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Periodic Solutions for a Second Order Partial Differential Equation

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1. - Introduction

Consider the following equation:

$$u_{tt} - \beta a(t) \sum_{i,j} (a_{ij}(x) u_{x_j})_{x_i} = f(t,x) \tag{1}$$

where $\beta \in \mathbb{R}$,

$$a(t) \text{ is continuous, } 2\pi\text{-periodic and strictly positive} \tag{2}$$

and $a_{ij}(x)$ are C^∞ functions such that

$$a_{ij}(x) = a_{ji}(x), \quad \sum a_{ij} \xi_i \xi_j \geq \nu |\xi|^2, \quad \nu > 0, \tag{3}$$

$$a_{ij}(x) \text{ is } 2\pi\text{-periodic in } x_1, \ldots, x_n. \tag{4}$$

Note that equation (1), according to the sign of β, is of hyperbolic ($\beta > 0$) or elliptic ($\beta < 0$) type.

Our aim here is to investigate the existence of solutions $u(t,x)$ to equation (1) which are *periodic* in t.

More precisely, we shall prove two different results. In the first one (see Theorem 1), we shall consider (1) together with the boundary conditions

$$u(t,x) \text{ is } 2\pi\text{-periodic in } t, \ x_1, \ldots, x_n \tag{5}$$

(and $f(t,x)$ satisfies a similar assumption). In the second one (see Theorem 2), assumption (5) is replaced by a Dirichlet type condition at the boundary of an open set $\Omega \subset \mathbb{R}^n$.

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Before stating our results, some comments about the problem are necessary. First of all, in the elliptic case $\beta < 0$, the existence of solutions to problem (1), (5) (or (1) with Dirichlet conditions) is well known. For instance, it can be obtained as an application of the abstract theory developed in [LM]. On the contrary, in the hyperbolic case $\beta > 0$, the problem is more subtle; it is not difficult to see that, for some values of the parameter β and a suitable choice of a periodic $f(t,x)$, (1) has no regular periodic solution, even if f is assumed to be of class C^∞ (see Remark 1 where an example is examined in detail).

A similar problem was studied by de Simon [d], who proved:

THEOREM ([d]). *Let Ω be an open bounded set in $\mathbb{R}^n$, s and integer greater than $n-1$. Then for almost any period $T > 0$, for every $f(t,x)$ of period T with respect to t, and having derivatives up to the order s in $L^2([0,T]\times\Omega)$, the wave equation*

$$u_{tt} - \Delta u = f(t,x) \tag{6}$$

has a (distribution) solution $u(t,x) \in L^2_{\text{loc}}(\mathbb{R}_t; L^2(\Omega))$, periodic of period T in the variable t.

Note that, by the change of variables $t' = 2\pi t/T$, (6) can be put into the form

$$u_{tt} - \beta\Delta u = f(t,x) \tag{7}$$

with a variable parameter β, and a fixed time period 2π. Thus the above result can be stated as follows: for almost any $\beta > 0$ and any $f(t,x)$, 2π-periodic in t and sufficiently regular, (7) has a solution 2π-periodic in t.

With an identical proof, the above result can be extended to the more general strictly hyperbolic equations of the form

$$u_{tt} - \beta\sum_{i,j}(a_{ij}(x)u_{x_j})_{x_i} = f(t,x) \tag{8}$$

with coefficients independent of time.

In this work, we prove similar results for equation (1), where the coefficients may depend also on t, though in a very particular form. We remark that the method of [d] does not allow to handle this case.

To state our first theorem, we introduce the spaces $H^{p,q}(\mathbb{T}^{n+1})$ (p integer, q real ≥ 0) defined as follows:

DEFINITION 1. $u(t,x) \in L^2_{\text{loc}}(\mathbb{R}_t \times \mathbb{R}^n_x)$ is said to belong to $H^{p,q}(\mathbb{T}^{n+1})$ if it is 2π-periodic in $t, x_1, \ldots, x_n$, and moreover $\partial_t^j u \in L^2_{\text{loc}}(\mathbb{R}_t; H^q_{\text{loc}}(\mathbb{R}^n_x))$ for $j \leq p$.

We have then:

THEOREM 1. *Assume* (2), (3), (4) *are fulfilled, and that* $a(t)$ *is of class* C^k. *Then for almost any* β, *for any function*

$$f(t,x) \in H^{k,m+s}(\mathbb{T}^{n+1}) \tag{9}$$

with $s > n-1$, $m \geq k \geq 0$, *satisfying* $\int_{\mathbb{T}^{n+1}} f \, \mathrm{d}t \, \mathrm{d}x = 0$, *problem* (1), (5) *has a solution* $u(t,x)$ *belonging to*

$$u \in H^{j,m-j}(\mathbb{T}^{n+1}), \quad j = 0, \dots, k+2, \tag{10}$$

which is unique up to the addition of arbitrary constants.

In the following theorem, we shall use the spaces $H_0^s(\Omega)$ (closure in $H^s(\Omega)$ of $C_0^\infty(\Omega)$, with induced product and norm) and the spaces $H^s(\Omega) \cap H_0^1(\Omega)$ which are Hilbert spaces with the product

$$(u,v) = \sum_{|\alpha|=s} \int_\Omega \partial^\alpha u \overline{\partial^\alpha v} \, \mathrm{d}x + \int_\Omega \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x. \tag{11}$$

THEOREM 2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with regular boundary, let* $a_{ij} \in C^\infty(\overline{\Omega})$, *and assume that* (2), (3) *are fulfilled and that* $a(t)$ *is of class* C^k. *Then for almost any* β, *for any function*

$$f \in H^k_{\mathrm{loc}}(\mathbb{R}_t; H_0^{s+m}(\Omega)) \tag{12}$$

2π*-periodic in* t, *with* $s > n-1$, $m \geq k \geq 0$, *equation* (1) *has a unique solution* $u(t,x)$ *belonging to*

$$u \in H^j(\mathbb{R}_t; H^{m-j}(\Omega) \cap H_0^1(\Omega)), \quad j = 0, \dots, k+2 \tag{13}$$

and 2π*-periodic in* t.

REMARK 1. What happens exactly when β belongs to the exceptional set of measure 0 alluded to in the above theorems? This question seems difficult; we recall here a well known elementary special case, which makes the difficulty evident.

Consider the wave equation in $\mathbb{R}_t \times \mathbb{R}_x$

$$u_{tt} - \beta u_{xx} = f(t,x) \tag{14}$$

with $\beta = \alpha^2$; $f(t,x)$ is 2π-periodic in t, x, and we look for solutions $u(t,x)$ which are 2π-periodic in t and in x. We shall write

$$f(t,x) = \sum f_{nm} e^{int} e^{imx}, \quad u(t,x) = \sum u_{nm} e^{int} e^{imx}. \tag{15}$$

If the constant β is of the form $\beta = n_0^2/m_0^2$, with n_0, m_0 integers, then evidently the wave operator $\partial_t^2 - \beta\partial_x^2$ is not suriective on $\mathcal{D}'(\mathbb{T}^2)$; to solve the equation for arbitrary f we must assume $f_{n_0m_0} = 0$. Thus, in the following, we shall restrict ourself to the case

$$\alpha = \sqrt{\beta} \notin \mathbb{Q}.$$

Then we can formally write

$$u_{nm} = \frac{f_{nm}}{n^2 - \alpha^2 m^2}, \quad (n,m) \neq (0,0). \tag{16}$$

It is evident that the formal expressions (16) represent the unique L^2 periodic solution to (14), when it exists.

Now assume $f \in L^2$, i.e. $\{f_{nm}\}$ is ℓ^2 summable. For the formal solution $u(t,x)$ to be at least in L^2, it would be necessary that $n^2 - \beta m^2 \geq c > 0$ for some c. But this is (almost) never the case. In fact, it is known that ([Kh] Theorem 32) for almost any real α the inequality

$$\left|\frac{n}{m} - \alpha\right| < \frac{1}{m^2 \ln m}$$

has an infinite number of solutions in integers n, m. Thus the same is true for the inequality (we can assume $\alpha > 0$ and thus n, $m > 0$)

$$|n^2 - \alpha^2 m^2| \leq \frac{n + \alpha m}{m \ln m} = \frac{n}{m \ln m} + \frac{\alpha}{\ln m};$$

we can therefore find two sequences n_k, m_k with

$$|n_k^2 - a^2 m_k^2| \leq \frac{3\alpha}{\ln m_k}$$

(we have used here the obvious fact that $n_k/m_k \to \alpha$). By extracting a subsequence, we can choose m_k so that $m_k \geq k$. Now if we define a function $f \in L^2$ through its Fourier coefficients as

$$f_{n_k m_k} = \frac{1}{k^{1/2} \ln k}$$

while the other coefficients are 0, we have immediately

$$|u_{n_k m_k}| \geq \frac{1}{3\alpha} \frac{\ln m_k}{\ln k} \frac{1}{k^{1/2}} \geq \frac{1}{3\alpha} \frac{1}{k^{1/2}}$$

and hence $u(t,x)$ is not in L^2.

The result of Theorem 1 corresponds in this situation to the following result: fixed $\epsilon > 0$, for almost any real α the inequality

$$\left|\frac{n}{m} - \alpha\right| \leq \frac{1}{m^{2+\epsilon}}$$

has only a finite number of solutions in integers n, m. This implies that, as soon as $f \in H^s$ for some s, for almost any α the formal solution will be in any H^r with $r < s$.

More generally, it is possible to construct a periodic function $f(t,x)$ in H^∞ (i.e. with all the derivatives in L^2) such that, for suitable values of α, the corresponding solution is not in L^2. This can be done proceeding as above, but using the following result: given any positive function of integer argument $\varphi(m)$, there exist irrational numbers α such that

$$\left|\frac{n}{m} - \alpha\right| < \varphi(m)$$

has an infinite number of integer solutions ([Kh] Theorem 22).

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2. - Proof of the theorems

Theorems 1 and 2 can be obtained as special cases of an abstract theorem in Hilbert spaces. We begin by fixing a suitable framework.

Let H be a separable Hilbert space, and A a selfadjoint (unbounded) operator on H. We shall assume that an orthonormal basis $(e_p)_{p\geq 1}$ of H exists, made of eigenvectors of A; precisely we assume that

$$Ae_p = \mu_p e_p \tag{16}$$

with

$$\mu_p \in \mathbb{R}, \quad |\mu_p| \geq c > 0. \tag{17}$$

With any sequence $\{c_p\}$ of complex numbers such that $|c_p| \geq$ const. > 0, we shall associate the subspace of H defined as

$$H\{c_p\} = \{v \in H : v = \sum v_p e_p, \ \{v_p c_p\} \in l^2\}, \tag{18}$$

endowed with the natural norm $\|v\| = (\sum |v_p c_p|^2)^{1/2}$. Of course, definition (18) depends on the choice of the basis $(e_p)_{p\geq 1}$.

REMARK 2. In particular, when Ω is a bounded open set with regular boundary, and A is a selfadjoint elliptic operator of order 2 on H with coefficients in $C^\infty(\overline{\Omega})$, we can find a sequence $(e_p)_{p\geq 1}$ of eigenvectors of A with the following properties: $(e_p)_{p\geq 1}$ is an orthonormal basis in L^2, $e_p \in H_0^1(\Omega) \cap C^\infty(\Omega)$, the eigenvalues μ_p are strictly positive and form an increasing unbounded sequence, and finally $(e_p/\mu_p^{1/2})$ is an orthonormal basis in $H_0^1(\Omega)$ under the product $(u,v) = \int \nabla u \cdot \nabla \overline{v}$ (this is classical, see e.g. [B, XI.8]). We have then

$$H\{\mu_p^{1/2}\} = H_0^1(\Omega).$$

More generally, for $s \geq 1$,

$$H_0^s(\Omega) \subseteq H\{\mu_p^{s/2}\} \subseteq H_0^1(\Omega) \cap H^s(\Omega). \tag{19}$$

The second inclusion is a consequence of the interior regularity for elliptic operators (see [GT]). Here is a sketchy proof of the first inclusion: let $f \in H_0^s(\Omega)$, then $A^{s/2}f \in L^2(\Omega)$ (recall that fractional powers of elliptic operators are well defined), thus $A^{s/2}f = \sum \lambda_p e_p$ for some sequence $\{\lambda_p\} \in \ell^2$, and $A^{s/2}(f - \sum \lambda_p \mu_p^{-s/2} e_p) = 0$. This implies $f - \sum \lambda_p \mu_p^{-s/2} e_p = 0$ (since it belongs to H_0^1).

In the periodic case the situation is simpler. Let

$$H = \left\{ v \in L^2(\mathbb{T}^n) : \int_{\mathbb{T}^n} v \, \mathrm{d}x = 0 \right\} \tag{20}$$

and A be an elliptic operator of order 2 of the form

$$Au = -\sum_{i,j} (a_{ij}(x) u_{x_j}) x_i, \tag{21}$$

with $a_{ij} \in C^\infty(\mathbb{R}_x^n)$ such that

$$a_{ij} = a_{ji}, \quad \sum a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \quad a_{ij} \ 2\pi\text{-periodic in } x_1, \dots, x_n \tag{22}$$

(for some $\nu > 0$). By well known results about elliptic operators on compact manifolds without boundary (see [T, Ch.XII]; [H, Ch.XVII]), we can find a basis (e_p) of H made of eigenvectors of A, with eigenvalues $0 < \mu_p \uparrow \infty$. We have now

$$H\{\mu_p^{s/2}\} = \left\{ v \in H^2(\mathbb{T}^n) : \int_{\mathbb{T}^n} v \, \mathrm{d}x = 0 \right\}. \tag{23}$$

In fact, it is easy to see that

$$H\{\mu_p^s\} = D(A^s)$$

for real $s \geq 0$.

Finally, we recall that in both cases the asymptotic behaviour of the eigenvalues is

$$\mu_p \sim p^{2/n} \tag{24}$$

as $p \to \infty$ ([A], [T], [H]).

In the following, we shall use the notation (k integer ≥ 0)

$$H_\sharp^k(0,T;H\{c_p\})$$

to denote the set of T-periodic functions $f(t)$, defined on $\mathbb{R}$ with values in $H\{c_p\}$, such that

$$f^{(j)}(t) = \frac{d^j}{dt^j} f(t) \in L^2(0,T;H\{c_p\}),\ \ 0 \leq j \leq k.$$

We have then

PROPOSITION. *Let $a(t)$ be a strictly positive T-periodic function of class C^k, $k \geq 0$. Let $\{\gamma_p\}$, $\{c_p\}$ be two sequences such that $c_p > c > 0$; assume that an operator A on the Hilbert space H is given, satisfying* (16), (17), *assume that*

$$\sum \frac{1}{c_p|\mu_p|^{1/2}} < \infty \tag{25}$$

and that

$$\gamma_p \geq c_p|\mu_p|^{k/2}. \tag{26}$$

Then there exists a set of measure zero $V \subset \mathbb{R}$ such that for any $\beta \in \mathbb{R}\backslash V$, and any function

$$f(t) \in H_\sharp^k(0,T;H\{\gamma_p\}) \tag{27}$$

the equation

$$u'' + \beta a(t)Au = f(t) \tag{28}$$

has a unique T-periodic solution, such that

$$u(t) \in H_\sharp^j(0,T;H\{\gamma_p c_p^{-1}|\mu_p|^{-j/2}\}) \tag{29}$$

for $j = 0, \ldots, k+2$.

We begin by proving the following lemma (cfr. [d]).

LEMMA. *Let $\{a_p\}$, $\{b_p\}$, $\{c_p\}$ be three sequences of real numbers, such that $a_p \neq 0$ $\forall p$, $c_p \geq c > 0$, and b_p has no finite point of accumulation. Define*

$$V = \{\beta \in \mathbb{R} : \forall\epsilon > 0\ \exists p,q |\beta a_p - b_q| \cdot c_p < \epsilon\}$$

and, fixed $s \geq r$ *real numbers,* $\delta > 0$,

$$\nu_\delta(p) = \text{ card } \{q \in \mathbb{N} : ra_p - \delta < b_q < sa_p + \delta\}.$$

If, for some δ*, the series*

$$\sum_{p \geq 0} \frac{\nu_\delta(p)}{c_p |a_p|}$$

converges, then

$$m(V \cap [r, s]) = 0.$$

PROOF. We can suppose $a_p > 0$. If not, observe that $V = V^+ \cap V^-$, where

$$V^\pm = \{\beta \in \mathbb{R} : \forall \epsilon > 0, \ \exists p, \ q, \ \pm a_p > 0, \ |\beta a_p - b_q| \cdot c_p < \epsilon\}$$

and consider the two cases separately.

It is easy to see that

$$V \cap [r, s] = \bigcap_{\epsilon > 0} \bigcup_{p \in \mathbb{N}} \bigcup_{q \in L_p(\epsilon)} J^\epsilon_{pq},$$

where

$$J^\epsilon_{pq} = \left] \frac{b_q}{a_p} - \frac{\epsilon}{c_p a_p}, \frac{b_q}{a_p} + \frac{\epsilon}{c_p a_p} \right[\cap [r, s],$$

$$L_p(\epsilon) = \left\{ q \in \mathbb{N} : \exists \beta \in [r, s], \ \beta a_p - \frac{\epsilon}{c_p} < b_q < \beta a_p + \frac{\epsilon}{c_p} \right\}.$$

The set $L_p(\epsilon)$ is finite, in virtue of the assumption about the sequence $\{b_q\}$, and its cardinality is not greater than $\nu_\delta(p)$ if $\epsilon < \delta c$. So that, for any $\epsilon > 0$ sufficiently small,

$$m(V \cap [r, s]) \leq \sum_{p \geq 0} \nu_\delta(p) \cdot \frac{2\epsilon}{c_p a_p} = C\epsilon. \qquad \square$$

PROOF OF THE PROPOSITION. Writing

$$f(t) = \sum f_p(t) e_p, \quad u(t) = \sum y_p(t) e_p,$$

we have to solve the (infinite) system of ordinary differential equations

$$y''_p + \beta \mu_p a(t) y_p = f_p(t), \quad p = 0, 1, 2, \ldots \tag{30}$$

with the conditions

$$y_p(0) = y_p(T), \quad y'_p(0) = y'_p(T).$$

To this end, consider the following family of operators, depending on the parameter σ:

$$T(\sigma) = \frac{d^2}{dt^2} + \sigma a(t)$$

on $L^2(0,T)$, with domain $H^2_\sharp(0,T)$ (T-periodic H^2 functions). We shall need some tools from the perturbation theory of operators, for which we refer to [K]. The family $T(\sigma)$ is a *holomorphic family of operators* (of type (A), according to [K] VII.2), that is to say

i) the domain is independent of σ;

ii) $T(\sigma)u$ is a holomorphic function of σ for any u in the domain;

and both conditions are trivially satisfied by our family (in fact, $T(\sigma)$ is an *entire* family of operators). Moreover, the operators of the family are selfadjoint and with compact resolvent for any σ.

Thus, by a theorem of Rellich (see [K], VII.3.9), the eigenvalues and the eigenfunctions of $T(\sigma)$ depend holomorphically on σ. More precisely, we can find two sequences of functions $\{\lambda_j^+(\sigma)\}_{j\geq 0}$, $\{\lambda_j^-(\sigma)\}_{j\geq 1}$, and correspondingly two sequences of L^2-valued functions $\{v_j^+(\sigma)\}_{j\geq 0}$, $\{v_j^-(\sigma)\}_{j\geq 1}$, with the following properties: all the functions are holomorphic in some neighbourhood (possibly depending on j) of the real axis, the $\lambda_j^\pm(\sigma)$ represent all the (repeated) eigenvalues of $T(\sigma)$, and the $v_j^\pm$ are the corresponding normalized eigenfunctions, which form a complete orthonormal family; moreover we have, for $\sigma = 0$,

$$\lambda_0^+(0) = 0, \qquad v_0^+(0) = T^{-1/2},$$

$$\lambda_j^+(0) = -\frac{4\pi^2}{T^2}j^2, \quad v_j^+(0) = \sqrt{\frac{2}{T}}\cos(2\pi jt/T), \quad j = 1,2,3,\ldots$$

$$\lambda_j^-(0) = -\frac{4\pi^2}{T^2}j^2, \quad v_j^-(0) = \sqrt{\frac{2}{T}}\sin(2\pi jt/T), \quad j = 1,2,3,\ldots.$$

We can say something more about the behaviour of the eigenvalues: note in fact that

$$\begin{aligned}\frac{d}{d\sigma}\lambda_j^\pm(\sigma) &= \frac{d}{d\sigma}\langle T(\sigma)v_j^\pm(\sigma), v_j^\pm(\sigma)\rangle \\ &= \langle T'(\sigma)v_j^\pm(\sigma), v_j^\pm(\sigma)\rangle + 2\,\mathrm{Re}\langle T(\sigma)v_j^\pm(\sigma), v_j^\pm(\sigma)'\rangle\end{aligned}$$

and the second term vanishes since $v_j^\pm(\sigma)$ is an eigenvector and has constant norm; to compute the first one, we remark that $T'(\sigma) = a(t)$, so that

$$\frac{d}{d\sigma}\lambda_j^\pm(\sigma) = \int_0^T a(t)|v_j^\pm(\sigma)(t)|^2\,dt.$$

Then, denoting with λ and Λ the minimum and the maximum of the function $a(t)$ respectively, we easily conclude that

$$\Lambda T \geq \frac{\mathrm{d}\lambda_j^{\pm}(\sigma)}{\mathrm{d}\sigma} \geq \lambda T > 0. \tag{31}$$

As a consequence of (31), λ_j^+ and λ_j^- have exactly one zero each on the entire real axis; moreover

$$\lambda T\sigma \geq \lambda_j^{\pm}(\sigma) + j^2 \geq \lambda T\sigma, \quad \sigma \in \mathbb{R}. \tag{32}$$

If we now denote by σ_{2q} the unique zero of λ_q^+, $q \geq 0$, and by σ_{2q-1} the unique zero of λ_q^-, $q \geq 1$, it follows from (32) that

$$\sigma_q \geq \text{const} \cdot q^2. \tag{33}$$

We can now solve the system (30). In terms of the $T(\sigma)$, (30) can be written

$$T(\beta\mu_p)y_p = f_p, \quad p = 1, 2, \dots . \tag{34}$$

To solve (34) for all p, a first necessary requirement is that

$$\beta\mu_p \neq \sigma_q \quad \forall p \geq 1, \ q \geq 0.$$

This excludes a countable set of possible values of β, namely

$$V_1 = \{\sigma_q/\mu_p\}_{p\geq 1, q\geq 0}.$$

If $\beta \notin V_1$, i.e. if 0 is not an eigenvalue of $T(\beta\mu_p)$, it is possible to solve the equations (34) simultaneously, obtaining a family of functions y_p of $H^2_\sharp(0,T)$ given by

$$y_p = T(\beta\mu_p)^{-1} f_p. \tag{35}$$

In order to obtain from these functions a solution to (28), we must ensure the convergence of the series $\sum y_p e_p$. We remark that this will also imply the uniqueness of the solution, since given any solution of (28), the sequence of its Fourier coefficients must solve (30), and hence must satisfy (35).

To this end, we shall estimate the norm of the bounded operator $T(\beta\mu_p)^{-1}$. This can be done as follows: since we know all its eigenvalues, and they are all different from 0 and form an increasing sequence, we can select the eigenvalue with minimal absolute value, say $\lambda^+_{k(p)}(\beta\mu_p)$ (the index $k(p)$ of course depends on p, and the case λ^- is completely analogous); then we have simply

$$\|T(\beta\mu_p)^{-1}\| = |\lambda^+_{k(p)}(\beta\mu_p)|^{-1}. \tag{36}$$

We apply now the Lemma to the sequences $a_p = \mu_p$, $b_p = \sigma_p$ (the third sequence c_p being exactly the sequence appearing in the statement of the Proposition). For any $r \le s$, $\delta \le 1$ we have

$$\begin{aligned}\nu_\delta(p) &= \operatorname{card}\{q : r\mu_p - \delta < \sigma_q < s\mu_p + \delta\} \\ &\le \operatorname{card}\{q : 0 \le \sigma_q < s'\mu_p\}\end{aligned}$$

since $\sigma_q > 0$, and choosing e.g. $s' = s + 1/c$. But then from the inequality (33) it follows that

$$\nu_\delta(p) \le \text{ const } \cdot |\mu_p|^{1/2}$$

and hence

$$\sum \frac{\nu_\delta(p)}{c_p|\mu_p|} \le \text{ const } \cdot \sum \frac{1}{c_p|\mu_p|^{1/2}} < \infty;$$

the constant here depends on r, s, but the series always converges, for all r, s (by (25)). Now the Lemma implies the existence of a set V_2, with zero measure, such that, for $\beta \notin V_2$,

$$|\beta\mu_p - \sigma_q| \cdot c_p \ge \epsilon(\beta) > 0, \quad p \ge 1, \ p \ge 0. \tag{37}$$

Set $V = V_1 \cup V_2$, and choose any $\beta \in \mathbb{R}\backslash V$. Since $\sigma_{2k(p)}$ is the unique zero of $\lambda^+_{k(p)}$, using (31) and (37) we get

$$|\lambda^+_{k(p)}(\beta\mu_p)| \ge |\beta\mu_p - \sigma_{2k(p)}| \cdot \lambda T \ge \epsilon(\beta) \cdot \lambda T \cdot c_p^{-1}$$

whence, by (36),

$$||T(\beta\mu_p)^{-1}|| \le \text{ const } \cdot c_p. \tag{38}$$

This is the desired estimate.

As a consequence, we have

$$||y_p||_{L^2(0,T)} \le \text{ const } \cdot c_p ||f_p||_{L^2(0,T)}. \tag{39}$$

Moreover, by (30) (we shall write for brevity $||\cdot|| = ||\cdot||_{L^2(0,T)}$),

$$||y''_p|| \le ||f_p|| + ||\beta\mu_p y_p||\lambda \le \text{ const } \cdot c_p|\mu_p| \cdot ||f_p||$$

and consequently

$$||y'_p|| \le \text{ const } \cdot c_p|\mu_p|^{1/2} \cdot ||f_p||; \tag{40}$$

to prove the last inequality, we have used the fact that, for any regular real valued T-periodic function $y(t)$, we have

$$||y'||^2 = \int_0^T y' \cdot y' \, \mathrm{d}t = -\int_0^T y \cdot y'' \, \mathrm{d}t \le ||y|| \cdot ||y''||$$

and hence

$$\|y'\| \le \|y\|^{1/2} \cdot \|y''\|^{1/2}$$

and the same holds for a complex valued $y(t)$, since we can write

$$\|y'\|^2 = \|(\operatorname{Re} y)'\|^2 + \|(\operatorname{Im} y)'\|^2.$$

Estimates for higher order derivatives of y_p can be obtained by simply derivating the equations (30) (up to k times, since $a(t)$ is of class C^k), and proceeding similarly. We obtain

$$\|y_p^{(j)}\| \le \text{const} \cdot c_p |\mu_p|^{j/2} \cdot \sum_{h=0}^{j-2} \|f_p^{(h)}\|, \quad 2 \le j \le k+2. \tag{41}$$

Now, assumption (27) can be formulated as

$$\{\|f_p^{(j)}\| \cdot \gamma_p\} \in \ell^2, \quad j = 0, \dots, k$$

since, for any function $f(t) = \sum f_p(t) e_p \in L^2(0,T; H\{\gamma_p\})$, we can write

$$\|f(t)\|^2_{L^2(0,T;H\{\gamma_p\})} = \int_0^T \|f^{(j)}(t)\|^2_{H\{\gamma_p\}} \, \mathrm{d}t = \sum_p \|f_p^{(j)}\|^2_{L^2(0,T)} \gamma_p^2;$$

thus, applying (41), we have

$$\{\|y_p^{(j)}\| \cdot c_p^{-1} |\mu_p|^{-j/2} \gamma_p\} \in \ell^2$$

for $j = 2, \dots, k+2$, and hence, recalling also (39) and (40), we get (29). This concludes the proof of the Proposition. □

PROOF OF THEOREMS 1, 2. To prove Theorem 1, we begin by showing that we can assume, for all t,

$$\int_{\mathbb{T}^n} f(t,x) \, \mathrm{d}x = 0. \tag{42}$$

In fact, let

$$\phi(t) = \int_{\mathbb{T}^n} f(t,x) \, \mathrm{d}x,$$

and note that, by assumption,

$$\int_0^{2\pi} \phi(t) \, \mathrm{d}t = 0.$$

Thus if we put

$$\Phi(t) = \int_0^t \int_0^s \phi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s - \frac{t}{2\pi}\int_0^{2\pi}\int_0^s \phi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \tag{43}$$

it is easy to verity that $\Phi(t)$ is 2π-periodic, and such that $\Phi'' = \phi$. It is then clear that we can firstly solve equation (1) with f replaced by $f - (2\pi)^{-n}\phi$, which satisfies a condition of the form (42) for all t; then, to obtain the solution to the original problem, it is sufficient to add to the solution thus obtained the function $\Phi(t)$.

Now choose H, A as in (20), (21), (22). Apply the Proposition with $T = 2\pi$,

$$c_p = |\mu_p|^{s/2}, \quad \gamma_p = |\mu_p|^{(s+m)/2}. \tag{44}$$

Since the eigenvalues μ_p of A grow as $\mu_p \sim p^{2/n}$ (see (24)), we have

$$c_p|\mu_p|^{1/2} \sim p^{(s+1)/n},$$

and as $s > n-1$, assumption (25) is fulfilled, while (26) is obviously verified since $m \geq k$.

Then a unique solution $u(t,x)$ exists, such that

$$u(t,x) \in H^j_\sharp(0,2\pi; H\{\mu_p^{(m-j)/2}\}), \quad 0 \leq j \leq k+2$$

(since $\gamma_p c_p^{-1}|\mu_p|^{-j/2} = \mu_p^{(m-j)/2}$) and this, by (23), implies (10).

The proof of Theorem 2 is completely analogous, with $H = L^2(\Omega)$, A an elliptic operator of the form (21) with coefficients in $C^\infty(\overline{\Omega})$, $e_p \in H^1_0(\Omega)$ (see Remark 2), and using (19) instead of (23). □

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