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On the existence of geodesics in static Lorentz manifolds with singular boundary


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1. - Introduction and statements of the results

In this paper we study some global geometric properties of certain Lorentz structures. More precisely we prove existence and multiplicity results about geodesics joining two given points in Lorentz manifolds having a singular boundary. We require that these geodesics do not touch the boundary.

Some particular solutions of the Einstein equations (for instance the Schwarzschild spacetime, see e.g. [9, page 149], and of the Einstein-Maxwell equations (for instance the Reissner-Nordström spacetime, see e.g. [9, page 156]) are examples of those Lorentz structures which we consider.

Before stating the definitions of the geometrical structures, we need to recall some basic notions which can be found for example in [14]. A pseudo-Riemannian manifold is a smooth manifold $\mathcal{G}$ on which a nondegenerate $(0, 2)$-tensor $g(z)[\cdot, \cdot] (z \in \mathcal{G})$ is defined. This tensor is called metric tensor. If $g$ is positive definite then $\mathcal{G}$ is a Riemannian manifold. A Lorentz manifold $\mathcal{L}$ is a pseudo-Riemannian manifold with the metric tensor $g$ having index 1 (i.e. every matrix representation of $g$ has exactly one negative eigenvalue). If a Lorentz manifold has dimension 4, it is called “spacetime”. If no ambiguity can occur, we denote by $\langle , \rangle_R$ the metric on a Riemannian manifold and by $\langle , \rangle_L$ the metric on a Lorentz manifold.

We recall that a geodesic on a Lorentz manifold $\mathcal{L}$ is a curve

$$\gamma : [a, b] \to \mathcal{L} \text{ solving } D_s^2 \gamma(s) = 0 \text{ for all } s,$$

where $a, b \in \mathbb{R}$, $\dot{\gamma}(s)$ is the derivative of $\gamma(s)$, and $D_s \gamma(s)$ is the covariant derivative of $\dot{\gamma}(s)$ with respect to the metric tensor $g$.

It is well known that a geodesic $\gamma$ is a critical point of the “energy”

\[\frac{d}{ds} \int_a^b \frac{1}{2} g(\dot{\gamma}(s), \dot{\gamma}(s)) ds = 0.\]
If $\gamma$ is a geodesic on $L$ there exists a constant $E_{\gamma} \in \mathbb{R}$ such that

$$E_{\gamma} = g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] \text{ for all } s.$$
massive body. This solution, called Schwarzschild metric, can be written (using polar coordinates) in the form:

\[
(1.4) \quad ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 - c^2 \left(1 - \frac{2m}{r}\right) dt^2,
\]

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta \cdot d\phi^2\) is the standard metric of the unit 2-sphere in the Euclidean 3-space, \(m = GM/c^2\), \(G\) is the universal gravitation constant, \(M\) is the mass of the body and \(c\) is the speed of the light.

The Schwarzschild spacetime is the Lorentz manifold

\[\mathcal{L} = M_0 \times \mathbb{R}, \quad M_0 = \{(r, \theta, \varphi) : r > 2m\}\]

equipped with the metric (1.4). Notice that the Riemannian metric

\[dx^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\]

has no meaning on the region

\[\partial M_0 = \{(r, \theta, \varphi) : r = 2m\}.
\]

Moreover it is easy to see that the radial geodesics \((r(s), \theta_0, \varphi_0) (\theta_0, \varphi_0 \in \mathbb{R})\) on \(M_0\) with respect to the Riemannian metric \(dx^2\) “reach” the region \(\{(r, \theta, \varphi) : r = 2m\}\) within a finite value of the parameter \(s\). Therefore \(M_0\) with the metric \(dx^2\) is not complete. Moreover

\[\beta(r) = c^2 \left(1 - \frac{2m}{r}\right) \to 0 \text{ as } r \to 2m.
\]

Then both conditions (i) and (ii) are not satisfied by the Schwarzschild spacetime.

The metric (1.4) is singular on \(\partial M_0 \times \mathbb{R}\) in the sense that it cannot be smoothly extended on \(\partial M_0 \times \mathbb{R}\). However the singularity is not intrinsic, but it is a consequence of the choice of the static coordinates.

In fact if we denote by \((K, g)\) the Kruskal spacetime (which is the maximal analytical extension of the Schwarzschild spacetime, cf. [11] or [9, pp. 153-155]), there is an injective isometry

\[\Psi : M_0 \times \mathbb{R} \to K,
\]

and \(g\) is not singular on \(\partial (\Psi(M_0 \times \mathbb{R}))\). However \(\partial (\Psi(M_0 \times \mathbb{R}))\) is not a smooth 3-manifold. In this way the singularity has been “transferred” from the metric to the geometry of the boundary, and this justifies the title of this paper.

We recall that (1.4) solves the Einstein equations for \(r > r_M\), \(r_M\) being the radius of the body responsible for the gravitational curvature. Then, it is physically meaningful to equip all \(M_0 \times \mathbb{R}\) with the metric (1.4) only if
\( r_M \leq 2m \). In this case the matter of the body is “contained” within the event horizon \( \{ r = 2m \} \) and we are in the presence of a universe with a black hole.

The name is justified by the fact that a light ray cannot leave the region \( \{ r \leq 2m \} \). If an astronaut “falls” in the black hole, he spends a finite “proper” time, but an observer far from the black hole does not see the astronaut to fall in it in a finite time. More precisely any time-like geodesic in the Schwarzschild spacetime can reach the region \( \{ r = 2m \} \) only if the time coordinate \( t \) goes to \( \pm \infty \) (see the appendix).

Having in mind, as model, the Schwarzschild spacetime we are led to introduce the following definitions:

**Definition 1.2.** Let \( U \) be an open connected subset of a manifold \( M \) and let \( \partial U \) be its topological boundary. \( U \) is said to be a *static universe* if

(i) \( U = M_0 \times \mathbb{R} \) is a static Lorentz manifold (see Definition 1.1);

(ii) \( \sup_{M_0} \beta < +\infty \), where \( \beta \) is the function in (1.3);

(iii) \( \lim_{k \to +\infty} \beta(x_k) = 0 \), for any \( (x_k, t_k) \to z \in \partial U \);

(iv) for every \( \delta > 0 \) the set \( \{ x \in M_0 : \beta(x) \geq \delta \} \) is complete (with respect to the Riemannian structure of \( M_0 \));

(v) for every time-like geodesic \( \gamma(s) = (x(s), t(s)) \) in \( U \) such that \( \lim_{s \to s_0} \beta(x(s)) = 0 \), we have \( \lim_{s \to s_0} \sup_{s \in [s_0] | t(s)| = +\infty} \).

**Remark 1.3.** Condition (v) says that if a material particle reaches the topological boundary of \( U \), an observer far from the boundary (Schwarzschild observer) does not see this event in a finite time, since his proper time is a reparametrization of the universal time \( t \). This condition justifies the name of the structure introduced in Definition 1.2.

Notice that in general (v) does not follow from (iii). In fact consider the Lorentz manifold

\[ \{ x \in \mathbb{R} : x > 1 \} \times \mathbb{R} \text{ with metric } ds^2 = dx^2 \beta(x)^2 dt^2, \]

where \( \beta \) is bounded and \( \beta(x) = x - 1 \) if \( x \leq 2 \).

A straightforward calculation shows that it does not satisfy (v).

In the Appendix we verify that the Schwarzschild spacetime satisfy (v) of Definition 1.2. Then, clearly, it is a static universe.

Another example of static universe is given by the Reissner-Nordström spacetime

\[ \left\{ r > m + \sqrt{m^2 - e^2} \right\} \times \mathbb{R} \]

when \( m^2 > e^2 \). Here \( m \) represents the gravitational mass and \( e \) the electric charge of the body responsible of the gravitational curvature. This can be seen by the same computations used for the Schwarzschild spacetime (see the Appendix).
Whenever $U$ is a static universe we have the following results about the existence of time-like geodesics joining two given events.

**Theorem 1.4.** Let $U = M_0 \times \mathbb{R}$ be a static universe (see Definition 1.2). Let $z_0 = (x_0, t_0)$ and $z_1 = (x_1, t_1)$ be events in $U$. There exists a time-like geodesic $\gamma$ in $U$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$ if and only if

$$\exists \, z \in C^1([0,1], M_0) : z(0) = x_0, \; z(1) = x_1 \text{ and}$$

$$(1.5) \quad \left[ \int_0^1 \frac{1}{\beta(z(s))} \, ds \right] \cdot \left[ \int_0^1 \langle \dot{z}(s), \ddot{z}(s) \rangle \, ds \right] < (t_1 - t_0)^2.$$ 

**Remark 1.5.** When we fix $x_0$ and $x_1$, the condition (1.5) is certainly satisfied if $|t_1 - t_0|$ is large enough, while it does not hold whenever $|t_1 - t_0|$ is small.

**Remark.** Condition (ii) of Definition 1.2 is essential to obtain our existence results. In fact the Anti-de Sitter space (see e.g. [9,16]) furnishes counterexamples to the existence of geodesics between two given events. However if $\beta(x)$ goes to $+\infty$ with a mild rate as $x$ goes to $\infty$, Theorems 1.4 and 1.6 still hold.

Now let $L = M_0 \times \mathbb{R}$ be a static Lorentz manifold and $(x_0, t_0), \; (x_1, t_1)$ two events in $L$. If $(x(s), t(s))$ is a geodesic joining $(x_0, t_0)$ and $(x_1, t_1)$, since the metric tensor is independent of $t$, $(x(s), t(s) + \tau)$ is a geodesic joining $(x_0, t_0 + \tau)$ and $(x_1, t_1 + \tau)$. Then the number of geodesics in $L$ joining two events $(x_0, t_0)$ and $(x_1, t_1)$ depends only on $x_0, x_1$ and $|t_1 - t_0|$.

We denote by $N(x_0, x_1, |t_1 - t_0|)$ the number of time-like geodesics in $U$ joining $(x_0, t_0)$ and $(x_1, t_1)$. If $U$ has a non-trivial topology we get the following multiplicity result of geodesics joining $z_0$ and $z_1$.

**Theorem 1.6.** Let $U = M_0 \times \mathbb{R}$ be a static universe and $(M_0, h)$ a $C^3$-Riemannian manifold which is not contractible in itself. Moreover assume that (1.5) holds.

Then

$$\lim_{|t_1 - t_0| \to +\infty} N(x_0, x_1, |t_1 - t_0|) = +\infty.$$ 

About other existence results for time-like geodesics joining two given points in Lorentz manifolds we refer to [1,18,19], where the Lorentz manifolds are assumed to be globally hyperbolic.

In this paper we deal also with the problem of the geodesical connectivity for a Lorentz manifold. We recall that

A Lorentz manifold $L$ is called geodesically connected if for every $z_0, z_1 \in L$ there exists a geodesic $\gamma : [0,1] \to L$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1.$
Clearly, for studying the geodesical connectivity it is necessary to consider also space-like geodesics which are more difficult to deal with. The geodesical connectivity has not been treated in the works [1,18,19], which deal only with time-like and null geodesics. This problem has been faced in [3,4] for stationary complete Lorentz manifolds\(^{(4)}\) without boundary. Here we consider the case of static Lorentz manifolds with singular boundary, in order to cover the case of the Schwarzschild spacetime.

For the study of the geodesical connectivity the condition of being a static universe (see Definition 1.2) is not appropriate. Indeed consider the Lorentz manifold

\[
(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \times \mathbb{R}
\]

with metric \(ds^2 = dx^2 + dy^2 - \beta(x, y)dt^2\),

where \(\beta\) is bounded and \(\beta(x, y) = \left(\sqrt{x^2 + y^2} - 1\right)^2 \text{ if } \sqrt{x^2 + y^2} \leq 2.\)

Simple calculations show that (1.7) is a static universe while it is not geodesically connected (the events of the type \((x_1, x_2, t_0)\) and \((-x_1, -x_2, t_0)\) cannot be joined by geodesics lying in the Lorentz manifold (1.7)). For this reason we introduce the following geometrical condition:

**DEFINITION 1.7.** Let \(L\) be an open connected subset of a manifold \(M\) and \(\partial L\) its topological boundary. \(L\) is said to be a static Lorentz manifold with convex boundary if

(i) \(L = M_0 \times \mathbb{R}\) is a static Lorentz manifold (see Definition 1.1));

(ii) \(\sup_{M_0} \beta < +\infty\), where \(\beta\) is the function in (1.3);

(iii) there exists \(\phi \in C^2(L, \mathbb{R}^+ \setminus \{0\})\) such that

\[
\lim_{(x,t) \to (x,t) \in \partial L} \phi(x,t) = 0 \text{ and } \phi(x,t) = \phi(x,0) \equiv \phi(x) \text{ } \forall (x,t) \in L;
\]

(iv) for every \(\eta > 0\) the set \(\{x \in M_0 : \phi(x) \geq \eta\}\) is complete (with respect to the Riemannian structure of \(M_0\));

(v) there exist \(N, M, \nu, \delta \in \mathbb{R}^+ \setminus \{0\}\) such that the function \(\phi\) of (iii) satisfies:

\(^{(4)}\) I.e. with the metric tensor not depending by the time variable.
In the appendix we prove that the Schwarzschild spacetime is a static Lorentz manifold with convex boundary using the function $\phi$ given by

$\phi(r, \theta, \varphi, t) = \sqrt{1 - \frac{2m}{r}}$.

Also the Reissner-Nordström spacetime $\{ r > m + \sqrt{m^2 - e^2} \} \times \mathbb{R}$ is a static Lorentz manifold with convex boundary provided that $m^2 > \frac{9}{5}e^2$, as we have proved in the appendix using the function

$\phi(r, \theta, \varphi, t) = \sqrt{1 - \frac{2m}{r} + \frac{e^2}{r^2}}$.

Definition 1.7 allows us to obtain the following result:

**THEOREM 1.8.** A static Lorentz manifold with convex boundary is geodesically connected.

**REMARK.** Notice that $\phi$ becomes zero on $\partial \mathcal{L}$, so (1.9) implies

$$\lim_{x \to x_0 \in \partial \mathcal{M}} \sup H^\phi_L(x)[v, v] \leq 0$$

(1.10)

for all $v$ such that $|(v, v)_L| \leq 1$.

However, in order to get the geodesic connectivity of $\mathcal{L}$, it seems we need a control of the rate for which the limit in (1.10) is achieved. The assumption (1.9) provides this control.

Whenever the topology of $\mathcal{L}$ is not trivial we get the following multiplicity result about space-like geodesics. This result has been proved in [4] in the case of stationary Lorentz manifolds $M_0 \times \mathbb{R}$ with $M_0$ compact.

**THEOREM 1.9.** Let $\mathcal{L} = M_0 \times \mathbb{R}$ be a static Lorentz manifold with convex boundary, and $(M_0, h)$ a $C^3$ Riemannian manifold which is not contractible.

(5) $\nabla_L \phi(z)$ denotes the gradient of the function $\phi$ with respect to the Lorentz structure, i.e. it is the unique vector field $F(z)$ on $\mathcal{L}$ such that $\langle F(z), v \rangle_L = \phi(v)$ for all $v \in T_z(\mathcal{M})$.

(6) $H^\phi_L(x)[v, v]$ denotes the Hessian of the function $\phi$ at $x$ in the direction $v$, i.e. $\frac{d^2}{ds^2} \langle \phi(\gamma(s)) \rangle |_{s=0}$ where $\gamma$ is a geodesic in $\mathcal{L}$ such that $\gamma(0) = x$ and $\gamma(0) = v$. 

Then, for every $z_0, z_1 \in \mathcal{L}$, there exists a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of space-like geodesics in $\mathcal{L}$ joining $z_0$ and $z_1$ such that

$$\lim_{n \to +\infty} E_{\gamma_n} = +\infty.$$  

**Remark 1.10.** Theorems 1.4 and 1.6 hold even for a static Lorentz manifold with convex boundary, while Theorems 1.8 and 1.9 in general do not hold for a static universe, as we can see using the Lorentz manifold (1.7).

The results proved in this paper have been announced in [6].

2. - Technical preliminaries

Let $\mathcal{L}$ be a static Lorentz manifold. Then (see Definition 1.1) $\mathcal{L}$ is isometric to $\mathcal{M}_0 \times \mathbb{R}$ equipped with the warped product

$$g(x)[\xi, \zeta] = h(x)[\xi, \xi] - \beta(x)r^2,$$

where $z = (x, t)$, $\zeta = (\xi, \tau) \in (T_x \mathcal{M}_0) \times \mathbb{R}$.

In the following we set for simplicity

$$(\xi, \xi)_R = h(x)[\xi, \xi]$$

and

$$(\xi, \xi)_L = g(x)[\xi, \xi].$$

Let $z_0 = (x_0, t_0)$, $z_1 = (x_1, t_1)$ be two events in $\mathcal{M}_0 \times \mathbb{R}$. We put

$$W^{1,2}([0, 1], \mathcal{M}_0)$$

$$= \left\{ x : [0, 1] \to \mathcal{M}_0, \text{ } x \text{ absolutely continuous}, \int_0^1 (x, x)_R < +\infty \right\},$$

and

$$\Omega^1 \equiv \Omega^1(\mathcal{M}_0, x_0, x_1) = \left\{ x \in W^{1,2}([0, 1], \mathcal{M}_0) : x(0) = x_0, \text{ } x(1) = x_1 \right\}.$$

$\Omega^1$ is a Hilbert manifold (see e.g. [10,17]) and its tangent space at $x \in \Omega^1$ is given by

$$T_x(\Omega^1)$$

$$= \left\{ \xi \in W^{1,2}([0, 1], T\mathcal{M}_0) \mid \xi(0) = \xi(1) = 0, \text{ } \xi(s) \in T_{x(s)}\mathcal{M}_0 \text{ for all } s \in [0, 1] \right\}.$$
Here $TM_0$ is the tangent bundle of $M_0$ while $W^{1,2}([0,1], TM_0)$ is the set of the absolutely continuous curves $\xi : [0,1] \to TM_0$, such that

$$\langle \xi, \xi \rangle_1 = \int_0^1 \langle D_s \xi(s), D_s \xi(s) \rangle_R < +\infty,$$

where $D_s$ is the covariant derivative with respect to the Riemannian structure. Notice that $\langle \cdot, \cdot \rangle_1$ is the Riemannian structure of $\Omega^1$ inherited by that of $M_0$.

We denote by $C([0,1], M_0)$ the space of the continuous curves $x : [0,1] \to M_0$ endowed with the metric

$$d(x, y) = \sup_{s \in [0,1]} d(x(s), y(s)),$$

where $d$ is the distance derived from the Riemannian metric on $M_0$. Consider now the Riemannian manifold

$$Z = \Omega^1 \times \left\{ t \in W^{1,2}([0,1], \mathbb{R}) \middle| t(0) = t_0, \ t(1) = t_1 \right\}.$$

It is easy to see that the “energy” functional

$$f(z) = \int_0^1 \langle \dot{z}(s), \dot{z}(s) \rangle_L ds$$

is $C^1$ on $Z$. The geodesics on $L$ joining $z_0$, $z_1$ are the critical points of $f$ on $Z$, namely $\gamma \in Z$ is a geodesic if and only if, for all

$$\zeta = (\xi, \tau) \in T_z \Omega^1 \times W^{1,2}_0([0,1], \mathbb{R}),$$

$$\langle f'(\gamma), \zeta \rangle_L = \int_0^1 \langle \gamma(s), D_s \zeta(s) \rangle_L ds = 0,$$

where $D_s \zeta(s)$ denotes the covariant derivative of $\zeta$ in the direction $\dot{\gamma}(s)$ with respect to the metric (2.1) and

$$W^{1,2}_0([0,1], \mathbb{R}) = \left\{ \tau \in W^{1,2}([0,1], \mathbb{R}) \middle| \tau(0) = \tau(1) = 0 \right\}.$$

We are interested in studying situations in which $(M_0, h)$ is not complete (see Section 1). In these cases also $C([0,1], M_0)$ and $\Omega^1$ are not complete. To overcome this lack of completeness we introduce a suitable penalization term in (2.5). More precisely we shall set

$$f_\varepsilon(z) = f(z) + \int_0^1 V_\varepsilon(\dot{z}) ds, \quad z = (x, t) \in Z,$$
where

\begin{equation}
\left\{ \begin{array}{ll}
\text{for all } \varepsilon \geq 0, & V_\varepsilon(x) = V_\varepsilon(x, 0) \equiv V_\varepsilon(x) \text{ is a non-negative } C^2 \\
\text{scalar field on } \mathcal{M} \text{ depending only on } x \in \mathcal{M}_0 \text{ and with } V_0 \equiv 0.
\end{array} \right.
\end{equation}

We shall specify the penalization function \( V_\varepsilon \) in Sections 3 and 4 where we shall prove Theorems 1.4, 1.6, 1.8 and 1.9.

Since the metric \( g \) is indefinite, the functional (2.7) is unbounded both from below and from above. Nevertheless the study of the critical points of \( f_\varepsilon (\varepsilon \geq 0) \) can be reduced to the study of the critical points of a suitable functional which is bounded from below when \( \beta \) is bounded from above.

In fact let \( z_0 = (x_0, t_0) \), \( z_1 = (x_1, t_1) \) be two points in \( \mathcal{L} \) and consider the functional

\begin{equation}
J_\varepsilon(x) = J(x) + \int_0^1 V_\varepsilon(x)ds, \quad x \in \Omega^1,
\end{equation}

\( J(x) \) being defined by

\[
J(x) = \int_0^1 \left( \dot{x}(s), \dot{x}(s) \right)_R - \Delta^2 \left[ \int_0^1 \frac{ds}{\beta(x(s))} \right]^{-1}
\]

where \( \Delta = t_1 - t_0. \) Observe that (2.9) is bounded from below if \( \beta \) is bounded from above.

The following theorem holds:

**Theorem 2.1.** Let \( z(s) = (x(s), t(s)) \in \mathcal{Z} \) (see (2.4)). Then the following statements are equivalent:

(i) \( z \) is a critical point of \( f_\varepsilon \) on \( \mathcal{Z} \);
(ii) \( x \) is a critical point of \( J_\varepsilon \) on \( \Omega^1 \), i.e.

\[
\{J'(x), \xi\} = \int_0^1 2\dot{x}(s)D_s \xi_R ds - \Delta^2 \left[ \int_0^1 \frac{ds}{\beta(x)} \right]^{-2} \cdot \int_0^1 \frac{(\beta'(x), \xi)_R}{\beta^2(x)} ds
\]

\begin{equation}
+ \int_0^1 \langle V_\varepsilon'(x), \xi \rangle_R ds = 0 \quad \text{for all } \xi \in \mathcal{T}_x \Omega^1
\end{equation}

\(^{(7)}\) Here \( \beta' \) and \( V_\varepsilon' \) denote the Riemann gradient of \( \beta \) and \( V_\varepsilon \), respectively.
and $t = t(s)$ solves the Cauchy problem

\begin{equation}
\begin{aligned}
\dot{t} &= \Delta \left[ \int_0^1 \frac{1}{\beta(x)} \, ds \right]^{-1} \frac{1}{\beta(x(s))} \\
\dot{t}(0) &= t_0.
\end{aligned}
\end{equation}

Moreover if (i) (or (ii)) is satisfied, we have

\begin{equation}
f_\varepsilon(z) = J_\varepsilon(x).
\end{equation}

In particular $z$ is a critical point of $f$ iff $x$ is a critical point of $J$.

When $\varepsilon = 0$, Theorem 2.1 has been proved in [5]. Nevertheless, for the convenience of the reader, we shall give here a proof of Theorem 2.1.

**Proof of Theorem 2.1.**

(i) $\Rightarrow$ (ii). Let $z(s) = (x(s), t(s))$ be a critical point of $f_\varepsilon$ on $Z$. Then

\begin{equation}
f_\varepsilon'(x) \begin{pmatrix} \xi \\ \tau \end{pmatrix} = 0.
\end{equation}

Taking $\xi = 0$ in (2.13) we get

\begin{equation}
\int_0^1 \beta(x) \dot{t} \, ds = 0 \quad \text{for all } \left( \begin{array}{c} \xi \\ \tau \end{array} \right) \in T_x Z.
\end{equation}

Then there exists a constant $K \in \mathbb{R}$ such that

\begin{equation}
\dot{t}(s) = \frac{K}{\beta(x(s))} \quad \text{for all } s \in [0, 1].
\end{equation}

Integrating in $[0, 1]$ we get

\begin{equation}
K = \Delta \left[ \int_0^1 \frac{1}{\beta(x)} \, ds \right]^{-1}, \quad \Delta = t_1 - t_0.
\end{equation}

By (2.16) and (2.15) we deduce that $t = t(s)$ solves (2.11). Now if we substitute (2.11) in (2.13) and choose $\tau = 0$, we see that (2.10) is satisfied.
(ii) ⇒ (i). Suppose that $x \in \Omega_1$ solves (2.10) and $t$ solves (2.11). Obviously $t$ solves (2.14). Now if in (2.10) we add (2.14) and substitute $\Delta^2 \left[ \int_0^{1} \frac{ds}{\beta(x)} \right]^{-2}$ by (2.11), we see that $z = (x, t)$ satisfies (2.13), namely it is a critical point of $f_\varepsilon$ on $Z$.

Finally (2.12) is immediately checked.

The following Lemma will be useful

**Lemma 2.2.** Let $z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Z$ be a critical point of $f_\varepsilon$. Then there exists $K_\varepsilon \in \mathbb{R}$ s.t.

\begin{equation}
K_\varepsilon = (\dot{z}(s), \dot{t}(s))_L - V_\varepsilon(z_\varepsilon(s)) \quad \text{for all } s \in [0, 1].
\end{equation}

Moreover

\begin{equation}
K_\varepsilon = J_\varepsilon(x_\varepsilon(s)) - 2 \int_0^{1} V_\varepsilon(x_\varepsilon(s))ds.
\end{equation}

**Proof.** Since $z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Z$ is a critical point of $f_\varepsilon$ we have

\begin{equation}
D_s \dot{z}_\varepsilon(s) - \frac{1}{2} \nabla_L V_\varepsilon(z_\varepsilon(s)) = 0 \quad \text{for all } s \in [0, 1],
\end{equation}

where $D_s$ and $\nabla_L$ denote respectively the covariant derivative and the gradient with respect to the Lorentz metric $(\cdot, \cdot)_L$ defined in (2.1). From (2.19) we deduce that, for all $s$,

\[\langle D_s \dot{z}_\varepsilon(s), \dot{z}_\varepsilon(s) \rangle_L - \frac{1}{2} \langle \nabla_L V_\varepsilon(z_\varepsilon(s)), \dot{z}_\varepsilon(s) \rangle_L = 0,\]

then, for all $s$,

\[\frac{d}{ds} \left[ \langle \dot{z}_\varepsilon(s), \dot{z}_\varepsilon(s) \rangle_L - V_\varepsilon(z_\varepsilon(s)) \right] = 0,
\]

from which we deduce (2.17). Now integrating (2.17) from 0 to 1 we have

\begin{equation}
K_\varepsilon = f(z_\varepsilon) - \int_0^{1} V_\varepsilon(z_\varepsilon(s))ds = f_\varepsilon(z_\varepsilon) - 2 \int_0^{1} V_\varepsilon(z_\varepsilon(s))ds.
\end{equation}

Then by using (2.12) of Theorem 2.1 we get (2.18).
We conclude this section with a lemma which allows us to overcome the difficulty due to the lack of completeness of $\mathcal{M}_0$.

**Lemma 2.3.** Let $\mathcal{L} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentz manifold and

$$\phi \in C^0(\mathcal{L} \cup \partial \mathcal{L}, \mathbb{R}^+ \cap C^1(\mathcal{L}, \mathbb{R}^*))$$

such that

$$\phi(x) = 0 \quad \text{iff} \quad x \in \partial \mathcal{L};$$

$$\phi(x, t) = \phi(x, 0) \equiv \phi(x) \quad \forall (x, t) \in \mathcal{L};$$

there exist $N, \delta \in \mathbb{R}^+ \setminus \{0\}$ such that

$$\delta \subseteq \mathcal{L}, \phi(x) < \delta \Rightarrow N \geq (\nabla L \phi(x), \nabla L \phi(x))_L = (\nabla R \phi(x), \nabla R \phi(x))_R^{(8)}.$$

Now let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\Omega^1(M_0, x_0, x_1)$ such that

$$\int_0^1 (\dot{x}_n(s), \dot{x}_n(s))_R \ ds \quad \text{is bounded}$$

and there exists $s_n \in [0, 1]$ such that

$$\lim_{n \to \infty} \phi(x_n(s_n)) = 0.$$

Then

$$\lim_{n \to \infty} \int_0^1 \frac{1}{\phi^2(x_n(s))} \ ds = +\infty.$$

The proof of Lemma 2.3 is essentially contained in [2]. Nevertheless for the convenience of the reader we shall give here the proof.

**Proof of Lemma 2.3.** From (2.24) we deduce that

$$\{x_n(s) : n \in \mathbb{N}, s \in [0, 1]\}$$

is a bounded subset of $\mathcal{M}_0$. Then, by using (2.23), we deduce that there exists a real constant $c_1$, independent of $n$, such that

$$\|\nabla_R \phi(x_n(s))\|_R^2 = (\nabla_R \phi(x_n(s)), \nabla_R \phi(x_n(s)))_R \leq c_1$$

(8) Here $\nabla_L \phi(x)$ denotes the gradient of the function $\phi$ with respect to the Lorentz structure $g(x)[\cdot, \cdot]$, while $\nabla_R \phi(x)$ denotes the gradient of the function $\phi$ with respect to the Riemann structure $h(x)[\cdot, \cdot]$. 
for all \( n \in \mathbb{N} \), for all \( s \in [0, 1] \). (Here \( \| \cdot \| \) denotes the norm induced by the Riemannian structure). From (2.25), for \( s > s_n \), we have

\[
\phi(x_n(s)) - \phi(x_n(s_n)) = \int_{s_n}^{s} \langle \nabla_R \phi(x_n(r)), \dot{x}_n(r) \rangle_R \, dr
\]

(2.26) \[
\leq \int_{s_n}^{s} \| \nabla_R \phi(x_n(r)) \| \cdot \| \dot{x}_n(r) \| \, dr \leq \sqrt{c_1} \cdot \sqrt{s - s_n} \cdot \left( \int_{0}^{1} \| \dot{x}_n(s) \|^2 \, ds \right)^{1/2}
\]

\[
\leq (\text{by } 2.24) \leq c_2 \cdot \sqrt{s - s_n},
\]

where \( c_2 \) is a constant independent of \( n \). Since \( \phi(x_n(1)) = \phi(x_1) > 0 \) for all \( n \in \mathbb{N} \), there exists \( \mu > 0 \) such that

\[
\forall \ n \in \mathbb{N}, \ \exists \tau_n > s_n : \phi(x_n(\tau_n)) - \phi(x_n(s_n)) \geq \mu > 0
\]

(2.27) independently of \( n \). Then, from (2.26) (with \( s = \tau_n \)) and (2.27), we get

(2.28) \[
\tau_n - s_n \geq \left( \frac{\mu}{c_2} \right)^2 > 0.
\]

Moreover using again (2.26) we get

\[
\int_{s_n}^{\tau_n} \frac{dr}{\phi(x_n(s_n)) + c_2 \sqrt{r - s_n}} \leq \int_{0}^{1} \frac{1}{\phi^2(x_n(r))} \, dr.
\]

(2.29) Since \( \lim_{n \to +\infty} \phi(x_n(s_n)) = 0 \), by (2.28) the left-hand side in (2.29) diverges, so by (2.29) we get the conclusion.

\[ \blacksquare \]

3. - Proof of Theorems 1.4, 1.6

In order to prove Theorems 1.4 and 1.6 we shall study the penalized functional \( J_\varepsilon \) defined by (2.9), i.e.

\[
J_\varepsilon(x) = \int_{0}^{1} \langle \dot{x}(s), \dot{x}(s) \rangle_R \, ds - \frac{\Lambda^2}{\int_{0}^{1} ds / \beta(x(s))} + \int_{0}^{1} V_\varepsilon(x(s)) \, ds, \ x \in \Omega^1
\]

(3.1)
where

$$V_\varepsilon(x) = \Psi_\varepsilon \left( \frac{1}{\phi^2(x)} \right);$$

$$\phi(x) = \phi(x, 0) \text{ with } \phi \text{ satisfying (iii) of Definition 1.7; }$$

and,

REMARK 3.1. By standard methods we can modify the function \( Q \) in order to get another function of class \( C^2 \) (which we continue to call \( Q \)) satisfying (2.23) and

In all this section we shall assume \( 0 \) to satisfy these properties.

Now consider

By (iv) of Definition 1.2 and Remark 3.1 it follows that

Now we shall prove a lemma which will play a fundamental role in the proof of Theorems 1.4 and 1.6.

LEMMA 3.2. Let \( U = M_0 \times \mathbb{R} \) be a smooth static universe and let \( J_\varepsilon \) be as in (3.1). For every \( \varepsilon \in ]0, 1] \), let \( x_\varepsilon \) be a critical point of \( J_\varepsilon \) on \( \Omega^1 = \Omega^1(x_0, x_1, M_0) \). Assume that

Then, if \( \varepsilon \) is small enough, \( x_\varepsilon \) is a critical point of \( J \) on \( \Omega^1 \) and \( J(x_\varepsilon) \leq -\mu \).

(9) Notice that we can choose \( \varphi: M_0 \times \mathbb{R} \to \mathbb{R} \) such that \( \varphi(x, t) = \beta(x) \) for all \( (x, t) \in M_0 \times \mathbb{R} \), because of (iii) of Definition 1.2.
PROOF. The penalization term $V_{\varepsilon}$ is zero when $\phi^2(x) \geq \varepsilon$ (see 3.2): then, in order to get the conclusion, it will be sufficient to prove that

there exists $\delta > 0$ such that, for $\varepsilon$ small enough,

$$
(3.5) \quad \phi(x_{\varepsilon}(s)) \geq \delta \quad \text{for all } s \in [0,1].
$$

Arguing by contradiction, assume that there exists a sequence $\varepsilon_n \to 0$ with the corresponding critical points $x_{\varepsilon_n}$ such that

$$
(3.6) \quad \phi(x_{\varepsilon_n}(s_n)) \to 0 \quad \text{as } n \to \infty,
$$

where $s_n \in [0,1]$ for all $n$. Let $\eta > 0$ such that $\phi(x_1), \phi(x_2) > \eta$. Then

$$
(3.7) \quad \phi(x_{\varepsilon_n}(s_n)) < \eta \quad \text{for } n \text{ large enough.}
$$

Let $s_n(\eta)$ be the “first” instant such that $\phi(x_{\varepsilon_n}(s_n(\eta))) = \eta$. Up to consider a subsequence we have

$$
s_n(\eta) \to s_{\eta} \quad \text{for } n \to \infty.
$$

Since $\beta$ is bounded from above, (3.4) implies that $\int_0^1 (\dot{x}_{\varepsilon_n}(s), \dot{x}_{\varepsilon_n}(s))_R \, ds$ is bounded (independently of $n$). Then for $n$ large enough we have that

$$
(3.8) \quad \phi(x_{\varepsilon_n}(s)) \geq \eta/2 \quad \text{for all } s \in [0,s_{\eta}].
$$

Moreover by virtue of the boundary conditions, we have that $x_{\varepsilon_n}$ is bounded in $W^{1,2}(0,1, M_0)$ (and therefore also in $C(0,1, M_0)$) so by virtue of Remark 3.1 there exists $\bar{\eta} > 0$ such that

$$
(3.9) \quad \beta(x_{\varepsilon_n}(s)) \geq \bar{\eta} \quad \text{for all } s \in [0,s_{\eta}].
$$

Since $x_{\varepsilon_n}$ is bounded in $W^{1,2}$ and $A_{\eta/2}$ is complete we have

$$
(3.10) \quad x_{\varepsilon_n} \to x_{\eta} \quad \text{weakly in } W^{1,2}([0,s_{\eta}], A_{\eta/2})
$$

and

$$
(3.11) \quad x_{\varepsilon_n} \to x_{\eta} \quad \text{in } C([0,s_{\eta}], A_{\eta/2}),
$$

where $C([0,s_{\eta}], A_{\eta/2})$ is the (complete) space of the continuous curves defined in $[0,s_{\eta}]$ and taking values in $A_{\eta/2}$, equipped with the distance (2.3). Now consider $t_{\varepsilon_n} = t_{\varepsilon_n}(s)$, $s \in [0,1]$, defined by

$$
(3.12) \quad \begin{cases}
  \dot{t}_{\varepsilon_n} = \Delta \left[ \int_0^1 \frac{ds}{\beta(x_{\varepsilon_n})} \right]^{-1} \frac{1}{\beta(x_{\varepsilon_n})} \\
  t_{\varepsilon_n} = t_0.
\end{cases}
$$
Since $\beta$ is bounded from above, \( \left[ \int_0^1 \frac{ds}{\beta(x_{\epsilon_n})} \right]^{-1} \) is bounded, so by (3.9) and (3.12) we deduce that \( \{t_n\} \) is bounded in $W^{1,2}([0, s_\eta], \mathbb{R})$ and therefore, passing to a subsequence, we have

(3.13) \[ t_{\epsilon_n} \to t_\eta \text{ weakly in } W^{1,2}([0, s_\eta], \mathbb{R}). \]

Since $x_{\epsilon_n}$ is a critical point of $J_{\epsilon_n}$ we have

(3.14) \[ D_s x_{\epsilon_n} = -\frac{\Delta^2}{\left( \int_0^1 \frac{1}{\beta(x_{\epsilon_n})} \right)^2} \cdot \frac{\beta'(x_{\epsilon_n})}{\beta^2(x_{\epsilon_n})} + V'_{\epsilon_n}(x_{\epsilon_n}). \]

Moreover $\left[ \int_0^1 \frac{ds}{\beta(x_{\epsilon_n})} \right]^{-1}$ is bounded; hence by (3.9) and (3.11), eventually passing to a subsequence, the right-hand side in (3.14) converges uniformly in $[0, s_\eta]$. Then

(3.15) \[ D_s x_{\epsilon_n} \to D_s x_\eta \text{ in } C([0, s_\eta], A_{\eta/2}). \]

Now, from (3.12) we have

(3.16) \[ |t_{\epsilon_n}| \leq |\Delta| \left[ \int_0^{s_\eta} \frac{ds}{\beta(x_{\epsilon_n}(s))} \right]^{-1} \frac{1}{\beta(x_{\epsilon_n})}, \]

so by (3.9) and (3.16) we deduce that

(3.17) \[ \{t_{\epsilon_n}\} \text{ is bounded in } L^\infty([0, s_\eta], \mathbb{R}). \]

Moreover from (3.12) we have

\[ \frac{d}{ds} (\beta(x_{\epsilon_n}(s)) \cdot \dot{t}_{\epsilon_n}(s)) = 0 \quad \text{for all } s \in [0, 1], \]

then

(3.18) \[ \frac{d^2}{ds^2} (t_{\epsilon_n}(s)) = -\frac{\langle \beta'(x_{\epsilon_n}(s)), \dot{x}_{\epsilon_n}(s) \rangle_R}{\beta(x_{\epsilon_n}(s))} \cdot \dot{t}_{\epsilon_n}(s) \]

for all $s \in [0, s_\eta]$.

Since $t_{\epsilon_n}(0) = t_0$ for all $n$, from (3.17), (3.18), (3.9) and (3.15) we easily have

(3.19) \[ t_{\epsilon_n} \to t_\eta \text{ in } C^2([0, s_\eta], \mathbb{R}). \]
Moreover from (3.12) we get that \( t_{e_n} \) is monotone. Then, if for instance \( t_0 \leq t_1 \),
\[
\text{(3.20)} \quad t_0 = t_{e_n}(0) \leq t_{e_n}(s) \leq t_{e_n}(1) = t_1 \quad \text{for all } n \in \mathbb{N} \text{ and } s \in [0, 1].
\]

Now, by Theorem 2.1, \( \gamma_{e_n} = (x_{e_n}, t_{e_n}) \) solves the equation

\[
D_s \dot{\gamma}_{e_n}(s) = \nabla_L V_{e_n}(\gamma_{e_n}(s)) \quad \text{for all } s \in [0, s_{e_n}].
\]

Then, by (3.8), we obtain, for \( n \) sufficiently large,
\[
\text{(3.21)} \quad D_s \dot{\gamma}_{e_n}(s) = 0 \quad \text{for all } s \in [0, s_{e_n}],
\]
so by (3.15) and (3.19) we get
\[
\text{(3.22)} \quad D_s \dot{\gamma}_n(s) = 0 \quad \text{for all } s \in [0, s_{e_n}],
\]
where \( \gamma_n = (x_\eta, t_\eta) \). Then \( \gamma_n \) is a geodesic in the interval \([0, s_\eta]\). In order to prove that \( \gamma_\eta \) is time-like, observe that, by virtue of (3.4), (2.17) and (2.18), we have, for all \( s \in [0, 1] \) and for all \( n \),
\[
-\mu \geq J_{e_n}(x_{e_n}) = \langle \dot{\gamma}_{e_n}(s), \gamma_{e_n}(s) \rangle_L - V_{e_n}(\gamma_{e_n}(s)) + 2 \int_0^1 V_{e_n}(\gamma_{e_n})
\]
\[
\text{(3.23)} \quad \geq \langle \dot{\gamma}_{e_n}(s), \gamma_{e_n}(s) \rangle_L - V_{e_n}(\gamma_{e_n}(s)).
\]

Taking in (3.23) the limit in the interval \([0, s_{e_n}]\), we get
\[
-\mu \geq \langle \dot{\gamma}_n(s), \gamma_n(s) \rangle_L \quad \text{for all } s \in [0, s_{e_n}],
\]
therefore \( \gamma_\eta \) is time-like.

Summarizing, in dependence of \( \eta > 0 \), we have constructed a subsequence (of \( \gamma_{e_n} = (x_{e_n}, t_{e_n}) \))
\[
\text{(3.24)} \quad \gamma_\eta^n = (x_\eta^n, t_\eta^n)
\]
which converges, in a suitable interval \([0, s_\eta] \subset [0, 1]\), to a time-like geodesic
\[
\gamma_\eta = (x_\eta, t_\eta),
\]
such that
\[
\gamma_\eta(0) = \gamma_0, \quad \phi(\gamma_\eta(s)) = \eta
\]
and
\[
t_0 \leq t_\eta(s) \leq t_1 \quad \text{for all } s \in [0, s_{e_n}].
\]
Repeating the above procedure in correspondence of $\eta/2$, we can select from (3.24) a subsequence
\[
\gamma_n^{\eta/2} = (x_n^{\eta/2}, t_n^{\eta/2})
\]
which approaches (in a suitable interval $[0, s_{\eta/2}]$ with $s_{\eta/2} > s_\eta$) a time-like geodesic
\[
\gamma_{\eta/2} = (x_{\eta/2}, t_{\eta/2})
\]
such that
\[
\gamma_{\eta/2} \text{ extends } \gamma_\eta,
\gamma_{\eta/2}(0) = \gamma_0, \quad \phi(\gamma_{\eta/2}(s_{\eta/2})) = \eta/2,
\]
and
\[
t_0 \leq t_{\eta/2}(s) \leq t_1 \quad \text{for all } s \in [0, s_{\eta/2}].
\]
Following this procedure we can find a geodesic for any $\eta/k$ ($k \in \mathbb{N}$). Taking the limit when $k$ goes to $+\infty$, we obtain a time-like geodesic
\[
\gamma = (x, t) : [0, s_0] \to \mathcal{U}
\]
such that
\[
s_0 = \sup \{ s_{\eta/k} : k \in \mathbb{N} \} < 1,
\gamma_{[0, s_0]} = \gamma_{\eta/k} \quad \text{for all } k \in \mathbb{N},
\gamma(0) = \gamma_0, \quad \phi(\gamma(s_{\eta/k})) = \eta/k,
\]
and
\[
(3.25) \quad t_1 \leq t(s) \leq t_2 \quad \text{for all } s \in [0, s_0].
\]
Now,
\[
\liminf_{s \to s_0} \phi(x(s)) = \lim_{k \to +\infty} \phi(\gamma(s_{\eta/k})) = \lim_{k \to +\infty} \phi(\gamma_{\eta/k}(s_{\eta/k})) = 0
\]
because of (3.6). Then, by virtue of Remark 3.1,
\[
(3.26) \quad \liminf_{s \to s_0} \beta(x(s)) = 0.
\]
Since $\gamma$ is time-like, (3.25) and (3.26) contradict property (v) of Definition 1.2. Then (3.5) (and therefore Lemma 3.2) is proved.

**Lemma 3.3.** Let $\varepsilon > 0$. Then for any $a \in \mathbb{R}$ the sublevels
\[
J^a_\varepsilon = \{ x \in \Omega^1(M_0, x_0, x_1) \mid J_\varepsilon(x) \leq a \}
\]
are complete metric spaces. Moreover if \((\mathcal{M}_0, h)\) is of class \(C^3\), \(J_\epsilon\) satisfies the Palais-Smale condition, i.e. any sequence \(\{x_n\}_{n \in \mathbb{N}} \subset \Omega^1\) such that

\[(3.27) \quad J_\epsilon(x_n) \text{ is bounded}\]

and

\[(3.28) \quad J_\epsilon'(x_n) \to 0\]

contains a subsequence convergent (in \(W^{1,2}\)) to \(x_\epsilon \in \Omega^1(\mathcal{M}_0, x_0, x_1)\).

PROOF. Clearly the sets

\[\left\{ \int_0^1 (\dot{x}, \dot{x})_R \, ds : x \in J_\epsilon^a \right\}, \quad \text{and} \quad \left\{ \int_0^1 \frac{ds}{\beta(x)} : x \in J_\epsilon^a \right\}\]

are bounded. Then by Lemma 2.3 we deduce that there exists \(\mu > 0\), such that

\[(3.29) \quad J_\epsilon^a \subset \Omega^1(A_\mu, x_0, x_1)\]

where

\[A_\mu = \left\{ x \in \mathcal{M}_0 \left| \phi(x) \geq \mu \right. \right\}.\]

Now \(A_\mu\), with the metric \((\cdot, \cdot)_R\), is complete. Then also the closed subset \(J_\epsilon^a\) of \(\Omega^1(A_\mu, x_0, x_1)\) (see 3.29) is complete. Now assume \((\mathcal{M}_0, h)\) to be of class \(C^3\) in order to use the Nash embedding theorem (see [13]). Let \(\{x_n\} \subset \Omega^1\) satisfy (3.27) and (3.28). Clearly \(\{x_n\} \subset J_\epsilon^a\) for some \(a \in \mathbb{R}\), then from (3.29)

\[\{x_n\} \subset \Omega^1(A_\mu, x_0, x_1) \text{ for some } \mu > 0.\]

By Nash embedding theorem (see [13]), \(A_\mu\) can be isometrically embedded into \(\mathbb{R}^N\) (with \(N\) sufficiently large) equipped with the Euclidean metric. Then, using Lemma (2.1) in [4] and arguing as in the proof of Theorem 1.1 in [5], we can deduce that \(\{x_n\}\) contains a subsequence convergent to

\[x_\epsilon \in \Omega^1(A_\mu, x_0, x_1) \subset \Omega^1(\mathcal{M}_0, x_0, x_1).\]

We are now ready to prove Theorems 1.4 and 1.6.

PROOF OF THEOREM 1.4. By virtue of Theorem 2.1 we see immediately that condition (1.5) is necessary to guarantee the existence of a time-like geodesic joining \(z_0\) and \(z_1\).

In order to prove the sufficiency, observe that by the assumption (1.5) there exists

\[\exists \in \Omega^1(\mathcal{M}_0, x_0, x_1) \equiv \Omega^1\]
such that
\[(3.30) \quad J(\vec{x}) < 0.\]

Since \(\phi(\vec{x}(\sigma)) > 0\) for all \(\sigma \in [0, 1]\), by the definition of \(V_{e}\), it is easy to see that
\[(3.31) \quad J_{e}(\vec{x}) = J(\vec{x}) < 0 \quad \text{for } e \text{ small enough.}\]

Clearly a minimizing sequence \(\{x_{n}\}\) for \(J_{e}\) is contained in some sublevel \(J_{e}^{a}\), which, by Lemma 3.3, is complete. Then, since \(J_{e}\) is weakly lower semicontinuous, the infimum of \(J_{e}\) on \(\Omega^{1}\) is attained at some \(x_{e} \in J_{e}^{a} \subset \Omega^{1}(M_{0}, x_{0}, x_{1})\). Moreover by (3.31) we have
\[J_{e}(x_{e}) \leq J_{e}(\vec{x}) = J(\vec{x}) < 0 \quad \text{for } e \text{ small.}\]

Then by Lemma 3.2 we deduce that \(x_{e}\), for \(e \) small enough, is also a critical point for \(J\) on \(\Omega^{1}\) and \(J(x_{e}) < 0\). Then, using Theorem 2.1, Theorem 1.4 is proved.

**Proof of Theorem 1.6.** Assume that \(M_{0}\) is not contractible in itself. Fadell and Husseini have recently proved that there exists a sequence of compact subsets of \(S^{1}\) such that
\[(3.32) \quad \lim_{m \to +\infty} \text{cat}_{\Omega^{1}}(K_{m}) = +\infty,\]
(see [7, Corollary 3.3] and [8, Remark 2.23]). Here \(\text{cat}_{\Omega^{1}}(K_{m})\) denote the Lusternik-Schnirelman category of \(K_{m}\) in \(\Omega^{1}\) (see e.g. [17]), i.e. the minimal number of closed, contractible subsets of \(\Omega^{1}\) covering \(K_{m}\). Now let \(m \in \mathbb{N}\). By (3.32) there exists a compact subset \(K \subset \Omega^{1}\) with
\[\text{cat}_{\Omega^{1}}(K) \geq m.\]
Clearly there exists \(\bar{\Delta}\) such that
\[\forall \ \Delta > \bar{\Delta} \text{ and } \forall \ e \in [0, 1], \quad \sup J_{e}(K) \leq \sup J_{1}(K) \leq -1.\]

Now for every \(e \in [0, 1]\) we set
\[c_{j_{e}} = \inf_{\text{cat}_{\Delta \geq j}} \sup J_{e}(A) \quad j = 1, \ldots, m.\]

Obviously
\[\forall \ e \in [0, 1], \quad \forall \ j \in \{1, \ldots, m\}, \quad c_{j_{e}} \geq \inf_{M_{0}} J \geq \inf_{M_{0}} J > -\infty\]
and
\[(3.33) \quad c_{j_{e}} \leq -1.\]
By Lemma 3.3 and well known methods in critical point theory (see e.g. [15, 17]) we deduce that every $c_{je}$ is a critical value for the functional $J_{e}$. Moreover if $c_{ie} = c_{je}$ for some $i \neq j$, there are infinitely many critical points of $J_{e}$ at the level $c_{je}$. Therefore

for every $\varepsilon \in [0, 1]$, there exist $x_{1\varepsilon}, \ldots, x_{m\varepsilon}$ distinct critical points of $J_{e}$, with critical values $\leq -1$.

Using Lemma 3.2 we deduce that, if $\varepsilon$ is small enough, $x_{1\varepsilon}, \ldots, x_{m\varepsilon}$ are also critical points of $J$ with critical values $\leq -1$. Then, since $m$ is arbitrary, (1.6) is obtained by virtue of Theorem 2.1.

4. - Proof of Theorems 1.8, 1.9

In order to prove Theorems 1.8 and 1.9 we shall study the penalized functional $J_{e}$ defined by (2.9) i.e.

$$J_{e}(x) = \int_{0}^{1} (\dot{x}(s), \dot{x}(s))_{\mathbb{R}} \, ds - \frac{\Delta^{2}}{2} \int_{0}^{1} \frac{1}{\beta(x(s))} \, ds + \int_{0}^{1} V_{e}(x(s)) \, ds,$$

where $x \in \Omega^{1} = \Omega^{1}(\mathcal{M}_{0}, x_{0}, x_{1})$,

$$V_{e}(x) = \frac{\varepsilon}{\phi^{2}(x)}, \quad \varepsilon \in [0, 1],$$

and $\phi$ satisfies (iii) and (v) of Definition 1.7.

Now we shall prove some preliminary lemmas.

**Lemma 4.1.** Let $\mathcal{L} = M_{0} \times \mathbb{R}$ be a static Lorentz manifold with convex boundary $\partial \mathcal{L}$ (see Definition 1.7). For any $\varepsilon \in [0, 1]$ let $x_{e}$ be a critical point of $J_{e}$ on $\Omega^{1}(\mathcal{M}_{0}, x_{0}, x_{1})$ and assume that there exists $c_{1} > 0$ such that

$$J_{e}(x_{e}) \leq c_{1}, \quad \text{for all } \varepsilon \in [0, 1].$$

Then

$$\phi(x_{e}(s)) \geq c_{0} > 0, \quad \text{for all } s \in [0, 1]$$

where $c_{0}$ is independent of $\varepsilon$.

**Proof.** Arguing by contradiction, assume that there exists a sequence $\{x_{e_{n}}\}_{n \in \mathbb{N}}$ of critical points of $J_{e_{n}}(e_{n} \to 0)$ such that

$$u_{n}(s_{n}) = \phi(x_{e_{n}}(s_{n})) \to 0 \quad \text{for } n \to +\infty$$
where \( s_n \) is a minimum point for the map

\[ u_n(s) \equiv \phi(x_{\varepsilon_n}(s)) \quad s \in [0, 1]. \]

Now we set

\[ \gamma_n(s) = (x_{\varepsilon_n}(s), t_{\varepsilon_n}(s)) \]

where \( t_{\varepsilon_n} = t_{\varepsilon_n}(s) \) is defined by

\[
\begin{aligned}
t_{\varepsilon_n}(s) &= \Delta \left[ \int_0^1 \frac{ds}{\beta(x_{\varepsilon_n}(s))} \right]^{-1} \frac{1}{\beta(x_{\varepsilon_n}(s))} \\
t_{\varepsilon_n}(0) &= t_0.
\end{aligned}
\]

By Theorem 2.1, \( \gamma_n \) is a critical point of \( f_{\varepsilon_n} \), then

\[ D_s \gamma_n(s) = \frac{1}{2} \nabla_L V_n(\gamma_n(s)), \quad \text{for all } s \in [0, 1] \]

(4.5)

where \( V_n(\gamma_n(s)) = V_{\varepsilon_n}(x_{\varepsilon_n}(s)) \).

Then for all \( s \in [0, 1] \) we have

\[
\frac{d^2}{ds^2} (u_n(s)) = \frac{d}{ds} \left[ (\nabla_L \phi(\gamma_n(s)), \gamma(s))_L \right]
\]

(4.6)

\[
= H^\phi_L(\gamma_n(s))[\gamma_n(s), \dot{\gamma}_n(s)] + (\nabla_L \phi(\gamma_n(s)), D_s \gamma_n(s))_L
\]

\[
= H^\phi_L(\gamma_n(s))[\gamma_n(s), \dot{\gamma}_n(s)] + \frac{1}{2} (\nabla_L \phi(\gamma_n(s)), \nabla_L V_n(\gamma_n(s)))_L.
\]

Eventually passing to a subsequence let

\[ s_0 = \lim_{n \to +\infty} s_n. \]

Since \( \phi(\gamma_n(0)) = \phi(x_0) > 0, \phi(\gamma_n(1)) = \phi(x_1) > 0 \), and \( \int_0^1 (x_{\varepsilon_n}, x_{\varepsilon_n})_R ds \) is bounded independently of \( n \), we have

\[ s_0 \neq 0, \ s_0 \neq 1. \]

Moreover, by (4.4) and the boundness of \( \int_0^1 (x_{\varepsilon_n}, x_{\varepsilon_n})_R ds \), there is \( \mu > 0 \) such that

\[ u_n(s) = \phi(\gamma_n(s)) < \delta \quad \text{for } s \in [s_0 - \mu, s_0 + \mu] \] and \( n \) large enough,
where $\delta > 0$ is introduced in (1.8) and (1.9). Then from (4.6), (1.8) and (1.9) we deduce that, for $n$ large enough and for all $s \in [s_0 - \mu, s_0 + \mu]$,

$$
\frac{d^2 u_n}{ds^2}(s) \leq M \phi(\gamma_n(s)) \langle \dot{\gamma}_n(s), \gamma_n(s) \rangle_L
$$

(4.7)

$$
- \varepsilon_n \langle \nabla L \phi(\gamma_n(s)), \nabla L \phi(\gamma_n(s)) \rangle_L - \frac{1}{\phi(\gamma_n(s))}
$$

$$
\leq M u_n(s) \langle \dot{\gamma}_n(s), \gamma(s) \rangle_L - \nu \varepsilon_n \cdot \frac{1}{u_n^2(s)}.
$$

Now by (2.17) and (2.18) in Lemma 2.2 we have, for all $s \in [0, 1]$,

(4.8) \quad \begin{align*}
|\langle \gamma_n(s), \gamma_n(s) \rangle_L| &= |J_{x_n}(x_{e_n}) - 2 \int_0^1 V_{e_n}(x_{e_n})ds + V_{e_n}(x_{e_n})|.
\end{align*}

Moreover

$$
\int_0^1 \langle \dot{x}_{e_n}(s), \dot{x}_{e_n}(s) \rangle_R ds - \Delta^2 \sup_{\mathcal{H}_0} \beta + \int_0^1 V_{e_n}(x_{e_n})ds
$$

(4.9)

$$
\leq \int_0^1 \langle \dot{x}_{e_n}(s), \dot{x}_{e_n}(s) \rangle_R ds - \frac{\Delta^2}{\beta(x_{e_n}(s))} + \int_0^1 V_{e_n}(x_{e_n})ds
$$

$$
= (\text{by (4.1)}) = J_{x_n}(x_{e_n}) \leq (\text{by (4.2)}) \leq c_1.
$$

From (4.9) we deduce that

$$
\int_0^1 V_{e_n}(x_{e_n}(s))ds \leq c_1 + \Delta^2 \sup_{\mathcal{H}_0} \beta \equiv c_2,
$$

so by (4.8) and (4.2) we have, for all $s \in [0, 1]$,

(4.10) \quad \begin{align*}
|\langle \gamma_n(s), \gamma_n(s) \rangle_L| &\leq c_1 + 2c_2 + \frac{\varepsilon_n}{u_n^2(s)}.
\end{align*}

Then, inserting (4.10) in (4.7), we get

$$
\frac{d^2 u_n}{ds^2}(s) \leq c_3 u_n(s) + c_4 \frac{\varepsilon_n}{u_n^2(s)} - \nu \varepsilon_n \cdot \frac{1}{u_n^2(s)}
$$

(4.11)

for all $s \in [s_0 - \mu, s_0 + \mu]$. 

\vspace{1cm}

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where \(c_3, c_4\) are real constants independent of \(n\). Since \(u_n(s) < \delta\), if \(\delta\) is small enough, we obtain from (4.11),

\[
\frac{d^2u_n}{ds^2}(s) \leq c_3u_n(s) \quad \text{for all } s \in [s_0 - \mu, s_0 + \mu].
\]

Then, since \(u_n'(s_n) = 0\), we get, by Gronwall Lemma, that \(u_n(s)\) converges uniformly to zero and this contradicts the boundary conditions

\[
u_n(0) = \phi(x_0) > 0, \quad u_n(1) = \phi(x_1) > 0.\]

By the same proof of Lemma 3.3 we have the following

**Lemma 4.2.** Let \(\mathcal{L} = M_0 \times \mathbb{R}\) be a static Lorentz manifold and denote by \(\partial\mathcal{L}\) the (topological) boundary of \(\mathcal{L}\). Assume that there exists a map \(\phi \in C^0(\mathcal{L} \cup \partial\mathcal{L}, \mathbb{R}^+) \cap C^1(\mathcal{L}, \mathbb{R}^+)\) satisfying (2.21), (2.22) and (2.23). Assume moreover that (ii) and (iv) of Definition 1.7 hold. Then for any \(a \in \mathbb{R}\) the sublevels

\[
J^a = \left\{ x \in \Omega^1(M_0, x_0, x_1) \left| J^a(x) \leq a \right. \right\}
\]

are complete metric spaces. Moreover, if \((M_0, h)\) is of class \(C^3\), \(J^a\) satisfies the Palais-Smale condition on \(\Omega^1(M_0, x_0, x_1)\).

**Lemma 4.3.** Assume (ii), (iii), (iv) of Definition 1.7. Assume moreover that (1.8) holds. Then, for any \(c \in \mathbb{R}\),

\[
\text{cat}_{\Omega^1}(J^c) < +\infty.
\]

**Proof.** For \(\mu > 0\), we set

\[
V_\mu = \left\{ x \in M_0 \left| \phi(x) < \mu \right. \right\}, \quad A_\mu = M_0 \setminus V_\mu.
\]

Now, if \(\mu > 0\) is small enough, there is a diffeomorphism

\[
\Psi : M_0 \to A_\mu
\]

which can be constructed by means of the solutions of the Cauchy problem

\[
\begin{cases}
\frac{d\eta}{ds} = \chi(\eta(s)) \frac{\phi'(\eta(s))}{1 + (\phi'(\eta(s)), \phi'(\eta(s)))^{1/2}} \\
\eta(0) = x \in M_0
\end{cases}
\]

where \(\chi\) is a real \(C^2\) map on \(M_0\) such that

\[
\chi(x) = \begin{cases} 
1 & \text{for } x \in V_{\delta}, \\
\chi(x) \in [0, 1] & \text{if } x \notin A_{\delta}, \\
0 & \text{for } x \in A_{\delta},
\end{cases}
\]
with $0 < \delta_2 < \delta_1 < \delta$, $\delta$ being introduced in (1.8). Denote by $\eta = \eta(s, x)$, $s \in \mathbb{R}$ the solution of (4.16). Then, by (1.8) and standard arguments (see e.g. [15, 17]), it can be shown that there exist $\bar{s}$ and $\mu > 0$ such that

\begin{equation}
\forall \ x \in \mathcal{M}_0 \quad \Psi(x) \equiv \eta(\bar{s}, x) \in A_\mu. \tag{4.17}
\end{equation}

Obviously we can choose $\mu > 0$ so small that

\begin{equation}
x_0, x_1 \in A_\mu. \tag{4.18}
\end{equation}

Let us set

\begin{equation}
\forall \ c \in \mathbb{R} \quad J^{c, \mu} = \left\{ x \in J^c \left| \forall \ s \in [0, 1] \ x(s) \in A_\mu \right. \right\}. \tag{4.19}
\end{equation}

It is not difficult to see that

\begin{equation}
\forall \ c \in \mathbb{R} \ \exists \ \alpha = \alpha(c) > 0 \ \text{s.t.} \ \forall \ x \in J^c, \quad \Psi(x) \in J^{c + \alpha, \mu}, \tag{4.20}
\end{equation}

because $\Psi$ is Lipschitz continuous and $\beta$ is bounded. Now consider the penalized functional

\begin{equation*}
J_*(x) = J(x) + \int_0^1 \frac{\vartheta(x)}{\varphi^2(x)} \, ds, \quad x \in \Omega^1(\mathcal{M}_0, x_0, x_1)
\end{equation*}

where $\vartheta$ is a $C^2$ positive scalar field on $\mathcal{M}_0$ such that

\begin{equation*}
\vartheta(x) = \begin{cases}
0 & \text{for } x \in A_{\mu/2} \\
\vartheta(x) \in [0, 1] & \text{for all } x \in \mathcal{M}_0,
\end{cases}
\end{equation*}

\begin{equation*}
1 & \text{for } x \in V_{\mu/3}
\end{equation*}

Clearly for all $x \in \Omega^1(A_\mu, x_0, x_1)$ we have

\begin{equation*}
J_*(x) = J(x),
\end{equation*}

then

\begin{equation}
J^{c + \alpha, \mu} \subset J_*^{c + \alpha} \equiv \left\{ x \in \Omega^1(\mathcal{M}_0, x_0, x_1) \left| J_*(x) \leq c + \alpha \right. \right\}. \tag{4.21}
\end{equation}

Therefore, from (4.20) and (4.21) we deduce that

\begin{equation}
\Psi(J^c) \subset J_*^{c + \alpha}. \tag{4.22}
\end{equation}

Now, as in Lemma 4.2, the penalized functional $J_*$ satisfies the Palais-Smale condition on $\Omega^1(\mathcal{M}_0, x_0, x_1)$ and its sublevels are complete. Then, arguing as in the proof of Theorem 1.1 in [5], we get

\begin{equation}
\text{cat}_{\Omega^1}(J_*^{c + \alpha}) < +\infty. \tag{4.23}
\end{equation}
At this point (4.22) and (4.23) and well known properties of the Ljusternik-Schnirelman category imply that
\[ \text{cat}_\Omega(J^c) \leq \text{cat}_\Omega(J^{c+\alpha}) \]
so
\[ \text{cat}_\Omega(J^c) < +\infty. \]

Now we are ready to prove Theorems 1.8 and 1.9.

**Proof of Theorem 1.8.** Since the sublevels of \( J_\varepsilon \) are complete (see Lemma 4.2) it is not difficult to show, as in the proof of Theorem 1.4 that, for all \( \varepsilon \in [0, 1] \), \( J_\varepsilon \) has a minimum point \( x_\varepsilon \) on \( \Omega^1(M_0, x_0, x_1) \). Clearly there exists \( c_1 \) (independent of \( \varepsilon \)) such that
\[ J_\varepsilon(x_\varepsilon) \leq c_1 \quad \text{for all } \varepsilon \in [0, 1]. \] (4.24)

Then, by Lemma 4.1, we obtain that there exists \( c_0 > 0 \) such that
\[ \forall \varepsilon \in [0, 1], \forall s \in [0, 1], \quad \phi(x_\varepsilon(s)) \geq c_0 > 0. \] (4.25)

Moreover, using again (4.24), we deduce that \( x_\varepsilon \) is bounded in the \( W^{1,2} \) norm independently of \( \varepsilon \). Since, by Lemma 4.2, \( J^{c_1} \) is complete, we obtain a sequence \( \{x_{\varepsilon_n}\}_{n \in \mathbb{N}} \) (\( \varepsilon_n \to 0 \)) such that
\[ x_{\varepsilon_n} \to \bar{x} \in \Omega^1(M_0, x_0, x_1) \text{ weakly in } W^{1,2}. \] (4.26)

Now, by (4.25) we can take the weak limit in the equation
\[ J'_{\varepsilon_n}(x_{\varepsilon_n}) = 0 \]
and we get, by (4.26), that \( \bar{x} \) is a critical point of \( J \) on \( \Omega^1(M_0, x_0, x_1) \). Then Theorem 1.8 is proved using Theorem 2.1 (with \( \varepsilon = 0 \)).

**Proof of Theorem 1.9.** Let \( \alpha \in \mathbb{R} \) and set
\[ J_\alpha = \left\{ x \in \Omega^1 \left| J(x) \geq \alpha \right. \right\}, \quad J_{\varepsilon, \alpha} = \left\{ z \in \Omega^1 \left| J_\varepsilon(z) \geq \alpha \right. \right\}. \]

By Lemma 4.3 there exists \( k = k(\alpha) \in \mathbb{N} \) such that
\[ B \cap J_\alpha \neq \emptyset \quad \text{if } B \subset \Omega^1 \text{ and } \text{cat}_\Omega(B) \geq k. \] (4.27)

Then, since \( J_\alpha \subset J_{\varepsilon, \alpha} \), we have
\[ B \cap J_{\varepsilon, \alpha} \neq \emptyset \]
(4.28)
for all \( \varepsilon > 0 \), for all \( B \subset \Omega^1 \) with \( \text{cat}_\Omega(B) \geq k \),
from which we deduce that
\[(4.29) \quad c_{k,\varepsilon} \equiv \inf \{ \sup J_\varepsilon(A) : \text{cat}_{\Omega^1}(A) \geq k \} \geq \alpha.\]

Since \(M_0\) is not contractible in itself, (3.32) holds, hence there exists a compact subset \(K\) of \(\Omega^1\) such that \(\text{cat}_{\Omega^1}(K) \geq k\). Therefore, for all \(\varepsilon \in ]0,1]\), we have
\[c_{k,\varepsilon} \leq \sup J_\varepsilon(K) \leq \sup J_1(K) \equiv c_1 < +\infty.\]

Therefore by Lemma 4.2 and well known arguments in critical point theory (see e.g. \([15, 17]\)), we deduce that every \(c_{k,\varepsilon}\) in (4.29) is a critical value of \(J_\varepsilon\), so for all \(\varepsilon \in ]0,1]\), there exists
\[x_\varepsilon \in \Omega^1 \text{ critical point of } J_\varepsilon \text{ s.t.} \]
\[(4.30) \quad c_1 \geq J_\varepsilon(x_\varepsilon) = c_{k,\varepsilon} \geq \alpha.\]

Now, by Lemma 4.1, we have that
\[(4.31) \quad \phi(x_\varepsilon(s)) \geq c_0 > 0 \quad \text{for all } \varepsilon \in ]0,1]\text{ and } s \in [0,1],\]

so, following the same arguments used in proving Theorem 1.8, we get the existence of a sequence \(\varepsilon_n \to 0\) with the corresponding critical point \(x_{\varepsilon_n}\) such that
\[(4.32) \quad x_{\varepsilon_n} \to \bar{x} \in \Omega^1(M_0, x_0, x_1) \text{ weakly in } W^{1,2}\]

and \(\bar{x}\) is a critical point of \(J\) on \(\Omega^1\).

Now we want to show that
\[(4.33) \quad J(\bar{x}) \geq \alpha.\]

Since \(\alpha\) is arbitrary, from (4.33) we easily get the conclusion using Theorem 2.1. Clearly, (4.33) is a consequence of (4.30) if we show that
\[(4.34) \quad x_{\varepsilon_n} \to \bar{x} \quad \text{in } C^1.\]

Since \(x_{\varepsilon_n}\) is a critical point of \(J_{\varepsilon_n}\) we have
\[(4.35) \quad 2D_s x_{\varepsilon_n} = \Delta^2 \left[ \int_0^1 \frac{ds}{\beta(x_{\varepsilon_n}(s))} \right]^{-2} \frac{\beta'(x_{\varepsilon_n})}{\beta^2(x_{\varepsilon_n})} - 2\varepsilon_n \frac{\phi'(x_{\varepsilon_n})}{\phi^3(x_{\varepsilon_n})} = F(x_{\varepsilon_n}),\]
where the derivatives are taken in the distributional sense. By (4.31), (4.32) and the completeness of $A_{\alpha}$, we obtain that

\begin{equation}
F(x_{\epsilon_n}) \to \Delta^2 \left[ \int_0^1 \frac{ds}{\beta(x)} \right]^{-2} \frac{\beta'(x)}{\beta^2(x)} \text{ uniformly.}
\end{equation}

Then from (4.35) we have that

$$\{D_s x_{\epsilon_n}\}_{n \in \mathbb{N}}$$

converges uniformly from which we deduce (4.34).

**Remark 4.4.** Clearly if two events $(x_0, t_0)$, $(x_1, t_1)$ are simultaneous (i.e. $t_1 = t_2$) we have

$$J(x) = \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_R \, ds, \quad x \in \Omega^1 \equiv \Omega^1(M_0, x_0, x_1),$$

then the critical points of $J$ are the geodesics on $M_0$ with respect to the Riemannian metric $\langle \cdot, \cdot \rangle_R$.

**Appendix**

In this appendix we will verify that the Schwarzschild spacetime is a static universe and a static Lorentz manifold with convex boundary. The same computations will show that, when $m^2 > e^2$, the Reissner-Nordström spacetime $\{r > m + \sqrt{m^2 - e^2}\} \times \mathbb{R}$ is a static universe and that it is a static Lorentz manifold with convex boundary provided that $m^2 > \frac{9}{5} \cdot e^2$.

**Proposition A.1.** The Schwarzschild spacetime is a static universe.

**Proof.** Clearly, in order to prove Proposition A.1, it suffices to prove that the Schwarzschild spacetime verifies (v) of Definition 1.2. To this aim let $\gamma(s) = (r(s), \varphi(s), \vartheta(s), t(s))$ a time-like geodesic with respect to the Lorentz metric (1.4); $\gamma$ is a critical point for the functional

\begin{equation}
f(\gamma) = \int_a^b \left[ \frac{1}{\beta'(r)} r^2 + r^2 (\vartheta^2 + \sin^2 \vartheta \cdot \varphi^2) - c^2 \beta(r) t^2 \right] \, ds,
\end{equation}

as
where $\beta(r) = 1 - \frac{2m}{r}$, on the space of the smooth curves $\gamma(s)$ on $L$. Then a geodesic $\gamma(s) = (r(s), \varphi(s), \theta(s), t(s))$ solves the following system of differential equations:

\begin{equation}
\begin{aligned}
- \frac{1}{\beta^2(r)} \beta'(r) \dot{r}^2 - \frac{d}{ds} \left( \frac{2}{\beta(r)} \dot{r} \right) + 2r(\dot{\theta}^2 + \sin^2 \theta \cdot \dot{\phi}^2) - c^2 \beta'(r) \dot{t}^2 &= 0 \\
- \frac{d}{ds} (2r^2 \ddot{\phi}) + 2 \sin \theta \cos \theta \cdot \dddot{\phi}^2 &= 0 \\
r^2 \sin^2 \theta \cdot \dot{\phi} &= L \\
\beta(r) \dot{t} &= K
\end{aligned}
\end{equation}

(A.2)

where $L$ and $K$ are real constants.

Let $\gamma(s) = (r(s), \varphi(s), \theta(s), t(s))$ be a solution of (A.2). Obviously, up to a rotation, we can assume $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$. Then by the second equation of (A.2) and the uniqueness of the Cauchy problem we have that

$$
\gamma(s) = (r(s), \varphi(s), \pi/2, t(s)) \quad \text{for all } s,
$$

and $(r(s), \varphi(s), t(s))$ solves

\begin{equation}
\begin{aligned}
- \frac{1}{\beta^2(r)} \beta'(r) \dot{r}^2 - \frac{d}{ds} \left( \frac{2}{\beta(r)} \dot{r} \right) + 2r \dot{\phi}^2 - c^2 \beta'(r) \dot{t}^2 &= 0 \\
r^2 \dot{\phi} &= L \\
\beta(r) \dot{t} &= K
\end{aligned}
\end{equation}

(A.3)

and therefore

\begin{equation}
\begin{aligned}
\frac{1}{\beta^2(r)} \beta'(r) \dot{r}^2 - \frac{2}{\beta(r)} \frac{d^2r}{ds^2} + 2r \ddot{\phi}^2 - c^2 \beta'(r) \dot{t}^2 &= 0 \\
r^2 \dot{\phi} &= L \\
\beta(r) \dot{t} &= K.
\end{aligned}
\end{equation}

(A.3)

Since $\gamma(s) = (r(s), \varphi(s), \pi/2, t(s))$ is a time-like geodesic we have

\begin{equation}
\frac{1}{\beta(r)} r^2 + r^2 \dot{\phi}^2 - c^2 \beta(r) \dot{t}^2 = E < 0
\end{equation}

(A.4)

so the constant $K$ in the third equation of (A.3) is different from zero. Obviously we can assume

\begin{equation}
K > 0.
\end{equation}

(A.5)
Now assume that the time-like geodesic $\gamma : [a, s_0] \to \mathcal{L}$ satisfies

$$\liminf_{s \to s_0} \beta(r(s)) = 0.$$ 

Now, replacing in (A.4) $\beta(r) \equiv K$ and $r^2 \phi$ with $L$, (see (A.3)) we see that there exists $\beta_0 > 0$ such that

$$\dot{r}(s) \neq 0 \text{ for all } s \text{ such that } \beta(r(s)) \leq \beta_0.$$ 

Moreover, $\beta'(r) \neq 0$ for all $r \neq 0$, hence the function $s \mapsto \beta(r(s))$ is monotone in a left neighbourhood of $s_0$ and therefore

$$\lim_{s \to s_0} \beta(r(s)) = 0,$$

i.e.

(A.6) \hspace{1cm} \lim_{s \to s_0} r(s) = 2m.

Now, by (A.4) and (A.6) we get $\lim_{s \to s_0} \dot{r}(s) = \pm Kc \neq 0$. So, since $r(s) > 2m$ if $s < s_0$, there exist a left neighbourhood $N^-$ of $s_0$ and two positive real constants $\delta_1$ and $\delta_2$ such that

(A.7) \hspace{1cm} -\delta_1 \leq \dot{r}(s) \leq -\delta_2 < 0 \text{ for all } s \in N^-.

To conclude the proof of Proposition A.1 we shall prove that

(A.8) \hspace{1cm} \lim_{s \to s_0} t(s) = +\infty. \quad \text{(*)}

To this aim, notice that, by the third equation of (A.3), we have

$$t(s) = t(a) + K \int_a^s \frac{1}{\beta(r(\tau))} \, d\tau,$$

and putting $r(\tau) - 2m = \sigma$ and $r_0 = r(a) - 2m > 0$, we have

$$t(s) = t(a) + K \int_{r(s) - 2m}^{r_0} \left(1 + \frac{2m}{\sigma}\right) \left(-\frac{1}{r(\tau)}\right) \, d\sigma.$$ 

Then by (A.5), (A.6) and (A.7), we get (A.8), because $r(s_0) - 2m > 0$ and $r(s) - 2m \to 0$ as $s \to s_0^-$. \quad \blacksquare

(10) We recall that we have assumed $K > 0$. 

\}

\}
REMARK. Consider the Reissner-Nordström spacetime and, when \( m^2 > e^2 \), put
\[
    r_+ = m + \sqrt{m^2 - e^2}, \quad \text{and} \quad r_- = m - \sqrt{m^2 - e^2}.
\]
By the same computations of the Proposition A.1, we see that
\[
    t(s) = t(a) + K \int_{r(a)-r_+}^{r(a)} \left( \frac{1}{\sigma} \right) \cdot \frac{(\sigma + r_+)^2}{(\sigma + r_+ - r_-)} \cdot \left( \frac{-1}{r(r)} \right) \, d\sigma,
\]
so \( t(s) \) diverges when a time-like geodesic approaches the topological boundary (where \( r = r_+ \)). Therefore the Reissner-Nordström spacetime \( \{ r > r_+ \} \times \mathbb{R} \) is a static universe.

PROPOSITION A.2. The Schwarzschild spacetime is a static Lorentz manifold with convex boundary according to Definition 1.7.

PROOF. Consider the function \( \phi \) given by
\[
    \phi(r, \theta, \varphi, t) = \sqrt{\beta(r)}.
\]
A simple calculation shows that \( \phi \) satisfies (1.8). Then, clearly, to prove Proposition A.2 it suffices to see that \( \phi \) satisfies (1.9). To this aim let \( \gamma(s) = (r(s), \varphi(s), \pi/2, t(s)) \) be a geodesic with respect to the Lorentz metric (1.4), i.e. a solution of (A.3). We have
\[
    \frac{d^2}{ds^2} \left( \phi(\gamma(s)) \right) = \frac{d^2}{ds^2} \left( \sqrt{\beta(r(s))} \right)
\]
where \( r \) solves the first equation of (A.3). Then
\[
\begin{align*}
    \frac{d^2}{ds^2} \left( \phi(\gamma(s)) \right) &= \frac{d}{ds} \left( \frac{1}{2} \beta^{-1/2}(r(s))\beta'(r(s))r'(s) \right) \\
    &= -\frac{1}{4} \beta^{-3/2}(\beta')^2 r^2 + \frac{1}{2} \beta^{-1/2} \beta'' r^2 + \frac{1}{2} \beta^{-1/2} \beta' \cdot \frac{d^2 r}{ds^2} \\
    &= \frac{1}{2} \beta^{-1/2} \beta'' r^2 + \frac{1}{4} \beta^{1/2} \beta' \left( 2r\dot{\varphi}^2 - c^2 \beta'(r)t^2 \right).
\end{align*}
\]
Moreover there exists \( E \in \mathbb{R} \) such that
\[
    \frac{1}{\beta(r)} \dot{r}^2 + \dot{\varphi}^2 - c^2 \beta(r)t^2 = E,
\]
so we have
\[
\frac{d^2}{ds^2} (\phi(\gamma(s))) = \frac{1}{2} \beta^{-1/2} \left( \beta'' - \frac{\beta'}{r} \right) \dot{r}^2 + \frac{e^2}{4} \beta^{1/2} \beta' \left( \frac{2\beta}{r} - \beta' \right) \dot{t}^2 + \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 E}{r} \leq \frac{e^2}{4} \beta^{1/2} \beta' \left( \frac{2\beta}{r} - \beta' \right) \dot{t}^2 + \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 E}{r}
\]
because \( \beta'' < 0 \) and \( \beta' > 0 \).
Moreover when \( r \leq 3m, \frac{2\beta}{r} - \beta' \leq 0 \). Therefore if \( \phi(\gamma) \leq \sqrt{1/3} \), we have
\[
\frac{d^2}{ds^2} (\phi(\gamma(s))) \leq \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 E}{r} \leq \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 |E|}{r}
\]
(\text{by (A.10)})
\[
\leq \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 E}{r} \leq \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 |E|}{r} \leq \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 |E|}{r}
\]
Now \( \frac{1}{\beta(r)} \dot{r}^2 + r^2 \dot{\varphi}^2 - c^2 \beta(r) \dot{t}^2 = \langle \dot{\gamma}, \dot{\gamma} \rangle_L \) with respect to the Lorentz structure (1.4) because \( \dot{\varphi} \equiv 0 \). Therefore we have, when \( \phi(\gamma) \leq \sqrt{1/3} \),
\[
\frac{d^2}{ds^2} (\phi(\gamma(s))) \leq \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 |\langle \gamma(s), \gamma(s) \rangle_L|}{r}
\]
\[
= \frac{1}{2r} c^2 |\langle \gamma(s), \gamma(s) \rangle_L| \cdot \phi(\gamma(s)).
\]
Finally, since \( r \geq 2m \) implies that \( \frac{\beta'}{r} = \frac{2m}{r^3} \), we get (1.9) and the proof of Proposition A.2.

\textbf{Remark.} Performing similar computations for the Reissner-Nordström spacetime \( \{ r > r_+ \} \times \mathbb{R} \), with
\[
\phi = \sqrt{1 - \frac{2m}{r} + \frac{\varphi^2}{r^2}},
\]
we obtain
\[
\frac{d^2}{ds^2} (\phi(\gamma(s))) = \frac{1}{2} \beta^{-1/2} \left( \beta'' - \frac{\beta'}{r} \right) \dot{r}^2 + \frac{e^2}{4} \beta^{1/2} \beta' \left( \frac{2\beta}{r} - \beta' \right) \dot{t}^2 + \frac{1}{2} \beta^{1/2} \beta' \frac{c^2 E}{r},
\]
where \( \beta = 1 - \frac{2m}{r} + \frac{\varphi^2}{r^2} \) and \( E = \frac{1}{\beta(r)} \dot{r}^2 + r^2 \dot{\varphi}^2 - e^2 \beta(r) \dot{t}^2 \). From this formula
we deduce that the Reissner-Nordström spacetime \( \{r > r_+, I \times \mathbb{R} \} \) is a static Lorentz manifold with convex boundary provided that \( m^2 \geq \frac{9}{5} \cdot e^2 \).

REFERENCES