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Symmetrization of hyperbolic systems with real constant coefficients


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Symmetrization of Hyperbolic Systems
with Real Constant Coefficients

TATSUO NISHITANI

Dedicated to Prof. J. Vaillant

1. - Introduction

Let $L(\xi)$ be an $m \times m$ matrix of real linear forms in $\xi \in \mathbb{R}^{n+1}$. The dimension of the linear subspace spanned by the linear forms in $L(\xi)$ is called the reduced dimension of $L(\xi)$.

In [6], Vaillant proved the following interesting result: assume that $L(\xi)$ is diagonalizable for every $\xi$ with real eigenvalues and that the reduced dimension of $L$ is not less than $m(m + 1)/2$; if the difference of any two diagonal forms does not belong to the subspace spanned by non-diagonal forms then $L(\xi)$ is symmetrizable by a non-singular constant matrix, that is the coefficient matrices of $L(\xi)$ are simultaneously symmetrizable (Proposition 3 in [6]).

In Section 3, we improve the above result and show that if $L(\xi)$ is diagonalizable with real eigenvalues for every $\xi \in \mathbb{R}^{n+1}$ and the reduced dimension of $L$ is not less than $m(m + 1)/2$, (which will be referred to as “maximal dimension”) then $L(\xi)$ is symmetrizable by a non-singular constant matrix (Theorem 3.4). The same result remains valid under less restrictive assumptions on the reduced dimension. Indeed, in Sections 4 and 5, we show that if $L(\xi)$ is diagonalizable for every $\xi$ with real eigenvalues and the reduced dimension of $L$ is not less than $m(m + 1)/2 - 1$, then the same result holds (Theorem 4.1).

Recently Oshime [4] has completely classified $3 \times 3$ strongly hyperbolic systems with real constant coefficients and he has listed up all possible forms of strongly $3 \times 3$ hyperbolic systems (see also [5]). By a result of [4] there is a $3 \times 3$ hyperbolic system which is diagonalizable (at every point), of reduced dimension $3(3 + 1)/2 - 2 = 4$ which is not symmetrizable by a non-singular constant matrix.
It would be interesting to determine the minimal reduced dimension $d(m)$ such that every diagonalizable $m \times m$ system with real eigenvalues is symmetrizable by a constant matrix. The results mentioned above imply that $d(3) = 5$ and $d(m) \leq m(m + 1)/2 - 1$ in general.

The interest in hyperbolic systems with constant coefficients of maximal reduced dimension comes on one hand from the fact that hyperbolic systems with variable coefficients are smoothly symmetrizable if $m = 2$ and the localizations have maximal reduced dimension (see Proposition 1.2 in [2]); on the other hand, diagonalizable systems with real eigenvalues appear naturally as the localizations at multiple characteristics of a class of strongly hyperbolic systems with variable coefficients ([3]).

2. - Preliminaries

Let $L(D)$ be a first order differential operator on $C^\infty(\mathbb{R}^{n+1}, \mathbb{C}^m)$:

$$L(D) = D_0 I + \sum_{j=1}^{m} A_j D_j,$$

where $I$ denotes the identity matrix of order $m$ and $A_j \in M(m, \mathbb{R})$, the set of all $m \times m$ real constant matrices. Let $L(\xi)$ be the symbol of $L(D)$:

$$L(\xi) = \xi_0 I + \sum_{j=1}^{m} A_j \xi_j.$$

Denoting $\xi = (\xi_0, \xi')$, $\xi' = (\xi_1, \ldots, \xi_n)$ we write $L(\xi)$ as

$$L(\xi) = (\phi^j_i(\xi))$$

where $\phi^j_i(\xi)$ denotes the $(i, j)$-th element of $L(\xi)$ so that $\phi^j_i(\xi) = \xi_0 + \psi_0(\xi')$ and $\phi^j_i(\xi) = \phi^j_i(\xi')$ if $i \neq j$. We say that $L(\xi)$ is diagonalizable if $L(\xi)$ is diagonalizable for every $\xi \in \mathbb{R}^{n+1}$. As in Vaillant [6] (see also [1]) we introduce the following definition.

**Definition 2.1.** Let $d(L) = \text{dim} \text{span}\{\phi^j_i(\xi)\}$. We call $d(L)$ the reduced dimension of $L$. In other terms $d(L) = \text{dim} \text{span}\{I, A_1, \ldots, A_n\}$.

**Remark.** Assume that $L(\xi)$ is diagonalizable with real eigenvalues; then it is clear that

$$d(L) \leq m(m + 1)/2.$$

Let us set

$$h(\xi) = \det L(\xi).$$
DEFINITION 2.2. We say that $\xi^o \in \mathbb{R}^{n+1}$ is a characteristic of order $r$ of $h$ (or of $L$) if

$$d^j h(\xi^o) = 0, \quad j < r, \quad d^r h(\xi^o) \neq 0$$

where $d^j h$ is the $j$-th differential of $h$.

Recall that a linear change of coordinates $\xi$ preserving the $\xi_0$ axis is induced by a linear change of coordinates $x$ preserving the $x_0$ coordinate and a similarity transformation of $L$ by a constant matrix is induced by a change of basis for $\mathbb{C}^m$. Note that the following holds:

**Lemma 2.1.** Under a similarity transformation and a linear change of coordinates $\xi$ preserving the $\xi_0$ axis, the reduced dimension and the diagonalizability of $L$ remain invariant.

Note that if $L(\xi^o)$ is diagonalizable and $\xi^o$ is a characteristic of order $m - r$ then every minor of order $r + 1$ of $L(\xi^o)$ vanishes.

**Lemma 2.2.** Let $L(\xi^o)$ be diagonalizable. Then we have

$$\text{span}\{\phi^j_i\} = \text{span}\{\phi^j_i | i \geq j\}.$$ 

In particular

$$d(L) = \dim \text{span}\{\phi^j_i(\xi^o) | i \geq j\}.$$ 

**Proof.** If the assertion were not true, we could find $p < q$ and $\xi^o \in \mathbb{R}^{n+1}$ such that

$$\phi^j_i(\xi^o) = 0, \quad i \geq j, \quad \phi^p_q(\xi^o) \neq 0.$$ 

Since $\xi^o$ is a characteristic of order $m$, $L(\xi^o)$ would vanish and hence a contradiction. \(\square\)

**Lemma 2.3.** Suppose that there is a non singular constant matrix $T$ such that

$$T^{-1}L(\xi)T$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$ and assume further that there is $\xi^o \in \mathbb{R}^{n+1}$ such that

$$\phi^j_i(\xi^o) = 0, \quad \phi^j_i(\xi^o) - \phi^j_i(\xi^o) \neq 0 \text{ for } i \neq j.$$ 

Then one can find a diagonal matrix $D = \text{diag}(d_1, \ldots, d_m)$ with $d_i > 0$ such that

$$D^{-1}L(\xi)D$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$.

**Proof.** Since $T^{-1}L(\xi)T$ is symmetric, it follows that

(2.1) \(L(\xi)H = H^T L(\xi)\)
with $H = T^*T$ where $^\prime T$ denotes the transposed matrix of $T$. Writing $H = (h^j_i)$, (2.1) implies that

$$(\phi^i_i(\xi^a) - \phi^j_j(\xi^a))h^j_i = 0$$

because $\phi^j_j(\xi^a) = 0$ for $i \neq j$. Hence $h^j_j = 0$ if $i \neq j$ and then

$$H = \text{diag}(h^1_1, \ldots, h^n_m)$$

where $h^1_i > 0$ because $H$ is positive definite. Since $T^{-1} = T^*H^{-1}$ the assumption implies that $^\prime TH^{-1}L(\xi)T$ is symmetric and hence $H^{-1}L(\xi)$ is also symmetric. We now define $D$ as

$$D = \text{diag} \left( \sqrt{h^1_1}, \ldots, \sqrt{h^n_m} \right).$$

Then it is clear that $D^{-1}L(\xi)D = \left( \sqrt{h^1_i}, \phi^j_j(\xi), \sqrt{h^n_m} \right)$ is symmetric since the condition that $H^{-1}L(\xi)$ is symmetric means that $h^{-1}_i\phi^j_j(\xi) = h^{-1}_j\phi^i_i(\xi)$. This completes the proof.

3. Case of maximal reduced dimension

The first step to prove the results stated in the Introduction is to transform $L(\xi)$, by a similarly transformation, to another $L(\xi') = (\phi^j_j(\xi'))$ in which $\phi^j_j, i \neq j$ are independent of diagonal forms. For later reference, we study a slightly more general case. Let us consider the following upper-triangular $m \times m$ matrix:

$$A(x) = \begin{pmatrix}
\phi_1(x) & \phi^1_2(x) & \phi^1_3(x) & \cdots & \phi^1_m(x) \\
0 & \phi_2(x) & \phi^2_3(x) & \cdots & \phi^2_m(x) \\
0 & 0 & \phi_3(x) & \cdots & \phi^3_m(x) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \phi_m(x)
\end{pmatrix}$$

where $\phi_j(x), \phi^j_j(x)$ are linear functions of $x = (x_1, \ldots, x_n)$.

**Lemma 3.1.** Assume that $A(x)$ is diagonalizable for every $x$. Then one can find a non singular $T \in M(m, \mathbb{C})$ such that

$$T^{-1}A(x)T = \text{diag}(\phi_1(x), \ldots, \phi_m(x)).$$

**Proof.** We first show that

$$\phi^p_{p+1} = c_p(\phi_p - \phi_{p+1})$$
for some constant \( c_p \in \mathbb{C} \). Consider

\[
\det(\lambda I + A(x) - \phi_{p+1}(x)I) = \prod_{j=1}^{m} (\lambda + \phi_j(x) - \phi_p(x) + \psi(x))
\]

where \( \psi(x) = \phi_p(x) - \phi_{p+1}(x) \). Let \( J(x) = \{ j | \phi_j(x) = \phi_p(x), j \neq p, p+1 \} \) and note that \( \lambda = 0 \) is an eigenvalue of \( A(x) - \phi_{p+1}(x)I \) with multiplicity \( |J(x)| + 2 \) if \( \psi(x) = 0 \). Observe that the \( (m - |J(x)| - 1) \)-th minor of \( \lambda I + A(x) - \phi_{p+1}(x)I \), obtained removing the \( i \)-th rows and columns for \( i \in J \) and the \( (p + 1) \)-th row and \( p \)-th column, is equal to

\[
\phi_{p+1}^p(x) \prod_{j \in J(x), j \neq p, p+1} (\lambda + \phi_j(x) - \phi_{p+1}(x))
\]

up to the sign. Since this must vanish when \( \lambda = 0 \) and \( \psi(x) = 0 \), and we conclude that

\[
\phi_{p+1}^p(x) = 0 \text{ if } \phi_p(x) = \phi_{p+1}(x).
\]

This proves the assertion. Now let us denote

\[
T_q^p(c) = I + Q_q^p(c)
\]

where every element of \( Q_q^p(c) \) is zero except for the \((p, q)\)-th element which is \( c \in \mathbb{C} \). Considering

\[
T_q^{m-1}(c_{m-1}) \cdots T_2^1(c_1)L(\xi) - T_2^1(-c_1) \cdots T_m^{m-1}(-c_{m-1})
\]

we may assume that \( \phi_{p+1}^p = 0 \) for \( 1 \leq p \leq m - 1 \). We proceed by induction on \( i - j = r \). Let \( q = p + r + 1 \) and suppose that

\[
\phi_j^i = 0 \text{ for } i < j \leq i + r.
\]

Set \( J(x) = \{ j | \phi_j(x) = \phi_p(x), j \neq p, q \} \) and consider the \( (m - |J(x)| - 1) \)-th minor of \( \lambda I + A(x) - \phi_q(x)I \) obtained removing the \( i \)-th rows and columns for \( i \in J \) and the \( q \)-th row and the \( p \)-th column. By the inductive hypothesis this is equal to

\[
\phi_q^p(x) \prod_{j \notin J(x), j \neq p, q} (\lambda + \phi_j(x) - \phi_p(x) + \psi(x))
\]

up to the sign where \( \psi(x) = \phi_p(x) - \phi_q(x) \). The same argument as before proves that

\[
\phi_q^p = c_{pq}(\phi_p - \phi_q)
\]

for some constant \( c_{pq} \). The rest of the proof is clear. \( \square \)

Recall that

\[
L(\xi) = (\phi_j^i(\xi)), \quad \phi_i^j(\xi) = \xi_0 + \psi_0(\xi').
\]
PROPOSITION 3.2. Assume that $L(\xi)$ is diagonalizable with real eigenvalues. Then there is a non singular $T \in M(m, \mathbb{R})$ such that

$$T^{-1}L(\xi)T = (\bar{\phi}_j^i(\xi))$$

verifies:

i) $\bar{\phi}_q^p \in V = \text{span}\{\bar{\phi}_j^i|i > j\}$ for $p < q$;

ii) $\bar{\phi}_i^i - (\xi_0 + \psi_i) \in V$ for $1 \leq i \leq m$.

PROOF. Let $J_1 \subset \{(i, j)|i > j\}$ be such that $\phi_j^i, (i, j) \in J_1$ are linearly independent and $\text{span}\{\phi_j^i|i > j\}$. Adding suitable $\phi_j^i, i \in J_2, J_2 \subset \{1, \ldots, m\}$ one can assume that $\phi_j^i, (i, j) \in J_1$ and $\phi_j^i, i \in J_2$ are linearly independent and $\text{span}\{\phi_j^i|i \geq j\}$. To simplify the notations we write

$\phi_j^i(\xi) = x_{ij}, (i, j) \in J_1,$ $\phi_j^i(\xi) = y_i, i \in J_2$

so that

$\phi_j^p(\xi) = l_p^q(y) + m_p^q(x), p < q,$

$\phi_j^i(\xi) = m_j^i(x), (i, j) \notin J_1, i > j,$

$\phi_j^i(\xi) = l_j^i(y) + m_j^i(x), i \notin J_2$

where $x = (x_{ij})_{i,j} \in J_1$ and $y = (y_i)_{i} \in J_2$. Then one can write

$L(\xi) = (l_j^i(y)) + (m_j^i(x))$

where $l_j^i(y) = y_i, i \in J_2,$ $m_j^i(x) = x_{ij}, (i, j) \in J_1$ and $l_j^i = 0$ if $i > j$. Since $(l_j^i(y))$ is diagonalizable for every $y$ there is $T \in M(m, \mathbb{C})$ by Lemma 3.1 such that

$$T^{-1}(l_j^i(y))T = \text{diag}(l_j^1(y), \ldots, l_j^m(y)).$$

On the other hand, setting

$$T(c)(m_j^i(x))T(-c) = (m_j^i(x))$$

it is clear that

$$\text{span}\{\bar{m}_j^i|i > j\} = \text{span}\{x_{ij}|(i, j) \in J_1\}$$

provided if $p < q$. Since $T$ is a product of several $T(c)$ with $p < q$, $T^{-1}L(\xi)T$ verifies the asserted properties.

PROPOSITION 3.3. Assume that $L(\xi)$ is diagonalizable with real eigenvalues. Suppose that $d(L) = m(m + 1)/2k(k + 1)/2$ and $\phi_j^i = 0$ for $i \geq j + m - k$. Then there is a non singular constant matrix $T$ such that $T^{-1}L(\xi)T = (\bar{\phi}_j^i(\xi))$ verifies that

i) $\bar{\phi}_q^p(\xi') \in V = \text{span}\{\bar{\phi}_j^i|i > j\}$ for $p < q$;
PROOF. From Lemma 2.2 and the assumptions it follows that \( \phi^i_j, 1 \leq i \leq m \) and \( \phi^j_i, j + m - k > i > j \) are linearly independent. Let us set

\[
\phi^j_i(\xi) = x_{ij}, \quad j + m - k > i > j, \quad \phi^i_j(\xi) = y_{ij}, \quad 1 \leq i \leq m.
\]

As in the proof of Proposition 3.2 one can write

\[
L(\xi) = (l^j_i(y)) + (m^j_i(x))
\]

where \( m^j_i = 0 \) if \( i \geq j + m - k \). Note that with

\[
T^p_q(c)(m^j_i(x))T^q_p(-c) = (\tilde{m}^j_i(x))
\]

we have \( \tilde{m}^j_i(x) = 0, \) \( i \geq j + m - k \) and \( \tilde{m}^j_i(x), \) \( i + m - k > i > j \) are linearly independent provided that \( p < q \). Then the same argument as in the proof of Proposition 3.2 proves the assertion.

Throughout this note we denote by

\[
L\left(\begin{array}{ccc}
i_1 & \cdots & i_k \\
j_1 & \cdots & j_k
\end{array}\right)(\xi)
\]

the minor of order \( k \) of \( L(\xi) \) composed of rows \( i_1 < \cdots < i_k \) and columns \( j_1 < \cdots < j_k \).

THEOREM 3.4. Assume that \( d(L) = m(m+1)/2 - k(k+1)/2 \) and \( \phi^j_i = 0 \) for \( i \geq j + m - k \). Suppose that \( L(\xi) \) is diagonalizable with real eigenvalues. Then \( L(\xi) \) is symmetrizable:

\[
T^{-1}L(\xi)T = S(\xi)
\]

where \( T \) is a non singular constant matrix and \( S(\xi) \) is real symmetric for every \( \xi \in \mathbb{R}^{n+1} \).

COROLLARY 3.5. Assume that \( d(L) = m(m+1)/2 \) and \( L(\xi) \) is diagonalizable with real eigenvalues. Then \( L(\xi) \) is symmetrizable by a constant non singular matrix.

PROOF OF THEOREM 3.4. From Proposition 3.3 it follows that we may assume that \( \phi^u_v \in V = \text{span}\{\phi^i_j|i > j\} \) for \( u < v \) and \( \phi^j_i = 0 \) for \( i \geq j + m - k \). Then we can follow exactly the same argument as in Vaillant [6, pp. 411-412]. Recall that

\[
\phi^u_v(\xi') = \sum_{p + m - k > q > p} C_{uq}^{vp} \phi^q_p(\xi')
\]
for $u < v$. The same induction on $q - p(m - k > q - p > 1)$ as in [6] shows that $C_{qq}^{pp} > 0$ and
\[ \forall (u, v), \ u < v, \ (u, v) \not\in \{p, q\} \Rightarrow C_{vq}^{up} = 0. \]
In particular, we have
\[ \phi_u^b = 0 \text{ if } u + m - k \leq v. \]
Thus we get
\[ \phi_q^p = C_{qq}^{pp} \phi_p^q, \ C_{qq}^{pp} > 0, \ p < q < p + m - k, \ \phi_q^p = 0, \ p + m - k \leq q. \]
We apply again the same reasoning as in [6, pp. 413-414]. Then we conclude that there is a diagonal matrix $D = \text{diag}(d_1, \ldots, d_m)$ with $d_i > 0$ such that
\[ D^{-1}L(\xi)D = S(\xi) \]
is symmetric for every $\xi \in \mathbb{R}^{n+1}$. This completes the proof.
\[ \square \]

4. - Case of less reduced dimension (1)

In this and the following sections we shall prove the following result.

**THEOREM 4.1.** Assume that $L(\xi)$ is diagonalizable with real eigenvalues and $d(L) = m(m + 1)/2 - 1$. Then $L(\xi)$ is symmetrizable:
\[ T^{-1}L(\xi)T = S(\xi) \]
where $T$ is a non singular constant matrix and $S(\xi)$ is real symmetric for every $\xi \in \mathbb{R}^{n+1}$.

To prove the theorem, we may assume that non diagonal forms are independent of the diagonal forms by Proposition 3.2. Then we look for characteristics of order $m - 2$ so that every 3-minor is zero by assumption. We choose suitable 3-minors to conclude, again after a similarity transformation, that $\phi_q^p$ depends only on $\phi_p^j$ for $p < q$:
\[ \phi_q^p = C_{qq}^{pp} \phi_p^q, \ C_{qq}^{pp} > 0. \]
Repeating again a similar argument we will show that
\[ C_{p}^{q}C_{p}^{p} = C_{q}^{q} \text{ for } 1 < p < q. \]
Then it is easy to find a symmetrizer following [6].

As noted above we assume, in what follows, that non diagonal forms of $L$ are independent of the diagonal forms. We divide the cases into two:
(a) \( \phi^i_{T_j}, \ (i > j) \) are linearly independent for every \( T \in M(m, \mathbb{R}) \) which exchanges some rows and the corresponding columns, where \( T^{-1}L(\xi)T = (\phi^i_{T_j}(\xi)) \).

(b) \( \phi^i_{T_j}, \ (i > j) \) are linearly dependent for some \( T \in M(m, \mathbb{R}) \) which exchanges some rows and the corresponding columns.

We study case (a) in this section and case (b) in the next section. From our assumptions we have

\[
\sum_{i \geq j} c^i_j \phi^i_j = 0.
\]

Assuming (a) it is clear that \( c^i_{j_0} \neq 0, \ c^i_{j_0} \neq 0 \) for some \( j_0 \neq i_0 \) because \( \sum_{i=1}^{m} c^i_i = 0 \).

Then exchanging columns and the corresponding rows we may assume that

\[
i_0 = 1, \ j_0 = m.
\]

Therefore \( \phi^i_{j}, \ 2 \leq i \leq m, \ \phi^i_{j}, \ i > j \) are linearly independent and the same is true for \( \phi^i_{j}, \ 1 \leq i \leq m - 1, \ \phi^j_{i}, \ i > j \). Set

\[
V = \text{span}\{\phi^i_{j}|i > j}\}.
\]

The following two lemmas are easily verified.

**Lemma 4.2.** We have

\[
\dim \text{span}\{\phi^i_{j} - \delta^i_j a(\xi')|i \geq j, (i,j) \neq (1,1)\} = m(m + 1)/2 - 1,
\]

\[
\dim \text{span}\{\phi^i_{j} - \delta^j_i a(\xi')|i \geq j, (i,j) \neq (m,m)\} = m(m + 1)/2 - 1
\]

for any linear form \( a(\xi') \), where \( \delta^i_j \) is the Kronecker delta.

**Lemma 4.3.** Let \( p \neq q \) and assume that either \( p, q \leq m - 1 \) or \( p, q \geq 2 \).

Then we have

\[
\phi^p_p - \phi^q_q \notin V.
\]

Recall that for \( u < v \)

\[
\phi^u_v = \sum_{i > j} C^{u|v}_{i|j} \phi^i_j.
\]

**Lemma 4.4.** Let \( u \geq 2 \) and \( u < v \). For \( p \geq 2 \) we have

\[
C^{up}_{u(v+1)} = 0 \text{ unless } (u, v) = (p, p + 1).
\]

Let \( v \leq m - 1 \) and \( u < v \). For \( p \leq m - 2 \) we have

\[
C^{up}_{u(v+1)} = 0 \text{ unless } (u, v) = (p, p + 1).
\]
PROOF. We may assume that \( \psi_2 = 0 \) as before. We follow Vaillant [6]. Let \( p \geq 2 \) and take \( \xi' \) so that \( \phi_j' (\xi') = 0, \ i > j, \ (i, j) \neq (p, p + 1) \) and \( \psi_1 (\xi') = 0, \ i \geq 3, \ i \neq p, \ p + 1 \). Then it is clear that

\[
h(\xi) = ((\xi_0 + \psi_p)(\xi_0 + \psi_{p+1}) - \phi_{p+1} \phi_{p+1}) (\xi_0 + \psi_1) \xi_0^{-3}.
\]

Note that \( \phi_{p+1} (\xi') = c \phi_{p+1} (\xi') \) with some \( c \geq 0 \) which follows from the hyperbolicity of \( h \). We show that \( c > 0 \). Assume \( c = 0 \). Take \( \xi' \) so that \( \gamma (\xi') = \psi_{p+1} (\xi') = 0, \ \phi_{p+1} (\xi') \neq 0 \). If \( \psi_1 (\xi') = 0 \), then \( (0, \xi') \) is a characteristic of order \( m \) and hence \( L(0, \xi') = 0 \) by the diagonalizability which gives an obvious contradiction. If \( \psi_1 (\xi') \neq 0 \) so that \( (0, \xi') \) is a characteristic of order \( m - 1 \), taking the 2-minor,

\[
L \begin{pmatrix} 1 & p + 1 \\ 1 & p \end{pmatrix} (0, \xi') = 0
\]

we also get a contradiction.

We now take \( \psi_p (\xi') = 1, \ \psi_{p+1} (\xi') = c \alpha^2, \ \phi_{p+1} (\xi') = \alpha \) so that \( (0, \xi') \) is a characteristic of order \( m - 1 \) (resp. \( m - 2 \)) if \( \psi_1 (\xi') = 0 \) (resp. \( \psi_1 (\xi') \neq 0 \)). When \( \psi_1 (\xi') = 0 \) every 2-minor of \( L(0, \xi') \) is zero. Since \( \alpha \) is arbitrary we conclude that

\[
C_{\phi_{p+1}}^{up} = 0 \text{ unless } (u, v) = (p, p + 1).
\]

When \( \psi_1 (\xi') \neq 0 \) every 3-minor of \( L(0, \xi') \) must vanish. Since

\[
L \begin{pmatrix} 1 & p1 & p2 \\ 1 & q1 & q2 \end{pmatrix} (0, \xi') = \psi_1 (\xi') L \begin{pmatrix} p1 & p2 \\ q1 & q2 \end{pmatrix} (0, \xi')
\]

every 2-minor of the \( (m-1) \times (m-1) \) right-lower submatrix of \( L(0, \xi') \) is zero and the proof is reduced to the preceding case. The second assertion can be proved by the same argument applied to the left-upper \( (m-1) \times (m-1) \) submatrix.

PROPOSITION 4.5. Let \( u \geq 2 \) and \( u < v \). For \( q > p \geq 2 \) we have

\[
C_{\phi_{p+1}}^{up} = 0 \text{ unless } (u, v) = (p, q).
\]

Let \( v \leq m - 1 \) and \( u < v \). For \( p < q \leq m - 1 \) we have

\[
C_{\phi_{p+1}}^{up} = 0 \text{ unless } (u, v) = (p, q).
\]

PROOF. The same arguments as in [6, pp. 411-412] with the modifications indicated in the proof of Lemma 4.4 show the assertions. 

By Proposition 4.5 we can write for \( u \geq 2, \ u < v \)

\[
(4.1) \quad \phi_u = C_{\phi_{p+1}}^{up} \phi_u + \sum_{i=2}^{m} C_{\phi_{p+1}}^{i} \phi_i
\]
LEMMA 4.6. There is a non singular matrix $T \in M(m, \mathbb{R})$ such that

$$T^{-1}L(\xi)T = (\delta_j^i(\xi))$$

verifies

$$\delta^i_j = 0, \delta^m_i = 0 \text{ for } (i, j) = (1, m - 1), (1, m), (2, m),$$

where $\delta^i_j = \sum_{j > i} \delta^i_j$. Furthermore $T^{-1}L(\xi)T$ verifies the conclusion of Proposition 4.5.

PROOF. Without restrictions we may assume that $\psi_2 = 0$. We divide the cases into two: $\phi_1 - \phi_m \notin V$ and $\phi_1 - \phi_m \in V$.

Case $\phi_1 - \phi_m \notin V$. This assumption implies that either $\partial \psi_m / \partial \psi_k \neq 0$ for some $k$, $3 \leq k \leq m - 1$ or $\partial \psi_m / \partial \psi_k = 0, 3 \leq k \leq m - 1$ and $\partial \psi_m / \partial \psi_1 \neq 0$. Let us assume the former case. Then $\psi_k$ is a linear combination of $\psi_1, \ldots, \psi_{k-1}, \psi_{k+1}, \ldots, \psi_m$ and $\phi_j, i > j$. Take $\phi_j^2(\xi^i) = 0, i > j, (i, j) \neq (2, 1), \phi_1^2(\xi') = \alpha$ and set

$$\lambda^\pm = \frac{-\psi_1 \pm \sqrt{\psi_1^2 + 4\alpha^2}}{2}, c = \frac{C_{22}}{2}.$$ 

Take $\psi_i$ so that $\psi_i = -\lambda^\pm, 3 \leq i \leq m, i \neq k$. Then $(\lambda^\pm, \xi')$ is a characteristic of order $m - 2$. Note that

$$\lambda^\pm + \psi_k = B_1\psi_1 + B_2\lambda^\pm + B_3\alpha$$

with some constants $B_i$. Take the 3-minor

$$L\left(\begin{array}{ccc} 1 & 2 & k \\ 2 & k & m \end{array}\right) \left(\lambda^\pm, \xi'\right) = \begin{bmatrix} c\alpha & 0 & C_m^1 \alpha \\ \lambda^\pm & 0 & C_m^2 \alpha \\ 0 & \lambda^\pm + \psi_k & C_m^k \alpha \end{bmatrix} = 0$$

where $C_i$ stand for $C_{ij}$ for simplicity and we have used Proposition 4.5 to conclude that $\phi_u^m$ is independent of $\phi_1^2$ when $1 < u < m, u < v$. Assume that $B_1 \neq 0$ and recall that (4.3) is equal to

$$cC_m^2B_3\alpha^3 - (cC_m^2B_2 - C_m^1B_3)\alpha^2\lambda^\pm - cC_m^2B_1\alpha^2\psi_1 + C_m^1B_2\alpha(\lambda^\pm)^2 + C_m^1B_1\alpha\lambda^\pm\psi_1 = 0.$$
Since \( \lambda^+ \to 0 \), \( \lambda^+ \psi_1 \to c\alpha^2/4 \) as \( \psi_1 \to \infty \) we obtain that \( C_m^2 = 0 \). Then (4.3) is reduced to

\[
C_m^1 B_3 \alpha^2 \lambda^\pm + C_m^1 B_2 \alpha (\lambda^\pm)^2 + C_m^1 B_1 \alpha \lambda^\pm \psi_1 = 0
\]

and hence we see that \( C_m^1 = 0 \). If \( B_1 = 0, \ B_2 \neq 0 \), noting that \( |\lambda^-| \to \infty \) as \( \psi_1 \to -\infty \) we get \( C_m^1 = 0 \) and then \( C_m^2 = 0 \). If \( B_1 = B_2 = 0, \ B_3 \neq 0 \), a similar argument shows that \( C_m^1 = C_m^2 = 0 \).

Let \( B_1 = B_2 = B_3 = 0 \). This means that \( (\lambda^\pm, \xi^\prime) \) is a characteristic of order \( m - 1 \). Then taking the 2-minor

\[
L \left( \begin{array}{cc} 1 & 2 \\ 2 & m \end{array} \right) (\lambda^\pm, \xi^\prime) = \begin{vmatrix} \frac{c\alpha}{\lambda^\pm} & C_m^1 \alpha \\ \frac{C^2_m}{\lambda^\pm} & C_m^2 \alpha \end{vmatrix} = 0
\]

we conclude that \( C_m^1 = C_m^2 = 0 \).

We turn to the latter case. We take \( \phi_j^m(\xi^\prime) = 0, \ i > j, \ (i,j) \neq (2,1), \ (i,j) \neq (m, m-1) \) and \( \phi_{m-1}^m = \beta \). Hence

\[
h(\xi) = (\xi_0(\xi_0 + \psi_1) - \alpha \phi_j^m) \times ((\xi_0 + \psi_{m-1})(\xi_0 + \psi_m) - \beta \phi_{m-1}^m) \prod_{j \neq 1, 2, m-1, m} (\xi_0 + \psi_j).
\]

Recall that \( \phi_j^m = C_m^1 \alpha + C_m^{m-1} \beta \) and \( \phi_{m-1}^m = C_m^{m-1} \alpha + C_m^{m-1} \beta \). Here it is clear that \( C_m^{m-1} \alpha + C_m^{m-1} \beta = 0 \) from the hyperbolicity of \( h \) because \( \{\psi_1, \psi_3, \ldots, \psi_{m-1}\} \) are linearly independent and so are \( \{\psi_3, \ldots, \psi_m\} \). Note that

\[
\psi_m = \delta \psi_1 + a\alpha + b\beta
\]

with \( \delta \neq 1, \ a = C_m^{m-1}, \ b = C_m^{m-1} \). Let \( \psi_{m-1}^\pm \) solve the equation

\[
(\lambda^\pm + \psi_{m-1}^\pm)(\lambda^\pm + \psi_1 + a\alpha + b\beta) = c_1 \beta^2
\]

which is a linear equation in \( \psi_{m-1}^\pm \) where \( c_1 = C_m^{m-1} \). Taking \( \psi_1 = -\lambda^\pm, \ i \neq 1, \ 2, \ m-1, \ m, \ (\lambda^\pm, \xi^\prime) \) turns out to be a characteristic of order \( m - 2 \). Consider the 3-minor

\[
L \left( \begin{array}{ccc} 1 & 2 & m \\ 1 & m-1 & m \end{array} \right) (\lambda^\pm, \xi^\prime) = \begin{vmatrix} \lambda^\pm + \psi_1 & C_{m-1}^1 \beta & C_m^1(\alpha, \beta) \\ \alpha & 0 & C_m^2(\alpha, \beta) \\ 0 & \beta & \lambda^\pm + \psi_m \end{vmatrix} = 0
\]

where \( C_m^1 = C_m^{m-1}, \ C_m^2(\alpha, \beta) = C_m^{m-1} \alpha + C_m^{m-1} \beta \) and \( C_m^2(\alpha, \beta) = C_m^{m-1} \alpha + C_m^{m-1} \beta \).

Here we have used \( C_m^{m-1} = 0, \ C_m^{m-1} = 0, \ C_m^{m-1} = 0 \) which follows from Proposition 4.5. Note that (4.4) is equal to

\[
(\delta C_{m-1}^1 \alpha \beta + C_m^2(\alpha, \beta) \beta) \psi_1 - (C_{m-1}^1 \alpha \beta + C_m^2(\alpha, \beta) \beta) \lambda^\pm
\]

\[
+ C_m^1(\alpha, \beta) \alpha \beta - (a\alpha + b\beta) C_{m-1}^1 \alpha \beta = 0.
\]
As before it follows that

$$\delta C^{1}_{m-1} \alpha \beta + C^{2}_{m} (\alpha, \beta) \beta = 0, \quad C^{1}_{m-1} \alpha \beta + C^{2}_{m} (\alpha, \beta) = 0.$$  

Since $\delta \neq 1$ we see that $C^{1}_{m-1} = 0$, $C^{2}_{m} (\alpha, \beta) = 0$. Hence $C^{1}_{m} (\alpha, \beta) = 0$. Thus we have proved that

$$C^{1}_{m2} = C^{2}_{m2} = 0.$$  

Repeating an analogous argument, exchanging $\psi_1$ and $\psi_m$, and noting that we may assume that $\psi_{m-1} = 0$ instead of $\psi_2 = 0$ we conclude that

$$C^{1m-1}_{mm} = C^{1m-1}_{m-1m} = 0.$$  

**Case $\phi^1_m - \phi^m_m \in V$.** Noting that $\partial \psi_m / \partial \psi_k = 0$, $3 \leq k \leq m - 1$, $\partial \psi_m / \partial \psi_1 = 1$, we take the same $\xi'$ as in the second case of $\phi^1_1 - \phi^m_1 \notin V$. Then (4.4) turns out to be

$$\beta (C^{1}_{m-1} \alpha + C^{2}_{m} (\alpha, \beta)) \psi_1 - \beta (C^{1}_{m-1} \alpha + C^{2}_{m} (\alpha, \beta)) \lambda^\pm$$  

$$+ \alpha \beta (C^{1}_{m} (\alpha, \beta) - C^{1}_{m-1} (a \alpha + b \beta)) = 0.$$  

Hence it follows that

$$(4.6) \quad C^{2}_{m} (\alpha, \beta) = -C^{1}_{m-1} \alpha, \quad C^{1}_{m} (\alpha, \beta) = (a \alpha + b \beta) C^{1}_{m-1}.$$  

Now we take $T = T^{1}_{m} (-C^{1}_{m-1})$ and set

$$T^{-1} L(\xi) T = (\tilde{\phi}^j_j (\xi)).$$  

Then it is clear that

$$(4.7) \quad \tilde{\phi}^1_m = \phi^1_m - C^{1}_{m-1} (\phi^m_m - \phi^1_1) - (C^{1}_{m-1})^2 \phi^m_1,$$  

$$\tilde{\phi}^1_{m-1} = \phi^1_{m-1} - C^{1}_{m-1} \phi^m_{m-1}, \quad \tilde{\phi}^2_m = \phi^2_m + C^{1}_{m-1} \phi^2_1.$$  

It is easy to see that $C^{1}_{j2} = \tilde{C}^{1m-1}_{j2} = 0$ for $(i, j) = (1, m - 1)$, $(1, m)$, $(2, m)$ by (4.6) and (4.7). Note that $\tilde{\phi}^j_j$, $1 \leq j \leq m - 2$, differs from $\phi^j_j$ only by a constant times $\phi^m_j$, and $\tilde{\phi}^i_i$, $i \geq 3$, differs from $\phi^i_i$ by a constant times $\phi^1_1$. This implies that Proposition 4.5 remains valid for $(\tilde{\phi}^j_j (\xi))$.

In what follows we assume that the original $L(\xi)$ verifies the conclusion of Lemma 4.6.

**Lemma 4.7.** For $2 \leq q \leq m - 1$ we have

$$C^{1}_{uq} = 0 \text{ if } u < m, \ u \neq 1, \ u \neq q,$$  

$$C^{1m-1}_{vm} = 0 \text{ if } 1 < v, \ v \neq m - q + 1, \ v \neq m.$$  

**Lemma 4.8.** For $2 \leq q \leq m - 1$ we have

$$C^{1}_{uq} = 0 \text{ if } u < m, \ u \neq 1, \ u \neq q,$$  

$$C^{1m-1}_{vm} = 0 \text{ if } 1 < v, \ v \neq m - q + 1, \ v \neq m.$$  

**Lemma 4.9.** For $2 \leq q \leq m - 1$ we have

$$C^{1}_{uq} = 0 \text{ if } u < m, \ u \neq 1, \ u \neq q,$$  

$$C^{1m-1}_{vm} = 0 \text{ if } 1 < v, \ v \neq m - q + 1, \ v \neq m.$$
PROOF. Without restrictions we may assume that $\psi_0 = 0$. Take $\xi'$ so that $\phi^i_j = 0$, $i > j$, $(i, j) \not\equiv (q, 1)$, $\phi^q_i = \alpha$ and $\psi_i = 0$, $i \geq 2$. By Proposition 4.5 we have $\phi^i_k = 0$ if $u < v$, $v < m - 1$, $(u, v) \not\equiv (1, q)$. Hence

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \phi^q_i\phi^i_q)\xi_0^{m-2}.$$ 

As before, we easily see that $\phi^i_q = c\phi^q_i$ with some $c > 0$. Then $(0, \xi')$ is a characteristic of order $m - 2$. Take the 3-minor, assuming for instance $q < u$,

$$L \left( \begin{array}{ccc} 1 & q & u \\ 1 & q & m \end{array} \right) (0, \xi') = \begin{vmatrix} \psi_1(\xi') & c\alpha & C^1_m\alpha \\ \alpha & 0 & C^q_m\alpha \\ 0 & 0 & C^q_m\alpha \end{vmatrix} = 0$$

where $C^q_i = C^{q1}_i$. Then we have $C^u_{mq} = 0$. Similarly we can prove the second assertion.

COROLLARY 4.8. We have for $u < v$

$$C^u_{vq} = 0 \text{ unless } (u, v) = (1, 2),$$

$$C^{um-1}_{vm} = 0 \text{ unless } (u, v) = (m, m - 1).$$

PROOF. The assertion easily follows from Lemmas 4.6 and 4.7.

LEMMA 4.9. Let $2 \leq q \leq m - 1$. Then we have for $u < v$

$$C^u_{vq} = 0 \text{ unless } (u, v) = (1, q),$$

$$C^{um-1-q+1}_{vm} = 0 \text{ unless } (u, v) = (m - q + 1, m).$$

PROOF. If $q = 2$ this is Corollary 4.8. Let $q \geq 3$. Take $\psi^i_j = 0$, $i > j$, $(i, j) \not\equiv (q, 1)$ and $\phi^q_i = \alpha$. Then from Proposition 4.5 and Lemma 4.7 it follows that for $u < v$

$$\phi^u_v = 0 \text{ unless } (u, v) = (1, q), (1, m), (q, m).$$

Without restrictions we can suppose that $\psi_q = 0$. We first study the case where $\partial\psi_m/\partial\psi_k \neq 0$ for some $k, k \neq 1, q, \leq m - 1$. Since

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - c\alpha^2)(\xi_0 + \psi_k) \prod_{j \neq 1, q, \ell} (\xi_0 + \psi_j)$$

with $c = C^q_1$, we can follow the same arguments proving Lemma 4.6 choosing $\psi_i = -\lambda^i$, $i \neq k, 1, q$. Assuming $q < k$ for instance, take the 3-minor,

$$L \left( \begin{array}{ccc} 1 & q & k \\ q & k & m \end{array} \right) (\lambda^i, \xi') = \begin{vmatrix} c\alpha & 0 & C^1_m\alpha \\ \lambda^i & 0 & C^q_m\alpha \\ 0 & \lambda^i + \psi_k & 0 \end{vmatrix} = 0.$$
The same reasoning as in the proof of Lemma 4.6 proves that \( C^1_m = C^q_m = 0 \) where \( C^1_m = C^{1q}_m \), \( C^q_m = C^{q1}_m \). We treat the remaining case \( \partial \psi_m / \partial \psi_k = 0 \), \( k \neq 1 \), \( k \leq m - 1 \). We first study the case \( q < m - 1 \). We take \( \phi_j(\xi') = 0 \), \( i > j \), \( (i, j) \neq (q, 1) \), \( (m, m - 1) \) and \( \phi^1_q = \alpha \), \( \phi^m_{m-1} = \beta \). From Proposition 4.5 and Lemmas 4.6, 4.7 it follows that

\[
\phi^v_v = 0 \text{ unless } (u, v) = (1, q), (1, m), (q, m), (m - 1, m)
\]

and

\[
\phi^1_q = C^{1q}_m \alpha, \quad \phi^m_{m-1} = C^{m-1m}_m \beta + C^{m-11}_m \alpha
\]

\[
\phi^1_m = C^{1m}_m \alpha, \quad \phi^q_m = C^{qm}_m \alpha.
\]

Since

\[
h(\xi) = (\xi_0(\xi_0 + \psi_1) - \alpha \phi^1_q) \times ((\xi_0 + \psi_m - 1)(\xi_0 + \psi_m) - \beta \phi^m_{m-1}) \prod_{j \neq 1, q, m-1, m} (\xi_0 + \psi_j)
\]

if follows from hyperbolicity that \( C^{mq}_m = 0 \). Choosing \( \psi^\pm_{m-1} \) and \( \psi_j, j \neq 1, q, m - 1, m \) as in the proof of Lemma 4.6 we consider the 3-minor

\[
L \begin{pmatrix} 1 & q & m \\ 1 & m-1 & m \end{pmatrix} = \begin{vmatrix} \lambda^\pm + \psi_1 & 0 & C^{11}_m \alpha \\ \alpha & 0 & C^{q1}_m \alpha \\ 0 & \beta & \lambda^\pm + \psi_m \end{vmatrix} = 0.
\]

Here we have used \( C^{1m-1}_m = 0 \) which follows from Lemma 4.6. Repeating the same arguments as in the proof of Lemma 4.6 we obtain that \( C^{1q}_m = 0 \) and \( C^{q1}_m = 0 \). Exchanging \( \psi_1 \) and \( \psi_m \) and repeating the same reasoning we conclude that

\[
C^{1m-q+1}_m = 0, \quad C^{1m-q+1}_m = 0.
\]

When \( q = m - 1 \) we take \( \phi^1_j = 0 \), \( i > j \), \( (i, j) \neq (q, 1) \), \( (2, 1) \) and \( \phi^q_1 = \alpha \), \( \phi^2_1 = \beta \). Without restrictions we may assume that \( \psi_2 = 0 \). It is easy to see that

\[
h(\xi) = (\xi_0 + \psi_m - 1)((\xi_0 + \psi_m - 1)(\xi_0 + \psi_1) - \beta \phi^2_1) - C^{1m-1}_m \alpha^2 \xi_0 \prod_{j \neq 1, 2, m-1, m} (\xi_0 + \psi_j)
\]

with \( C^{1m-1}_m = C^{1m-1}_m \). Note that \( \psi_m \neq 0 \) by Lemma 4.2. Take \( \psi_{m-1} \) such that

\[
(\psi_m - \psi_m)(\psi_m(\psi_1 - \psi_m) + \beta \phi^2_1) + C^{1m-1}_m \psi_m \alpha^2 = 0
\]

and \( \psi_j = \psi_m, j \neq 1, 2, m - 1, m \) so that \( (-\psi_m, \xi') \) is a characteristic of order \( m - 2 \). We consider the 3-minor

\[
L \begin{pmatrix} 1 & 2 & m-1 \\ 2 & m-1 & m \end{pmatrix} = \begin{vmatrix} c \beta & C^{m-1}_m \alpha & C^{1}_m \alpha \\ \psi_1 & 0 & 0 \\ 0 & \psi_{m-1} - \psi_m & C^{m-1}_m \alpha \end{vmatrix} = 0.
\]
This gives that \( C^1_m = C^{11}_{m-1} = 0 \), \( C^{m-1}_m = C^{m-11}_{mm-1} = 0 \) because \( \phi^1_j \neq 0 \), \( C^1_{m-1} = C^{11}_{m-1m-1} \neq 0 \) and \( \beta, \psi_m \) are arbitrary provided \( \psi_m(\psi_1 - \psi_m) + \beta \phi^1_2 \neq 0 \). Working in the \((m-1) \times (m-1)\) right-lower submatrix, similar arguments show that
\[
C^{12}_{mm} = C^{12}_{2m} = 0
\]
which completes the proof. □

**Lemma 4.10.** We have \( C^{1p}_{mq} = 0 \) for \( 1 < p < q < m \).

**Proof.** Let \( q < m - 1 \). Take \( \phi^j_i = 0, i > j, (i, j) \neq (q, p), (m, m - 1), \phi^q_i = \alpha, \phi^m_{m-1} = \beta \). From Proposition 4.5 and Lemmas 4.6, 4.9 we see that \( \phi^1_m = C^{1p}_{mq} \phi^p_q \) and \( \phi^m_{m-1} - C^{m-1m-1}_{mm-1} \phi^m_{m-1} \). Without restriction we may assume that \( \psi_q = 0 \) and hence \( \psi_1 \neq 0 \) by Lemma 4.2. Then it is clear that
\[
h(\xi) = (\xi_0 + \psi_1)((\xi_0 + \psi_p)\xi_0 - C^p_q \alpha^2) \\
\times ((\xi_0 + \psi_{m-1})\xi_0 + \psi_m) - C^{m-1}_{m} \beta^2 \prod_{j \neq 1, p, q, m-1, m} (\xi_0 + \psi_j)
\]
where \( C^p_q = C^{pq}_{qq}, C^{m-1}_{m-1} = C^{m-1m-1}_{mm-1} \). Recall that \( \psi_m = l_1(\psi_1) + l_2(\alpha, \beta) \). Let \( \psi_p, \psi_{m-1} \) solve the equations
\[-(\psi_p - \psi_1)\psi_1 = C^p_q \alpha^2, (\psi_{m-1} - \psi_1)(\psi_m - \psi_1) = C^{m-1}_{m} \beta^2.
\]
With this choice of \( \psi_p \) and \( \psi_{m-1}, \) \((-\psi_1, \xi')\) is a characteristic of order \( m - 2 \) choosing \( \psi_1 = \psi_1, i \neq 1, p, q, m - 1, m \). Observe the 3-minor
\[
L \left( \begin{array}{ccc}
1 & p & m - 1 \\
q & m - 1 & m
\end{array} \right) = \left| \begin{array}{ccc}
0 & 0 & C^1_m \alpha \\
C^p_q \alpha & 0 & 0 \\
0 & \psi_{m-1} - \psi_1 & C^{m-1}_{m} \beta
\end{array} \right| = 0
\]
where \( C^1_m = C^{1p}_{mq} \). This shows that \( C^{1p}_{mq} = 0 \) because \( C^p_q \neq 0 \) if \( \psi_m - \psi_1 \neq 0 \). When \( \psi_m - \psi_1 = 0 \) taking \( \phi^m_{m-1} = 0 \) we get
\[
(4.8) \quad h(\xi) = (\xi_0 + \psi_1)^2((\xi_0 + \psi_p)\xi_0 - C^p_q \alpha^2) \prod_{j \neq 1, p, q, m} (\xi_0 + \psi_j).
\]
Choosing \( \psi_p, \psi_j \) such that
\[-(\psi_p - \psi_1)\psi_1 = C^p_q \alpha^2, \psi_j = \psi_1, j \neq 1, p, q, m
\]
\((-\psi_1, \xi')\) is a characteristic of order \( m - 1 \). Thus taking the 2-minor
\[
L \left( \begin{array}{cc}
1 & p \\
p & m
\end{array} \right) = \left| \begin{array}{cc}
0 & C^1_m \alpha \\
\psi_p - \psi_1 & 0
\end{array} \right| = 0
\]
we conclude that $C_{m}^{1} = 0$.

When $q = m - 1$ it is clear that

$$h(\xi) = (\xi_0 + \psi_1)((\xi_0 + \psi_m)(\xi_0 + \psi_m) - C_{m}^{m-1}\beta^2)$$

$$- C_{m-1}^{p}\alpha^2(\xi_0 + \psi_m) \prod_{j \neq 1, p, m-1, m} (\xi_0 + \psi_j)$$

where $C_{m-1}^{p} = C_{m-1}^{m-1}$. Then if $\psi_m - \psi_1 \neq 0$, choosing $\psi_{m-1}$, $\psi_p$, $\psi_j$ so that

$$(\psi_p - \psi_1)((\psi_m - \psi_1)\psi_1 + C_{m}^{m-1}\beta^2) + C_{m-1}^{p}\alpha^2(\psi_m - \psi_1) = 0$$

and $\psi_j = \psi_1$, $j \neq 1$, $p$, $m - 1$, $m$ it is enough to take the 3-minor

$$L \left( \begin{array}{ccc} 1 & m-1 & m \\ m-2 & m-1 & m \end{array} \right) = \begin{vmatrix} 0 & 0 & C_{m}^{1}\alpha \\ \alpha & \psi_{m-1} - \psi_1 & C_{m}^{m-1}\beta \\ 0 & \beta & \psi_m - \psi_1 \end{vmatrix} = 0$$

to get $C_{m}^{1} = C_{m-1}^{m-1} = 0$. If $\psi_m - \psi_1 = 0$, taking $\beta = 0$, $h(\xi)$ coincides with (4.8) and then the proof is clear.

By (4.1), (4.2) and Lemma 4.9 it follows that

$$\phi_{u} = C_{uv}^{u} \phi_{u} + C_{uv}^{v} \phi_{m}, \ (u, v) \neq (1, m), \ u < v$$

and from Lemmas 4.9 and 4.10 we see that

$$\phi_{m}^{1} = C_{m}^{m} \phi_{m}^{m}.$$

**LEMMA 4.11.** We have

$$C_{uv}^{1} = 0 \text{ unless } (u, v) = (1, m).$$

**PROOF.** Recall that $\phi_{u} = C_{vv}^{u} \phi_{u} + C_{mm}^{u} \phi_{m}$ for $u < v$. Since $C_{vv}^{u} > 0$ we choose $\xi'$ so that $\phi_{m}^{m} = \alpha$, $\phi_{m-1}^{m} = \beta$ and

$$\phi_{u} = \frac{-C_{uv}^{1} \alpha}{C_{vv}^{u}}, \ u \geq 2, \ (u, v) \neq (m - 1, m), \ \phi_{m}^{v} = 0, \ 2 \leq v \leq m - 1.$$

Without restrictions we may assume that $\psi_{m-1} = 0$. It is clear that

$$h(\xi) = \{\xi_{0}(\xi_{0} + \psi_{1})(\xi_{0} + \psi_{m}) - \beta C(\alpha, \beta)(\xi_{0} + \psi_{1})$$

$$+ \alpha(C_{m-1}^{1} C(\alpha, \beta) - C_{m}^{1} \alpha \xi_{0}) \prod_{j \neq 1, m-1, m} (\xi_{0} + \psi_{j})$$

...
where $C_{m-1}^{1} = C_{m-2}^{1}$, $C_{m}^{1} = C_{m-1}^{1}$, $C(\alpha, \beta) = C_{m}^{\alpha-1} \alpha + C_{m-1}^{\alpha-1} \beta$. Take $\psi_{j} = -y, j \neq 1, m - 1, m$ and let $\psi_{m}$ solve the equation

\[ y(y + \psi_{1})(y + \psi_{m}) - \beta C(\alpha, \beta)(y + \psi_{1}) + C_{m-1}^{1}(\alpha, \beta) \alpha^{2} - C_{m}^{1} \alpha^{2} y = 0. \] (4.10)

Then clearly $(y, \xi')$ is a characteristic of order $m - 2$. Note that $y$ and $\psi_{1}$ are arbitrary provided that $y(y + \psi_{1}) \neq 0$. Let us take the 3-minor $(2 \leq q \leq m - 2)$

\[ L\left(\begin{array}{ccc} 1 & m - 1 & m \\ 1 & \alpha & m - 1 \end{array}\right) = \left|\begin{array}{ccc} y + \psi_{1} & \phi_{q}^{1} & C_{m-1}^{1} \alpha \\ 0 & \phi_{q}^{m-1} & y \\ \alpha & \phi_{q}^{m} & \beta \end{array}\right| = 0. \]

Since $y, \psi_{1}, \beta, \alpha$ are arbitrary and

\[ \phi_{q}^{1} = C_{qm}^{1} \alpha, \quad \phi_{q}^{m-1} = -C_{m-1}^{1} \alpha/C_{m-1}^{m-1}, \quad \phi_{q}^{m} = -C_{qm}^{m-1} / C_{qm}^{m} \]

by (4.9) it follows that

\[ C_{qm}^{1} = C_{mm}^{1} = C_{mm}^{m} = 0, \quad 2 \leq q \leq m - 2. \] (4.11)

Take $\psi_{j} = -y, j \neq 1, m - 1, m$ and let $\psi_{m} = \psi_{1}(\psi_{m}, y)$ solve equation (4.10). In this case $y$ and $\psi_{m}$ are arbitrary provided that $y(y + \psi_{m}) - \beta C(\alpha, \beta) \neq 0$ and $(y, \xi')$ is a characteristic of order $m - 2$ again. Consider the 3-minor $(2 \leq p < q \leq m - 2)$

\[ L\left(\begin{array}{ccc} q & m - 1 & m \\ p & \alpha & m - 1 \end{array}\right) = \left|\begin{array}{ccc} \phi_{p}^{1} & 0 & 0 \\ \phi_{p}^{m-1} & y & C(\alpha, \beta) \\ \phi_{p}^{m} & \beta & y + \psi_{m} \end{array}\right| = 0. \]

Hence $\phi_{p}^{1} = -C_{qm}^{1} \alpha/C_{qm}^{m} = 0$ and then

\[ C_{qm}^{1} = 0, \quad 2 \leq p < q \leq m - 2. \] (4.12)

We next choose $\xi'$ such that $\phi_{1}^{m} = \alpha, \phi_{1}^{2} = \beta$ and

\[ \phi_{v}^{u} = -C_{um}^{1} \alpha/C_{vu}^{m}, \quad u < v, \quad 3 \leq v \leq m - 1, \]

\[ \phi_{v}^{m} = 0, \quad 2 \leq u \leq m - 1. \]

Then similar arguments as above prove that

\[ C_{qm}^{1} = C_{qm}^{1} = C_{qm}^{2} = 0 \quad \text{for} \quad 3 \leq q \leq m - 1, \]

\[ C_{mm}^{1} = 0 \quad \text{for} \quad 3 \leq p < q \leq m - 1. \] (4.13)

From (4.11), (4.12) and (4.13) we get the desired assertion. \qed
PROPOSITION 4.12. There is a non singular $T \in \mathcal{M}(m, \mathbb{R})$ such that

$$T^{-1}L(\xi)T = (\tilde{\phi}_j^i(\xi))$$

verifies for $u < v$ that

$$C_{uv}^{pq} = 0 \text{ unless } (u, v) = (p, q)$$

where $\tilde{\phi}_u^v = \sum_{i\geq j} C_{uv}^{ji} \phi_i^j$.

To simplify the notation we set

$$C_q^p = C_{qq}^{pq}, \quad p < q$$

which are positive. By Proposition 4.12 we know that

$$\phi_q^p = C_q^p \phi_p^q \quad \text{for } p < q.$$  

We recall some facts.

LEMMA 4.13 (Oshime [4]). Let $m = 3$ and $d(L) = 3(3+1)/2 - 1 = 5$. Suppose that $L(\xi)$ is diagonalizable with real eigenvalues. Then $L(\xi)$ is symmetrizable by a non singular constant matrix.

Let us consider the matrix

$$A(x) = \begin{pmatrix} \psi(x) & \alpha x_2 & \gamma x_4 \\ x_2 & 0 & \beta x_3 \\ x_4 & x_3 & x_1 \end{pmatrix}$$

where $\psi(x)$ is linear in $x = (x_1, \ldots, x_4)$.

LEMMA 4.14. Assume that $A(x)$ is diagonalizable with real eigenvalues for every $x$. Then we have $\alpha, \beta, \gamma > 0$ and $\alpha \beta = \gamma$.

PROOF. The assertion that $\alpha, \beta, \gamma > 0$ is easily verified. Recall that $x_0 I + A(x)$ has reduced dimension 5 and hence is symmetrizable by Lemma 4.13: there is $T$ such that $T^{-1}A(x)T$ is symmetric for every $x$. As in the proof of Lemma 2.1, setting $H = T^\dagger T$, we have

$$A(x)H = H^\dagger A(x).$$

From this we easily see that $H$ is diagonal with positive elements. Then a simple observation proves that $\alpha \beta = \gamma$. \qed
We next consider the matrix
\[ A(x) = \begin{pmatrix} \phi(x) & \alpha x_3 & \gamma x_5 \\ x_3 & 0 & \beta x_4 \\ x_5 & x_4 & x_2 \end{pmatrix} \]
where \( \phi(x) \) is a linear function in \( x = (x_1, \ldots, x_5) \) and \( \partial \phi / \partial x_1 \neq 0 \).

**Lemma 4.15** (Vaillant [6]). Assume that the eigenvalues of \( A(x) \) are all real. Then we have \( \alpha, \beta, \gamma > 0 \) and \( \alpha \beta = \gamma \).

**Proof.** It is easy to see that \( \alpha, \beta, \gamma > 0 \). We take \( x_2 = 0, x_3 = 1/\sqrt{\alpha}, x_4 = 1/\sqrt{\beta}, x_5 = 1/\sqrt{\gamma} \) and \( x_1 \) so that \( \phi(x) = 0 \). Then it is clear that
\[
\det(\lambda + A(x)) = \lambda^3 - 3\lambda + \sqrt{\alpha \beta / \gamma} + \sqrt{\gamma / \alpha \beta}.
\]
The discriminant is
\[
27 \left\{ -4 + \left( \sqrt{\alpha \beta / \gamma} + \sqrt{\gamma / \alpha \beta} \right)^2 \right\}
\]
which must be non-positive. Hence \( \alpha \beta / \gamma = 1 \). \( \Box \)

**Proposition 4.16.** For \( 1 < p < q \) we have
\[ C_p^1 C_q^p = C_q^1. \]

**Proof.** Let \( q < m \). Take \( \xi' \) so that
\[
\phi_j = 0, \quad i > j, \quad (i, j) \neq (p, 1), (q, 1), (q, p).
\]
Without restriction we may assume that \( \psi_p = 0 \). Since \( L(\xi) \) has only real eigenvalues it is clear that
\[
\begin{pmatrix} \psi_1 & C_p^1 \phi_p^q & C_q^1 \phi_q^q \\ \phi_p^q & \psi_p & C_p^q \phi_p^q \\ \phi_q^q & \phi_p^q & \psi_q \end{pmatrix}
\]
has only real eigenvalues. Since \( q < m \) we can take \( \psi_1, \psi_q, \phi_p^q, \phi_q^q, \phi_p^q \) as independent forms and then we apply Lemma 4.15 to get \( C_p^1 C_q^p = C_q^1 \). When \( q = m \) we take \( \xi' \) so that
\[
\phi_j = 0, \quad i > j, \quad (i, j) \neq (p, 1), (m, 1), (m, p).
\]
Consider the \( 3 \times 3 \) matrix
\[
A = \begin{pmatrix} \psi_1 & C_p^1 \phi_p^m & C_m^1 \phi_m^m \\ \phi_p^m & \psi_p & C_m^p \phi_p^m \\ \phi_m^m & \phi_p^m & \psi_m \end{pmatrix}
\]
where we may assume that $\psi_p = 0$. Note that, after an exchange of rows and of the corresponding columns, $L(\xi)$ becomes a direct sum $A \oplus B$ where the diagonal forms of $B$ are $\xi_0 + \psi_i$ ($i \neq 1, m, p$). Then it is clear that $A$ is diagonalizable with real eigenvalues since $\psi_i$ ($i \neq 1, p, m$) are independent of $\psi_1$, $\psi_m$, $\phi_1^p$, $\phi_1^m$, $\phi_p^m$. Thus applying Lemma 4.14 we obtain $C_m^r C_m^g = C_m^l$.

**Theorem 4.17.** Assume that $d(L) = m(m+1)/2 - 1$ and that $L(\xi) = (\phi_i^j(\xi))$ is diagonalizable with real eigenvalues. Suppose that $\phi_j^i$, $i < j$, are independent of diagonal forms and that $L$ verifies the property (a) stated at the beginning of the present section. Then there is a non singular matrix $T$ such that

$$T^{-1}L(\xi)T$$

is symmetric for every $\xi \in \mathbb{R}^{n+1}$.

**Proof.** Using the same notation as in the proof of Proposition 4.16 we set

$$d_1 = 1, \quad d_q = 1/\sqrt{C_q^q} \quad \text{for} \quad q > 1.$$ 

Then with $T = \text{diag} \,(d_1, \ldots, d_m)$ we have $T^{-1}L(\xi)T = (d_i^{-1}\phi_j^i d_j)$. When $i < j$ we see that

$$d_i^{-1}\phi_j^i d_j = d_j^{-1}\phi_i^j d_i$$

which proves the assertion. \qed

5. - Case of less reduced dimension (2)

In this section we study the case (b) described at the beginning of the previous section. Recall that

$$\phi_{i_0}^{i_0} = \sum_{i > j, (i,j) \neq (i_0,j_0)} C_{i_0 j}^{i_0 j} \phi_j^i$$

with some $i_0 > j_0$. The following lemma is easily verified.

**Lemma 5.1.** We have

$$\text{dim span}\{\phi_i^j - \delta_i^j a(\xi) | i \geq j, (i,j) \neq (i_0,j_0)\} = m(m+1)/2 - 1$$

for every linear form $a(\xi)$.

If $\phi_{i_0}^{i_0} = 0$ then exchanging rows and the corresponding columns we may assume that $\phi_1^m = 0$. Then we can apply Theorem 3.2 with $k = 1$ and hence $T^{-1}L(\xi)T$ becomes symmetric for every $\xi$ for some non singular $T$. Thus in what
follows we assume that \( \phi_{\rho}^0 \neq 0 \). Again exchanging rows and the corresponding columns we may assume that \((i_0, j_0) = (2, 1)\). Set

\[
I_1 = \{(i, j) | i > j, (i, j) \neq (2, 1)\}
\]

and note that \( \phi_i^j = 0 \), \( (i, j) \in I_1 \) implies \( \phi_i^j = 0 \).

**PROPOSITION 5.2.** Assume that \( L(\xi) = (\phi_j(\xi)) \) is diagonalizable with real eigenvalues. Then we have

\[
C_{\text{up}}^{\text{up}} = 0 \text{ unless } (u, v) = (1, 2), \ (p, p + 1).
\]

To prove this proposition, without restriction, we may assume \( \psi_2 = 0 \). We first establish some lemmas.

**LEMMA 5.3.** \( \phi_j^i = 0 \), \( i > j \), \( (i, j) \neq (3, 1), (3, 2) \) so linear combination of \( \phi_j^i \)'s, \( (i, j) = (3, 1), (3, 2) \). Then

\[
A_{11} = \begin{pmatrix}
\psi_1 & \phi_1^1 & \phi_1^3 \\
\phi_1^2 & 0 & \phi_1^3 \\
\phi_1^3 & \phi_2^3 & \psi_3
\end{pmatrix}
\]

is diagonalizable with real eigenvalues.

**PROOF.** Let \( \phi_j^i = 0 \), \( i > j \), \( (i, j) \neq (3, 1), (3, 2) \) and set

\[
L = \begin{pmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{pmatrix}.
\]

Then \( \psi_4, \ldots, \psi_m \) are eigenvalues of \( A_{22} \). Since \( \psi_4, \ldots, \psi_m \) are independent of \( \psi_1, \psi_3, \phi_1^1, \phi_2^3 \) one can separate the eigenvalues of \( A_{22} \) from those of \( A_{11} \). Then it follows that \( A_{11} \) is diagonalizable with real eigenvalues. \( \square \)

Slightly changing notations we consider the following matrix:

\[
A(x) = \begin{pmatrix}
x_1 & b(x_3, x_4) & d(x_3, x_4) \\
a(x_3, x_4) & 0 & c(x_3, x_4) \\
x_4 & x_3 & x_2
\end{pmatrix}, \ x = (x_1, x_2, x_3, x_4).
\]

**LEMMA 5.4.** Assume that \( A(x) \) is diagonalizable with real eigenvalues. Then we have

\[
b(x_3, x_4) = \alpha a(x_3, x_4), \ c(x_3, x_4) = \beta x_3, \ d(x_3, x_4) = \gamma x_4
\]

with positive constants \( \alpha, \beta, \gamma > 0 \) such that \( \alpha \beta = \gamma \).
PROOF. It suffices to repeat the proof of Lemma 4.14.  

PROOF OF PROPOSITION 5.2.

**First step.** Take \( \xi' \) so that \( \phi_i' = 0, (i, j) \in I_1, (i, j) \neq (3, 2) \) and \( \phi_3^2 = 1 \). Then from Lemmas 5.3, 5.4 it follows that

\[
\phi_1^2 = a, \quad \phi_2^1 = a a, \quad \phi_3^2 = \beta, \quad \phi_3^3 = 0
\]

with \( \alpha, \beta > 0 \) and \( a \in \mathbb{R} \). Then it is clear that

\[
h(\xi) = \{(\xi_0 + \psi_1)\xi_0(\xi_0 + \psi_3) - a a^2(\xi_0 + \psi_3) + \beta(\xi_0 + \psi_4)\}
\]

with \( a, \xi \geq 0 \).

Taking \( \xi_0 = y \) we consider the equation

\[
y_2 + \psi_1 y - a a^2 \psi_3 + (y^2 - \beta)\psi_1 + y^3 - a a^2 y^2 - \beta y = 0.
\]

For every \( \psi_1, y \) with \( y^2 + \psi_1 y - a a^2 \neq 0 \) one can solve equation (5.1) with respect to \( \psi_3 \), that is \( \psi_3 = \psi_3(y, \psi_1) \). Take \( \psi_i = -y, i \geq 4 \) so that \( (y, \xi') \) is a characteristic of order \( m - 2 \) and hence every 3-minor is zero. Recall again that

\[
\phi_u^w = C_{u^3}, \quad u < v.
\]

We divide the cases into two; \( a = 0 \) and \( a \neq 0 \).

**Case \( a \neq 0 \).** Let \( v \geq 4 \). Take the 3-minor

\[
L \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & v \end{pmatrix} = \begin{vmatrix} y + \psi_1 & a a & C_0^1 \\ a & y & C_0^2 \\ 0 & 1 & C_0^3 \end{vmatrix} = 0
\]

where \( C_0^1 = C_0^2 = C_0^3 \). Since \( y, \psi_1 \) are arbitrary provided that \( y^2 + \psi_1 y - a a^2 \neq 0 \) it follows that

\[
C_0^1 = C_0^2 = C_0^3 = 0 \quad \text{for} \quad v \geq 4.
\]

When \( v > u > 3 \) we take the 3-minor

\[
L \begin{pmatrix} 2 & 3 & u \\ 2 & 1 & v \end{pmatrix} = \begin{vmatrix} a & y & C_0^2 \\ 0 & 1 & C_0^3 \\ 0 & 0 & C_0^u \end{vmatrix} = 0
\]

with \( C_j^1 = C_j^2 \) to conclude that

\[
(5.2) \quad C_{u^3}^u = 0.
\]
We turn to the case $a = 0$. In this case it is clear that

$$h(\xi) = (\xi_0(\xi_0 + \psi_3) - \beta) \prod_{i \neq 2,3} (\xi_0 + \psi_i).$$

Take $\psi_3$ so that $1 + \psi_3 = \beta$ and $\psi_i = -1, i \neq 2, 3$. Then $(1, \xi')$ is a characteristic of order $m - 1$ and every 2-minor is zero. This shows that

$$(5.3) \quad C_{u2}^{u2} = 0 \text{ unless } (u, v) = (2, 3).$$

By (5.2) and (5.3) we obtain the desired assertion when $p = 2$.

Second step. Now we study $C_{vp+1}^{np}$, $p \geq 3$. Take $\phi_p^{p+1} = 1, \phi_j^i = 0, (i, j) \in I_1, (i, j) \neq (p + 1, p)$. Recall that

$$\phi_1^2 = a, \quad \phi_v^a = C_{vp+1}^{np}$$

and $\phi_2^1 = \alpha a$, $\phi_{p+1}^p = \beta$. Then it is clear that

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \alpha a^2) \prod_{i \neq 1,2,p,p+1} (\xi_0 + \psi_i)$$

where $\alpha, \beta \geq 0$ which follows from hyperbolicity. Before going further we have:

**Lemma 5.5.** Let $a \neq 0$. Then we have

$$\alpha > 0, \quad \beta > 0.$$

**Proof.** We first show that $\beta > 0$. If $\beta = 0$ we take $\psi_p = \psi_{p+1} = 0, \psi_i = 0, i \neq 1, 2, p, p+1$ so that $(0, \xi')$ is a characteristic of order $m - 2$. Take the 3-minor

$$L \left( \begin{array}{ccc} 1 & 2 & p+1 \\ 1 & 2 & p \\ \end{array} \right) = \begin{vmatrix} \psi_1 & a & C_p^1 \\ \alpha & 0 & C_p^2 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

with $C_p^i = C_{pp+1}^{ip}$. This means that $\alpha = 0$ and hence $(0, \xi')$ is a characteristic of order $m$ taking $\psi_1 = 0$. This gives a contradiction. We next show that $\alpha > 0$. If $\alpha = 0$, taking $\psi_i = 0, i \neq 2, (0, \xi')$ is a characteristic of order $m - 2$. Take the 3-minor

$$L \left( \begin{array}{ccc} 2 & p & p+1 \\ 1 & p & p+1 \end{array} \right) = \begin{vmatrix} a & C_p^2 & C_{p+1}^2 \\ \beta & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$
which gives $\beta = 0$ and hence a contradiction again. 

We continue to study $C_{vp+1}^{up}$. We first investigate the case $a \neq 0$: Recall that, taking $\phi_{p+1}^{vp} = \mu$,

$$h(\xi) = (\xi_0(\xi_0 + \psi_1) - \alpha a^2 \mu^2) \prod_{i \neq 1, 2, p, p+1} (\xi_0 + \psi_i).$$

Let us set

$$\lambda^\pm = -\frac{\psi_1}{2} \pm \sqrt{\frac{\psi_1^2 + 4\alpha a \mu^2}{4}}$$

and note that $\lambda^+ \to 0$ as $\psi_1 \to \infty$. We take $\psi_p^\pm = 1 - \lambda^\pm$, $\psi_{p+1}^\pm = \beta \mu^2 - \lambda^\pm$ so that

$$(\lambda^\pm + \psi_p^\pm)(\lambda^\pm + \psi_{p+1}^\pm) = \beta \mu^2.$$

Taking $\psi_i^\pm = -\lambda^\pm$, $i \neq 1, 2, p, p+1, (0, \xi')$ will be a characteristic of order $m - 2$.

When $u > p + 1$ we take the 3-minor

$$L \begin{pmatrix} 2 & p + 1 & u \\ 2 & p + 1 & v \end{pmatrix} = \begin{vmatrix} \lambda^\pm & C_{p+1}^2 & C_v^2 \\ 0 & \beta \mu^2 & C_{p+1}^v \\ 0 & 0 & C_v^u \end{vmatrix} = 0$$

with $C_j = C_{jp+1}^{vp}$ to conclude that $C_{vp+1}^{up} = 0$. When $u = p$ and $v \neq p + 1$ or $u = p + 1$ we take the 3-minor

$$L \begin{pmatrix} 2 & p & \max\{u, p + 1\} \\ 2 & p + 1 & v \end{pmatrix} = \begin{vmatrix} \lambda^\pm & C_{p+1}^2 \mu & C_v^2 \mu \\ 0 & \beta \mu & C_v^2 \mu \\ 0 & \beta \mu^2 & C_v^2 \mu \end{vmatrix} = 0.$$

Since $\mu$ is arbitrary we get $C_{vp+1}^{up} = 0$. Similarly we get $C_{vp+1}^{up} = 0$ when $u < p$.

Case $a = 0$. It is clear that

$$h(\xi) = \xi_0(\xi_0 + \psi_1)(\xi_0 + \psi_{p+1}) - \beta \mu^2 \prod_{i \neq 2, p, p+1} (\xi_0 + \psi_i).$$

Taking $\psi_i$ as in the proof of the first step, $(0, \xi')$ is a characteristic of order $m - 1$. Then every 2-minor is zero. Thus it is easy to see that

$$C_{vp+1}^{up} = 0$$

unless $(u, v) = (p, p+1)$.

This completes the proof of Proposition 5.2.

Lemma 5.6. Assume that

$$\phi_i^j = 0, \ i > 2, \ i \neq q, \ j = 1, \ 2, \ \phi_i^i = 0, \ i > q, \ \phi_q^i = 0, \ 2 < i < q.$$
and the other $\phi_j$'s ($i > j$) verify $\phi_j = a_j \phi_i$. Then

$$A_{11} = \begin{pmatrix} \phi_1^1 & \phi_1^2 & \phi_1^q \\ \phi_2^1 & \phi_2^2 & \phi_2^q \\ \phi_q^1 & \phi_q^2 & \phi_q^q \end{pmatrix}$$

is diagonalizable with real eigenvalues.

**Proof.** Interchanging the third and $q$-th rows and the corresponding columns we arrive at

$$L(\xi) = \begin{pmatrix} A_{11} & * \\ O & A_{22} \end{pmatrix}.$$ 

Since the diagonal forms of $A_{22}$ are $\phi_i^i$, $i \neq 1, 2, q$, the same argument as in the proof of Lemma 5.3 proves the assertion. \qed

**Proposition 5.7.** We have for $u < v$ that

$$C_{vq}^{up} = 0 \text{ unless } (u, v) = (1, 2), (p, q).$$

**Proof.** We proceed by induction on $q - p = r$. When $q - p = 1$ this is Proposition 5.2. Assume that for $p < q \leq p + r$ we have

$$C_{vq}^{up} = 0 \text{ unless } (u, v) = (1, 2), (p, q).$$

Let $q = p + r + 1$. We may assume $\psi_2 = 0$ without restrictions.

**First step.** Let $p = 1$. Take $\phi_1^1 = 0$, $\phi_2^1 = 0$, $i > 2$, $i \neq q$ and $\phi_j^i = 0$, $i > j$, $i > q$. Recall that

$$\phi_u^u = C_{vq}^{u1} \phi_1^1 + C_{vq}^{uq} \phi_q^u \text{ for } 3 \leq u < v \leq q$$

by the inductive hypothesis. We take $\phi_1^q$ so that

$$(5.4) \quad \phi_u^u = -\frac{C_{vq}^{u1} \phi_1^q}{C_{vq}^{uq}}.$$ 

Thus $\phi_u^u = 0$ for $3 \leq u < v \leq q$. Applying Lemmas 5.5 and 5.4 we get

$$\phi_1^2 = \alpha \phi_1^1, \quad \phi_1^q = \gamma \phi_1^q, \quad \phi_q^2 = \beta \phi_q^q$$

with $\alpha, \beta, \gamma > 0$. Take $\phi_1^q = 1$, $\phi_q^2 = 0$ and hence $\phi_q^2 = \alpha \in \mathbb{R}$. Then it is easy to see that

$$h(\xi) = \{(\xi_0 + \psi_q)(\xi_0 + \psi_1) - \alpha \alpha^2 (\xi_0 + \psi_q) - \gamma \xi_0 \} \prod_{i \neq 1, 2, q} (\xi_0 + \psi_i).$$
Setting $\xi_0 = y$ we consider the equation

$$
(y^2 + \psi_1 y - \alpha a^2)\psi_q + y^2 \psi_1 + y^3 - (\alpha a^2 + \gamma)y = 0
$$

with respect to $\psi_q$. For every given $y$, $\psi_1$ with $y^2 + \psi_1 y - \alpha a^2 \neq 0$ we can solve (5.5) with respect to $\psi_q$; $\psi_q = \psi_q(y, \psi_1)$. Taking $\psi_i = -y$, $i \neq 1, 2, q$, $(y, \xi')$ is a characteristic of order $m - 2$.

**Case $a \neq 0$.** A repetition of the argument in the proof of Proposition 5.2 shows that

$$
C_{\xi y}^{21} = C_{\xi y}^{11} = 0 \text{ for } v > 2.
$$

For $2 < u < v \leq q$ take the 3-minor

$$
L \left( \begin{array}{ccc} 1 & 2 & v \\ 1 & 2 & u \end{array} \right) = \begin{vmatrix} y + \psi_1 & \alpha a & \phi_u^1 \\ a & y & \phi_u^2 \\ \phi_1^u & 0 & \phi_u^v \end{vmatrix} = 0
$$

to conclude that $\phi_u^v = 0$. Recalling (5.4) we get

$$
C_{\xi y}^{u1} = 0 \text{ for } 2 < u < v \leq q.
$$

When $q < u < v$, arguments similar to those in the proof of Proposition 5.2 (first step, case $a \neq 0$) prove that

$$
C_{\xi y}^{u1} = 0 \text{ unless } (u, v) = (1, 2), (1, q).
$$

With (5.6) and (5.7) this shows the assertion in the case $a \neq 0$.

**Case $a = 0$.** In this case we have

$$
h(\xi) = \xi_0 \{ (\xi_0 + \psi_q)(\xi_0 + \psi_1) - \gamma \} \prod_{i \neq 1, 2, q} (\xi_0 + \psi_i).
$$

Taking $\psi_i = 0$, $i \neq 1, 2, q$ and $\psi_1 = 1$, $\psi_q = \gamma$, $(0, \xi')$ is a characteristic of order $m - 1$. Hence every 2-minor is zero. This shows that

$$
C_{\xi y}^{u1} = 0 \text{ unless } (u, v) = (1, 2), (1, q).
$$

**Second step.** We study the case $p = 2$, $q = p + r + 1$. Take $\phi_i^1 = 0$, $\phi_i^2 = 0$, $i > 2 \ i \neq q$ and $\phi_j^1 = 0$, $i > j$, $i > q$. Recall that

$$
\phi_u^v = C_{\xi y}^{u1} \phi_1^1 + C_{\xi y}^{u2} \phi_2^2 + C_{\xi y}^{u3} \phi_u^v
$$
for $2 < u < v \leq q$ by the inductive hypothesis. Choose $\phi^u_v$ so that

$$\phi^u_v = -\frac{C_{uv}^2 \phi^q_v + C_{uv}^1 \phi^1_v}{C_{uv}^0}$$

and hence $\phi^u_v = 0$ for $2 < u < v \leq q$. It follows from Lemma 5.6 that

$$\begin{pmatrix}
\phi^1_1 & \phi^2_1 & \phi^1_q \\
\phi^1_2 & \phi^2_2 & \phi^2_q \\
\phi^1_q & \phi^2_q & \phi^3_q
\end{pmatrix}$$

is diagonalizable with real eigenvalues. Then by Lemma 5.4 we see that

$$\phi^1_2 = \alpha \phi^2_1, \quad \phi^2_2 = \beta \phi^2_q, \quad \phi^1_q = \gamma \phi^q_1$$

with $\alpha, \beta, \gamma > 0$. Thus choosing $\phi^q_2 = \mu$, $\phi^q_1 = 0$ we have

$$h(\xi) = \{(\xi_0 + \psi_q)(\xi_0 + \psi_1) - (\xi_0 + \psi_q)\alpha^2 \mu^2 - \beta \mu^2(\xi_0 + \psi_1)\} \prod_{i \neq 1,2,q} (\xi_0 + \psi_i).$$

The rest of the proof is almost the same as in the first step.

**Third step.** We finally treat the case $p \geq 3$, $q = p + r + 1$. It follows from the inductive hypothesis that

$$\phi^u_v = C_{uv}^p \phi^q_v + C_{uv}^r \phi^p_v$$

for $p < u < v \leq q$.

We take

$$\phi^v_u = -\frac{C_{uv}^p \phi^q_p}{C_{uv}^p}$$

so that $\phi^p_u = 0$ unless $p < u < v \leq q$, $(v,u) = (2,1)$, $(q,p)$. Then it is easy to and

$$h(\xi) = (\xi_0 + \psi_1 - \phi^2_1 \phi^1_2)$$

$$\times ((\xi_0 + \psi_p)(\xi_0 + \psi_q) - \phi^p_\psi \phi^p_q) \prod_{i \neq 1,2,p,q} (\xi_0 + \psi_i).$$

We first establish the following implication:

$$\phi^1_2 = 0 \Rightarrow \phi^2_2 = 0.$$
Assume that $\phi_1^2 = 0$. Take $\psi_1 = 0$, $i \neq 2$, $p$, $q$, $\psi_p = 1$, $\psi_q = \phi_p \phi_q$ and $\phi_p \neq 0$. If $\phi_q = 0$ then $(0, \xi')$ is a characteristic of order $m$ and hence a contradiction. If $\phi_q \neq 0$ then $(0, \xi')$ is a characteristic of order $m-1$. Take the 2-minor

$$L \begin{pmatrix} 1 & p \\ 2 & p \end{pmatrix} = \begin{vmatrix} \phi_2^1 & \phi_p^1 \\ 0 & 1 \end{vmatrix} = 0$$

which gives $\phi_2^1 = 0$. Take $\phi_p^q = \mu$. Since $\phi_1^2 = a \phi_1^2$ we have

$$\phi_p^q = \mu, \quad \phi_1^2 = a \mu, \quad \phi_2^1 = \alpha \mu$$

with some $\alpha$, $\beta$, $\alpha \in \mathbb{R}$. By hyperbolicity of $h(\xi)$ we have $\alpha \geq 0$, $\beta \geq 0$. Arguments similar to those proving Lemma 5.5 show the following:

**Lemma 5.8.** Assume that $a \neq 0$. Then we have

$$\alpha > 0, \quad \beta > 0.$$

Let us recall that

$$\lambda^\pm = -\frac{\psi_1}{2} \pm \sqrt{\frac{\psi_1^2 + 4 \alpha^2 \mu^2}{4}}.$$ 

Let $a \neq 0$. If $u \leq p$ or $u \geq q$ then a repetition of the arguments in the proof of Proposition 5.2 (second step) proves that

$$C_{uq}^{np} = 0.$$

When $p < u < v \leq q$ we take the 3-minor

$$L \begin{pmatrix} 2 & p & v \\ 2 & p & u \end{pmatrix} = \begin{vmatrix} \lambda^\pm & C_p^2 \mu & C_q^2 \mu \\ 0 & 1 & C_q^2 \mu \\ 0 & \phi_p^u & \phi_q^v \end{vmatrix} = 0$$

with $C_j^i = C_{jq}^{ip}$. This shows that $\phi_v^u = 0$. Recalling (5.8) we get

$$C_{uq}^{np} = 0 \text{ for } p < u < v \leq q.$$ 

For $v > q$ it is enough to take

$$L \begin{pmatrix} 2 & p & u \\ 2 & p & v \end{pmatrix} = \begin{vmatrix} \lambda^\pm & C_p^2 \mu & C_v^2 \mu \\ 0 & 1 & C_v^2 \mu \\ 0 & 0 & C_v^2 \mu \end{vmatrix} = 0$$

to conclude that $C_v^u = C_{uq}^{np} = 0$.

We also have

$$C_{uq}^{np} = 0 \text{ for } u < v$$
when \( a = 0 \) by the same arguments as in the proof of Proposition 5.2 (second case). Thus we have for \( u < v, \ q = p + r + 1 \) that
\[
C_{uv}^{wp} = 0 \quad \text{unless} \quad (u, v) = (1, 2), \ (p, q).
\]

Now the proof follows from induction on \( r \).

**LEMMA 5.9.** \( \phi_1^2 \) and \( \phi_2^1 \) are collinear, that is there is \( k > 0 \) such that
\[
\phi_2^1 = k\phi_1^2.
\]

**PROOF.** It is enough to show that \( \phi_2^1 = 0 \) implies \( \phi_2^1 = 0 \). Let
\[
\phi_1^2 = \sum_{(i, j) \in I_1} C_{ij}^l \phi_j^1.
\]
Since \( \phi_1^2 \neq 0 \) there is \((i_0, j_0) \in I_1 \) with \( C_{i_0}^{j_0} \neq 0 \). Hence we can take \( \phi_1^2 \) as an independent form so that \( \phi_1^2 \) is a linear combination of the other \( \phi_j^1 \)'s \( (i > j) \). After exchanging rows and the corresponding columns we may assume that \((i_0, j_0) = (2, 1)\). We denote by \((\tilde{\phi}_i^j)\) the resulting matrix. Note that this operation acts on the diagonal as a permutation and transforms a symmetric pair with respect to the diagonal to another symmetric pair. Repeating the same reasoning as in the proof of Proposition 5.7 we conclude that
\[
\tilde{\phi}_i^u = C_{i}^{uv} \tilde{\phi}_u^v \quad \text{for} \quad u < v, \ (u, v) \neq (1, 2).
\]
This proves that \( \phi_1^2 = 0 \Rightarrow \phi_2^1 = 0 \) and hence the assertion.

To simplify the notation we write \( \phi_i^1 = \phi_i^1 \) which are positive. Then from Proposition 5.7 and Lemma 5.9 it follows that
\[
\phi_p^q = C_q^{pq} \phi_p^q, \quad p < q.
\]

We now prove that
\[
C_p^q = C_q^1, \quad p < q.
\]

We first show the following lemma.

**LEMMA 5.10.** Let
\[
A(x) = \begin{pmatrix}
    x_1 & a \phi(x') & 0 & 0 \\
    \phi(x') & 0 & \beta x_4 & \delta x_6 \\
    0 & x_4 & x_2 & \gamma x_5 \\
    0 & x_6 & x_5 & x_3
\end{pmatrix}
\]
where \( \phi(x') \) is a linear function in \( x' = (x_4, x_5, x_6) \) and \( a, \beta, \gamma, \delta > 0 \). Assume that the eigenvalues of \( A(x) \) are all real. Then \( \beta \gamma = \delta \).
COROLLARY 5.11. Assume that
\[ A(x) = \begin{pmatrix}
  x_1 & \alpha(x') & \beta x_4 & \delta x_6 \\
  \phi(x') & 0 & 0 & 0 \\
  0 & x_4 & x_2 & \gamma x_5 \\
  x_6 & 0 & x_5 & x_3
\end{pmatrix} \]
has only real eigenvalues and \( \alpha, \beta, \gamma, \delta > 0 \). Then \( \beta \gamma = \delta \).

PROOF. Interchanging the first and second rows and the corresponding columns the proof is reduced to that of Lemma 5.10.

PROOF OF LEMMA 5.10. Set \( h(\lambda, x) = \det(\lambda I + A(x)) \). Then it is easy to see that, with \( x_2 = x_3 = 0 \),
\[ h(\lambda, x) = (\lambda + x_1) \left( \lambda^3 - (\beta x_4^2 + \gamma x_2^2 + \delta x_6^2) \lambda \\
+ (\beta \gamma + \delta)x_2 x_5 x_6 \right) - \alpha \phi(x')^2 (\lambda^2 - \gamma x_2^2). \]
Here we take \( x_4 = 1/\sqrt{\beta}, x_5 = 1/\sqrt{\gamma}, x_6 = 1/\sqrt{\delta} \) so that \( h(\lambda, x) \) turns out to be
\[ h(\lambda, x) = (\lambda + x_1) \left\{ \lambda^3 - 3\lambda + \sqrt{\beta \lambda / \delta} + \sqrt{\delta / \beta \gamma} \right\} \\
- \alpha \phi \left( 1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta} \right)^2 (\lambda^2 - 1). \]
We divide the cases into two.

Case \( \phi \left( 1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta} \right) = 0 \). The same arguments proving Lemma 4.15 show the assertion.

Case \( \phi \left( 1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta} \right) \neq 0 \). Let us set
\[ f(\lambda) = (\lambda + x_1)(\lambda^3 - 3\lambda + A), \quad g(\lambda) = C(\lambda^2 - 1) \]
where \( A = \sqrt{\beta \gamma / \delta} + \sqrt{\delta / \beta \gamma} \geq 2, C = \alpha \phi \left( 1/\sqrt{\beta}, 1/\sqrt{\gamma}, 1/\sqrt{\delta} \right)^2. \) Recall that \( A = 2 \) implies the assertion. Assume \( A > 2 \) and hence \( f(\lambda) = 0 \) has only two real roots. Let \( \lambda^\pm(x_1), \lambda^-(x_1) < \lambda^+(x_1) \) be the real roots of \( f''(\lambda) = 0 \) so that \( \lambda^+(x_1) \downarrow 0 \) as \( x_1 \to +\infty \). Since \( f'(1) = A - 2 > 0 \), taking \( x_1 \) so that \( \lambda^+(x_1) < 1 \), it follows that \( f(\lambda) \) is increasing in \( \lambda \geq 1 \). Thus
\[ f(\lambda) \geq f(1) = (1 + x_1)(A - 2), \quad 1 \leq \lambda \leq 2. \]
For \( \lambda \geq 2 \) we see that \( f'(\lambda) > x_1 \lambda > 2C \lambda = g'(\lambda) \) taking \( x_1 > 2C \) and hence \( f(\lambda)g(\lambda) \) is increasing in \( \lambda \geq 2 \). Noting that \( f(1) > g(2) \) for \( x_1 \) large we conclude that
\[ f(\lambda) - g(\lambda) > 0 \text{ for } \lambda \geq 1. \]
On the other hand the two real roots of \( f(A) = 0 \) are \(-x_1\) and \(k - 1\). Then it is clear that \( f(A) \) is increasing in the interval \((k, -1)\) and \( f(A) > 0 \) for \(\lambda > k\). With (5.10) we can easily conclude that \( f(A) - g(A) = 0 \) has only two real roots taking \(x_1\) large enough. This contradicts the assumption. \(\square\)

**Lemma 5.12.** Let

\[
A(x) = \begin{pmatrix}
x_1 & ax_4 & 0 & \delta x_6 \\
ax_4 & 0 & 0 & \gamma x_5 \\
0 & 0 & x_2 & \beta x_4 \\
x_6 & x_5 & x_4 & x_3
\end{pmatrix}
\]

where \(\alpha, \beta, \gamma, \delta > 0\). Assume that all eigenvalues of \(A(x)\) are real. Then we have \(\alpha \beta = \delta\).

**Proof.** We first exchange columns and the corresponding rows so that the resulting matrix is

\[
\begin{pmatrix}
x_1 & \alpha x_4 & \delta x_6 & 0 \\
ax_4 & 0 & \gamma x_5 & 0 \\
x_6 & x_5 & x_4 & x_3 \\
0 & 0 & \beta x_4 & x_2
\end{pmatrix}
\]

Taking \(x_1 = x_3 = 0, x_4 = 1/a\sqrt{\alpha}, x_5 = 1/\sqrt{\gamma}, x_6 = 1/\sqrt{\delta}\), the same reasoning as in the proof of Lemma 5.11 proves that \(\alpha \beta = \delta\). \(\square\)

**Lemma 5.13.** There is \(p > 2\) such that

\[
C_1^2 C_p^1 = C_p^1.
\]

**Proof.** Recall that \(\phi_1^2 \neq 0\) and hence there is \(p > 2\) such that \(\partial \phi_1^2 / \partial \phi_k^p \neq 0\) with some \(k < p\). When \(k = 1\) or \(2\) we take \(\xi'\) so that \(\phi_1^j = 0, i > j, (i, j) \neq (p, 1), (p, 2)\). Recall again that \(\phi_2^1 = C_2^1 \phi_1^1, \phi_2^2 = C_2^2 \phi_2^2, \phi_1^1 = C_1^1 \phi_1^p\) and \(\phi_2^1\) is linear in \(\phi_1^p, \phi_2^p\). Note that

\[
\begin{pmatrix}
\phi_1^1 & \phi_2^1 & \phi_1^1 \\
\phi_1^2 & \phi_2^2 & \phi_2^1 \\
\phi_1^p & \phi_2^p & \phi_2^p
\end{pmatrix}
\]

is diagonalizable with real eigenvalues by Lemma 5.6. We apply Lemma 5.4 to get

\[
C_2^1 C_p^1 = C_p^1.
\]

When \(\partial \phi_2^2 / \partial \phi_1^k = \partial \phi_2^2 / \partial \phi_2^k = 0\) and \(\partial \phi_2^2 / \partial \phi_2^p \neq 0\) with some \(2 < q < p\) we take \(\xi'\) so that

\[
\phi_1^j = 0, i > j, (i, j) \neq (p, q), (p, 1), (p, 2).
\]
Then it is clear that
\[
\begin{pmatrix}
\phi_1^1 & \phi_2^1 & 0 & \phi_p^1 \\
\phi_1^2 & \phi_2^2 & 0 & \phi_p^2 \\
0 & 0 & \phi_q^p & \phi_p^p \\
\phi_1^p & \phi_2^p & \phi_p^p & \phi_p^p
\end{pmatrix}
\]
has only real eigenvalues. Note that $\phi_1^p$ is linear in $\phi_p^p$. From Lemma 5.12 and (5.9) the assertion follows easily.

**Lemma 5.14.** We have

\[C_q^1 C_p^p = C_q^1, \quad 3 \leq p < q, \quad C_q^2 C_q^3 = C_q^2, \quad 3 < q.\]

**Proof.** Take $\xi'$ so that $\phi_j^0 = 0$, $(i, j) \neq (p, 1), (q, 1), (q, p)$. Then it is clear that

\[
\begin{pmatrix}
\phi_1^1 & \phi_2^1 & \phi_p^1 & \phi_q^p \\
\phi_1^2 & \phi_2^2 & 0 & \phi_p^p \\
\phi_1^p & 0 & \phi_p^p & \phi_p^p \\
\phi_1^q & 0 & \phi_q^p & \phi_p^p
\end{pmatrix}
\]

has only real eigenvalues. Noting (5.9) we apply Corollary 5.11 to get the first assertion. We turn to the second assertion. Take $\xi'$ so that $\phi_j^0 = 0$, $(i, j) \neq (3, 2), (q, 2), (q, 3)$. Then the eigenvalues of

\[
\begin{pmatrix}
\phi_1^1 & \phi_2^1 & 0 & 0 \\
\phi_1^2 & \phi_2^2 & \phi_3^2 & \phi_3^2 \\
0 & \phi_3^3 & \phi_3^3 & \phi_3^3 \\
0 & \phi_2^q & \phi_3^q & \phi_q^q
\end{pmatrix}
\]

are all real. Then the assertion follows from Lemma 5.10.

**Lemma 5.15.** Assume that

\[C_q^1 C_q^3 = C_q^1, \quad C_q^2 C_q^3 = C_q^2, \quad \text{for } q > 3\]

and

\[C_p^1 C_p^2 = C_p^1, \quad \text{for some } p > 2.\]

Then we have

\[C_q^1 C_q^3 = C_q^1, \quad q > 2.\]

**Proof.** Since $C_q^2 = (C_q^3)^{-1} C_p^2$, $C_q^1 = (C_p^2)^{-1} C_p^1$ it follows that $C_q^1 C_q^3 = (C_p^2)^{-1} C_p^1 = C_q^1$ and hence

\[C_q^2 C_q^3 = C_q^1 C_q^3 = C_q^1 C_q^2 = C_q^1.\]
From Lemmas 5.13, 5.14 and 5.15 it follows that:

**Proposition 5.16.** We have

\[ C_p^1 C_q^p = C_q^1 \text{ for } 1 < p < q. \]

**Theorem 5.17.** Assume that \( d(L) = m(m + 1)/2 - 1 \) and \( L(\xi) = (\phi_i(\xi)) \) is diagonalizable with real eigenvalues. Suppose that \( \phi_i(\xi) \) are independent of the diagonal forms and \( L \) verifies (b). Then there is a non singular constant matrix \( T \) such that

\[ T^{-1} L(\xi) T \]

is symmetric for every \( \xi \in \mathbb{R}^{n+1}. \)

**Proof.** The proof is a repetition of that of Theorem 4.17.

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**References**


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