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XI-PING ZHU

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Multiple Convex Hypersurfaces with Prescribed Mean Curvature

XI-PING ZHU(*)

1. - Introduction

In Yau's problem section (Problem 59 in [13]) it was asked when a function G defined in $\mathbb{R}^3$ is the mean curvature of a closed surface with prescribed genus. Let X be a smooth closed hypersurface embedded in $\mathbb{R}^{n+1}$ and oriented with respect to its inner normal. The mean curvature $\sigma_1(X)$ of X is the sum of its principal curvatures, i.e. $\sigma_1(X) = \kappa_1(X) + \cdots + \kappa_n(X)$. The problem is to find reasonable conditions on G such that the equation

$$\sigma_1(p) = G(p), \qquad p \in X \tag{1.1}$$

has a solution for a closed embedded hypersurface X with prescribed genus. It was proposed to minimize the functional

$$I(X) = \frac{1}{n} \int_X 1 - \int_{\tilde{X}} G \tag{1.2}$$

($\tilde{X}$ is the subset bounded by X) among all hypersurfaces X of the same genus. However, it is not clear how the minimum, if it ever exists, should have the same genus.

For the prescribed function G it is assumed that: (a) there exist R_1 and R_2, $0 < R_1 < R_2$, such that $G(x) > \frac{1}{R_1}$ on $|x| = R_1$ and $G(x) < \frac{1}{R_2}$ on $|x| = R_2$; (b) $\frac{\partial}{\partial \rho} \rho G(\rho x) \leq 0$, $x \in S^n$, $\rho > 0$. Then it was shown by Bakelman-Kantor [2], Treibergs-Wei [11] and Caffarelli-Nirenberg-Spruck [4] that there exists a unique starsharped hypersurface lying in $A = \{x \in R^{n+1} : 0 < R_1 < |x| < R_2\}$ whose mean curvature is equal to G. The monotonicity condition (b) is used not only in characterizing uniqueness but also in the proof of existence.

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More recently, the problem has been studied by K.S. Chou [6] via the negative gradient flow associated with I:

$$\frac{\partial X}{\partial t} = -(\sigma_1(X) - G(X))\nu \tag{1.3}$$
$$X(\cdot, 0) \text{ given}$$

where ν is the outer unit normal at $X(\cdot, t)$. Under the conditions that G is a concave function which becomes negative outside a large ball and that there exists a convex hypersurface Y with $I(Y) \leq 0$, he proved that (1.1) admits a convex hypersurface solution.

Here we are mainly interested in the question when equation (1.1) has multiple solutions. In [6], under the additional condition that G is invariant under a sufficiently large orthogonal subgroup Γ of $O(n+1)$, Chou has also obtained multiple solutions for (1.1). He looked for the multiple critical points of I by a mountain pass lemma. Since the functional I is not continuous in the appropriate space, his argument is directly on the flow (1.3) within the space of all Γ-symmetric convex hypersurfaces which prevent the sought-after hypersurfaces of (1.3) from deviating too much from the round spheres.

The purpose of the present paper is to relax this additional symmetry condition. Our approach is similar to that of Chou [5], [7], Urbas [12], and it involves studying the evolution equation satisfied by the support function of the hypersurfaces $X(\cdot, t)$, rather than working directly with (1.3). The main ingredients of this paper are to provide a lower positive bound for the principal curvatures of the flow $X(\cdot, t)$ and the global existence for the solution of (1.3) except when $X(\cdot, t)$ shrinks to a point. Our main result is the following:

THEOREM. *Let G be a smooth function which satisfies:*

(H_1) $\Omega = \{x : G(x) > 0\}$ *is bounded and simply-connected;*

(H_2) *G is uniformly concave on Ω, i.e. there exists a positive constant c_0 such that*

$$-\sum_{i,j=1}^{n+1} \frac{\partial^2 G}{\partial x_i \partial x_j} \xi^i \xi^j \geq c_0 |\xi|^2, \qquad \textit{for } \xi \in R^{n+1}.$$

Then (1.1) *admits two convex solutions if there exists a convex hypersurface X lying inside Ω and satisfying $I(X) \leq 0$.*

This paper consists of four sections. In Section 2 we shall derive the equation for the support function of the hypersurfaces $X(\cdot, t)$. In Section 3 we derive the a priori estimates on the first derivative in t and second derivatives in x of the solutions. The main theorem will be proved in the final Section.

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2. - Preliminary results

Let X be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$. We may assume that X is parametrized by the inverse Gauss map

$$X : S^n \to \mathbb{R}^{n+1}.$$

The support function H of X is defined by

$$H(x) = \sup\{\langle x, p\rangle : p \in X\}, \qquad x \in R^{n+1}\backslash\{0\}.$$

H is differentiable and X can be recovered from H by

$$p^i(x) = \frac{\partial H(x)}{\partial x_i}, \ \ x \in S^n, \ \ i = 1, \cdots, n+1$$

where $p = (p^1, \cdots, p^{n+1})$ is the point on X with outer unit normal x. Thus all geometric quantities can be described in terms of H.

Let $e_1, \cdots, e_n$ be a smooth local orthonormal frame field on S^n. From the computations in [12], we know that the second fundamental form of X is:

$$b_{ij} = \nabla_{ij} H + \delta_{ij} H, \tag{2.1}$$

and that the metric of X is:

$$g_{ij} = \sum_{k=1}^{n} b_{ik} b_{jk}. \tag{2.2}$$

The principal radii of curvature are the eigenvalues of matrix $\sum_{k=1}^{n} b^{ik} g_{jk}$, (here b^{ik} is the inverse of b_{ik}), which, by virtue of (2.1) and (2.2), is given by

$$\sum_{k=1}^{n} b^{ik} g_{jk} = b_{ij} = \nabla_{ij} H + \delta_{ij} H. \tag{2.3}$$

Now, suppose that (1.3) has a solution $X(\cdot, t)$ which is uniformly convex for each t. Let $H(\cdot, t)$ be the support function of $X(\cdot, t)$, and let $\nu_t : S^n \to S^n$ be the Gauss map of $X(\cdot, t)$. We define a new parametrization $\bar{X}(\cdot, t)$ by

$$\bar{X}(x, t) = X(\nu_t^{-1}(x), t).$$

Then (1.3) becomes

$$\begin{aligned}\frac{\partial \bar{X}}{\partial t} &= \left\langle \frac{\partial X}{\partial x}, \frac{\partial \nu_t^{-1}}{\partial t} \right\rangle + \frac{\partial X}{\partial t} \\ &= \left\langle \frac{\partial X}{\partial x}, \frac{\partial \nu_t^{-1}}{\partial t} \right\rangle - (\sigma_1 - G)(\nu_t^{-1}(x), t) x.\end{aligned}$$

Thus

$$\frac{\partial H}{\partial t} = \left\langle \frac{\partial \bar{X}}{\partial t}(x,t), x \right\rangle = -(\sigma_1 - G). \tag{2.4}$$

Therefore, the support function satisfies the initial value problem

$$\begin{aligned} \frac{\partial H}{\partial t} &= -\frac{1}{f(R_1,\cdots,R_n)} + G(DH) \\ &= -\frac{1}{F(\nabla^2 H + HI)} + G(DH), \qquad \text{on } S^n \times [0,\infty) \\ H(\cdot,0) &= H_0, \end{aligned} \tag{2.5}$$

where H_0 is the support function of $X(\cdot,0)$, $D = (D_1,\cdots,D_{n+1})$ is the gradient on $\mathbb{R}^{n+1}$, $R_1,\cdots,R_n$ are the principal radii of curvature of $X(\cdot,t)$,

$$f(R_1,\cdots,R_n) = \frac{R_1\cdots R_n}{\sum\limits_{i_1<\cdots<i_{n-1}} R_{i_1}\cdots R_{i_{n-1}}},$$

and $F(a_{ij}) = f(\mu_1,\cdots,\mu_n)$; here $\mu_1,\cdots,\mu_n$ are the eigenvalues of $[a_{ij}]$.

Conversely, it is not hard to show that (2.5) together with the condition

$$\nabla^2 H + HI > 0 \quad \text{for } (x,t) \in S^n \times [0,\infty]$$

implies (1.3). (This fact for similar equations is established in [5], [12].)

Next, let us make some remarks about the function

$$f(\mu_1,\cdots,\mu_n) = \frac{\mu_1\cdots\mu_n}{\sum\limits_{i_1<\cdots<i_{n-1}} \mu_{i_1}\cdots\mu_{i_{n-1}}}, \qquad \mu_i > 0,\ i = 1,\cdots,n.$$

It is easy to prove that the function is homogeneous of degree 1 and that $\frac{\partial f}{\partial \mu_i} > 0$ for $i = 1,\cdots,n$. From the inequalities of Marcus and Lopes [10] (or see Section 33 in [1]), we know that f is concave on $\{(\mu_1,\cdots,\mu_n) \in \mathbb{R}^n | \mu_i > 0$ for all $i\}$.

We also see that F is homogeneous of degree 1 on the cone of real symmetric positive $n \times n$ matrices. In [3] it is proved that the eigenvalues of $[F_{ij}] = \left[\frac{\partial F}{\partial a_{ij}}\right]$ are $\frac{\partial f}{\partial \mu_1},\cdots,\frac{\partial f}{\partial \mu_n}$, and therefore

$$[F_{ij}] > 0 \tag{2.6}$$

for all symmetric positive $n \times n$ matrices, which shows that equation (2.5) is parabolic. It is also proved in [3] that the concavity of f implies the concavity of F, i.e.

$$\sum F_{ij,kl}\,\eta_{ij}\,\eta_{kl} \le 0 \tag{2.7}$$

for all real symmetric $n \times n$ matrices $[\eta_{ij}]$, where $F_{ij,kl} = \dfrac{\partial^2 F}{\partial a_{kl} \partial a_{ij}}$.

We shall need the following inequality which follows directly from the monotonicity and the concavity of f (see Lemma 3.2 in [12])

$$\mathcal{T} \equiv \sum_{i=1}^{n} F_{ii} \geq \frac{1}{n} \tag{2.8}$$

for all symmetric positive $n \times n$ matrices.

In this paper, we always assume that G is a smooth function which satisfies:

(H_1) $\Omega = \{x : G(x) > 0\}$ is bounded and simply-connected;

(H_2) G is uniformly concave on Ω, i.e. there exists a c_0 such that

$$-\sum_{i,j}^{n+1} \frac{\partial^2 G}{\partial x_i \partial x_j} \xi^i \xi^j \geq c_0 |\xi|^2, \quad \text{for all } \xi \in R^{n+1}, \ x \in \Omega.$$

3. - A priori estimates on H

We begin with an upper bound estimate for H. Since G is smooth and uniformly concave, we can find a small positive ε_0 such that the principal curvatures of the boundary of the set $\tilde{\Omega} = \{x : G(x) > \varepsilon_0\}$ are greater than ε_0.

LEMMA 3.1. *Suppose $X(\cdot, t)$ is a solution of* (1.3) *on* $S^n \times [0, T)$ *with $X(\cdot, 0)$ lying in $\tilde{\Omega}$. Then $X(\cdot, t)$ remains in $\tilde{\Omega}$ for all $t \in [0, T)$.*

PROOF. Assume by contradiction that X touches $\partial\tilde{\Omega}$ from inside at a first time at a point. Without loss of generality, we may assume that $X(\cdot, t)$ encloses the origin. Then, for the normal ν at this point, $\left\langle \dfrac{\partial X}{\partial t}, \nu \right\rangle$ is non-negative. However, on the other hand, we have

$$\left\langle \frac{\partial X}{\partial t}, \nu \right\rangle = -(\sigma_1(X) - G(X)) < -n\varepsilon_0 + \varepsilon_0 < 0$$

at this point. A contradiction holds. ∎

Now, we prove a lower positive bound for the principal curvatures of the flow $X(\cdot, t)$. From (2.3), we know that the principal radii of curvature of $X(\cdot, t)$ are the eigenvalues of the matrix $\nabla^2 H + HI$. We only need to derive an upper bound for the eigenvalues of $\nabla^2 H + HI$.

LEMMA 3.2. *Suppose H is a solution of* (2.5) *on $S^n \times [0,T]$ with $X(\cdot,0)$ contained inside Ω. If at $t=0$ we have*

$$\nabla^2 H + HI \le KI$$

for a positive constant K then there exists a constant C such that

$$\nabla^2 H + HI \le CI, \qquad \text{on } S^n \times [0,T),$$

here C depends only on n, c_0, K, and the C^1*-norm of G on Ω.*

PROOF. Suppose that the maximum eigenvalue of the matrix $[b_{ij}] = \nabla^2 H(x,t) + H(x,t)I$ on $S^n \times [0,T]$ is attained at $(x_0,t_0) \in S^n \times [0,T]$ with unit eigenvector $\xi \in T_{x_0}S^n$. Without loss of generality, we may assume $t_0 > 0$ and that x_0 is the south pole. By a suitable choice of coordinates we may assume that all directions of the orthonormal frame $e_1, \cdots, e_n$ are in the principal directions and that ξ is in the direction e_1 at the point (x_0,t_0).

At the point (x_0,t_0), we have

$$\frac{\partial b_{11}}{\partial t} \ge 0, \tag{3.1}$$

$$\nabla_{ij} b_{11} \le 0. \tag{3.2}$$

Let us differentiate equation (2.5) to get

$$\frac{\partial}{\partial t}\nabla_l H = \sum_{ij} F_{ij}F^{-2}\nabla_l(\nabla_{ij}H + \delta_{ij}H) + \nabla_l G, \tag{3.3}$$

$$\begin{aligned} \frac{\partial}{\partial t}\nabla_{kl} H &= \sum_{ij} F_{ij}F^{-2}\nabla_{kl}(\nabla_{ij}H + \delta_{ij}H) \\ &\quad + \sum_{ij,rs} F_{ij,rs}F^{-2}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\nabla_k(\nabla_{rs}H + \delta_{rs}H) \\ &\quad - 2F^{-3}\sum_{ij,rs} F_{ij}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\cdot F_{rs}\nabla_k(\nabla_{rs}H + \delta_{rs}H) + \nabla_{kl}G. \end{aligned} \tag{3.4}$$

Let R^j_{ikl} be the Riemann curvature tensor of S^n. By the Gauss equations, we have

$$R^j_{ikl} = R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Then we interchange the order of covariant differentiation to get

$$\begin{aligned} \nabla_{klij}H &= \nabla_{ijkl}H + (\nabla_k R^m_{jli} + \nabla_i R^m_{lkj})\nabla_m H + R^m_{lkj}\nabla_{im}H \\ &\quad + R^m_{lki}\nabla_{jm}H + R^m_{jli}\nabla_{km}H + R^m_{jki}\nabla_{lm}H \\ &= \nabla_{ijkl}H + 2\delta_{kl}\nabla_{ij}H - 2\delta_{ij}\nabla_{kl}H + \delta_{jk}\nabla_{il}H - \delta_{il}\nabla_{jk}H. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla_{kl}H &= F^{-2}\sum_{ij}(F_{ij}\nabla_{ijkl}H + 2\delta_{kl}F_{ij}\nabla_{ij}H) \\
&\quad - \mathcal{T}F^{-2}\nabla_{kl}H + F^{-2}\sum_{i}(F_{ik}\nabla_{il}H - F_{il}\nabla_{ik}H) \\
&\quad + \sum_{ij,rs}F^{-2}F_{ij,rs}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\nabla_k(\nabla_{rs}H + \delta_{rs}H) \\
&\quad - 2F^{-3}\sum_{ij,rs}F_{ij}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\cdot F_{rs}\nabla_k(\nabla_{rs}H + \delta_{rs}H) + \nabla_{kl}G.
\end{aligned}
\tag{3.6}
$$

Using the degree 1 homogeneity of F,

$$
\delta_{kl}\frac{\partial H}{\partial t} = -F^{-2}\delta_{kl}\sum_{ij}F_{ij}\nabla_{ij}H - F^{-2}\delta_{kl}\mathcal{T}H + G\delta_{kl}. \tag{3.7}
$$

Thus, adding (3.6) and (3.7), we see that $b_{kl} = \nabla_{kl}H + \delta_{kl}H$ satisfies the equation

$$
\begin{aligned}
\frac{\partial}{\partial t}b_{kl} &= F^{-2}\sum_{ij}F_{ij}\nabla_{ij}b_{kl} - \mathcal{T}F^{-2}b_{kl} + F^{-2}\sum_{i}(F_{ik}\nabla_{il}H - F_{il}\nabla_{ik}H) \\
&\quad + \sum_{ij,rs}F^{-2}F_{ij,rs}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\nabla_k(\nabla_{rs}H + \delta_{rs}H) \\
&\quad - 2F^{-3}\sum_{ij,rs}F_{ij}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\cdot F_{rs}\nabla_k(\nabla_{rs}H + \delta_{rs}H) + \nabla_{kl}G + \delta_{kl}G.
\end{aligned}
\tag{3.8}
$$

Now, letting $k = l = 1$, by the concavity of F, we obtain

$$
\frac{\partial}{\partial t}b_{11} \le F^{-2}\sum_{ij}F_{ij}\nabla_{ij}b_{11} - \mathcal{T}F^{-2}b_{11} + \nabla_{11}G + G. \tag{3.9}
$$

Using (3.1) and (3.2), we further have

$$
0 \le -\mathcal{T}F^{-2}b_{11} + \nabla_{11}G + G, \tag{3.10}
$$

at (x_0, t_0).

We use the Gauss-Weingarten relations

$$
\frac{\partial^2 X}{\partial x_i \partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial X}{\partial x_k} - b_{ij}\nu
$$

to conclude that:

$$\begin{aligned}\nabla_{11}G &= \sum_{\alpha,\beta=1}^{n+1} \frac{\partial^2 G}{\partial X_\alpha \partial X_\beta} \frac{\partial X_\alpha}{\partial x_1} \frac{\partial X_\beta}{\partial x_1} - b_{11} \sum_{\alpha=1}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \\ &\leq -c_0 \left\langle \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_1} \right\rangle - b_{11} \sum_{\alpha=1}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \\ &= -c_0 g_{11} - b_{11} \sum_{\alpha=1}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha .\end{aligned} \tag{3.11}$$

From (2.2), we know that:

$$g_{11} = \sum_{k=1}^{n} b_{1k}^2 \geq b_{11}^2. \tag{3.12}$$

By putting (3.11) and (3.12) into (3.10), we see that

$$0 \leq -\mathcal{T} F^{-1} b_{11} - c_0 b_{11}^2 - b_{11} \sum_{\alpha=1}^{n+1} \frac{\partial G}{\partial x_\alpha} \nu_\alpha + G. \tag{3.13}$$

Clearly this implies an upper bound for b_{11}, the conclusion of the lemma. ∎

Next, we derive a bound for H in the $\tilde{C}^2$ norm, which involves the first derivative in t and second derivatives in x.

LEMMA 3.3. *Suppose H is a solution of (2.5) on $S^n \times [0,T]$, $X(\cdot,0)$ being contained inside $\tilde{\Omega}$. Suppose further that $H > r$ for some positive r. Then there exist constants C and C' such that*

$$\|H\|_{\tilde{C}^2(S^n\times[0,T])} \leq C$$

and

$$\nabla^2 H + HI \geq C'I \quad \text{on } S^n \times [0,T].$$

Here C and C' depend on n, r, diam($\tilde{\Omega}$), c_0, C^1-norm of G in $\tilde{\Omega}$ and the initial data. Furthermore, for each $k \geq 2$ and t_0 with $0 < t_0 < T$, there exists C_k which also depends on higher derivatives of G and t_0^{-1} such that

$$\|H\|_{\tilde{C}^k(S^n\times[t_0,T])} \leq C_k.$$

Recall that $\|H\|_{\tilde{C}^k(S^n\times[t_0,T])} = \max_{S^n\times[t_0,T]} \sum_{2l+m\leq k} |\nabla^m D_t^l H|$.

PROOF. From Lemma 3.1, it is clear that H and DH are bounded by diam($\tilde{\Omega}$). Now we adapt the method of Chou [7] to our lemma.

Step 1. $\dfrac{\partial H}{\partial t} \geq -C$.

Consider the function $\dfrac{\partial H}{\partial t}\left(H - \dfrac{r}{2}\right)^{-1}$. Suppose that it has a negative minimum which is attained at $(x_1, t_1) \in S^n \times [0, T]$. Without loss of generality we may assume that $t_1 > 0$ and that x_1 is the south pole. The mapping

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, -(1 - |x^2|)^{\frac{1}{2}})$$

maps $\{x \in R^n : |x| < 1\}$ onto S^n_-, and gives a coordinate system for S^n_-.

Let

$$\eta(x,t) = \frac{\dfrac{\partial H}{\partial t}(x, -(1-|x|^2)^{\frac{1}{2}}, t)}{H(x, -(1-|x|^2)^{\frac{1}{2}}, t) - \dfrac{r}{2}},$$

for $x \in R^n$ with $|x| < 1$. At (x_1, t_1), we have, for i, $j = 1, \cdots, n$.

$$0 = \frac{\partial \eta}{\partial x_i} = \frac{\nabla_i \dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}} - \frac{\dfrac{\partial H}{\partial t}\nabla_i H}{\left(H - \dfrac{r}{2}\right)^2};$$

$$\begin{aligned}
0 \leq \frac{\partial^2 \eta}{\partial x_i \partial x_j} &= \frac{1}{H - \dfrac{r}{2}}\left[\nabla_{ij}\frac{\partial H}{\partial t}\right.\\
&\quad - \frac{\left(\nabla_i \dfrac{\partial H}{\partial t}\nabla_j H + \nabla_j \dfrac{\partial H}{\partial t}\nabla_i H\right)}{H - \dfrac{r}{2}}\\
&\quad \left. + \frac{2\dfrac{\partial H}{\partial t}\nabla_i H \nabla_j H}{\left(H - \dfrac{r}{2}\right)^2} - \frac{\dfrac{\partial H}{\partial t}\nabla_{ij} H}{\left(H - \dfrac{r}{2}\right)}\right]\\
&\quad + \frac{\left(\dfrac{\partial^2 H}{\partial x_{n+1}\partial t}\left(H - \dfrac{r}{2}\right) - \dfrac{\partial H}{\partial t}\dfrac{\partial H}{\partial x_{n+1}}\right)}{\left(H - \dfrac{r}{2}\right)^2}\delta_{ij}\\
&= \frac{1}{H - \dfrac{r}{2}}\left[\nabla_{ij}\frac{\partial H}{\partial t} - \frac{\dfrac{\partial H}{\partial t}\nabla_{ij} H}{\left(H - \dfrac{r}{2}\right)} + \frac{\dfrac{r}{2}\dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}}\delta_{ij}\right],
\end{aligned}$$

(in the last passage we have used the homogeneity of H);

$$0 \geq \frac{\partial \eta}{\partial t} = \frac{\dfrac{\partial^2 H}{\partial t^2}}{H - \dfrac{r}{2}} - \frac{\left(\dfrac{\partial H}{\partial t}\right)^2}{\left(H - \dfrac{r}{2}\right)^2}.$$

We differentiate equation (2.5) to get

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} &= \frac{1}{F^2} \sum_{i,j}^{n} F_{ij} \left(\nabla_{ij} \frac{\partial H}{\partial t} + \delta_{ij} \frac{\partial H}{\partial t} \right) + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial t} \\ &= \frac{1}{F^2} \sum_{i,j}^{n} F_{ij} \left(\nabla_{ij} \frac{\partial H}{\partial t} + \delta_{ij} \frac{\partial H}{\partial t} \right) + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \frac{\partial H}{\partial t}. \end{aligned} \tag{3.14}$$

Then, at (x_1, t_1),

$$\begin{aligned} \frac{\left(\dfrac{\partial H}{\partial t}\right)^2}{H - \dfrac{r}{2}} &\geq \frac{\partial^2 H}{\partial t^2} \\ &\geq \frac{1}{F^2} \sum_{i,j}^{n} F_{ij} \left(\frac{\dfrac{\partial H}{\partial t} \nabla_{ij} H}{H - \dfrac{r}{2}} - \frac{\dfrac{r}{2} \dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}} \delta_{ij} + \delta_{ij} \frac{\partial H}{\partial t} \right) \\ &\quad + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \frac{\partial H}{\partial t} \\ &= \frac{1}{F^2} \sum_{i,j}^{n} F_{ij} \left(\frac{\dfrac{\partial H}{\partial t} (\nabla_{ij} H + \delta_{ij} H)}{H - \dfrac{r}{2}} - \frac{\dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}} \delta_{ij} r \right) \\ &\quad + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \frac{\partial H}{\partial t} \\ &= \frac{1}{F} \frac{\dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}} - \mathcal{T} \frac{r}{F^2} \frac{\dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}} + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \frac{\partial H}{\partial t}; \end{aligned} \tag{3.15}$$

here we have used monotonicity and homogeneity of F.

Applying (2.8) and (2.5), (3.15) becomes

$$\frac{\left(\dfrac{\partial H}{\partial t}\right)^2}{H - \dfrac{r}{2}} \geq \frac{\dfrac{\partial H}{\partial t}}{H - \dfrac{r}{2}} \left(G - \frac{\partial H}{\partial t} \right) - \frac{r \dfrac{\partial H}{\partial t}}{n \left(H - \dfrac{r}{2} \right)} \left(G - \frac{\partial H}{\partial t} \right)^2 + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \frac{\partial H}{\partial t}.$$

Clearly this implies a lower bound for $\dfrac{\partial H}{\partial t}$.

Step 2. $\dfrac{\partial H}{\partial t} \le C$.

Consider the function $H^{-1}\dfrac{\partial H}{\partial t}$. Suppose it has a positive maximum at (x_2, t_2). We may assume that $t_2 > 0$, that x_2 is the south pole and that the local coordinate system is the same as in Step 1. Let

$$\varsigma(x,t) = \frac{\dfrac{\partial H}{\partial t}(x, -(1-|x|^2)^{\frac{1}{2}}, t)}{H(x, -(1-|x|^2)^{\frac{1}{2}}, t)},$$

for $x \in R^n$ with $|x| < 1$. At (x_2, t_2) we have, for i, $j = 1, \cdots, n$,

$$0 = \frac{\partial \varsigma}{\partial x_i} = \frac{\nabla_i \dfrac{\partial H}{\partial t}}{H} - \frac{\dfrac{\partial H}{\partial t}\nabla_i H}{H^2},$$

$$0 \ge \frac{\partial^2 \varsigma}{\partial x_i \partial x_j} = \frac{1}{H}\left[\nabla_{ij}\frac{\partial H}{\partial t} - \frac{\dfrac{\partial H}{\partial t}\nabla_{ij}H}{H}\right],$$

$$0 \le \frac{\partial \varsigma}{\partial t} = \frac{\dfrac{\partial^2 H}{\partial t^2}}{H} - \frac{\left(\dfrac{\partial H}{\partial t}\right)^2}{H^2}.$$

Putting in (3.14), we obtain

$$\begin{aligned}
\frac{\left(\dfrac{\partial H}{\partial t}\right)^2}{H} &\le \frac{\partial^2 H}{\partial t^2} \\
&\le \frac{1}{F^2}\sum_{i,j}^{n} F_{ij}\left(\frac{\dfrac{\partial H}{\partial t}\nabla_{ij}H}{H} + \delta_{ij}\frac{\partial H}{\partial t}\right) + \sum_{\alpha}^{n+1}\frac{\partial G}{\partial X_\alpha}\nu_\alpha\frac{\partial H}{\partial t} \\
&= \frac{1}{F^2}\sum_{i,j}^{n} F_{ij}\left(\frac{\dfrac{\partial H}{\partial t}(\nabla_{ij}H + \delta_{ij}H)}{H}\right) + \sum_{\alpha}^{n+1}\frac{\partial G}{\partial X_\alpha}\nu_\alpha\frac{\partial H}{\partial t} \\
&= \frac{1}{FH}\frac{\partial H}{\partial t} + \sum_{\alpha}^{n+1}\frac{\partial G}{\partial X_\alpha}\nu_\alpha\frac{\partial H}{\partial t}
\end{aligned} \tag{3.16}$$

at (x_2, t_2).

Using equation (2.5), (3.16) becomes

$$\frac{\left(\frac{\partial H}{\partial t}\right)^2}{H} \leq \frac{\frac{\partial H}{\partial t}}{H}\left(G - \frac{\partial H}{\partial t}\right) + \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha \frac{\partial H}{\partial t}$$

i.e.

$$2\left(\frac{\partial H}{\partial t}\right)^2 \leq \frac{\partial H}{\partial t}\left(G + H \sum_{\alpha}^{n+1} \frac{\partial G}{\partial X_\alpha} \nu_\alpha\right)$$

which implies an upper bound for $\frac{\partial H}{\partial t}$.

Step 3. Completion of the proof of Lemma 3.3.

Combining Lemma 3.2, Step 1 and Step 2, we obtain immediately from equation (2.5) that

$$\nabla^2 H + HI \geq C' I.$$

As a result, equation (2.5) is uniformly parabolic by virtue of (2.6). Thus Hölder continuity estimates for $\nabla^2 H$ and $\frac{\partial H}{\partial t}$ now follow from the results of Krylov and Safonov [8], [9], and once we have these, estimates for higher derivatives follow from the standard theory of linear uniformly parabolic equations. ∎

4. - The existence theorems

Let us consider the functional I given by (1.2):

$$I(X) = \frac{1}{n} \int_X 1 - \int_{\tilde{X}} G$$

for a closed (connected) hypersurface X, where $\tilde{X}$ is the bounded component of $\mathbb{R}^{n+1} \backslash X$. For a smooth variation vector field on X, it is known that I has the following first variation formulas

$$\delta I(X)\xi = \int_X (\sigma_1 - G)\langle \xi, \nu \rangle. \tag{4.1}$$

Therefore, its negative gradient flow is given by (1.3):

$$\frac{\partial X}{\partial t} = -(\sigma_1(X) - G(X))\nu$$
$$X(\cdot, 0) = X_0 \text{ given.}$$

Along the gradient flow which solves (1.3), I satisfies

$$\frac{d}{dt} I(X(\cdot, t)) = \delta I(X(\cdot, t)) \frac{dX}{dt} = -\int_{X(\cdot,t)} (\sigma_1 - G)^2 \leq 0, \tag{4.2}$$

and

$$I(X(\cdot, 0)) - I(X(\cdot, T)) = \int_0^T \int_{X(\cdot,t)} (\sigma_1 - G)^2. \tag{4.3}$$

THEOREM 4.1. *Let G be a smooth function which satisfies* (H_1) *and* (H_2). *Suppose that X_0 is a convex hypersurface lying inside Ω. Then there exists a solution $X(\cdot, t)$ of* (1.3) *on a maximal interval* $[0, T^*)$ *with $T^* > 0$ and satisfy the following:*

Each $X(\cdot, t)$ is a convex hypersurface lying in Ω, and either $X(\cdot, t)$ shrinks to a point as $t \to T^*$ or there exists a sequence $t_j \to T^* = \infty$ such that $\{X(\cdot, t_j)\}$ converges smoothly to a solution of (1.1).

PROOF. Without loss of generality, we may assume that X_0 lies inside $\tilde{\Omega}$. Let H_0 be the support function of X_0. A standard argument using the implicit function theorem yields the existence of a unique smooth solution of (2.5) on $S^n \times [0, T)$ for some small positive T. Consequently we have the local existence of (1.3). Applying Lemma 3.1 and Lemma 3.2, we know that the solution $X(\cdot, t)$ preserves convexity and lies in Ω.

From Lemma 3.2, the principal curvatures of $X(\cdot, t)$ have a lower positive bound for all $t \in [0, T^*)$. We claim that $D(X(\cdot, t))$ (the diameter of $X(\cdot, t)$) and $r_{in}(X(\cdot, t))$ (the inradius of $X(\cdot, t)$) tend to zero simultaneously if this ever happens. For, if $\tilde{X}(\cdot, t)$ collapses into a degenerate convex body but not a point as time evolves, there would be some point on X with arbitrarly small principal curvature along some direction, which contradicts Lemma 3.2. Therefore, we may assume that the inradius $r_{in}(X(\cdot, t))$ of $X(\cdot, t)$ has a uniform positive lower bound for all $t \in [0, T^*)$.

Let $H(\cdot, t)$ be the support function of $X(\cdot, t)$ which solves (2.5). We define the support center of $X(\cdot, t)$ to be

$$z(t) = \int_{S^n} H(x, t)x. \tag{4.4}$$

Then, it is easy to see that there exists $\delta > 0$, such that $B(z(t), \delta)$ is contained inside $X(\cdot, t)$ for all $t \in [0, T^*)$.

From (4.2), we know that $I(X(\cdot, t))$ has a lower bound for $t \in [0, T^*)$ and that it is a nonincreasing function on $[0, T^*)$. So we can choose $t_0 \in (0, T^*)$ such that

$$I(X(\cdot, t_0)) - \lim_{t \to T^*} I(X(\cdot, t)) = \int_{t_0}^{T^*} \int_{X(\cdot, t)} (\sigma_1 - G)^2 < \frac{1}{\tau_n} \left(\frac{\delta}{4}\right)^2 \tag{4.5}$$

where τ_n is the area of S^n.

For t and t' in $[t_0, T^*)$ with $|t - t'| \leq 1$, we have

$$\begin{aligned} |z(t) - z(t')| &\leq \left| \int_{t'}^{t} \int_{S^n} \left| \frac{\partial H}{\partial t} \right| \right| \\ &\leq \left| \int_{t'}^{t} \int_{\{|\frac{\partial H}{\partial t}| \leq \frac{\delta}{4\tau_n}\}} \left| \frac{\partial H}{\partial t} \right| \right| + \int_{t_0}^{T^*} \int_{\{|\frac{\partial H}{\partial t}| > \frac{\delta}{4\tau_n}\}} \left| \frac{\partial H}{\partial t} \right|. \end{aligned}$$

From (4.5), it follows that

$$\frac{1}{\tau_n} \left(\frac{\delta}{4}\right)^2 > \int_{t_0}^{T^*} \int_{\{|\frac{\partial H}{\partial t}| > \frac{\delta}{4\tau_n}\}} \left| \frac{\partial H}{\partial t} \right|^2 \geq \left(\frac{\delta}{4\tau_n}\right) \int_{t_0}^{T^*} \int_{\{|\frac{\partial H}{\partial t}| > \frac{\delta}{4\tau_n}\}} \left| \frac{\partial H}{\partial t} \right|.$$

Thus, we have

$$|z(t) - z(t')| < \left| \int_{t'}^{t} \int_{\{|\frac{\partial H}{\partial t}| \leq \frac{\delta}{4\tau_n}\}} \left| \frac{\partial H}{\partial t} \right| \right| + \frac{\delta}{4} \leq \frac{\delta}{2}.$$

In other words, the ball $B\left(z(t), \dfrac{\delta}{2}\right)$ is contained in $X(\cdot, t')$ for all t', $t' \geq t_0$ with $|t' - t| \leq 1$. Applying Lemma 3.3, by a standard argument, we can obtain uniform estimates of H of all orders in $[t_0, T^*)$. Then, if T^* is finite, one can extend H beyond T^*, a contradiction to the maximality of T^*. Hence T^* must be infinity. In view of the uniform estimates on H of all orders in $[t_0, +\infty)$, and of (4.5), we can choose a sequence $\{X(\cdot, t_j)\}$, $(t_j \to +\infty)$, which converges smoothly to a solution of (1.1). ∎

THEOREM 4.2. *Suppose that G is a smooth function which satisfies* (H_1) *and* (H_2). *Then there exist at least two convex hypersurfaces solving* (1.1) *if there is a convex hypersurface X lying inside Ω and satisfying $I(X) \leq 0$.*

PROOF. By (4.2) and Theorem 4.1, it is clear that there exists a convex solution X_1 of (1.1) with $I(X_1) \leq 0$. Now we want to prove that there exists another solution X_2 with $I(X_2) > 0$.

Without loss of generality, we may assume $X_1 \subset \tilde{\Omega}$ and that X_1 encloses the origin O of $\mathbb{R}^{n+1}$. We observe that, for any convex hypersurface X,

$$I(X) = \frac{1}{n} \int_X 1 - \int_{\tilde{X}} G \geq \frac{1}{n} |X| (\max_{\Omega} G) \text{ vol } \tilde{X}$$

where we denote the area of X by $|X|$. Then, by the isoperimetric inequality, we have

$$I(X) \geq \frac{1}{n} |X| - c_n (\max_{\Omega} G) |X|^{\frac{n+1}{n}} \tag{4.6}$$

for some positive constant c_n depending on n only.

For fixed $\alpha > 0$ small enough, by (4.6), we have

$$I(X) \geq \alpha \left(\frac{1}{n} - c_n (\max_{\Omega} G) \alpha^{\frac{1}{n}} \right) > 0 \tag{4.7}$$

for all convex X with $|X| = \alpha$. It is clear that we may choose $\rho > 0$ small enough such that the area of the sphere $S_\rho = \{x \in R^{n+1} : |x| = \rho\}$ is less than α and

$$I(S_\rho) < \frac{\alpha}{2} \left(\frac{1}{n} - c_n (\max_{\Omega} G) \alpha^{\frac{1}{n}} \right).$$

Let Γ be the class of all continuous γ is from $[0, 1]$ to the topological space of all convex hypersurfaces (endowed with the Hausdorff distance) such that $\gamma(0) = S_\rho$ and $\gamma(1) = X_1$. Set

$$c = \inf_{\gamma \in \Gamma} \max_{s} I(\gamma(s)) \geq \alpha \left(\frac{1}{n} - c_n (\max_{\Omega} G) \alpha^{\frac{1}{n}} \right) > 0$$

We pick a $\gamma \in \Gamma$ and we solve (1.3) with initial curve $\gamma(s)$ to obtain a family of solution $\gamma(t, s)$. By Theorem 4.1, $\gamma(t, s)$ ceases to exist only when $I(\gamma(t, s))$ tends to zero. Thus we know that $\gamma(t, s)$ exists as long as $I(\gamma(t, s)) \geq \frac{c}{2}$.

We define $t^*(s) = \sup \left\{ t : I(\gamma(t, s)) \geq \frac{c}{2} \right\}$ (set $t^*(s)$ to be zero if $I(\gamma(0, s)) < \frac{c}{2}$). Notice that t^* cannot be continuous, otherwise $\gamma(t^*(s), s)$ would define a curve in Γ, and this yields a contradiction, to the definition of c. Let s_0 be a point of discontinuity; we claim that $t^*(s_0) = \infty$. Suppose on the contrary there exist $\{s_j\}$ such that

$$s_j \to s_0 \text{ and } t^*(s_j) \to t_1 \neq t^*(s_0).$$

For any $t < t^*(s_0)$ we have $I(\gamma(t, s_0)) > \frac{c}{2}$ (otherwise $t^*(s_0)$ would be infinity). Hence, for large j, $I(\gamma(t, s_j)) > \frac{c}{2}$, so $t_1 > t^*(s_0)$. For any t_2 satisfying $t_1 > t_2 > t^*(s_0)$ we have $I(\gamma(t_2, s_0)) < \frac{c}{2}$ (remark that $\gamma(t^*(s_0), s_0)$ cannot be a critical point otherwise $t^*(s_0)$ would be infinity). By continuity, $I(\gamma(t_2, s_j)) < \frac{c}{2}$ for large j. But this implies $t^*(s_j) \le t_2$, i.e. $t_1 = \lim_j t^*(s_j) \le t_2$, a contradiction. Thus $t^*(s_0)$ must be infinity.

So, (1.3) with initial hypersurface $\gamma(0, s_0)$ has a solution $\gamma(t, s_0)$ on $[0, +\infty)$ satisfying $I(\gamma(t, s_0)) \ge \frac{c}{2} > 0$ for all t. Then, by Theorem 4.1, we get a convex solution X_2 of (1.1) with $I(X_2) > 0$. ∎

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Department of Mathematics
Zhongshan University
Guangzhou 510275
People's Republic of China