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Introduction

Let $S$ be a minimal surface of general type defined over $\mathbb{C}$. We call $S$ a canonical surface if the rational map associated with $|K|$ is birational onto its image. Assume that $S$ is a canonical surface with a non-linear pencil, and let $f : S \to B$ be the corresponding fibration. Since $S$ is canonical, any general fibre of $f$ is a non-hyperelliptic curve. A natural question is then: what is the genus of a general fibre? This leads us to studying the slope of non-hyperelliptic fibrations. For a hyperelliptic fibration of genus $g$, $4 - 4/g$ is the best possible lower bound of the slope by [P] and [H1]. Later, Xiao [X] showed that the slope is not less than $4 - 4/g$ even when non-hyperelliptic. But, for non-hyperelliptic fibrations, it may not be the best bound. In fact, we showed in [K2] that the slope is not less than 3 when $g = 3$ (see also [H2] and [R2]), and Xiao himself conjectured that the slope is strictly greater than $4 - 4/g$ for non-hyperelliptic fibrations ([X, Conjecture 1]).

At present, we have two methods for studying the slope. The first is Xiao’s method [X] of relative projections and the second is counting relative hyperquadrics which is still at an experimental stage (see [R2] and [K2]). Combining these two, we show that the slope is not less than $24/7$ for $g = 4$ and give a bound $40/11$ for $g = 5$ (Theorems 4.1 and 5.1). We also answer affirmatively to Xiao’s conjecture referred above (Proposition 2.6).

As an application, we show in Section 6 that, for an irregular canonical surface $S$ (with a non-linear pencil), the canonical image cannot be cut out by quadrics when $K^2 \leq (10/3) \chi(\mathcal{O}_S)$. For irregular surfaces, Reid’s conjecture [R1, p. 541] may be shown along the same line if we can sufficiently develop the second method.

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1. Relative hyperquadrics

Let $B$ be a non-singular projective curve of genus $b$, and let $\mathcal{E}$ be a locally free sheaf on $B$. We put $\mu(\mathcal{E}) = \deg(\mathcal{E})/\text{rk}(\mathcal{E})$. According to [HN], $\mathcal{E}$ has a uniquely determined filtration by its sub-bundles $\mathcal{E}_i$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

which satisfies

(i) $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable for $1 \leq i \leq \ell$,

(ii) $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ for $1 \leq i \leq \ell - 1$.

As usual, we call such a filtration the Harder-Narashimhan filtration of $\mathcal{E}$. Put $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ and $r_i = \text{rk}(\mathcal{E}_i)$. Then

$$\deg(\mathcal{E}) = \sum_{i=1}^{\ell-1} r_i(\mu_i - \mu_{i+1}) + r_\ell \mu_\ell.$$

Let $\pi : \mathbb{P}(\mathcal{E}) \to B$ be the projective bundle associated with $\mathcal{E}$. We denote by $T_\mathcal{E}$ and $F$ a tautological divisor such that $\pi_*\mathcal{O}(T_\mathcal{E}) = \mathcal{E}$ and a fibre of $\pi$, respectively. Note that for any $\mathbb{R}$-divisor $D$ on $\mathbb{P}(\mathcal{E})$, there are real numbers $x, y$ satisfying $D \equiv xT_\mathcal{E} + yF$, where the symbol $\equiv$ means numerical equivalence.

The following can be found in [N].

**Lemma 1.1.** An $\mathbb{R}$-divisor which is numerically equivalent to $T_\mathcal{E} - xF$ is pseudo-effective if and only if $x \leq \mu_1$. It is nef if and only if $x \leq \mu_\ell$.

Assume that $\ell \geq 2$. For $1 \leq i \leq \ell - 1$ let

$$\rho_i : W_i \to \mathbb{P}(\mathcal{E})$$

denote the blowing-up along $B_i = \mathbb{P}(\mathcal{E}/\mathcal{E}_i)$. Then $W_i$ has a projective space bundle structure $\pi_i : W_i \to \mathbb{P}(\mathcal{E}_i)$. We put $\mathcal{E}_i = \rho_i^{-1}(B_i)$. Then $\pi_i^*T_\mathcal{E}$ is linearly equivalent to $\rho_i^*T_\mathcal{E} - \mathcal{E}_i$. Furthermore, $\mathcal{E}_i$ is isomorphic to the fibre product $\mathbb{P}(\mathcal{E}_i) \times_B B_i$. Let $p_1 : \mathcal{E}_i \to \mathbb{P}(\mathcal{E})$ be the projection map onto the first factor. Then $p_1 = \pi_i|_{\mathcal{E}_i}$. Similarly, if $p_2 : \mathcal{E}_i \to B_i$ denotes the projection to the second factor, then $p_2 = \rho_i|_{\mathcal{E}_i}$. In particular, $[-\mathcal{E}_i]|_{\mathcal{E}_i}$ is given by $p_1^*T_\mathcal{E} - p_2^*T_\mathcal{E}/\mathcal{E}_i$.

The following is essentially the same as [N, Claim (4.8)].

**Lemma 1.2.** Assume that an $\mathbb{R}$-divisor $Q \equiv p_1^*T_\mathcal{E} + p_2^*T_\mathcal{E}/\mathcal{E}_i - xF$ on $\mathcal{E}_i$ is pseudo-effective. Then $x \leq \mu_1 + \mu_\ell + \deg(\mathcal{E}_{i-1}/\mathcal{E}_i)$.

**Proof.** Since $T_\mathcal{E}/\mathcal{E}_i - \mu_\ell F$ is nef on $B_i$, $H_y = T_\mathcal{E}/\mathcal{E}_i - (\mu_\ell - y)F$ is ample for any positive rational number $y$. Let $m$ be a sufficiently large positive integer such that $mH_y$ is a very ample $\mathbb{Z}$-divisor, and choose $s - 1$ general members $H_j \in |mH_y|$ so that $C = \cap_j H_j$ is an irreducible non-singular
curve, where $s = \text{rk}(\mathcal{E}/\mathcal{E}_t)$. Let $\tau : C \to B$ denote the natural map. Then 
\[\mathbb{P}(\mathcal{E}_t) \times_B C \simeq \mathbb{P}(\tau^*\mathcal{E}_t).\]
Since the restriction of $Q$ to this space is numerically equivalent to
\[T\tau*\mathcal{E}_t - \mu_1(\tau^*\mathcal{E}_t)F_C + \{(T\tau*\mathcal{E}_t + (\mu_1 - x)F) \cdot C\}F_C,
\]
where $F_C$ denotes a fibre of $\mathbb{P}(\tau^*\mathcal{E}_t) \to C$, and since it must be pseudo-effective, it follows from Lemma 1.1 that 
\[(T\tau*\mathcal{E}_t + (\mu_1 - x)F) \cdot C \geq 0,\]
that is, 
\[(T\tau*\mathcal{E}_t + (\mu_1 - x)F)H_y^{s-1} \geq 0.\]
Letting $y \downarrow 0$, we get
\[x \leq \text{deg}(\mathcal{E}/\mathcal{E}_t) - s\mu_2 + \mu_1 + \mu_\ell = \text{deg}(\mathcal{E}_{t-1}/\mathcal{E}_t) + \mu_1 + \mu_\ell.\]

An effective divisor $Q$ on $\mathbb{P}(\mathcal{E})$ is called a relative hyperquadric if it is numerically equivalent to $2T\tau - xF$ for some $x \in \mathbb{Z}$. It is said to be of rank $r$, \(rk(Q) = r\), if it induces a hyperquadric of rank $r$ on a generic fibre of $\mathbb{P}(\mathcal{E})$.

**Lemma 1.3.** Assume that $\ell \geq 2$ and consider a relative hyperquadric $Q \equiv 2T\tau - xF$ on $\mathbb{P}(\mathcal{E})$. If $Q$ is not singular along $B_{t-1}$, then $x \leq \mu_1 + \mu_\ell$.

**Proof.** We may assume that $x > 2\mu_\ell$. Then, by Lemma 1.1, $Q$ vanishes on $B_{t-1}$, since $Q|_{B_{t-1}} \equiv 2T\tau/\mathcal{E}_{t-1} - xF$. However, since $Q$ is not singular along $B_{t-1}$, it cannot vanish twice along $B_{t-1}$. Let $\tilde{Q}$ be the proper transform of $Q$ by $\rho_{t-1}$. Then
\[\tilde{Q} \equiv \rho_{t-1}^*(2T\tau - xF) - \mathbb{E}_{t-1} = \rho_{t-1}^*T\tau + \pi_{t-1}^*T\mathcal{E}_{t-1} - xF.\]
Hence $\tilde{Q}|_{\mathbb{E}_{t-1}} \equiv \rho_{t-1}^*T\mathcal{E}_{t-1} + \pi_{t-1}^*T\mathcal{E}_{t-1} - xF$. Since it must be effective, we get $x \leq \mu_1 + \mu_\ell$ by Lemma 1.2.

**Lemma 1.4.** Let $Q \equiv 2T\tau - xF$ be a relative hyperquadric on $\mathbb{P}(\mathcal{E})$. If $x > \mu_1 + \mu_\ell$, then $rk(Q) \leq r_{t-1}$ and $Q$ is singular along $B_{t-1}$.

**Proof.** Since $x > \mu_1 + \mu_\ell$, it follows from Lemma 1.3 that $Q$ is singular along $B_{t-1}$. Let $\tilde{Q}$ be the proper transform of $Q$ by $\rho_{t-1}$. Then
\[\tilde{Q} \equiv \rho_{t-1}^*(2T\tau - xF) - 2\mathbb{E}_{t-1} = \pi_{t-1}^*(2T\mathcal{E}_{t-1} - xF).\]
Hence there exists a relative hyperquadric $Q_{t-1} \equiv 2T\mathcal{E}_{t-1} - xF$ on $\mathbb{P}(\mathcal{E}_{t-1})$ satisfying $rk(Q) = rk(Q_{t-1}) \leq r_{t-1}$. Now, the assertion can be shown by induction.

**Lemma 1.5.** Let $Q \equiv 2T\tau - xF$ be a relative hyperquadric on $\mathbb{P}(\mathcal{E})$. If $rk(Q) \geq 3$, then the following hold.

1. If $r_1 \geq 3$, then $x \leq 2\mu_1$.
2. If $r_1 = 2$, then $x \leq \mu_1 + \mu_2$.
3. If $r_1 = 1$ and $r_2 \geq 3$, then $x \leq 2\mu_2$. 
PROOF. (1) follows from Lemma 1.1 applied to a Q-divisor Q/2. We only have to show that \( x \leq 2\mu_2 \) in (3) and (4), since the other assertions follow from Lemma 1.4. Assume that \( r_1 = 1 \). Then \( B_1 \) is a relative hyperplane on \( \mathbb{P}(\mathcal{E}) \). Since \( \text{rk}(Q) \geq 3 \), we see that \( Q \) cannot vanish identically on \( B_1 \). Note that \( 0 \subseteq \mathcal{E}_2/\mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}/\mathcal{E}_1 \) is the Harder-Narasimhan filtration of \( \mathcal{E}/\mathcal{E}_1 \). Since \( Q|_{B_1} \equiv 2T_{\mathcal{E}/\mathcal{E}_1} - xF \), we get \( x \leq 2\mu_2 \) by Lemma 1.1.

**LEMMA 1.6.** Let \( Q \equiv 2T_{\mathcal{E}} - xF \) be a relative hyperquadric on \( \mathbb{P}(\mathcal{E}) \). If \( \text{rk}(Q) \geq 4 \), then the following hold.

1. If \( r_1 \geq 4 \), then \( x \leq 2\mu_1 \).
2. If \( r_1 = 3 \), then \( x \leq \mu_1 + \mu_2 \).
3. If \( r_1 = 2 \) and \( r_2 \geq 4 \), then \( x \leq \mu_1 + \mu_2 \).
4. If \( r_1 = 2 \) and \( r_2 = 3 \), then \( x \leq \mu_1 + \mu_3 \).
5. If \( r_1 = 1 \) and \( r_2 \geq 4 \), then \( x \leq 2\mu_2 \).
6. If \( r_1 = 1 \) and \( r_2 = 3 \), then \( x \leq \min\{2\mu_2, \mu_1 + \mu_3\} \).
7. If \( r_1 = 1, r_2 = 2 \) and \( r_3 \geq 4 \), then \( x \leq \mu_2 + \mu_3 \).
8. If \( r_1 = 1, r_2 = 2 \) and \( r_3 = 3 \), then \( x \leq \min\{\mu_2 + \mu_3, \mu_1 + \mu_4\} \).

**PROOF.** We show that \( x \leq \mu_2 + \mu_3 \) in (7) and (8). Assume by contradiction that \( x > \mu_2 + \mu_3 \). Since \( r_1 = 1 \), \( B_1 \) is a relative hyperplane on \( \mathbb{P}(\mathcal{E}) \). We have \( Q|_{B_1} \equiv 2T_{\mathcal{E}/\mathcal{E}_1} - xF \). Since \( x > \mu_2 + \mu_3 \), it follows from Lemma 1.4 that \( Q|_{B_1} \) is singular along \( B_2 \) which is a relative hyperplane of \( B_1 \). This implies that, on \( F \cong \mathbb{P}^{r-1} \), \( Q \) is defined by \( X_1L(X_1, \ldots, X_r) + CX_2^2 = 0 \) with a system of homogeneous coordinates \((X_1, \ldots, X_r)\) on \( F \) satisfying \( B_1|_F = (X_1) \), where \( L \) is a linear form and \( c \) is a constant. In particular, \( Q \) cannot be of rank \( \geq 4 \). Hence \( x \leq \mu_2 + \mu_3 \).

The other assertions can be shown similarly as in Lemma 1.5.

**REMARK 1.7.** Put \( \nu_j = \mu_i \) when \( r_{i-1} < j \leq r_i \) \((1 \leq i \leq \ell)\). Then \( \nu_1 \geq \cdots \geq \nu_r \), \( r = \text{rk}(\mathcal{E}) \), and \( \deg(\mathcal{E}) = \sum \nu_j \). With this notation, the conditions in Lemma 1.5 (resp. Lemma 1.6) can be written as \( x \leq \min\{2\nu_2, \nu_1 + \nu_3\} \) (resp. \( x \leq \min\{\nu_2 + \nu_3, \nu_1 + \nu_4\} \)).

2. - Some inequalities

Let \( f : S \to B \) be a surjective holomorphic map of a non-singular projective surface \( S \) onto a non-singular projective curve \( B \) with connected fibres. We always assume that \( f \) is relatively minimal, that is, no fibre of \( f \) contains a \((-1)\)-curve. If a general fibre of \( f \) is a \((n)\)-hyperelliptic curve of genus \( g \geq 2 \), we call \( f \) a \((n)\)-hyperelliptic fibration of genus \( g \). Let \( K_{S/B} \) be the relative
canonical bundle. It is nef by Arakelov’s theorem [B].

**LEMMA 2.1.** Let \( f : S \to B \) be a relatively minimal fibration of genus \( g \geq 2 \), and put \( b = g(B) \). Then \( f_*\omega_{S/B} \) is a locally free sheaf of rank \( g \) and degree \( \Delta(f) := \chi(\mathcal{O}_S) - (g - 1)(b - 1) \). Furthermore, the following hold.

1. \( \Delta(f) > 0 \) unless \( f \) is locally trivial.
2. Every locally free quotient of \( f_*\omega_{S/B} \) has nonnegative degree.

**PROOF.** \( \text{rk}(f_*\omega_{S/B}) \) equals the genus of a fibre. The assertion about the degree follows from the Riemann-Roch theorem (on \( S \) and \( B \)) and the Leray spectral sequence, since we have \( R^1f_*\omega_{S/B} = f_*\mathcal{O}_S \) by the relative duality theorem. (1) and (2) can be found in [B] and [F], respectively. \( \square \)

When \( f \) is not locally trivial, we put \( \lambda(f) = K^2_{S/B}/\Delta(f) \) and call it the **slope** of \( f \).

**NOTATION 2.2.** Let \( f : S \to B \) be a relatively minimal fibration of genus \( g > 2 \). Put \( c = \sum_{i=1}^\ell \mathcal{E}_i \) and let \( c_1 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell \subset \mathcal{E} \) be its Harder-Narasimhan filtration. The natural sheaf homomorphism \( f^*\mathcal{E} \to \omega_{S/B} \) induces a rational map \( h : S \to \mathbb{P}(\mathcal{E}) \). The image \( V = h(S) \) is called the relative canonical image. To be more precise, let \( \mathcal{A} \) be a sufficiently ample divisor on \( B \), and put \( L(\mathcal{A}) = K_{S/B} + f^*\mathcal{A} \). Let \( \sigma : \tilde{S} \to S \) be a composition of blowing-ups such that the variable part \( |\sigma^*L(\mathcal{A})| \) is free from base points. We assume that \( \sigma \) is the shortest among those with such a property. Let \( Z \) be the fixed part of \( |\sigma^*L(\mathcal{A})| \) and let \( E \) be an exceptional divisor with \( K = \sigma^*K + [E] \), where \( K \) is the canonical bundle of \( \tilde{S} \). Since \( \mathcal{A} \) is sufficiently ample, we can assume that \( Z \) has no horizontal components. In particular, we see that \( M(\mathcal{A}) \) induces a canonical divisor on a general fibre \( D \) of the induced fibration \( \tilde{f} : \tilde{S} \to B \). The holomorphic map associated with \( M(\mathcal{A}) \) factors thorough \( \mathbb{P}(\mathcal{E}) \) and we have a holomorphic map \( \tilde{h} : \tilde{S} \to \mathbb{P}(\mathcal{E}) \) over \( h \) which satisfies \( M(\mathcal{A}) = \tilde{h}^*(T_\mathcal{E} + \pi^*\mathcal{A}) \). Then \( V = \tilde{h}(\tilde{S}) \). When \( f \) is non-hyperelliptic, \( V \) is birational to \( S \) and any general fibre of \( V \to B \) can be identified with a canonical curve of genus \( g \).

Put \( M = \tilde{h}^*T_\mathcal{E} \). Since \( M - \mu_\ell D \) is nef by Lemma 1.1 and since \( \mu_\ell \geq 0 \) by Lemma 2.1, (2), we see that \( M \) is nef.

We have (at least) two methods for studying the slope of non-hyperelliptic fibrations, which we recall below.

**I (I) Relative projections ([X])**

Here we recall Xiao’s method. For each \( 1 \leq i \leq \ell \), the natural sheaf homomorphism \( f^*\mathcal{E}_i \subset f^*f_*\omega_{S/B} \to \omega_{S/B} \) induces a rational map \( h_i : S \to \mathbb{P}(\mathcal{E}_i) \) over \( B \). We let \( \sigma_i : S_i \to S \) be a composition of blowing-ups which eliminates the indeterminacy of \( h_i \). We choose a non-singular model \( S^* \) which dominates all the \( S_i \)'s, and we denote by \( \rho : S^* \to S \) the natural map. Let \( M_i \) be the pull-back to \( S^* \) of \( T_\mathcal{E_i} \). Let \( D^* \) be a general fibre of the induced fibration
\(S^* \to B\) and put \(N_i = M_i - \mu_i D^*\), \(Z_i = \rho^* K_{S/B} - M_i\). Then \(Z_i\) is effective and, by Lemma 1.1, \(N_i\) is a nef \(Q\)-divisor. Note that, modulo exceptional curves, \(Z_i\) corresponds to \(Z\). In particular, we see that \(Z_i D^* = 0\). Note also that \(Z_i - Z_i\) corresponds to the inverse image of the center \(B_i\) of a relative projection \(\mathbb{P}(E) \to \mathbb{P}(E_i)\).

Put \(d_i = N_i D^* (1 \leq i \leq \ell)\). Note that \(d_i = 2g - 2\). For \(1 \leq i \leq \ell - 1\), \(d_i\) is the degree of an \(r_i - 1\) dimensional linear system \(|M_i||D^*|\) and hence Clifford’s theorem shows that \(d_i \geq 2r_i - 1\) unless \((d_i, r_i) = (0, 1)\) when \(f\) is non-hyperelliptic. We recall two inequalities which follow from [X, Lemma 2].

\[
(2.1) \quad K_{S/B}^2 \geq \sum_{i=1}^{\ell-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 4(g-1)\mu_{\ell},
\]

\[
(2.2) \quad K_{S/B}^2 \geq (d_1 + 2g - 2)(\mu_1 - \mu_\ell) + 4(g-1)\mu_\ell.
\]

(II) Counting relative hyperquadrics

Let \(f : S \to B\) be a non-hyperelliptic fibration. We can assume that \(\mathcal{A}\) is taken so that the holomorphic map associated with \(|T_\xi + \pi^* \mathcal{A}|\) gives a quadratically normal embedding of \(\mathbb{P}(E)\). Then we have

\[
(2.3) \quad h^0(2M(\mathcal{A})) \geq h^0(2T_\xi + 2\pi^* \mathcal{A}) - h^0(\delta_\xi(2T_\xi + 2\pi^* \mathcal{A}))
\]

where \(\delta_\xi\) denotes the ideal sheaf of \(V\) if \(\mathbb{P}(E)\). Since the restriction map \(H^0(M(\mathcal{A})) \to H^0(K_D)\) is surjective, we can lift all the quadric relations in \(S^2H^0(K_D)\) to \(S^2H^0(M(\mathcal{A}))\). Since \(H^0(M(\mathcal{A})) \simeq H^0(T_\xi + \pi^* \mathcal{A})\), it follows that \(H^0(\delta_\xi(2T_\xi + \pi^* \mathcal{A})) \to H^0(\delta_\xi(2))\) is surjective, where \(\delta_\xi\) is the ideal sheaf of \(D' = \delta(D)\) in \(F \simeq \mathbb{P}^{g-1}\). Since \(f\) is non-hyperelliptic, we have \(h^0(\delta_\xi(2)) = (g-2)(g-3)/2\). Put

\[
x_i = \max\{\deg \delta | r_k \{H^0(\delta_\xi(2T_\xi + 2\pi^* \mathcal{A})) \to H^0(\delta_\xi(2))\} \geq i\},
\]

where \(\delta\) ranges over \(\text{Pic}(B)\). Then \(x_1 \geq x_2 \geq \cdots \geq x_k\), where \(k = (g-2)(g-3)/2\). We can find a set of divisors \(\{\delta_i\}\) with \(\deg \delta_i = x_i(1 \leq i \leq k)\) and relative hyperquadrics \(Q_i\) linearly equivalent to \(2T_\xi \pi^* \delta_i\) such that they induce a basis for \(H^0(\delta_\xi(2))\). Furthermore, we can assume that \(H^0(\delta_\xi(2T_\xi + 2\pi^* \mathcal{A}))\) is generated by them in the sense that

\[
H^0(\delta_\xi(2T_\xi + 2\pi^* \mathcal{A})) = \bigoplus_i H^0(2\mathcal{A} + \delta_i)Q_i.
\]

Since \(\mathcal{A}\) is sufficiently ample, \(2\mathcal{A} + \delta_i\) cannot be a special divisor. Hence

\[
h^0(\delta_\xi(2T_\xi + 2\pi^* \mathcal{A})) = \sum_i x_i + (g-2)(g-3)(2a + 1 - b)/2,
\]
where \( a = \deg \mathcal{A} \). We have

\[
h^0(2T_\ell + 2\pi^* A) = (g + 1)\Delta(f) + g(g + 1)(2a + 1b)/2
\]

by the Riemann-Roch theorem. Therefore, we can re-write (2.3) as

\[
(2.4) \quad h^0(2M(\mathcal{A})) \geq (g + 1)\Delta(f) + 3(g - 1)(2a + 1 - b) - \sum x_i.
\]

**LEMMA 2.3.** \( h^1(E + Z - M(\mathcal{A})) \leq M(E + Z)/2 \), where \( \mathcal{A} = 2\mathcal{A} - K_B \).

**PROOF.** Since \( E + Z \) has no horizontal components with respect to \( \tilde{f} \), we can find an effective divisor \( A_1 \) on \( B \) satisfying \( \tilde{f}^* A_1 \geq E + Z \). We assume that \( \deg A_1 \) is minimal among those divisors with such a property, and put \( L_1 = \tilde{f}^* A_1 \). Since \( \mathcal{A} \) is sufficiently ample, there exists an irreducible non-singular member \( L_2 \in |M(\mathcal{A} - A_1)| \). Put \( L_3 = (L_1 - E - Z) + L_2 \). Since \( L_3 \geq L_2 \), we can assume that \( |L_3| \) induces a birational map of \( S \) onto the image. Then, by Ramanujam’s theorem, we get \( h^1(-L_3) = h^0(O_{L_3}) - 1 \). Consider the cohomology long exact sequences for

\[
0 \rightarrow O_{L_3} \rightarrow O_{L_1 + L_3}(E + Z) \rightarrow O_{E + Z}(E + Z) \rightarrow 0,
\]

\[
0 \rightarrow O_{L_1}(E + Z - L_2) \rightarrow O_{L_1 + L_2}(E + Z) \rightarrow O_{L_2}(E + Z) \rightarrow 0.
\]

From these, we get

\[
h^0(O_{L_3}) \leq h^0(O_{L_1 + L_3}(E + Z)) \leq h^0(O_{L_1}(E + Z - L_2)) + h^0(O_{L_2}(E + Z)).
\]

Since, on fibres, \( E + Z \) is trivial and \( L_2 \) looks like a canonical divisor, we have that

\[
h^0(O_{L_1}(E + Z - L_2)) = h^0(O_{L_1}(-L_2)) = 0.
\]

Hence we get

\[
h^1(-L_3) \leq h^0(O_{L_3}(E + Z)) - 1 \leq L_2(E + Z)/2 = M(E + Z)/2.
\]

by Clifford’s theorem. \( \square \)

Since \( \chi(2M(\mathcal{A})) = M^2 + \Delta(f) + 3(g - 1)(2a + 1 - b) - M(E + Z) \) by the Riemann-Roch theorem, and since we have \( h^1(2M(\mathcal{A})) = h^2 - h(E + Z - M(\mathcal{A})) \), it follows from (2.4) and Lemma 2.3 that

\[
(2.5) \quad M^2 \geq g\Delta(f) - \sum_{i=1}^{(g-2)(g-3)/2} x_i + \frac{1}{2} M(E + Z).
\]
Since $K_{S/B}^2 = M^2 + (\sigma^*K_{S/B} + M)Z$, we have in particular
\begin{equation}
(2.6) \quad K_{S/B}^2 \geq g\Delta(f) - \sum_{i=1}^{(g-2)(g-3)/2} x_i.
\end{equation}

REMARK 2.4. There is another version due to Reid [R2]. It is easy to see that $f_*(\omega^2_{S/B})$ is a locally free sheaf of rank $3g - 3$ and degree $K_{S/B}^2 + \Delta(f)$. If $f$ is non-hyperelliptic, then the sheaf homomorphism $S^2(f_*(\omega_{S/B})) \rightarrow f_*(\omega^2_{S/B})$ is generically surjective by Max Noether’s theorem. Hence we have an exact sequence of sheaves on $B$:
\begin{equation}
(2.7) \quad 0 \rightarrow \mathcal{R} \rightarrow S^2(f_*(\omega_{S/B})) \rightarrow f_*(\omega^2_{S/B}) \rightarrow \mathcal{T} \rightarrow 0,
\end{equation}
where $\mathcal{T}$ is a torsion sheaf and $\mathcal{R}$ is a locally free sheaf of rank $(g - 2)(g - 3)/2$. Since $\deg S^2(f_*(\omega_{S/B})) = (g + 1)\Delta(f)$, it follows from (2.7) that
\begin{equation}
(2.8) \quad K_{S/B}^2 = g\Delta(f) - \deg \mathcal{R} + \text{length } \mathcal{T} \geq g\Delta(f) - \deg \mathcal{R}.
\end{equation}

We close the section giving an application of method (II).

LEMMA 2.5. Let $f : S \rightarrow B$ be a non-hyperelliptic fibration of genus $g$. Suppose that $f_*\omega_{S/B}$ is semi-stable. Then
\begin{equation}
(2.9) \quad K_{S/B}^2 \geq \left(5 - \frac{6}{g}\right)\Delta(f).
\end{equation}

PROOF. We give here two proofs using (2.6) and (2.8), respectively.
(1) Since $Q_i = 2T_i - x_iF$ is effective, it follows from Lemma 1.1 that $x_i \leq 2\Delta(f)/g$ since $f_*\omega_{S/B}$ is semi-stable. Hence we get (2.9) from (2.6).
(2) Since $f_*\omega_{S/B}$ is semi-stable, so is $S^2(f_*\omega_{S/B})$ (see, e.g., [G]). Hence we have $\mu(\mathcal{R}) \leq \mu(S^2(f_*\omega_{S/B}))$, that is, $g\deg \mathcal{R} \leq (g - 2)(g - 3)\Delta(f)$. Substituting this in (2.8) we get (2.9). \qed

PROPOSITION 2.6. Let $f : S \rightarrow B$ be a non-hyperelliptic fibration of genus $g$, and assume that it is not locally trivial. Then $\lambda(f) > 4 - 4/g$. Hence the conjecture of Xiao [X, Conjecture 1] is true.

PROOF. Xiao [X, Theorem 2] showed that $\lambda(f) > 4 - 4/g$ when $f_*\omega_{S/B}$ is not semi-stable, by using (2.1) and (2.2). Hence we can assume that $f_*\omega_{S/B}$ is semi-stable. But then, we have a stronger inequality (2.9). \qed

LEMMA 2.7. Let $f : S \rightarrow B$ be a non-hyperelliptic fibration of genus $g \geq 4$. Assume that the Harder-Narashimhan filtration of $f_*\omega_{S/B}$ is $0 \subset \mathcal{E}_1 \subset f_*\omega_{S/B}$ and $\text{rk}(\mathcal{E}_1) = 1$. Then (2.9) holds without equality.

PROOF. Since all the $Q_i$’s have rank $\geq 3$, we have $x_i \leq 2\mu_2 < 2\Delta(f)/g$ by Lemma 1.5. Hence (2.6) implies (2.9). \qed
3. - The case $g = 3$

In this section, we consider non-hyperelliptic fibrations of genus 3 in order to supplement [K2] and give a geometric interpretation of length $T$ in (2.8). Some results here overlap with [H3].

Let $f : S \to B$ be a non-hyperelliptic fibration of genus 3 and let the notation be as in 2.2. The relative canonical image $V$ is a divisor on $P(\mathcal{E})$ linearly equivalent to $4T_\xi - \pi^*A_0$ for some divisor $A_0$ on $B$. Put $a = \deg A$ and $a_0 = \deg A_0$. Since $\tilde{h}$ is a birational holomorphic map onto the image and since $M(A) = \tilde{h}^*(T_\xi + \pi^*A)$, we have

$$M(A)^2 = (T_\xi + \pi^*A)^2(4T_\xi - \pi^*A_0) = 4\Delta(f) + 8a - a_0.$$ 

Hence

$$(3.1) \quad M^2 - 3\Delta(f) = \Delta(f) - a_0.$$ 

Since $K_{S/B}^2 = M^2 + (\sigma^*K_{S/B} + M)Z$, (3.1) is equivalent to

$$(3.2) \quad K_{S/B}^2 - 3\Delta(f) = \Delta(f) - a_0 + (\sigma^*K_{S/B} + M)Z.$$ 

In view of (2.8), the right hand side of (3.2) is nothing but length $T$ (since $\mathcal{R} = 0$).

Let $C$ be a general member of $|M(A)|$. Then

$$2g(C) - 2 = M(A)(\tilde{K} + M(A))$$

$$= 8\Delta(f) + 12a - 2a_0 + 8(b - 1) + M(E + Z).$$

On the other hand, the arithmetic genus of $C' = \tilde{h}(C)$ is given by

$$2p_a(C') - 2 = (T_\xi + \pi^*A)(4T_\xi - \pi^*A_0)(2T_\xi + \pi^*(\det \mathcal{E} + \omega_B + A - A_0))$$

$$= 12\Delta(f) + 8(b - 1) + 12a - 6a_0.$$ 

Hence

$$(3.3) \quad p_a(C') - g(C) = 2\Delta(f) - 2a_0 - M(E + Z)/2 \geq 0.$$ 

Note further that the conductor of $C \to C'$ is given by

$$(3.4) \quad \tilde{h}^*\omega_{C'} - \omega_C = \tilde{f}^*(\det \mathcal{E} - A_0)|_C - (E + Z)|_C.$$ 

The following is a refinement of [K2, Theorem 1.2].

**Lemma 3.1.** Let the notation be as above. For a non-hyperelliptic fibration $f : S \to B$ of genus 3, $K_{S/B}^2 \geq M^2 \geq 3\Delta(f)$ holds. If $M^2 = 3\Delta(f)$, then $K_{S/B}^2 = 3\Delta(f)$. 


PROOF. It follows from (3.3) that $\Delta(f) \geq a_0$. Hence we have $M^2 \geq 3\Delta(f)$ by (3.1). Assume that $M^2 = 3\Delta(f)$, that is, $a_0 = \Delta(f)$. Then, by (3.3), we have $M(E + Z) = 0$. Since $0 \leq (\sigma^*K_{S/B})Z = MZ + Z^2 = Z^2$, Hodge’s index theorem shows that $Z = 0$. Hence (3.2) implies that $K_{S/B}^2 = 3\Delta(f)$. □

The above equalities are sometimes useful in determining the singularity of $V$.

THEOREM 3.2. When $K_{S/B}^2 = 3\Delta(f)$, $V$ has at most rational double points, and it is linearly equivalent to $4T_\mathcal{E} - \pi^*\det \mathcal{E}$. When $K_{S/B}^2 > 3\Delta(f)$, $V$ is non-normal. In particular, if $K_{S/B}^2 = 3\Delta(f) + 1$, $V$ has at most rational double points except for a double conic curve described in [K1, § 9].

PROOF. Assume first that $K_{S/B}^2 = 3\Delta(f)$. Then $a_0 = \Delta(f)$, and $|L(\mathcal{A})|$ has no base locus as we saw in the proof of Lemma 3.1. We have $p_a(C') = g(C)$ by (3.3). It follows that $V$ has at most isolated singular points. We have

$$\chi(\mathcal{O}_V) = \chi(\mathcal{O}_{\mathcal{O}(\mathcal{E})}) - \chi(-V)$$
$$= 1 - b + \chi(T_\mathcal{E} + \pi^*(\det \mathcal{E} + A_0))$$
$$= \Delta(f) + 2b - 2 = \chi(\mathcal{O}_S).$$

Hence $V$ has at most rational singular points. Since $V$ is a hypersurface of a non-singular 3-fold $\mathbb{P}(\mathcal{E})$, it has at most rational double points. In particular, we have $\omega_{S/B} = h^*\omega_{V/B}$. Since $\omega_{V/B}$ is induced from $T_\mathcal{E} + \pi^*(\det \mathcal{E} - A_0)$ and $K_{S/B} = h^*T_\mathcal{E}$, we see that $f^*(\det \mathcal{E} - A_0)$ is linearly equivalent to zero. That is, $A_0 = \det \mathcal{E}$.

It follows from (2.5), (3.1) and (3.3) that $p_a(C') - g(C) \geq M^2 - 3\Delta(f)$. Hence, by Lemma 3.1, we have $p_a(C') - g(C) > 0$ when $K_{S/B}^2 > 3\Delta(f)$. Since $C'$ is obtained by cutting $V$ by a general member of $|T_\mathcal{E} + \pi^*\mathcal{A}|$, it follows that $V$ has more than isolated singular points.

Assume that $K_{S/B}^2 = 3\Delta(f) + 1$. By Lemma 3.1, we must have $M^2 = K_{S/B}^2$. It follows that $\Delta(f) = a_0 + 1$ and that $|L(\mathcal{A})|$ has no base locus. By (3.3) and (3.4), we have $p_a(C') - g(C) = 2$ and $h^*\omega_C - \omega_C = f^*(\det \mathcal{E} - A_0)|_C$. Hence $C'$ has two double points contained in a unique fiber. Since $V$ has no horizontal singular locus, we see that $V$ has a double curve along a conic traced out by the singular points of $C'$. The rest follows from an argument in [K1, § 9]. □

REMARK 3.3. Horikawa [H2] announced that he classified degenerate fibres in genus 3 pencils. Though a part of it can be found in [H3], the whole body has not appeared yet.
4. - The case \( g = 4 \)

In this section we show the following theorem with several lemmas.

**THEOREM 4.1.** \( f : S \to B \) be a non-hyperelliptic fibration of genus 4. Then

\[
K_{S/B}^2 \geq \frac{24}{7} \Delta(f).
\]

If a general fibre of \( f \) has two distinct \( g_1^1 \)'s, then

\[
K_{S/B}^2 \geq \frac{7}{2} \Delta(f).
\]

For the proof of Theorem 4.1, we freely use the notation of the previous sections. In particular, we set \( \mathcal{E} = f_* \omega_{S/B} \) and let \( 0 \subset E_1 \subset \cdots \subset E_\ell = \mathcal{E} \) be the Harder-Narashimhan filtration. By § 2, (II), there exists a relative hyperquadric \( Q \equiv 2T_\ell - xF \) through the relative canonical image \( V \) and

\[
K_{S/B}^2 \geq 4\Delta(f) - x.
\]

Since \( \text{rk}(Q) = 4 \) if and only if a general fibre of \( f \) has two distinct \( g_1^1 \)'s, the second part of Theorem 4.1 is nothing but the following:

**LEMMA 4.2.** If \( \text{rk}(Q) = 4 \), then (4.2) holds.

**PROOF.** In view of (4.3), we only have to check that \( x \leq \Delta(f)/2 \). But this is straightforward applying Lemma 1.6. Let \( \nu_1, \ldots, \nu_4 \) be as in Remark 1.7. Then it follows from Lemma 1.6 that \( x \leq \min\{\nu_2 + \nu_3, \nu_1 + \nu_4\} \). Hence

\[
2x \leq \sum_{j=1}^4 \nu_j = \Delta(f).
\]

**LEMMA 4.3.** If \( x \leq \mu_1 + \mu_\ell \), then (4.1) holds.

**PROOF.** By (2.2), we have \( K_{S/B}^2 \geq (d_1 + 6)(\mu_1 - \mu_\ell) + 12\mu_\ell \geq 6(\mu_1 + \mu_\ell) \). Hence (4.1) holds if \( \mu_1 + \mu_\ell \geq (4/7)\Delta(f) \). Assume that \( \mu_1 + \mu_\ell \leq (4/7)\Delta(f) \). Then \( x \leq \mu_1 + \mu_\ell \leq (4/7)\Delta(f) \) and we get (4.1) from (4.3).

Recall that a canonical curve of genus 4 cannot meet the vertex of the quadric through it, if the quadric is of rank 3.

**LEMMA 4.4.** If \( x > \mu_1 + \mu_\ell \), then \( \tau_{\ell-1} = 3 \) and \( d_{\ell-1} = 6 \).

**PROOF.** If \( x > \mu_1 + \mu_\ell \) then, by Lemma 1.3, \( Q \) is singular along \( B_{\ell-1} \). Since \( \text{rk}(Q) \geq 3 \) and \( \tau_\ell = 4 \), we must have \( \tau_{\ell-1} = 3 \) by Lemma 1.4.

We have \( d_{\ell-1} = 6 - Z_{\ell-1}D^* \). Since \( \text{rk}(Q) = 3 \) and since \( B_{\ell-1} \) is the (relative) vertex of \( Q \), we see that any general fibre of \( V \to B \) cannot meet \( B_{\ell-1} \). Since \( Z_{\ell-1} - Z_\ell \) corresponds to \( B_{\ell-1} \cap V \) as we remarked in § 2, (I), we have \( (Z_{\ell-1} - Z_\ell)D^* = 0 \). It follows that \( d_{\ell-1} = 6 \), since we always have \( Z_\ell D^* = 0 \). 

\[\square\]
We complete the proof of Theorem 4.1 with the following:

**Lemma 4.5.** Even if \( x > \mu_1 + \mu_2 \), (4.1) holds.

**Proof.** We can assume that \( \ell = 3 \) and \( d_{\ell-1} = 6 \) by Lemma 4.4.

Assume that \( \ell = 2 \). Since \( r_1 = 3 \), we get \( x \leq 2\mu_1 \) by Lemma 1.5. On the other hand, since \( d_1 = 6 \), it follows from (2.1) that \( K_{S/B}^2 \geq 12\mu_2 + 12\mu_2 = 12\mu_1 \). Hence, if \( \mu_1 \geq (2/7)A(f) \), we get (4.1). If \( \mu_1 \leq (2/7)A(f) \), then \( x \leq (4/7)A(f) \) and (4.1) follows from (4.3).

Assume that \( \ell = 3 \). Since \( r_1 \leq 2 \) and \( r_2 = 3 \), we have \( x \leq \mu_1 + \mu_2 \) by Lemma 1.5. Since \( d_2 = 6 \), it follows from (2.1) that

\[
K_{S/B}^2 \geq (d_1 + 6)(\mu_1 - \mu_2) + 12(\mu_2 - \mu_3) + 12\mu_3 \geq 6(\mu_1 + \mu_2).
\]

Hence we can show (4.1) as we did in Lemma 4.3.

Assume that \( \ell = 4 \). By Lemma 1.5, we have \( x \leq \min\{2\mu_2, \mu_1 + \mu_3\} \). Since \( d_3 = 6 \), it follows from (2.1) that

\[
K_{S/B}^2 \geq 3(\mu_1 - \mu_2) + 9(\mu_2 - \mu_3) + 12(\mu_3 - \mu_4) + 12\mu_4 = 3(\mu_1 + 2\mu_2 + \mu_3).
\]

Hence \( K_{S/B}^2 \geq 6 \min\{2\mu_2, \mu_1 + \mu_3\} \) and we can show (4.1) as we did in Lemma 4.3.

\[\square\]

5. - The case \( g = 5 \)

In this section we show the following theorem with several lemmas.

**Theorem 5.1.** Let \( f : S \to B \) be a non-hyperelliptic fibration of genus 5. When a general fibre of \( f \) is non-trigonal we have:

\[
(5.1) \quad K_{S/B}^2 \geq M^2 \geq 4A(f).
\]

When a general fibre is trigonal we have:

\[
(5.2) \quad K_{S/B}^2 \geq \frac{40}{11}A(f).
\]

By (II), there are three relative hyperquadrics \( Q_i \equiv 2T_\ell - x_iF, \ 1 \leq i \leq 3 \), through \( V \) satisfying \( x_1 \geq x_2 \geq x_3 \) and

\[
(5.3) \quad K_{S/B}^2 \geq 5A(f) - x, \quad x = \sum_{i=1}^3 x_i.
\]

**Lemma 5.2.** Let \( f : S \to B \) be a non-hyperelliptic, non-trigonal fibration of genus 5. Then \( K_{S/B}^2 \geq M^2 \geq 4A(f) \). If \( M^2 = 4A(f) \) then \( K_{S/B}^2 = 4A(f) \).
PROOF. Since a general fibre of $f$ is non-trigonal, the relative canonical image $V$ is an irreducible component of $\cap_{i=1}^3 Q_i$. Hence, comparing degrees, we get $M(\mathcal{A})^2 \leq (T_\varepsilon + \pi^* \mathcal{A})^3 \Pi (2T_\varepsilon - x_i F)$, that is, $M^2 \leq 8\Delta(f) - 4x$. Eliminating $x$ from (2.5) using this, we get

$$M^2 \geq 4\Delta(f) + \frac{2}{3}M(E + Z)$$

from which the assertion follows immediately. \hfill \Box

In the rest of the section, we assume that $f : S \to B$ is a trigonal fibration of genus 5. Recall that, for a suitable choice of homogeneous coordinates $(X_0, \ldots, X_4)$ on $\mathbb{P}^4$, any quadric through a trigonal canonical curve of genus 5 can be written as $c_1(X_1^2 - X_0X_2) + c_2(X_0X_3 + X_1X_3) + c_3(X_2X_3 - X_1X_4)$. Hence there is only one quadric of rank 3, and the vertices of any two independent members cannot meet. Without losing generality, we can assume that $\text{rk}(Q_1) \geq 3$, $\text{rk}(Q_2) \geq \text{rk}(Q_3) \geq 4$.

**Lemma 5.3.** If $r_i = 2$ then $x_3 \leq 2\mu_{i+1}$.

**Proof.** Assume contrarily that $x_3 > 2\mu_{i+1}$. Then all the $Q_j$’s vanish identically on $B_i$, which is a $\mathbb{P}^2$-bundle on $B$. This contradicts the fact that $\cap Q_j$ induces a Hirzebruch surface on a general fibre of $\mathbb{P}(\mathcal{E}) \to B$. \hfill \Box

**Lemma 5.4.** Assume that there are rational numbers $y_1$ and $y_2$ satisfying $x \leq y_1$, $K_{S/B}^2 \geq y_2$ and $8y_1 \leq 3y_2$. Then (5.2) holds. In particular, (5.2) holds when $x \leq 3(\mu_1 + \mu_\ell)$.

**Proof.** It follows from (5.3) that $K_{S/B}^2 \geq 5\Delta(f) - y_1$. Hence (5.2) holds when $y_1 \leq (15/11)\Delta(f)$. Assume that $y_1 \geq (15/11)\Delta(f)$. Since $3y_2 \geq 8y_1$, we have $K_{S/B}^2 \geq y_2 \geq (8/3)y_1$. Hence (5.2) holds. In particular, since we have $K_{S/B}^2 \geq 8(\mu_1 + \mu_\ell)$ by (2.2), we get (5.2) if $x \leq 3(\mu_1 + \mu_\ell)$. \hfill \Box

We can assume that $x > 3(\mu_1 + \mu_\ell)$. Then $x_1 > \mu_1 + \mu_\ell$.

**Lemma 5.5.** Assume that $x_1 > \mu_1 + \mu_\ell$. Then $x_i \leq \mu_1 + \mu_\ell$ for $i = 2, 3$ and $r_{\ell-1} \geq 3$. If $r_{\ell-1} = 3$ then $d_{\ell-1} = 6$. If $r_{\ell-1} = 4$ then $d_{\ell-1} = 7$.

**Proof.** Since $x_1 > \mu_1 + \mu_\ell$, $Q_1$ is singular along $B_{\ell-1}$ by Lemma 1.3. Since $\text{rk}(Q_1) \geq 3$, we have $r_{\ell-1} \geq 3$. Furthermore, $Q_2$ and $Q_3$ cannot be singular along $B_{\ell-1}$ as we remarked just before Lemma 5.3. Hence $x_2, x_3 \leq \mu_1 + \mu_\ell$ by Lemma 1.3 again. If $r_{\ell-1} = 3$, then $\text{rk}(Q_1) = 3$. Since a trigonal curve of genus 5 meets the vertex of rank 3 quadric through it at two points, we get $d_{\ell-1} = 8 - 2 = 6$. If $r_{\ell-1} = 4$ then $d_{\ell-1} \geq 7$ by Clifford’s theorem. \hfill \Box

**Lemma 5.6.** Assume that $\ell = 2$ and $x_1 > \mu_1 + \mu_2$. Then $K_{S/B}^2 \geq (15/4)\Delta(f)$. 


PROOF. Since we have $x_1 \leq 2\mu_1$ by lemma 1.5 and $x_i \leq \mu_1 + \mu_2$ for $i = 2, 3$ by Lemma 5.5, we get $x \leq 4\mu_1 + 2\mu_2$.

Assume that $r_1 = 3$. We have $K_{S/B}^2 \geq 5\Delta(f) - 2(2\mu_1 + \mu_2)$ by (5.3). On the other hand, it follows from (2.2) that $K_{S/B}^2 \geq 14\mu_1 + 2\mu_2$, since $d_1 = 6$ by Lemma 5.5. Since $\Delta(f) = 3\mu_1 + 2\mu_2$, these inequalities imply $K_{S/B}^2 \geq (15/4)\Delta(f)$.

Assume that $r_1 = 4$. Since $\Delta(f) = 4\mu_1 + \mu_2$, we have $x \leq \Delta(f) + \mu_2 < \Delta(f) + \Delta(f)/5$. Hence we get $K_{S/B}^2 > (19/5)\Delta(f)$ from (5.3).

We assume that $\ell \geq 3$ in the sequel.

LEMMA 5.7. Assume that $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ and $r_{\ell-1} = 3$. Then (5.2) holds.

PROOF. We have $\ell = 3$ or 4. Note that $\text{rk}(Q_1) = 3$ and $\text{rk}(Q_i) \geq 4$ for $i = 2, 3$.

We have $x_1 \leq \mu_1 + \mu_{\ell-1}$ by Lemma 1.5, $x_2 \leq \mu_1 + \mu_\ell$ by Lemma 5.5 and $x_3 \leq 2\mu_{\ell-1}$ by Lemmas 1.6 and 5.3. Hence $x \leq 2\mu_1 + 3\mu_{\ell-1} + \mu_\ell$. On the other hand, applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_{\ell-1}, \mu_\ell\}$, we get

$$K_{S/B}^2 \geq 6(\mu_1 - \mu_{\ell-1}) + 14(\mu_{\ell-1} - \mu_\ell) + 16\mu_\ell = 6\mu_1 + 8\mu_{\ell-1} + 2\mu_\ell,$$

since $d_1 \geq 0$, $d_{\ell-1} = 6$ and $d_\ell = 8$. We have $\mu_1 > \mu_\ell$. It follows that

$$8(2\mu_1 + 3\mu_{\ell-1} + \mu_\ell) < 3(6\mu_1 + 8\mu_{\ell-1} + 2\mu_\ell).$$

Applying Lemma 5.4, we see that (5.2) holds without equality.

LEMMA 5.8. Assume that $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ and $r_{\ell-1} = 4$. If $r_{\ell-2} \leq 2$, then (5.2) holds.

PROOF. We have $\ell = 3$ or 4. Since $r_{\ell-2} \leq 2$, it follows from Lemma 1.4 that $x_1 \leq \mu_1 + \mu_{\ell-1}$. We have $x_2 \leq \mu_1 + \mu_\ell$ by Lemma 5.5. Furthermore, we can assume that $x_3 \leq 2\mu_{\ell-1}$ by Lemmas 1.6 and 5.3. Hence $x \leq 2\mu_1 + 3\mu_{\ell-1} + \mu_\ell$. On the other hand, applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_{\ell-1}, \mu_\ell\}$, we get

$$K_{S/B}^2 \geq 7(\mu_1 - \mu_{\ell-1}) + 15(\mu_{\ell-1} - \mu_\ell) + 16\mu_\ell = 7\mu_1 + 8\mu_{\ell-1} + \mu_\ell,$$

since $d_1 \geq 0$, $d_{\ell-1} \geq 7$ and $d_\ell = 8$. It follows from $\mu_1 > \mu_\ell$ that

$$8(2\mu_1 + 3\mu_{\ell-1} + \mu_\ell) < 3(7\mu_1 + 8\mu_{\ell-1} + \mu_\ell).$$

Hence, as in the previous lemma, we see that (5.2) holds without equality.

LEMMA 5.9. Assume that $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ and $r_{\ell-1} = 4$. If $r_{\ell-2} = 3$ and $x_1 > \mu_1 + \mu_{\ell-1}$, then (5.2) holds.
PROOF. Since $x_1 > \mu_1 + \mu_{\ell-1}$, $B_{\ell-2}$ is the relative vertex of $Q_1$ and it follows that $d_{\ell-2} = 6$.

Assume that $\ell = 3$. Since $d_1 = 6$, we have $K_{S/B}^2 \geq 14\mu_1 + 2\mu_3$ by (2.2). By Lemmas 1.5 and 5.5, we have $x_1 \leq 2\mu_1$ and $x_2, x_3 \leq \mu_1 + \mu_3$. Hence $x \leq 4\mu_1 + 2\mu_3$. We can show that $K_{S/B}^2 > (15/4)\Delta(f)$ using (5.3).

Assume that $\ell = 4$ or 5. We have $x_1 \leq \mu_1 + \mu_{\ell-2}$ and $x_2 \leq \mu_1 + \mu_4$ by Lemmas 1.5 and 5.5, respectively. Furthermore, we have $x_3 \leq 2\mu_{\ell-2}$ by Lemmas 1.6 and 5.3. Hence $x \leq 2\mu_1 + 3\mu_{\ell-2} + \mu_4$. On the other hand, applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_{\ell-2}, \mu_4\}$, we get

$$K_{S/B}^2 \geq 6(\mu_1 - \mu_{\ell-2}) + 14(\mu_1 - \mu_4) + 16\mu_4 = 6\mu_1 + 8\mu_{\ell-2} + 2\mu_4,$$

since $d_1 \geq 0, d_{\ell-2} = 6$ and $d_{\ell} = 8$. Hence we see that (5.2) holds without equality as in the proof of Lemma 5.7.

We finish the proof of Theorem 5.1 with the following:

**LEMMA 5.10.** Assume that $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ and $\tau_{\ell-1} = 4$. If $\tau_{\ell-2} = 3$ and $x_1 \leq \mu_1 + \mu_{\ell-1}$, then (5.2) holds.

**PROOF.** Assume that $\ell = 3$. Since $x \leq (\mu_1 + \mu_2) + 2(\mu_1 + \mu_3) = 3\mu_1 + \mu_2 + 2\mu_3$ and $\Delta(f) = 3\mu_1 + \mu_2 + \mu_3$, it follows from (5.3) that $K_{S/B}^2 > (19/5)\Delta(f)$, which is stronger than (5.2).

Assume that $\ell = 4$ and $\tau_1 = 1$. Then $x_1 \leq 2\mu_2$ and $x_2, x_3 \leq \mu_1 + \mu_4$ by Lemmas 1.5 and 5.5. Since $x_1 > \mu_1 + \mu_4$, we have in particular $\mu_1 + \mu_4 < 2\mu_2$. We have $x \leq 2(\mu_1 + \mu_2 + \mu_4)$. Applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_2, \mu_4\}$ we get

$$K_{S/B}^2 \geq 5(\mu_1 - \mu_2) + 13(\mu_2 - \mu_4) + 16\mu_4 = 5\mu_1 + 8\mu_2 + 3\mu_4,$$

since $d_1 \geq 0, d_2 \geq 5$ and $d_4 = 8$. Since $6(\mu_2 - \mu_4) + (2\mu_2 - \mu_1 - \mu_4) > 0$, we have $3(5\mu_1 + 8\mu_2 + 3\mu_4) > 16(\mu_1 + \mu_2 + \mu_4)$ and therefore (5.2) holds without equality.

Assume that $\ell = 4$ and $\tau_1 = 2$. We get $x_1 \leq \mu_1 + \mu_3$ and $x_2, x_3 \leq \mu_1 + \mu_4$ by Lemma 5.5. Hence $x \leq 3\mu_1 + \mu_3 + 2\mu_4$. Applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_3, \mu_4\}$ we get

$$K_{S/B}^2 \geq 10(\mu_1 - \mu_3) + 15(\mu_3 - \mu_4) + 16\mu_4 > 8\mu_1 + 7\mu_3 + \mu_4,$$

since $d_1 \geq 3$, $d_3 \geq 7$ and $d_4 = 8$. Since $\mu_3 > \mu_4$, we have $3(8\mu_1 + 7\mu_3 + \mu_4) > 8(3\mu_1 + \mu_3 + 2\mu_4)$ and, therefore, (5.2) holds without equality.

Assume that $\ell = 5$. We have $x_1 \leq \min\{2\mu_2, \mu_1 + \mu_4\}$, $x_2 \leq \min\{\mu_2 + \mu_3, \mu_1 + \mu_3\}$ and $x_3 \leq \min\{2\mu_3, \mu_1 + \mu_3\}$ by Lemmas 1.5, 1.6, 5.3 and 5.5. If $\mu_2 + \mu_3 \leq \mu_1 + \mu_5$, then we get $x \leq 2\mu_2 + (\mu_1 + \mu_5) + 2\mu_3 \leq 3(\mu_1 + \mu_5)$ which contradicts the assumption of the lemma. Hence $\mu_2 + \mu_3 > \mu_1 + \mu_5$. Then we have $x \leq (\mu_1 + \mu_4) + (\mu_1 + \mu_5) + 2\mu_2 = 2\mu_1 + 2\mu_3 + \mu_4 + \mu_5$. Note that we have $11x \leq 15\Delta(f) = 15\sum_{i=1}^{5} \mu_i$ when $7(\mu_1 + \mu_3) \leq 15\mu_2 + 4(\mu_4 + \mu_5)$. In particular, (5.2)
will follow from (5.3) if $2\mu_2 \geq \mu_1 + \mu_3$. So, we may assume that $2\mu_2 < \mu_1 + \mu_3$.

Then, since $\mu_3 - \mu_5 > \mu_1 \mu_2$ and $\mu_1 - \mu_2 > \mu_2 - \mu_3$, we get

$$3(\mu_3 - \mu_5) > (\mu_1 - \mu_2) + (\mu_2 - \mu_3) + \mu_3 - \mu_5 = \mu_1 - \mu_5 > \mu_1 - \mu_4.$$ 

We apply [X, Lemma 2] for the sequence $\{\mu_1, \mu_3, \mu_4, \mu_5\}$ to get

$$K^2_{S/B} \geq 5(\mu_1 - \mu_3) + 12(\mu_3 - \mu_4) + 15(\mu_4 - \mu_5) + 16\mu_5 = 5\mu_1 + 7\mu_3 + 3\mu_4 + \mu_5,$$

since $d \geq 0$, $d_3 \geq 5$, $d_4 \geq 7$ and $d_5 = 8$. Note that we have

$$3(5\mu_1 + 7\mu_3 + 3\mu_4 + \mu_5) = 8(\mu_1 + \mu_4) + 8(\mu_1 + \mu_5) + 16\mu_3 + 5(\mu_3 - \mu_5) - (\mu_1 - \mu_4) > 8x + 2(\mu_3 - \mu_5).$$

Hence (5.2) can be shown using Lemma 5.4. \(\Box\)

Inequality (5.1) gives us a hope that the following holds.

**CONJECTURE.** $K^2_{S/B} \geq 4\Delta(f)$ holds for a Petri general fibration.

**6. - Application**

Let $S$ be a canonical surface and $X$ its canonical image. The intersection of all hyperquadrics through $X$ is called the quadric hull of $X$ and denoted by $Q(X)$. The dimension of the irreducible component of $Q(X)$ containing $X$ is called the *quadric dimension* of $S$. A conjecture of Miles Reid [R1] states that every canonical surface with $K^2 < 4pg - 12$ has quadric dimension 3.

**THEOREM 6.1.** Let $S$ be an irregular canonical surface and assume that the image of the Albanese map of $S$ is a curve. Then $K^2 \geq 3\chi(O_S) + 10(q - 1)$. When $K^2 \leq (10/3)\chi(O_S) + (122/7)(q - 1)$, the Albanese pencil is a non-hyperelliptic fibration of genus 3. When $K^2 \leq \min \{ (10/3)\chi(O_S) + (122/7)(q - 1), 4p_g - 12 + q \}$, the quadric dimension of $S$ is 3 and the irreducible component of $Q(X)$ containing the canonical image $X$ is birationally a threefold scroll over a curve.

**PROOF.** The first inequality was remarked in [K2]. By the assumption, the Albanese map induces a non-hyperelliptic fibration $f : S \to B$, where $B$ is the Albanese image and hence $g(B) = q$. If $f$ has genus $g$, then it follows from Proposition 2.6 that $K^2_{S/B} > (4 - 4/g)\Delta(f)$, that is, $K^2 > (4 - 4/g)\chi(O_S) + (g+1)(q-1)$. We have $g \leq 5$ when $K^2 \leq (10/3)\chi(O_S) + (122/7)(q-1)$. The cases $g = 4$ and $g = 5$ can be excluded by Theorems 4.1 and 5.1, respectively. Hence we have $g = 3$. As for the last assertion, we remark that the restriction map
LEMMA 6.2. Let $S$ be a minimal surface of general type with a non-linear pencil. If $K^2 < 4\chi(\mathcal{O}_S)$ then the base of the pencil is a curve of genus $g(S)$. If $S$ is a canonical surface with a non-linear pencil, then

$$K^2 \geq \min\{4\chi(\mathcal{O}_S), 3\chi(\mathcal{O}_S) + 10(q - 1)\}$$

**Proof.** Let $f : S \to B$ be the fibration associated with the non-linear pencil. If $q > b = g(B)$, then it follows from [X, Theorem 1] that $K_{S/B}^2 \geq 4\Delta(f)$ which implies that $K^2 \geq 4\chi(\mathcal{O}_S)$ since $b > 0$. Hence we have $b = q$ when $K^2 < 4\chi(\mathcal{O}_S)$.

Assume that $S$ is a canonical surface. Then $f$ is non-hyperelliptic. Hence we have $K_{S/B}^2 \geq 3\Delta(f)$ by Corollary 2.6 and Lemma 3.1. When $K^2 < 4\chi(\mathcal{O}_S)$, this implies that $K^2 \geq 3\chi(\mathcal{O}_S) + 10(q - 1)$, since $b = q$ and $g \geq 3$.

THEOREM 6.3. Let $S$ be a canonical surface with a non-linear pencil. If $K^2 \leq \min\{(10/3)\chi(\mathcal{O}_S), 4p_g - 12 + q\}$ then $S$ has quadric dimension 3.

**Proof.** Let $f : S \to B$ be the fibration associated with the non-linear pencil. By Lemma 6.2, we have $g(B) = q$. Since $K^2 \leq (10/3)\chi(\mathcal{O}_S)$, one can show that $f$ is a non-hyperelliptic fibration of genus 3 as in Theorem 6.1. The rest follows from [K4, Theorem 8.3].

COROLLARY 6.4. Let $S$ be a canonical surface with $q = 1$ and $K^2 \leq (10/3)\chi(\mathcal{O}_S)$. Then the Albanese map gives a non-hyperelliptic fibration of genus 3. If $K^2 \leq \min\{(10/3)\chi, 4\chi - 11\}$ then $S$ has quadric dimension 3.

This and Theorem 3.2 give a picture of canonical surfaces with $q = 1$ and $K^2 = 3\chi$ or $3\chi + 1$, which is quite similar to the regular case (see [AK] and [K1]): they have a pencil of non-hyperelliptic curves of genus 3. Another “similar” result is the following theorem which will be shown in the next section (see [K3] for the regular case).

**Theorem 6.5.** The moduli space of even canonical surfaces with $K^2 = 3\chi(\mathcal{O}_S) + 1$ and $q = 1$ is non-reduced.

**Remark 6.6.** Ashikaga [A] constructed a series of canonical surfaces with a non-hyperelliptic fibration of genus 3. See also [K2].

7. - Proof of Theorem 6.5

In this section we show Theorem 6.5. Though the proof is essentially the same as in [K3], there is one point which is unclear: a vector bundle on an elliptic curve is not necessarily decomposable.
Let $S$ be a canonical surface with $K^2 = 3\chi(\mathcal{O}_S) + 1$, $q(S) = 1$ and let $f : S \to B = \text{Alb}(S)$ be the Albanese map. By Corollary 6.4, any general fibre $D$ of $f$ is a non-hyperelliptic curve of genus 3. Assume further that $S$ is an even surface, that is, there is a line bundle $L$ with $K = 2L$. Since $L^2$ is even and $K^2 = 4L^2$, there exists a non-negative integer $n$ satisfying

\[(7.1)\quad \chi = 8n + 5, \quad L^2 = 6n + 4.\]

By the Riemann-Roch theorem, we have

\[(7.2)\quad 2h^0(L) - h^1(L) = -L^2/2 + \chi = 5n + 3.\]

Since $D$ is of genus 3 we have $LD = 2$. Since $D$ is non-hyperelliptic, we have $h^0(L|_D) = 1$ by Clifford’s theorem. It follows that the rational map $\Phi_L$ associated with $|L|$ factors through $f : S \to B$. Hence there is a divisor $\mathcal{L}$ on $B$ such that $L = [f^*\mathcal{L} + Z_L]$, where $Z_L$ is the fixed part of $|L|$. We have $h^0(\mathcal{L}) \ge h^0(L) \ge (5n + 3)/2$ by (7.2). Hence $\deg \mathcal{L} \ge (5n + 3)/2$. Since $LD = 2$, we have $L^2 = 2\deg \mathcal{L} + LZ_L$, that is,

\[(7.3)\quad LZ_L = 6n + 4 - 2\deg \mathcal{L}.\]

Put $\mathcal{E} = f_*\omega_S/B = f_*\omega_S$ and let $\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$ be the Harder-Narashimhan filtration of $\mathcal{E}$ as usual. Let $\pi : \mathbb{P}(\mathcal{E}) \to B$ be the associated projective bundle. As we have seen in Section 3, we have a holomorphic map $h : S \to \mathbb{P}(\mathcal{E})$ satisfying $K = h^*T_\mathcal{E}$, and $V = h(S)$ is linearly equivalent to $4T_\mathcal{E} - \pi^*\mathcal{A}_0$, $\deg \mathcal{A}_0 = \chi - 1$.

**Lemma 7.1.** The vector bundle $f_*\omega_S$ splits as a direct sum of line bundles. More precisely, there are three line bundles $\mathcal{L}_i(0 \le i \le 2)$ on $B$ satisfying $f_*\omega_S = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ and $\deg \mathcal{L}_0 \le n + 1$, $\deg \mathcal{L}_1 \ge 2n + 1$, $\deg \mathcal{L}_2 \ge 5n + 3$.

**Proof.** Since $K = 2L = [2f^*\mathcal{L} + 2Z_L]$, we see that $|K - 2f^*\mathcal{L}|$ contains an effective divisor. Since $H^0(K) \simeq H^0(T_\mathcal{E})$, it follows that $H^0(T_\mathcal{E} - 2\pi^*\mathcal{L}) \neq 0$. Then, by Lemma 1.1, we get

\[\mu_1 \ge 2\deg \mathcal{L} \ge \begin{cases} 5n + 3 & \text{if } n \text{ is odd}, \\ 5n + 4 & \text{if } n \text{ is even}. \end{cases}\]

Since $\deg \mathcal{E} = \chi = 8n + 5$ and since $\deg \mathcal{E} \ge \deg \mathcal{E}_1 = r_1\mu_1$, we must have $r_1 = 1$. Recall that $V$ is numerically equivalent to

\[4T_\mathcal{E} - (\chi - 1)F = 4(T_\mathcal{E} - (2n + 1)F).\]

Since $V$ cannot vanish identically on $\mathbb{P}(\mathcal{E}/\mathcal{E}_1)$, it follows from Lemma 1.1 that $\mu_1(\mathcal{E}/\mathcal{E}_1) \ge 2n + 1$. We have

\[\deg(\mathcal{E}/\mathcal{E}_1) = 8n + 5 - \deg \mathcal{E}_1 = 8n + 5 - \mu_1.\]
Hence \( \deg(\mathcal{E}/\mathcal{E}_1) \leq 3n + 2 \) if \( n \) is odd, and \( \deg \mathcal{E}/\mathcal{E}_1 \leq 3n + 1 \) if \( n \) is even. Since \( \mu(\mathcal{E}/\mathcal{E}_1) < \mu(\mathcal{E}/\mathcal{E}_1) \), we see in particular that \( \mathcal{E}/\mathcal{E}_1 \) is not semi-stable. Let \( 0 \subset \mathcal{F}_1 \subset \mathcal{E}/\mathcal{E}_1 \) be the Harder-Narasimhan filtration of \( \mathcal{E}/\mathcal{E}_1 \), and put \( \mathcal{F}_2 = (\mathcal{E}/\mathcal{E}_1)/\mathcal{F}_1 \). Then \( \deg \mathcal{F}_1 \geq 2n + 1 \) and we have \( \deg \mathcal{F}_2 \leq n + 1 \) if \( n \) is odd, and \( \deg \mathcal{F}_2 \leq n \) if \( n \) is even. Hence \( \deg \mathcal{F}_1 - \deg \mathcal{F}_2 > 0 \) and \( H^1(\mathcal{F}_1 - \mathcal{F}_2) = 0 \). This implies that \( \mathcal{E}/\mathcal{E}_1 = \mathcal{F}_1 \oplus \mathcal{F}_2 \).

Since \( \mathcal{E}_1 \) and \( \mathcal{F}_1 \) are of positive degree, we have \( h^1(\mathcal{E}) = h^1(\mathcal{E}/\mathcal{E}_1) = h^1(\mathcal{F}_2) \) from the cohomology long exact sequence for

\[
0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{E}_1 \rightarrow 0.
\]

On the other hand, since \( \mathcal{E} = f_*\omega_S \), we have \( h^1(\mathcal{E}) = 0 \). Hence \( h^1(\mathcal{F}_2) = 0 \) and we have \( \deg \mathcal{F}_2 \geq 0 \). Then

\[
\deg \mathcal{E}_1 - \deg \mathcal{F}_1 \geq \deg \mathcal{E}_1 - \deg \mathcal{E}/\mathcal{E}_1 \geq 2n + 1.
\]

It follows that \( H^1((\mathcal{E}/\mathcal{E}_1)^* \otimes \mathcal{E}_1) = 0 \). This implies that \( \mathcal{E} = \mathcal{E}_1 \otimes (\mathcal{E}/\mathcal{E}_1) \). Now, put \( L_0 = \mathcal{F}_2, L_1 = \mathcal{F}_1 \) and \( L_2 = \mathcal{E}_1 \).

**Lemma 7.2.** Let the notation be as in Lemma 7.1. Then \( n \) is odd, \( \deg L_0 = n + 1 \), \( \deg L_1 = 2n + 1 \) and \( \deg L_2 = 5n + 3 \). Furthermore, \( V \) is linearly equivalent to \( 4t\mathcal{E} - 4\pi^*L_1 \).

**Proof.** We can find sections \( X_i \) of \( [T_t - \pi^*L_1] \) such that \( (X_0, X_1, X_2) \) forms a system of homogeneous coordinates on fibres of \( \pi \). Assume that \( V \) is linearly equivalent to \( 4t\mathcal{E} - 4\pi^*A_0 \) as in Section 3, and recall that \( \deg A_0 = \chi - 1 = 8n + 4 \). Then the equation of \( V \) can be written as

\[
\sum \phi_{ij}X_0^iX_1^jX_2^j = 0,
\]

where \( \phi_{ij} \) is a section of \( L_{ij} = (4 - i - j)L_0 + iL_1 + jL_2 - A_0 \). If \( \deg L_{01} < 0 \), then \( V \) has a multiple curve along \( X_1 = X_2 = 0 \). Hence \( \deg L_{01} \geq 0 \), that is, \( 3\deg L_0 + \deg L_2 \geq 8n + 4 \). Since \( \deg L_0 + \deg L_1 + \deg L_2 = 8n + 5 \), we get \( 2\deg L_0 \geq \deg L_1 - 1 \). Since \( \deg L_0 \leq n + 1 \) and \( \deg L_1 \geq 2n + 1 \), we have either

(i) \( \deg L_0 = n, \deg L_1 = 2n + 1, \deg L_2 = 5n + 4 \), or
(ii) \( \deg L_0 = n + 1, \deg L_1 = 2n + 1, \deg L_2 = 5n + 3 \).

We show that (i) is impossible. Assume by contradiction that (i) is the case. Note that \( V \) contains an elliptic curve \( B' \) defined by \( X_1 = X_2 = 0 \). We have \( \deg L_{01} = 0 \). If \( \phi_{01} = 0 \), then \( V \) would have a multiple curve along \( B' \), which is impossible. Hence \( L_{01} \) must be trivial and \( \phi_{01} \) is a non-zero constant. But then \( V \) is non-singular in a neighbourhood of \( B' \). This is impossible, since \( V \) is singular along a fibre which meets \( B' \).
Hence we have (ii). In particular, it follows from the proof of Lemma 7.1 that \( n \) is odd. We know that \( V \) is defined by an equation of the form

\[
\phi_{40}X_1^4 + X_2(\phi_{01}X_0^3 + \cdots + \phi_{04}X_2^3) = 0.
\]

Since \( \deg L_{40} = 0 \) and \( \phi_{40} \) cannot be zero, \( L_{40} \) is a trivial bundle, which means that \( A_0 \) is linearly equivalent to \( 4L_1 \). \( \square \)

Put \( n = 2k - 1 \).

**Lemma 7.3.** \( L_2 = 2L, LZ_L = 2k, DZ_L = 2 \) and \( Z_L^2 = -8k + 2 \).

**Proof.** In the proof of Lemma 7.1, we have

\[
\deg L_2 = \mu_1 \geq 2 \deg L = 5n + 3.
\]

Since \( \deg L_2 = 5n + 3 = 10k - 2 \), we get \( \deg L = 5k - 1 \). Recall that \( H^0(TL - 2\pi^* L) \neq 0 \). Since any element of \( H^0(TL - 2\pi^* L) \) can be written as \( \psi X_2 \) with \( \psi \in H^0(L_2 - 2L) \), and since \( L_2 - 2L \) is of degree 0, we see that \( L_2 = 2L \).

Since \( \deg L = 5k - 1 \), it follows from (7.3) that \( LZ_L = n + 1 = 2k \). Since \( LD = 2 \), we have \( DZ_L = 2 \). We have \( 2k = LZ_L = (\deg L)DZ_L + Z_L^2 \). Hence \( Z_L^2 = -8k + 2 \). \( \square \)

Note that we have \( K = h^*(X_2) + \pi^* L_2 = h^*(X_2) + 2f^* L \). Hence \( (X_2) \) corresponds \( 2Z_L \). We can show the following as in [K3, Lemma 2.3] using (7.4).

**Lemma 7.4.** \( Z_L = 2G_0 + G_1 \), where \( G_0 \) is a non-singular elliptic curve and \( G_1 \) is a \((-2)\)-curve.

Since every even canonical surface with \( K^2 = 3\chi + 1 \) and \( q = 1 \) has a \((-2)\)-curve \( G_1 \), we have Theorem 6.5 by a result of Burns-Wahl [BW] (see [K3, Proof of Theorem 1.5]).

**Example.** Let \( \mathcal{M} \) be a line bundle of degree 2 on an elliptic curve \( B \) which induces the double covering \( B \to \mathbb{P}^1 \). Choose a point \( P \in B \) with \( 2P \in |\mathcal{M}| \). Put \( \mathcal{L}_0 = k\mathcal{M}, \mathcal{L}_1 = (2k - 1)\mathcal{M} + [P], \mathcal{L}_2 = (5k - 1)\mathcal{M} \) and \( \mathcal{E} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \). Let \( \xi \in H^0([P]) \) define \( P \), and choose sufficiently general members \( \Phi_0 \in H^0(2T\xi - 2\pi^* \mathcal{L}_1) \) and \( \Phi_1 \in H^0(3T\xi \pi^*(4\mathcal{L}_1 - \mathcal{L}_2 + 2[P])) \). We consider a surface defined in the total space of \([2T\xi - \pi^*(2\mathcal{L}_1 + [P])] \to \mathbb{P}(\mathcal{E}) \) by

\[
\xi w - \Phi_0 = w^2 - X_2 \Phi_1 = 0.
\]

where \( w \) is a fibre coordinate. It is easy to see that it has only one rational double point of type \( A_1 \) and the minimal resolution is an even canonical surface with \( K^2 = 3\chi + 1, q = 1 \) and \( \chi = 16k - 3 \) (see [K3]).
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