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Is an Operator on Weak L^p which Commutes with Translations a Convolution?

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Let S be a linear translation invariant operator with domain the space of test functions on the real line $\mathbb{R}$, and bounded with respect to the norm of *Weak*-$L^p(\mathbb{R}) = L^{p,\infty}(\mathbb{R})$, $1 < p < +\infty$, id est $\|Sf\|_{p,\infty} \leq c\|f\|_{p,\infty}$. To this operator S it is naturally associated a tempered distribution u such that for every test function f one has $Sf = u * f$. (See e.g. [8].) Since the test functions are not dense in $L^{p,\infty}(\mathbb{R})$, it is natural to ask if there exists an extension of this operator from the space of test functions to all of $L^{p,\infty}(\mathbb{R})$, and whenever an extension is possible, if it is unique.

Of course similar questions can be asked for operators on $L^{p,\infty}(\mathbb{G})$, with $\mathbb{G}$ a group, for example the integers $\mathbb{Z}$, or the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

It turns out that the extension of the operator from the space of test functions to the whole space is always possible, but the question of uniqueness of the extension is more subtle. Let us consider what happens in the limiting cases $p = 1$ and $p = +\infty$.

Using the Hahn-Banach theorem it is easy to prove that there exist invariant means on $L^\infty(\mathbb{R})$ which are not identically zero but vanish on test functions. A classical example is given by a suitable limit of the sequence of functionals $\Lambda_n f = \dfrac{1}{2n}\displaystyle\int_{-n}^{+n} f(x)dx$. By multiplying the limit of these functionals by a constant function one obtains a non trivial translation invariant linear operator bounded on $L^\infty(\mathbb{R})$ and vanishing on test functions.

In general W. Rudin has shown in [5] that when $\mathbb{G}$ is an infinite locally compact amenable group, then on $L^\infty(\mathbb{G})$ there exist many invariant means.

P. Sjögren has proved in [7] that there exist translation invariant linear operators bounded on $L^{1,\infty}(\mathbb{R})$, which are not zero, but vanish on $L^1(\mathbb{R})$. His

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example is a suitable limit of the sequence of operators

$$S_n f(x) = \frac{1}{n\log(n)} \sum_{k=0}^{n-1} \left(f \cdot \chi_{\{1/n<|f|<n\}}\right)(x - k/n).$$

Similar constructions work as well on other groups.

By "interpolation" between 1 and $+\infty$ it is then natural to conjecture that these "singular" translation invariant operators exist also on the spaces $L^{p,\infty}(\mathbb{G})$, $1 < p < +\infty$. But let us try to construct one of these operators on the torus $\mathbb{T}$.

The most natural way to proceed is the following. It is known that there exist non trivial continuous linear functionals on $L^{p,\infty}(\mathbb{T})$, $1 < p < +\infty$, which vanish on simple functions. (M. Cwikel has given in [3] a description of the dual space of *Weak-L^p*.) By averaging over $\mathbb{T}$ these functionals can be made translation invariant, and by multiplying by a constant function, which belongs to $L^{p,\infty}(\mathbb{T})$ and is invariant under translations, one produces continuous linear translation invariant operators on $L^{p,\infty}(\mathbb{T})$ which are zero on simple functions. The surprise is that the invariant functionals and the associated operators are identically zero. To give an idea of this let us consider the following example.

Let Λ be a suitable limit, as $\varepsilon \to 0_+$, of the functionals $\Lambda_\varepsilon f = \varepsilon^{1/p-1} \int_0^\varepsilon f(x)dx$. This functional Λ kills all bounded functions, but if $g(x) = |x|^{-1/p}$ then $\Lambda g = \frac{p}{p-1}$. However for every non trivial translation $\tau_y g$ one has $\Lambda\tau_y g = 0$, so that by averaging over $\mathbb{T}$ one obtains zero. (By the way, the limit as $\varepsilon \to +\infty$ of the above functionals $\{\Lambda_\varepsilon\}$ is a non trivial translation invariant functional on $L^{p,\infty}(\mathbb{R})$.)

Indeed we shall see that there is a difference between the torus (a compact group) and the (non compact) groups of the integers or the line. Every translation invariant linear operator bounded on $L^{p,\infty}(\mathbb{T})$ which vanishes on simple functions is identically zero. In some sense all translation invariant operators on $L^{p,\infty}(\mathbb{T})$ are convolutions. On the contrary, on $\ell^{p,\infty}(\mathbb{Z})$ and on $L^{p,\infty}(\mathbb{R})$ there are continuous translation invariant linear operators which are not zero, but vanish on simple functions. In some sense these operators are not convolutions.

Let us give now some definitions, and then a precise statement of our results.

Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. *Weak-$L^p(\mathbb{X}, \Sigma, \mu)$* $= L^{p,\infty}(\mathbb{X}, \Sigma, \mu)$, $0 < p < +\infty$, is the space of (equivalence classes of) measurable functions satisfying $\sup_{t>0} t^p \cdot \mu(\{x : |f(x)| > t\}) < +\infty$. See [8].

$L^{p,\infty}(\mathbb{X}, \Sigma, \mu)$ when $1 < p < +\infty$ is a Banach space, and, under reasonable assumptions on the measure space, a norm is given by

$$\|f\|_{p,\infty} = \sup_{E\in\Sigma} \mu(E)^{1/p-1} \int_E |f(x)|d\mu(x).$$

Closely related to the norm $\|\cdot\|_{p,\infty}$ we also have two seminorms,

$$N_0(f) = \lim_{\mu(E)\to 0_+} \sup \mu(E)^{1/p-1} \int_E |f(x)|d\mu(x),$$

$$N_\infty(f) = \lim_{\mu(E)\to +\infty} \sup \mu(E)^{1/p-1} \int_E |f(x)|d\mu(x).$$

In some sense the seminorm N_0 measures the "peaks", and the seminorm N_∞ measures the "tails" of a function. The closure in $L^{p,\infty}(\mathbb{X},\Sigma,\mu)$ of the subspace of simple functions is characterized by the conditions $N_0(f) = N_\infty(f) = 0$.

In the sequel we shall write $L^{p,\infty}(\mathbb{G})$ to denote the space *Weak-L^p* on a locally compact group $\mathbb{G}$ equipped with left Haar measure, and we shall write $|E|$ to denote the measure of a measurable set E. Through this paper we shall use the additive notation $x+y$ to denote the group operation. We do not use the multiplicative notation $x\cdot y$ which is perhaps more common when the group is non commutative since the groups we have in mind are the integers, the reals and the torus $\mathbb{R}/\mathbb{Z}$. The operator of left translation τ_y, $y\in\mathbb{G}$, is defined by $\tau_y f(x) = f(-y+x)$. We say that an operator S is translation invariant if it commutes with left translations, that is $S\tau_y = \tau_y S$ for all y in $\mathbb{G}$.

Our first problem is the extension of a linear operator defined on a weak-$*$ dense subspace to all the space, and then the decomposition of an operator into an "absolutely continuous" part and a "singular" part.

THEOREM 1. i) *Let S be a linear operator defined on simple functions, such that for every simple function $\|S\varphi\|_{p,\infty} \le c\|\varphi\|_{p,\infty}$, $1<p<+\infty$. Then this operator can be naturally extended to a linear operator S_1 bounded from all $L^{p,\infty}(\mathbb{G})$ into $L^{p,\infty}(\mathbb{G})$. Moreover if the operator on simple functions is translation invariant, then also the extension is translation invariant.*

ii) *Let S be a linear operator bounded from $L^{p,\infty}(\mathbb{G})$ into $L^{p,\infty}(\mathbb{G})$, $1<p<+\infty$. Then this operator S can be decomposed into a sum $S = S_0+S_1+S_\infty$, where*

$$S_0 f = \lim_{n\to+\infty} S\left(f\cdot\chi_{\{|f|\ge n\}}\right),$$

$$S_1 f = \lim_{n\to+\infty} S\left(f\cdot\chi_{\{1/n<|f|<n\}}\right),$$

$$S_\infty f = \lim_{n\to+\infty} S\left(f\cdot\chi_{\{|f|\le 1/n\}}\right).$$

These limits exist in the weak-$*$ topology of $L^{p,\infty}(\mathbb{G}) = (L^{p/(p-1),1}(\mathbb{G}))^*$. Moreover, the operator S_1 is the extension defined in i) *of the restriction to simple functions of the operator S. The operators S_0 and S_∞ vanish on simple functions, and are bounded with respect to the seminorms N_0 and N_∞ respectively, $\|S_0 f\|_{p,\infty} \le cN_0(f)$ and $\|S_\infty f\|_{p,\infty} \le cN_\infty(f)$.*

If the operator S is translation invariant, then also the three operators S_0, S_1, S_∞ are translation invariant.

It is clear that on a compact group the seminorm N_∞ is zero, so that in the decomposition of an operator into $S_0+S_1+S_\infty$ the term S_∞ is absent. Similarly on a discrete group the seminorm N_0 is zero, and S_0 is the zero operator. In general, apart from these quite trivial cases, in the decomposition of an operator into $S_0+S_1+S_\infty$ it is not clear a priori which terms can be present. Indeed it is not even clear if on $L^{p,\infty}(\mathbb{G})$ there exist translation invariant singular operators. The following theorems give an answer to these questions.

THEOREM 2. *Let $\mathbb{G}$ be a unimodular group. If a translation invariant linear operator bounded from $L^{p,\infty}(\mathbb{G})$ into $L^{p,\infty}(\mathbb{G})$, $1 < p < +\infty$, vanishes on the bounded functions, then it also vanishes on all $L^{p,\infty}(\mathbb{G})$. In other words, if $\mathbb{G}$ is unimodular the only translation invariant linear operator on $L^{p,\infty}(\mathbb{G})$ which is bounded with respect to the seminorm N_0 is the zero operator.*

THEOREM 3. *Let $\mathbb{G}$ be a non compact σ - compact unimodular group. Then there exist translation invariant linear operators bounded from $L^{p,\infty}(\mathbb{G})$ into $L^{p,\infty}(\mathbb{G})$, $1 < p < +\infty$, which are not zero but vanish on the closure in $L^{p,\infty}(\mathbb{G})$ of the subspace of functions with support of finite measure. In other words, if $\mathbb{G}$ is unimodular and non compact there are non zero translation invariant linear operators on $L^{p,\infty}(\mathbb{R})$ which are bounded with respect to the seminorm N_∞.*

THEOREM 4. *Let $\mathbb{G}$ be a non unimodular group. Then there exist non zero translation invariant linear operators on $L^{p,\infty}(\mathbb{G})$, $1 < p < +\infty$, which are bounded with respect to the seminorm N_0, and also operators which are bounded with respect to the seminorm N_∞.*

All the paper is essentially self contained. However the proof of ii) in Theorem 1 is similar to the decomposition of the dual space of *Weak-L^p* into an absolutely continuous part and a singular part. See [3].

We have a simple "pseudo" proof of Theorem 2, at least for compact groups, which is based on the study of invariant linear functionals on *Weak-L^p*, but this uses the characterization of the dual space of *Weak-L^p*. The proof of Theorem 2 presented in this paper, although a bit cumbersome, relies only on some measure theory and is quite elementary.

The proof of the Theorems 3 and 4 is in the spirit of the construction of a "Banach Limit", i.e. an invariant mean on $L^\infty(\mathbb{G})$. These left translation invariant singular operators are suitable limits of right translations. The arguments are not constructive and rely on the axiom of chioce. It is noteworthy that the construction of singular operators on non unimodular groups is very simple, even simpler than for unimodular groups.

A final remark on operators on the non Banach spaces $L^{p,\infty}(\mathbb{G})$ with $0 < p < 1$ and $p = 1$.

When $p = 1$ it is possible to prove that for any infinite group $\mathbb{G}$ there exist singular translation invariant linear operators bounded on $L^{1,\infty}(\mathbb{G})$, which are not zero, but vanish on $L^1(\mathbb{G})$. The idea for a construction, for non discrete groups, is essentially due to P. Sjögren (see [7]).

It is not difficult to prove that the operators

$$T_n f(x) = \frac{1}{\log(n)} f(x) \cdot \chi_{\{1/n<|f|<n\}}(x)$$

are bounded from $L^{1,\infty}(\mathbb{G})$ into $L^1(\mathbb{G})$ uniformly with respect to n, $\|T_n f\|_1 \leq c\|f\|_{1,\infty}$, and also $\|T_n(f+g)-T_n f - T_n g\|_1 \leq \dfrac{c(f,g)}{\log(n)}$, $\|T_n(\lambda f)-\lambda T_n f\|_1 \leq \dfrac{c(\lambda,f)}{\log(n)}$. Then a suitable weak-$*$ limit $Tf = \lim_{n\to+\infty} T_n f$ defines a translation invariant linear operator from $L^{1,\infty}(\mathbb{G})$ into the space $M(\mathbb{G})$ of bounded Borel measures on $\mathbb{G}$. It is clear that if f is in $L^1(\mathbb{G})$ then $Tf = 0$, and also there exist functions f in $L^{1,\infty}(\mathbb{G})$ with $Tf = \delta_0$, the point mass at the origin.

The desired singular operator on $L^{1,\infty}(\mathbb{G})$ can be obtained by composing the operator T with a convolution operator, e.g. $Sf(x) = \int_{\mathcal{V}} Tf(x+y)dy$ with $\mathcal{V}$ open and relatively compact.

Consider now the case of a discrete group such as the integers $\mathbb{Z}$. Then a singular translation invariant operator bounded from $\ell^{1,\infty}(\mathbb{Z})$ into $\ell^1(\mathbb{Z})$ can be obtained as a suitable limit of the sequence of operators

$$S_n f(j) = \begin{cases} \dfrac{1}{\log(n)} \displaystyle\sum_{k=n+1}^{2n} f(j+k!) & \text{if } |j| \leq n, \\ 0 & \text{if } |j| > n. \end{cases}$$

See also the proof of Theorem 3.

The case of the spaces $L^{p,\infty}(\mathbb{G})$ with $0 < p < 1$ is different. This has been proved by M. Cwikel when the group is discrete, while the general case is due to N.J. Kalton ([4] Theorem 6.4).

In particular it has been proved by M. Cwikel in [2] that when $0 < p < 1$ the dual of the space of sequences $\ell^{p,\infty}(\mathbb{Z})$ can be naturally identified with $\ell^\infty(\mathbb{Z})$, and, contrary to the case $1 \leq p \leq +\infty$, there exist no singular linear functionals. This immediately implies that there do not exist singular linear operators bounded from $\ell^{p,\infty}(\mathbb{Z})$ into $\ell^{p,\infty}(\mathbb{Z})$. In fact if S is such a singular linear operator, then for every j in $\mathbb{Z}$ the operator that associates to the sequence α the number $S\alpha(j)$ is a singular linear functional. Observe that in these arguments the invariance under translations plays no role.

Indeed it can be proved that when $0 < p < 1$ the continuous linear operators from $\ell^{p,\infty}(\mathbb{Z})$ into $\ell^{p,\infty}(\mathbb{Z})$ which commute with translations are precisely the convolutions with sequences in $\ell^p(\mathbb{Z})$, i.e. operators of the form $S\alpha(j) = \sum_{j=-\infty}^{+\infty} b(k)\alpha(j-k)$ with $\sum_{j=-\infty}^{+\infty} |b(k)|^p < +\infty$. See [1] and [6].

Proof of the Theorems

PROOF OF THEOREM 1. To prove i) we start by observing that the operator S can be extended by continuity to the closure in $L^{p,\infty}(\mathbb{G})$ of the subspace of simple functions. This subspace has as dual the Lorentz space $L^{p/(p-1),1}(\mathbb{G})$, so that we can define an adjoint operator S^* bounded from $L^{p/(p-1),1}(\mathbb{G})$ into $L^{p/(p-1),1}(\mathbb{G})$. Since $(L^{p/(p-1),1}(\mathbb{G}))^* = L^{p,\infty}(\mathbb{G})$, the adjoint of the operator S^* is the required extension of S. It is easily seen that the above construction is equivalent to defining $S_1 f$ as a weak-$*$ limit of a sequence $\{Sf_n\}$, where $\{f_n\}$ is a sequence of simple functions which converges weak-$*$ to f.

To prove ii) we first define S_1 as the extension of the restriction to simple functions of the operators S, that is

$$S_1 f = \lim_{n\to+\infty} S\left(f \cdot \chi_{\{1/n<|f|<n\}}\right).$$

Then, for some $t > 0$ define

$$S_0 f = (S - S_1)\left(f \cdot \chi_{\{|f|\geq t\}}\right), \qquad S_\infty f = (S - S_1)\left(f \cdot \chi_{\{|f|<t\}}\right).$$

Since the operator $S - S_1$ is a bounded linear operator on $L^{p,\infty}(\mathbb{G})$ which kills all functions in the closure of the subspace generated by simple functions, it is easily seen that this definition is independent of t. To show that the two operators S_0 and S_∞ are linear, one has only to check that for every function f and g, and scalar λ and ν, the following functions are bounded and supported on sets of finite measure, and thus killed by $S - S_1$:

$$(\lambda f + \nu g)\cdot \chi_{\{|\lambda f+\nu g|\geq t\}} - \lambda\left(f \cdot \chi_{\{|f|\geq t\}}\right) - \nu\left(g \cdot \chi_{\{|g|\geq t\}}\right),$$

$$(\lambda f + \nu g)\cdot \chi_{\{|\lambda f+\nu g|< t\}} - \lambda\left(f \cdot \chi_{\{|f|< t\}}\right) - \nu\left(g \cdot \chi_{\{|g|< t\}}\right).$$

Finally, since $(S - S_1)\left(f \cdot \chi_{\{|f|\geq t\}}\right)$ and $(S - S_1)\left(f \cdot \chi_{\{|f|< t\}}\right)$ are independent of t, and since in the weak-$*$ topology $\lim_{t\to+\infty} S_1\left(f \cdot \chi_{\{|f|\geq t\}}\right) = \lim_{t\to 0_+} S_1\left(f \cdot \chi_{\{|f|< t\}}\right) = 0$, one has

$$S_0 f = \lim_{t\to+\infty}(S - S_1)\left(f \cdot \chi_{\{|f|\geq t\}}\right) = \lim_{t\to+\infty} S\left(f \cdot \chi_{\{|f|\geq t\}}\right),$$

$$S_\infty f = \lim_{t\to 0_+}(S - S_1)\left(f \cdot \chi_{\{|f|< t\}}\right) = \lim_{t\to 0_+} S\left(f \cdot \chi_{\{|f|< t\}}\right).$$

The proof that if the operator S is translation invariant, then also the operators S_0, S_1, S_∞ are translation invariant is immediate. □

PROOF OF THEOREM 2. Suppose that S is a translation invariant linear operator which vanishes on bounded functions, and assume that the norm of S from $L^{p,\infty}(\mathbb{G})$ into $L^{p,\infty}(\mathbb{G})$ is one. Then there exists an f with $\|f\|_{p,\infty} = 1$, such that $\|Sf\|_{p,\infty} \approx 1$. Since S vanishes on bounded functions, if we define $f_N(x) = \begin{cases} f(x) & \text{if } |f(x)| > N, \\ 0 & \text{if } |f(x)| \leq N, \end{cases}$ then for every N we have $Sf = Sf_N$, and since S commutes with translations, for every y we also have $S(f_N + \tau_y f_N) = Sf_N + \tau_y Sf_N$.

The idea is to prove that for all sufficiently small translations τ_y one has $\|S(f_N + \tau_y f_N)\|_{p,\infty} = \|Sf + \tau_y Sf\|_{p,\infty} \approx \|2 \cdot Sf\|_{p,\infty} \approx 2$, while for a big N the supports of the two functions f_N and $\tau_y f_N$ are essentially disjoint, so that $\|f_N + \tau_y f_N\|_{p,\infty} \approx 2^{1/p}$. This contradicts the assumption that the norm of the operator S is equal to one.

Although the idea of the proof is simple, the technical details, given in the following lemmas, are a bit cumbersome.

LEMMA 2.1. *Let g be a function in $L^{p,\infty}(\mathbb{G})$, $1 < p < +\infty$, and let $\varepsilon > 0$. Then there exists a neighbourhood of the origin $\mathcal{V} = \mathcal{V}(\varepsilon, g)$ in $\mathbb{G}$ such that for every y in $\mathcal{V}$ one has $\|g + \tau_y g\|_{p,\infty} \geq (2 - \varepsilon)\|g\|_{p,\infty}$.*

PROOF. There exists an open relatively compact set E with

$$|E|^{1/p-1} \int_E |g(x)| dx \geq (1 - \varepsilon/3)\|g\|_{p,\infty},$$

and one has

$$\begin{aligned} &|E|^{1/p-1} \int_E |g(x) + g(-y + x)| dx \\ &\geq 2|E|^{1/p-1} \int_E |g(x)| dx - |E|^{1/p-1} \int_E |g(-y + x) - g(x)| dx. \end{aligned}$$

Let U be a relatively compact neighbourhood of the origin. Then the function $g \cdot \chi_{-U+E}$ is in $L^1(\mathbb{G})$, and if $y \in U$,

$$\int_E |g(-y + x) - g(x)| dx \leq \int_{\mathbb{G}} |g(-y + x)\chi_{-U+E}(-y + x) - g(x)\chi_{-U+E}(x)| dx.$$

By the continuity of translations in the space $L^1(\mathbb{G})$ the last term tends to zero when $y \to 0$, and the lemma follows. □

LEMMA 2.2. *Let f be a function in $L^{p,\infty}(\mathbb{G})$, $1 < p < +\infty$, and let $\varepsilon > 0$. Define $f_N(x) = \begin{cases} f(x) & \text{if } |f(x)| > N, \\ 0 & \text{if } |f(x)| \leq N. \end{cases}$ Then to almost every y in $\mathbb{G}$ we can associate an $N = N(\varepsilon, y, f)$ such that*

$$\|f_N + \tau_y f_N\|_{p,\infty} \leq (2^{1/p} + \varepsilon)\|f\|_{p,\infty}.$$

PROOF. Without loss of generality we can assume f non negative and $\|f\|_{p,\infty} = 1$. Decompose the support of this function into a family $\{A_j\}$ of disjoint sets, with $|A_j| = 2^{-j}$, and with $\operatorname{ess\,sup}\{f(x) : x \in A_j\} \leq \operatorname{ess\,inf}\{f(x) : x \in A_{j+1}\}$. Observe that since $\|f\|_{p,\infty} = 1$ and $|A_j| = 2^{-j}$ we must have $\operatorname{ess\,sup}\{f(x) : x \in A_j\} \leq 2^{1/p}2^{j/p}$.

Fix a large integer m, and set $\bigcup_{k=-2m}^{2m} A_{j+k} = B_j$. Then with suitable N and n write

$$\begin{aligned} f_N(x) + f_N(-y+x) &= \sum_{j>n} f(x)\chi_{A_j}(x) + \sum_{j>n} f(-y+x)\chi_{A_j}(-y+x) \\ &= \sum_{j>n} [f(x)\chi_{A_j\setminus(y+B_j)}(x) + f(-y+x)\chi_{(y+A_j)\setminus B_j}(x)] \\ &\quad + \sum_{j>n} [f(x)\chi_{A_j\cap(y+B_j)}(x) + f(-y+x)\chi_{(y+A_j)\cap B_j}(x)] \\ &= F(x) + G(x). \end{aligned}$$

To complete the proof we only have to estimate the norms of these two functions in $L^{p,\infty}(\mathbb{G})$. This is done in the following two (sub)lemmas.

LEMMA 2.3. *If the integer m in the definition of the sets $\{B_j\}$ is large enough we have that $\|F\|_{p,\infty} < 2^{1/p} + \varepsilon/2$.*

PROOF. We have to estimate $|E|^{1/p-1} \int_E F(x)dx$, where E is an arbitrary measurable set with measure less than or equal to the measure of the support of F. Let $2^{-h-1} < |E| \leq 2^{-h}$, and split the series which defines the function F into four pieces:

$$\sum_{j>n} [f(x)\chi_{A_j\setminus(y+B_j)}(x) + f(-y+x)\chi_{(y+A_j)\setminus B_j}(x)] = Z(x) + W(x) + V(x) + U(x),$$

with

$$\begin{aligned} Z(x) &= \sum_{n<j\leq h-m} [f(x)\chi_{A_j\setminus(y+B_j)}(x) + f(-y+x)\chi_{(y+A_j)\setminus B_j}(x)], \\ W(x) &= \sum_{j\geq h+m} [f(x)\chi_{A_j\setminus(y+B_j)}(x) + f(-y+x)\chi_{(y+A_j)\setminus B_j}(x)], \\ V(x) &= \sum_{h-m<j<h+m} f(-y+x)\chi_{(y+A_j)\setminus B_j}(x), \\ U(x) &= \sum_{h-m<j<h+m} f(x)\chi_{A_j\setminus(y+B_j)}(x). \end{aligned}$$

Then, since $|E| \approx 2^{-h}$,

$$|E|^{1/p-1} \int_E Z(x)dx \leq |E|^{1/p} \|Z\|_\infty \leq |E|^{1/p} 2^{1+1/p} \sum_{n<j\leq h-m} 2^{j/p} \leq c2^{-m/p}.$$

Also

$$|E|^{1/p-1} \int_E W(x)dx \leq 2|E|^{1/p-1} \sum_{j\geq h+m} \int_{A_j} f(x)dx$$

$$\leq 2|E|^{1/p-1} \sum_{j\geq h+m} |A_j|^{1-1/p} \leq c2^{(1/p-1)m}.$$

Therefore when m is big the contribution of the two functions Z and W to the integral $|E|^{1/p-1} \int_E F(x)dx$ is small.

Finally observe that the two functions V and $U(= \tau_y V)$ are dominated by f and $\tau_y f$ respectively, hence $\|V\|_{p,\infty} \leq 1$ and $\|U\|_{p,\infty} \leq 1$. Moreover V and U have disjoint supports, so that $\|V + U\|_{p,\infty} = (\|V\|^p_{p,\infty} + \|U\|^p_{p,\infty})^{1/p} \leq 2^{1/p}$. □

LEMMA 2.4. i) *Let* $|A_j| = 2^{-j}$ *and* $B_j = \bigcup_{k=-2m}^{2m} A_{j+k}$. *Given* $\varepsilon > 0$, *to almost every* y *it is possible to associate an* n *such that if* $j > n$, *then* $|A_j \cap (y + B_j)| < \varepsilon 2^{-j}$.

ii) *Let* $C_j(y) = [A_j \cap (y + B_j)] \cup [(y + A_j) \cap B_j]$. *Then, given* $\varepsilon > 0$, *to almost every* y *it is possible to associate an* n *such that* $\left\| \sum_{j>n} 2^{j/p} \chi_{C_j(y)} \right\|_{p,\infty} < \varepsilon.$

In particular, for almost every y *if* n *is big enough we have* $\|G\|_{p,\infty} < \varepsilon/2$.

PROOF.

$$\int_{\mathbb{G}} \sum_{j>n} 2^j |A_j \cap (y + B_j)| dy = \sum_{j>n} 2^j \int_{\mathbb{G}} \chi_{A_j} * \chi_{-B_j}(y) dy$$

$$= \sum_{j>n} |-B_j| = \sum_{j>n} |B_j| \leq \sum_{j>n} 2^{2m+1-j}.$$

Observe that for the equality $|-B_j| = |B_j|$ we need the group $\mathbb{G}$ to be unimodular. Therefore $\left\{ \sum_{j>n} 2^j |A_j \cap (y + B_j)| \right\}$ is a pointwise monotone sequence which converges to zero as $n \to +\infty$ in the norm of $L^1(\mathbb{G})$. We can conclude that this sequence converges to zero also for almost every y in $\mathbb{G}$, and i) follows. ii) is an immediate consequence of i). □

PROOF OF THEOREM 3. This Theorem will be obtained from the following simple lemmas.

LEMMA 3.1. *Let* $\{\mathcal{A}_n\}$ *be an increasing family of open relatively compact sets with* $\bigcup_{n=1}^{+\infty} \mathcal{A}_n = \mathbb{G}$. *Then there exists a sequence of points* $\{y_n\}$ *such that for every* n *the sets* $\mathcal{A}_n$ *and* $(\mathcal{A}_n + y_{n+1})$, $(\mathcal{A}_n + y_{n+2})$, $(\mathcal{A}_n + y_{n+3})$, ... *are mutually disjoint.*

PROOF. Observe that every y_n must satisfy only a finite numbers of conditions. Indeed it is enough to choose recursively the point y_n outside the relatively compact set $\left[\bigcup_{i<j<n} [-\mathcal{A}_i + (\mathcal{A}_i + y_j)]\right] \cup \left[\bigcup_{k<n} [-\mathcal{A}_k + \mathcal{A}_k]\right]$. □

LEMMA 3.2. i) *The operator*

$$S_n f(x) = \begin{cases} n^{1/p-1} \displaystyle\sum_{k=n+1}^{2n} f(x+y_k) & \text{if } x \in \mathcal{A}_n, \\ 0 & \text{if } x \notin \mathcal{A}_n, \end{cases}$$

maps $L^{p,\infty}(\mathbb{G})$ *into* $L^{p,\infty}(\mathbb{G})$, *with norm bounded by one.*

ii) *If* g *is a function in* $L^{p,\infty}(\mathbb{G})$ *with support of finite measure, then the sequence* $\{S_n g\}$ *converges to zero in the weak-$*$ topology of* $L^{p,\infty}(\mathbb{G}) = (L^{p/(p-1),1}(\mathbb{G}))^*$.

iii) *Let* $\mathcal{V}$ *be an open relatively compact set contained in* $\mathcal{A}_1$, *and let* $h(x) = \begin{cases} k^{-1/p} & \text{if } x \in \mathcal{V} + y_k, \\ 0 & \text{otherwise.} \end{cases}$ *Then if* $x \in \mathcal{V}$ *we have* $\lim_{n\to+\infty} S_n h(x) = \frac{p}{p-1}(2^{1-1/p} - 1)$.

PROOF. Let E be a measurable subset of $\mathbb{G}$. We must show that

$$|E|^{1/p-1} \int_E |S_n f(x)| dx \leq \|f\|_{p,\infty}.$$

Since $S_n f(x) = 0$ when $x \notin \mathcal{A}_n$, we may assume that $E \subseteq \mathcal{A}_n$. Then by the previous Lemma the sets $\{E + y_k\}_{k>n}$ are mutually disjoint, and the set $\bigcup_{k=n+1}^{2n} (E + y_k) = D$ has measure $n|E|$. Hence

$$|E|^{1/p-1} \int\limits_E \left| n^{1/p-1} \sum_{k=n+1}^{2n} f(x+y_k) \right| dx$$

$$\leq (n|E|)^{1/p-1} \sum_{k=n+1}^{2n} \int\limits_{E+y_k} |f(x)|dx = |D|^{1/p-1} \int\limits_D |f(x)|dx \leq \|f\|_{p,\infty}.$$

Observe that we have used the unimodularity of the group $\mathbb{G}$. This proves i).

The proof of ii) is immediate if the support of the function g is also compact. In general, when the the function g is in $L^{p,\infty}(\mathbb{G})$ and has support of finite measure, then g is also in $L^1(\mathbb{G})$. Let b be a simple function. Then we have

$$\left| \int\limits_{\mathbb{G}} S_n g(x) b(x) dx \right| \leq n^{1/p-1} \|g\|_1 \|b\|_\infty,$$

and the integrals $\int\limits_{\mathbb{G}} S_n g(x) b(x) dx$ converge to zero as $n \to +\infty$. Since by i) the sequence $\{S_n g\}$ is bounded in $L^{p,\infty}(\mathbb{G})$, by standard approximation arguments we have that this sequence converges to zero in the weak-$*$ topology of $L^{p,\infty}(\mathbb{G})$.

To prove iii) observe that if $x \in \mathcal{V}$, then when n is big,

$$n^{1/p-1} \sum_{k=n+1}^{2n} h(x+y_k) = \frac{1}{n} \sum_{k=n+1}^{2n} \left(\frac{k}{n}\right)^{-1/p} \approx \int\limits_1^2 s^{-1/p} dx. \qquad \square$$

Of course, the operator in the statement of Theorem 2 is a suitable limit of the sequence of operators $\{S_n\}$.

Let $\mathcal{U}$ be an ultrafilter in $\mathbb{N}$ containing all subsets of $\mathbb{N}$ of the form $\{n : n \geq m\}$. Then for every f in $L^{p,\infty}(\mathbb{G})$ the image of this ultrafilter under the map $n \mapsto S_n f$ defines an ultrafilter base in a closed ball of $L^{p,\infty}(\mathbb{G})$. Since this ball is weak-$*$ compact, the ultrafilter base generated by $\mathcal{U}$ via $\{S_n f\}$ converges in the weak-$*$ topology to an element of $L^{p,\infty}(\mathbb{G})$ which we denote by Sf.

Of course we have $\|Sf\|_{p,\infty} \leq \|f\|_{p,\infty}$, and since the operators $\{S_n\}$ are linear, it can be proved that also S is linear.

The operators $\{S_n\}$ are not translation invariant, nevertheless the limit operator S commutes with left translations. To see this, observe that if χ is the characteristic function of a compact set and if y is a point of $\mathbb{G}$, then for every n big enough $\chi S_n \tau_y = \chi \tau_y S_n$. Hence at the limit $\chi S \tau_y = \chi \tau_y S$. Since this equality holds for every cut off function χ, we have that $S\tau_y = \tau_y S$. $\square$

PROOF OF THEOREM 4. Let $\mathbb{G}$ be a non unimodular group and denote by Δ its modular function. It is easy to verify that for every y in $\mathbb{G}$ the right translation operator $S_y f(x) = \Delta(y)^{1/p} f(x+y)$ commutes with left translations and is an isometry of $L^{p,\infty}(\mathbb{G})$.