset B(x_i^k, 5K^2r^k(x_i^k)); \]

17) $\Omega^k \subset \bigcup_{i=1}^{\infty} B(x_i^k, 5K^2r^k(x_i^k));$

18) $r^k(x_i^k) < \frac{r_0}{M^{k+1}};

19) m_{B(x_i^k, r^k(x_i^k))}(|f|) > t\lambda_k \geq m_{B(x_i^k, M^{k+1}|f|)}(|f|);

20) Given $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that
\[ B(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \subset B(x_i^k, 5K^2r^k(x_i^k)). \]

21) Given $i \in \mathbb{N}$, set
\[ J_i = \{j \in \mathbb{N} : B(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \subset B(x_i^k, 5K^2r^k(x_i^k)) \text{ and } B(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \not\subset B(x_i^k, 5K^2r^k(x_i^k)) \}
\]
for $\ell = 1, 2, \ldots, i - 1$.

Then $J_i \cap J_h = \emptyset$ for $i \neq h$ and $\mathbb{N} = \bigcup_{i \in \mathbb{N}} J_i$.

PROOF. Applying the covering lemma (4) to the family $B^k$, we obtain a sequence $\{x_i^k : i \in \mathbb{N}\}$ satisfying (15) to (19). In order to prove (20), observe that $x_j^{k+1} \in \Omega_{i}^{k+1} \subset \Omega^k$, then $B(x_j^{k+1}, r^k(x_j^{k+1})) \in B^k$, thus, from (16) there exists $i \in \mathbb{N}$ such that $B(x_j^{k+1}, r^k(x_j^{k+1})) \subset B(x_i^k, r^k(x_i^k))$. Now, since $r^{k+1}(x_j^{k+1}) \leq r^k(x_j^k)$ from (13), we get (20).
PROOF OF THE THEOREM. Let us first prove the “if” part of the theorem. Computing the integral of $|f - m_B(f)|$ using its non-increasing rearrangement, we get

$$\frac{1}{\mu(B)} \int_B |f - m_B(f)| d\mu \leq \frac{1}{\mu(B)} \int_0^{\mu(B)} \psi_B(t) dt$$

$$\leq \frac{\beta}{\mu(B)} \int_0^{\gamma_{\mu(B)/2}} \int_0^{\gamma \mu(B)/2} \frac{\phi(\xi)}{\xi} d\xi dt + (1 - \gamma/2) \psi_B \left( \frac{\gamma}{2} \mu(B) \right)$$

$$\leq \frac{\beta}{\mu(B)} \left\{ \int_0^{\gamma \mu(B)/2} \frac{\phi(\xi)}{\xi} d\xi \right\}$$

$$+ \left( 1 - \frac{\gamma}{2} \right) \int_0^{\gamma \mu(B)/2} \frac{\phi(\xi)}{\xi} d\xi.$$

The first and the last terms on the right hand side are bounded by a constant times $\log 2^{\alpha} K^2 \phi(r)$. For the second term we have the bound

$$\beta \gamma (5K^2)^{1/\alpha} \frac{1}{r^{1/\alpha}} \left( \int_0^{r \gamma \mu(B)/2} \frac{d\xi}{\xi^{1/\alpha-1}} \right) \phi \left( \frac{r}{2^{\alpha} 5K^2} \right),$$

which, using condition $A_2$, is actually bounded by a constant times $\phi(r)$. The desired result follows now from (2). In order to prove the “only if” part of the theorem let us first assume that $m_{B(2K,2K)}(f) = 0$. Applying the first inequality in (19) for $k+1$, (21), the second inequality in (19), (15), the fact that $f \in BMO(\phi)$
and (18), we get the following inequalities
\[
\begin{align*}
    t\lambda_{k+1} \sum_{j \in \mathbb{N}} \mu(B(x_{j}^{k+1}, r^{k+1}(x_{j}^{k+1}))) &\leq \sum_{i \in \mathbb{N}} \sum_{j \in J_{i}} \int_{B(x_{j}^{k+1}, r^{k+1}(x_{j}^{k+1}))} |f| \, d\mu \\
    &\leq \sum_{i \in \mathbb{N}} \sum_{j \in J_{i}} \int_{B(x_{j}^{k+1}, r^{k+1}(x_{j}^{k+1}))} |f - m_{B(x_{j}^{k+1}, r^{k+1}(x_{j}^{k+1}))}(f)| \, d\mu \\
    &\quad + t\lambda_{k} \sum_{j} \mu(B(x_{j}^{k+1}, r^{k+1}(x_{j}^{k+1}))) \\
    &\leq DC_{4} \phi \left( \frac{\tau_{0}}{M^{k}} \right) \sum_{i} \mu(B(x_{i}^{k}, r^{k}(x_{i}^{k}))) \\
    &\quad + t\lambda_{k} \sum_{j} \mu(B(x_{j}^{k+1}, r^{k+1}(x_{j}^{k+1}))),
\end{align*}
\]
where $C_{4}$ depends only on $A$, $K$ and $C$. Set $\lambda = \sum_{k} \mu(B(x_{j}^{k}, r^{k}(x_{j}^{k})))$. Then
\[
    t(\lambda_{k+1} - \lambda_{k}) \sum_{k+1} \leq DC_{4} \phi \left( \frac{\tau_{0}}{M^{k}} \right) \sum_{k}.
\]
From the definition of the sequence $\{\lambda_{k}\}$, taking $t = 2C_{4}D$, we get
\[
    \sum_{k+1} \leq \frac{1}{2} \sum_{k} \quad \text{for every } k.
\]
By iteration
\[
    \sum_{n} \leq \frac{1}{2^{n-1}} \sum_{k=1}^{n} \leq \frac{C_{5}}{2^{n}} \mu(B_{0}),
\]
consequently
\[
    \mu(\Omega_{n}) \leq \frac{C_{6}}{2^{n}} \mu(B_{0}).
\]
Since continuous functions are dense in $L^{1}(X, d)$, Lebesgue theorem on differentiation of integrals holds, so that
\[
    \{x \in B_{0} : |f(x)| > t\lambda_{n}\} \subset \Omega_{n}^{n}.
\]
Thus
\[
    \mu(\{x \in B_{0} : |f(x)| > t\lambda_{n}\}) \leq \frac{C_{7}}{2^{n}} \mu(B_{0}).
\]
Given $s \in (0, 1)$, take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < s \leq \frac{1}{2^{n-1}}$, then, for the rearrangement $\psi_{B_0}$ of $|f|$ on $B_0$, we have

$$
\psi_{B_0}(sC_9\mu(B_0)) \leq t\lambda_n \leq DC_9 \sum_{k=1}^{n} \phi\left(\frac{r_0}{M^k}\right)
\leq DC_9 \int_{\mathbb{R}^n} \frac{\phi(\xi)}{\xi} d\xi \leq DC_9 \int_{\mathbb{R}^n} \frac{\phi(\xi)}{\xi} d\xi.
$$

This finishes the proof of the theorem for the case $m_{B(x_0,2K\tau_0)}(f) = 0$. For the general case we use (3) and we apply the previous result to $f - m_{B(x_0,2K\tau_0)}(f)$.

PROOF OF PART (b) OF THE COROLLARY. Let $x$ and $y$ be different points in $X$ and take $B = B(x,2d(x,y))$. Since $|f(x) - m_B(f)|$ and $\psi_B$ have the same distribution function and $\psi_B$ is non-increasing, we have

$$
|f(x) - f(y)| \leq |f(x) - m_B(f)| + |f(y) - m_B(f)| \leq 2\psi_B(0),
$$

now, applying the theorem we get the desired result.

REFERENCES


