ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

KUNIHIKO KAJITANI
KAORU YAMAGUTI

On global real analytic solutions of the degenerate Kirchhoff equation

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 2 (1994), p. 279-297
<http://www.numdam.org/item?id=ASNSP_1994_4_21_2_279_0>

On Global Real Analytic Solutions of the Degenerate Kirchhoff Equation

KUNIHIKO KAJITANI - KAORU YAMAGUTI

1. - Introduction

We shall consider the problem of existence and uniqueness of real analytic solutions of the Cauchy problem for the degenerate Kirchhoff equation

$$\text{(1.1)}\qquad \begin{cases} \partial_t^2 u + M((Au,u))Au = f(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $Au(t,x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(t,x))$, $D_j = \dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x_j}$, $(Au(t,\cdot),u(t,\cdot))$ is an inner product of $Au(t,x)$ and $u(t,x)$ in $L^2(\mathbb{R}^n_x)$ and $M(\eta)$ is a non-negative function in $C^1([0,\infty))$.

When $A = \sum_{j=1}^{n} D_j^2$ the equation (1.1) is called the Kirchhoff equation, which has been studied by many authors (cf. [1], [2], [3], [8], [9] and [10]). In this paper, we shall treat the case where A is degenerate elliptic, that is, $[a_{ij}(x); i,j = 1,\dots,n]$ is a real symmetric matrix and

$$\text{(1.2)}\qquad a(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0$$

for $x \in \mathbb{R}^n$ and $\xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$. Moreover we assume that there are $c_0 > 0$ and $\rho_0 > 0$ such that

$$\text{(1.3)}\qquad |D_x^\alpha a_{ij}(x)| \leq c_0 \rho_0^{-|\alpha|}|\alpha|!$$

for $x \in \mathbb{R}^n_x$, $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ and $i, j = 1,\dots,n$, and that $M(\eta) \in C^1([0,\infty)]$ and

$$\text{(1.4)}\qquad M(\eta) \geq 0$$

Pervenuto alla Redazione il 20 Luglio 1993.

for $\eta \in [0,\infty)$. We introduce some functional spaces. For a topological space X and an interval $I \subset \mathbb{R}^1$, we denote by $C^k(I;X)$ the set of functions from I to X which are k times continuously differentiable with respect to $t \in I$ in X. For $s \in \mathbb{R}$ and $\rho > 0$ we define a Hilbert space $H^s_\rho = \{u(x) \in L^2(R^n_x); \langle\xi\rangle^s e^{\rho\langle\xi\rangle}\hat{u}(\xi) \in L^2(\mathbb{R}^n_\xi)\}$, where $\hat{u}(\xi)$ stands for Fourier transform of u and $\langle\xi\rangle = (1+\xi_1^2+\cdots\xi_n^2)^{1/2}$. For $\rho < 0$ we define H^s_ρ as the dual space of $H^{-s}_{-\rho}$. For $\rho = 0$ we denote by $H^s = H^s_0$ the usual Sobolev space. Then note that the dual space of H^s_ρ becomes $H^{-s}_{-\rho}$ for any s, $\rho \in \mathbb{R}$.

For $\rho \in \mathbb{R}$ define an operator $e^{\rho\langle D\rangle}$ from H^s_ρ to H^s as follows:

$$e^{\rho\langle D\rangle}u(x) = \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi+\rho\langle\xi\rangle}\hat{u}(\xi)\tilde{d\xi}$$

for $u \in H^s_\rho$, where $\tilde{d}\xi = (2\pi)^{-n}d\xi$. Note that $(e^{\rho\langle D\rangle})^{-1} = e^{-\rho\langle D\rangle}$ is a mapping from H^s to H^s_ρ.

We prove the following result:

MAIN THEOREM. *Assume that* (1.2) *through* (1.4) *are valid. Let* $0 < \rho_1 < \rho_0/\sqrt{n}$. *Put* $\rho(t) = \rho_1 e^{-\gamma t}$ *for* $\gamma > 0$. *Then there exists* $\gamma > 0$ *such that for any* $u_0 \in H^2_{\rho_1}$, $u_1 \in H^1_{\rho_1}$ *and for any* $f(t,x)$ *satisfying* $e^{\rho(t)\langle D\rangle}f \in C^0([0,\infty);H^1)$, *the Cauchy Problem* (1.1) *has the unique solution* $u(t,x)$ *satisfying* $e^{\rho(t)\langle D\rangle}u \in \bigcap_{j=0}^{2} C^{2-j}([0,\infty);H^j)$.

The idea in the proof of our main theorem is based on the method introduced in [5] in order to find the global real analytic solution of the Cauchy problem for a Kowalevskian system. Roughly speaking, we transform an unknown function u such as $v = e^{\rho(t)\langle D\rangle}u$ and then change the hyperbolic equation (1.1) of the unknown function u into the parabolic equation of v. Thanks to parabolicity, we can prove local existence of a solution v of the modified problem in the usual Sobolev spaces by the use of the principle of a contraction mapping. Finally we can show the existence of a time global solution of the original equation (1.1) modifying the energy estimate which was introduced in [3] in the case of $A = -\Delta$.

2. - Preliminaries

Let S^m be the *class of symbols of pseudo-differential operators of order* m whose element $a(x,\xi)$ in $C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ satisfies

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$$

for x, $\xi \in \mathbb{R}^n$ and for all multi-indeces α, $\beta \in \mathbb{N}^n$, where $a^{(\alpha)}_{(\beta)}(x,\xi) =$

$\left(\frac{\partial}{\partial \xi}\right)^{\alpha}\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}\right)^{\beta} a(x,\xi)$. We define a pseudo-differential operator $a(x,D)$ as usual

$$a(x,D)u(x) = \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\tilde{d}\xi$$

for $u \in S$ where S denotes the *Schwartz space of rapidly decreasing functions* in $\mathbb{R}^n$. Then we have the following well-known fact:

PROPOSITION 2.1. (i) *For $a(x,\xi) \in S^m$ and $s \in \mathbb{R}$, there is $C_s > 0$ such that*

$$\|a(x,D)u\|_s \leq C_s \|u\|_{s+m} \tag{2.1}$$

for $u \in H^{s+m}$.

(ii) *Assume $a(x,\xi) \in S^2$ is non-negative. Then there are positive numbers C_1 and C_2 such that*

$$\Re(a(x,D)u,u)_s \geq -C_1\|u\|_s \tag{2.2}$$

and

$$\begin{aligned}&\sum_{|\alpha|=1}\{\|a_{(\alpha)}(x,D)u\|_{s-1}^2 + \|a^{(\alpha)}(x,D)u\|_s^2\} \\ &\leq C_2(2C_1\|u\|_s^2 + \Re(a(x,D)u,u)_s)\end{aligned} \tag{2.3}$$

for $u \in H^{s+2}$.

For a proof refer to [6] and [4] for (i) and (ii) respectively.

Now let us state some properties of the Hilbert space H^s_ρ.

LEMMA 2.2. (i) *Let $\rho > 0$. Then it holds that*

$$\|D_x^\alpha w\|_{H^s} \leq \|w\|_{H^s_\rho}\rho^{-|\alpha|}|\alpha|! \tag{2.5}$$

and

$$|D_x^\alpha w(x)| \leq C_n\|w\|_{H^s_\rho}\rho^{-(|\alpha|+n+|s|)}(|\alpha|+n+|s|)! \tag{2.6}$$

for $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$ and $w \in H^s_\rho$.

(ii) *Let $u(x)$ be a function in H^∞ and $s \in \mathbb{R}$. If u satisfies*

$$\|D_x^\alpha u\|_{H^s} \leq c_0\rho_1^{-|\alpha|}|\alpha|! \tag{2.7}$$

for every multi-index $\alpha \in \mathbb{N}^n$, then $u(x)$ belongs to H^s_ρ for $\rho < \rho_1/\sqrt{n}$.

PROOF. (i) It is easy to verify (2.5) using the fact that $|\xi^\alpha| \le \langle\xi\rangle^{|\alpha|}$. Representing $D_x^\alpha w$ by Fourier transformation, we get

$$\begin{aligned}|D_x^\alpha w(x)| &= \left|\int e^{ix\cdot\xi}\xi^\alpha \hat{w}(\xi)\tilde{d}\xi\right| \\ &\le \int \langle\xi\rangle^{|\alpha|}|\hat{w}(\xi)|\tilde{d}\xi \\ &\le \left\{\int (\rho^{-\rho\langle\xi\rangle}\langle\xi\rangle^{|\alpha|+|s|})^2\tilde{d}\xi\right\}^{\frac{1}{2}}\|w\|_{H^2_\rho}\end{aligned}$$

which implies (2.6).

(ii) Since

$$\xi^\alpha\hat{u}(\xi) = \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi}D_x^\alpha u(x)dx,$$

we have the estimate by virtue of (2.7) that

$$\|\langle\xi\rangle^j\hat{u}\|^2_{H^s} \le (c_n c_0(\sqrt{n}\rho_1^{-1})^j j!)^2$$

for any j. Hence we obtain

$$\|e^{\rho\langle\xi\rangle}\hat{u}(\xi)\|^2_{H^s} \le \sum_{j=0}^\infty \left\|\frac{\rho^j\langle\xi\rangle^j}{j!}\hat{u}\right\|^2_{H^s} \le \frac{c_n^2c_0^2}{1-\sqrt{n}\rho\rho_1^{-1}}$$

if $\sqrt{n}\rho\rho_1^{-1} < 1$. □

Let $a(x)$ be a real analytic function in $\mathbb{R}^n$ satisfying

$$|D_x^\alpha a(x)| \le c_0\rho_0^{-|\alpha|}|\alpha|! \tag{2.8}$$

for all $x \in \mathbb{R}^n$ and for all multi-indices $\alpha \in \mathbb{N}^n$. Define a multiplier $a\cdot$ as $(a\cdot u)(x) = a(x)u(x)$. Let us define $a(\rho; x, D)u(x) = e^{\rho\langle D\rangle}a\cdot e^{-\rho\langle D\rangle}u(x)$ for $u \in L^2(\mathbb{R}^n)$ and denote its symbol by $a(\rho; x, \xi)$.

PROPOSITION 2.3. *Suppose that $a(x)$ satisfies (2.8).*

(i) *If a function u belongs to the class $H^s_{\rho_1}$ and $0 < \rho_1 < \rho_0$, then $a\cdot u$ belongs to the class H^s_ρ for $0 < \rho < \rho_1/\sqrt{n}$.*

(ii) *$a(\rho; x, D)$ is a pseudodifferential operator of order 0 and its symbol has the representation*

$$a(\rho; x, \xi) = a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi), \tag{2.10}$$

where

$$a_1(x, \xi) = -\sum_{j=1}^n D_{x_j}a(x)\xi_j\langle\xi\rangle^{-1}, \tag{2.11}$$

and a_2 and r respectively satisfy

$$|a^{(\alpha)}_{2(\beta)}(\rho; x, \xi)| \leq C_{\alpha\beta\rho_0}\langle\xi\rangle^{-|\alpha|}, \tag{2.12}$$

$$|r^{(\alpha)}_{(\beta)}(\rho; x, \xi)| \leq C_{\alpha\beta\rho_0}\langle\xi\rangle^{-1-|\alpha|} \tag{2.13}$$

for x, $\xi \in \mathbb{R}^n$, $|\rho| < n^{-1}\rho_0$ and α, $\beta \in \mathbb{N}^n$.

PROOF. (i) Assume $\rho > 0$. Taking into account the fact that

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha - \alpha'|!|\alpha'|!\rho_0^{-|\alpha'|}\rho_1^{-|\alpha-\alpha'|} \leq \frac{\rho_0}{\rho_0 - \rho_1}\rho_1^{-|\alpha|}|\alpha|!$$

if $\rho_1 < \rho_0$, we have the estimate that

$$\begin{aligned} \|D^\alpha_x(a \cdot u)\|_{H^s} &= \left\|\sum_{\alpha' < \alpha} \binom{\alpha}{\alpha'} D^{\alpha'}_x a \cdot D^{\alpha-\alpha'}_x u(\cdot)\right\|_{H^s} \\ &\leq c_0\|u\|_{H^s_{\rho_1}} \sum \binom{\alpha}{\alpha'} \rho_0^{-|\alpha'|}|\alpha'|!\rho_1^{-|\alpha-\alpha'|}|\alpha - \alpha'|! \\ &\leq c_0 \frac{\rho}{\rho_0 - \rho_1} \|u\|_{H^s_{\rho_1}} \rho^{-|\alpha|}|\alpha|! \end{aligned}$$

from (2.5) and (2.7). Therefore it follows from (ii) of Lemma 2.2 that $a \cdot u \in H^s_\rho$ for $\rho < \rho_1/\sqrt{n}$.

(ii) For $u \in S$ and $\varepsilon > 0$ we put $\hat{u}_\varepsilon(\xi) = e^{-\varepsilon|\xi|^2}\hat{u}(\xi)$. $u_\varepsilon(x)$ denotes the inverse Fourier transformation of $\hat{u}_\varepsilon(\xi)$. Then $u_\varepsilon(x)$ is in H^s_τ for every $\tau > 0$ and $e^{-\rho\langle D\rangle}u_\varepsilon(x)$ is also in H^s_τ for all $\tau > 0$ and $\rho \in \mathbb{R}^1$. Therefore it follows from (i) that $a \cdot e^{-\rho\langle D\rangle}u_\varepsilon$ is in H^s_τ if $\tau < \rho_0/\sqrt{n}$. Note that $a \cdot e^{-\rho\langle D\rangle}u_\varepsilon \in L^1(\mathbb{R}^n_x)$ and $e^{\rho\langle\xi\rangle}\mathcal{F}[a \cdot e^{-\rho\langle D\rangle}u_\varepsilon](\xi) \in L^1(\mathbb{R}^n_\xi)$ for $|\rho| < \rho_0/\sqrt{n}$ and $\varepsilon > 0$. So we can write

$$\begin{aligned} &e^{\rho\langle D\rangle}(a \cdot e^{-\rho\langle D\rangle}u_\varepsilon)(x) \\ &= \int e^{ix\cdot\eta+\rho\langle\eta\rangle}\tilde{d}\eta \int e^{-iy\cdot\eta}(a \cdot e^{-\rho\langle D\rangle}u_\varepsilon)(y)dy \\ &= \lim_{\delta\to+0} \int e^{ix\cdot\eta+\rho\langle\eta\rangle-\delta|\eta|^2}\tilde{d}\eta \int e^{-iy\cdot\eta-\delta|x-y|^2}(a \cdot e^{-\rho\langle D\rangle}u_\varepsilon)(y)dy \\ &= \lim_{\delta\to+0} \int\int\int e^{i(x-y)\cdot\eta+\rho\langle\eta\rangle-\delta|x-y|^2-\delta|\eta|^2}a(y)e^{iy\cdot\xi-\rho\langle\xi\rangle}\hat{u}_\varepsilon(\xi)\tilde{d}\eta dy\tilde{d}\xi \\ &= \lim_{\delta\to+0} \int e^{ix\cdot\xi}a_\delta(x, \xi)\hat{u}_\varepsilon(\xi)\tilde{d}\xi, \end{aligned}$$

where $a_\delta(x, \xi)$ is given by

$$a_\delta(x, \xi) = \int\int e^{-iy\cdot\eta-\delta|y|^2-\delta|\xi+\eta|^2+\rho(\langle\xi+\eta\rangle-\langle\xi\rangle)}a(x+y)dy\tilde{d}\eta.$$

Putting

$$\langle \xi+\eta\rangle - \langle\xi\rangle = \sum_{j=1}^{n} \eta_j \int_0^1 (\xi_j+\theta\eta_j)\langle \xi+\theta\eta\rangle^{-1} d\theta$$
$$= \eta\cdot w(\xi,\eta),$$

we can re-write $a_\delta(x,\xi)$ using Stokes formula:

$$a_\delta(x,\eta) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y-i\rho w(\xi,\eta))\cdot\eta-\delta|y|^2-\delta|\xi+\eta|^2} a(x+y)dy\tilde{d}\eta$$
$$= \int_{\mathbb{R}^n} \tilde{d}\eta \int_{\mathbb{R}^n - iw(\xi,\eta)} e^{-iz\cdot\eta-\delta(z+i\rho w(\xi,\eta))^2\delta|\xi+\eta|^2} a(x+z+i\rho w(\xi,\eta))dz$$
$$= \int_{\mathbb{R}^n} \tilde{d}\eta \int_{\mathbb{R}^n} e^{-iy\cdot\eta-\delta(y+i\rho w(\xi,\eta))^2-\delta|\xi+\eta|^2} a(x+y+i\rho w(\xi,\eta))dy$$

for $\rho < \rho_0/n$, where we write $z^2 = \sum_{j=1}^{n} z_j^2$ for $z\in C^n$. Thus, by Taylor's expansion, we obtain

$$\lim_{\delta\to+0} a_\delta(x,\xi) = \mathrm{Os} - \int\int e^{-y\cdot\eta} a(x+y+i\rho w(\xi,\eta))dy\tilde{d}\eta$$
$$= a(x+i\rho w(\xi,0)) + r(\rho;x,\xi),$$

where

$$r(\rho;x,\xi) = \lim_{\delta\to 0}\int\int \Bigg(e^{-iy\cdot\eta-\delta(y+i\rho w(\xi,\eta)^2-\delta|\xi+\eta|^2}$$
$$\sum_{|\alpha|=1} \partial_\eta^\alpha\{D_y^\alpha a(x+y+i\rho w(\xi,\eta))\}\Bigg) dy\tilde{d}\eta$$

satisfies (2.13) (See Lemma 2.4 in [6]). Another application of Taylor's expansion yields

$$a(x+i\rho\omega(\xi,0))$$
$$= a(x+i\rho\xi\langle\xi\rangle^{-1})$$
$$= a(x) + \rho a_1(x,\xi) + \rho^2 a_2(\rho;x,\xi),$$

where $a_1(x,\xi)$ and $a_2(\rho;x,\xi)$ satisfy (2.11) and (2.12) respectively. Since

$$u_\varepsilon(x) \to u(x) \text{ in } \mathcal{S} \text{ as } \varepsilon\to+0,$$

we have

$$(e^{\rho\langle D\rangle} a \cdot e^{-\rho\langle D\rangle} u)(x) = \lim_{\varepsilon \to +0} a(\rho; x, D) u_\varepsilon(x),$$

if $u \in \mathcal{S}$. □

Let $P(t) = [p_{ij}(t, x, D)]_{i,j=1,\dots,d}$ be a matrix consisting of pseudo-differential operators whose symbols $p_{ij}(t, x, \xi)$ belong to the class $C([0, T]; S^1)$. Let us consider the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt} U(t) = P(t)U(t) + F(t), \ \ t \in (0, T), \\ U(0) = U_0, \end{cases} \tag{2.14}$$

where $U(t) = {}^t(U_1(t, x), \dots, U_d(t, x))$ is an unknown vector-valued function and $F(t) = {}^t(F_d(t, x), \dots, F_d(t, x))$, $U_0 = {}^t(U_{01}, \dots, U_{0d})$ are known vector-valued functions. Then we have:

PROPOSITION 2.4. *Suppose that* $\det(\lambda - p(t, x, \xi)) \neq 0$ *for* $\lambda \in C^1$ *with* $\Re\lambda > -c_0\langle\xi\rangle$, $t \in [0, T]$ *and* $|\xi| \gg 1$. *Take an arbitrary real number s. Then for any* $U_0 \in (H^{s+1}(\mathbb{R}^n))^d$ *and for any* $F(t) \in C^0([0, T]; (H^{s+1})^d))$, *there exists a unique solution* $U(t) \in C^1([0, T]; (H^s)^d \cap C^0([0, T]; (H^{s+1})^d)$ *of* (2.14).

This proposition will be used in Section 4 to prove existence of local solutions of the Cauchy problem (1.6). The proof of this proposition is given in Proposition 4.5 in [7].

3. - A priori estimates of solutions for the linear problem

Let $0 < T < \infty$ and $m(t)$ be a non-negative function in $C^0([0, T])$ and $\rho(t)$ a positive function in $C^1([0, T)) \cap C^0([0, T])$ such that $\rho_t(t) < 0$ for $t \in [0, T)$. Consider the following Cauchy Problem,

$$\begin{cases} (\partial_t - \Lambda_t)^2 v(t) + m(t) A_\Lambda v(t) = g(t), \ \ t \in (0, T) \\ v(0) = v_0, \\ \partial_t v(0) = v_1, \end{cases} \tag{3.1}$$

where $\Lambda(t) = \rho(t)\langle D\rangle$, $\Lambda_t(t) = \rho_t(t)\langle D\rangle$ and $A_\Lambda = e^{\Lambda(t)} A e^{-\Lambda(t)}$. Then by (ii) of Proposition 2.3 we have

$$A_\Lambda = A + \rho(t) a_1(x, D) + \rho(t)^2 a_2(\rho(t); x, D) + r(\rho(t); x, D), \tag{3.2}$$

where

$$a(x,\xi)=\sum_{i,j} a_{ij}(x)\xi_i\xi_j,$$

$$a_1(x,\xi)=-\sum_{|\alpha|=1} a_{(\alpha)}(x,\xi)\xi^\alpha\langle\xi\rangle^{-1}\in C^0([0,T];S^2),$$

$$a_2(t;x,\xi)\in C^0([0,T];S^2),$$

and

$$r(\rho(t);x,\xi)\in C^0([0,T];S^1).$$

Let $\tilde{m}(t)$ and $\lambda(t)$ be positive functions in $C^1([0,T])$ and assume $\lambda'(t)\le 0$ for $t\ge 0$. Define

$$E(t)^2=\frac{1}{2}\{\|(\partial_t-\Lambda_t)v(t)\|_s^2+\lambda(t)\|v(t)\|_{s+1}^2+\tilde{m}(t)(A\langle D\rangle^s v(t),\langle D\rangle^s v(t))_{L^2}\} \tag{3.3}$$

for $t\in[0,T)$, where $(\cdot,\cdot)_s$ and $\|\cdot\|_s$ stand for an inner product and a norm of H^s respectively.

Assume that $v(t)\in\bigcap_{j=0}^{2} C^{2-j}([0,T);H^{j+s})$ is a solution of (3.1). Differentiating (3.3) we have

$$\begin{aligned}
2E'(t)E(t)&=\Re(-m(t)A_\Lambda v+g,(\partial_t-\Lambda_t)v)_s\\
&\quad+\rho_t(t)\|(\partial_t-\Lambda_t)v\|_{s+\frac{1}{2}}^2\\
&\quad+\tilde{m}_t(t)(A\langle D\rangle^s v,\langle D\rangle^s v)_{L^2}\\
&\quad+\Re((\partial_t-\Lambda_t)v,v)_{s+1}\lambda(t)+\lambda'(t)\|v(t)\|_{s+1}^2\\
&\quad+\tilde{m}(t)\{\Re(\langle D\rangle^{-s}A\langle D\rangle^s v,(\partial_t-\Lambda_t)v)_s+\Re(\Lambda_t\langle D\rangle^{-s}Av,v)_s\}\\
&\quad+\rho_t(t)\|v\|_{s+\frac{3}{2}}^2\lambda(t)\\
&\le \Re(g,(\partial_t-\Lambda_t)v)_s+\tilde{m}(t)\Re(\Lambda_t\langle D\rangle^{-s}A\langle D\rangle^s v,v)_s\\
&\quad+|\tilde{m}_t(t)|((A\langle D\rangle^s v,\langle D\rangle^s v)_s\\
&\quad+\Re((\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^s-m(t)A_\Lambda)v,(\partial_t-\Lambda_t)v)_s\\
&\quad+\frac{1}{2}\rho_t(t)\|(\partial_t-\Lambda_t)v\|_{s+\frac{1}{2}}^2+\lambda(t)\left\{\rho_t+\frac{\lambda}{|\rho_t|}\right\}\|v\|_{s+\frac{3}{2}}^2\\
&\le \|g(t)\|_sE(t)+\tilde{m}(t)\Re(\Lambda_t)\langle D\rangle^{-s}A\langle D\rangle^s v,v)_s\\
&\quad+\frac{|\tilde{m}(t)|}{\tilde{m}(t)}E(t)^2\\
&\quad+\||\Lambda_t|^{-\frac{1}{2}}(\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^s-m(t)A_\Lambda)v\|_s^2\\
&\quad+\frac{1}{4}\rho_t(t)\|(\partial_t-\Lambda_t)v\|_{s+\frac{1}{2}}^2+\lambda\left\{\rho_t+\frac{\lambda}{|\rho_t|}\right\}\|v\|_{s+\frac{3}{2}}^2
\end{aligned} \tag{3.4}$$

for $t \in [0, T)$. Since A is a positive operator, by taking into account (2.2) we have

$$\begin{aligned}
&\Re(\Lambda_t\langle D\rangle^{-s}A\langle D\rangle^{s}v, v)_s \\
&= \rho_t(t)\Re(\langle D\rangle^{1-s}A\langle D\rangle^{s}v, v)_s \\
&\le \rho_t(t)(A\langle D\rangle^{s+\frac{1}{2}}v, \langle D\rangle^{s+\frac{1}{2}}v)_{L^2} + c|\rho_t(t)|\|v\|_{s+1}^2 \\
&\le c\,\frac{|\rho_t|}{\lambda(t)}\,E(t)^2
\end{aligned} \tag{3.5}$$

where c is a positive constant depending only on s and A.

The equality

$$\begin{aligned}
&\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^{s} - m(t)A_\Lambda \\
&= (\tilde{m}(t) - m(t))\langle D\rangle^{-s}A\langle D\rangle^{s} + m(t)(A - A_\Lambda) + m(t)(\langle D\rangle^{-s}A\langle D\rangle^{s} - A)
\end{aligned} \tag{3.6}$$

and (3.2) lead us to the estimate

$$\begin{aligned}
&\||\Lambda_t(t)|^{-\frac{1}{2}}(\tilde{m}(t)\langle D\rangle^{-s}A\langle D\rangle^{s} - m(t)A_\Lambda)v\|_s \\
&\le |\tilde{m}(t) - m(t)|\||\Lambda_t|^{-\frac{1}{2}}\langle D\rangle^{-s}A\langle D\rangle^{s}v\|_s \\
&\quad + m(t)\{\rho(t)\||\Lambda_t|^{-\frac{1}{2}}a_1v\|_s + \rho(t)^2\||\Lambda_t|^{-\frac{1}{2}}a_2v\|_s \\
&\qquad + \||\Lambda_t|^{-\frac{1}{2}}rv\|_s + cm(t)\|v\|_{s+1}\} \\
&\le |\rho_t(t)|^{-\frac{1}{2}}\Big\{c|\tilde{m}(t) - m(t)|\|v\|_{s+\frac{3}{2}} + m(t)\rho(t)\|a_1v\|_{s-\frac{1}{2}} \\
&\qquad + cm(t)\rho(t)^2\|v\|_{s+\frac{3}{2}} + \frac{cm(t)}{\sqrt{\lambda(t)}}\,E(t)\Big\}
\end{aligned} \tag{3.7}$$

for $t \in [0, T)$. Besides, by virtue of (2.3) we have

$$\begin{aligned}
\|a_1v\|_{s-\frac{1}{2}}^2 &\le c\{2c\|v\|_{s+\frac{1}{2}}^2 + \Re(Av, v)_{s+\frac{1}{2}}\} \\
&\le c\{3c\|v\|_{s+\frac{1}{2}}^2 + (A\langle D\rangle^{s+\frac{1}{2}}v, \langle D\rangle^{s+\frac{1}{2}}v)\},
\end{aligned} \tag{3.8}$$

where c is a positive constant depending only on s and the coefficients of A. Therefore, from (3.4) through (3.8), we come to the conclusion that

$$\begin{aligned}
2E'(t)E(t) &\le \|g(t)\|_sE(t) \\
&\quad + \left\{\frac{c|\rho_t(t)|}{\lambda(t)} + \frac{m(t)^2\rho^2}{|\rho_t(t)|\lambda(t)} + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)} + \frac{m(t)^2}{\lambda(t)|\rho_t(t)|}\right\}E(t)^2 \\
&\quad + \frac{\rho_t(t)}{4}\,\|(\partial_t - \Lambda_t)v\|_{s+\frac{1}{2}}^2
\end{aligned} \tag{3.9}$$

$$+\left\{\lambda(t)\left(\rho_t(t)+\frac{\lambda(t)}{|\rho_t(t)|}\right)+c\,\frac{|\tilde{m}(t)-m(t)|^2}{|\rho_t(t)|}+cm(t)^2\,\frac{\rho(t)^4}{|\rho_t(t)|}\right\}\|v\|^2_{s+\frac{3}{2}}$$

$$+\left\{\tilde{m}(t)\rho_t(t)+cm(t)^2\,\frac{\rho(t)^2}{|\rho_t(t)|}\right\}(A\langle D\rangle^{s+\frac{1}{2}},\langle D\rangle^{s+\frac{1}{2}}v)_{L^2}$$

for $t\in[0,T)$.

PROPOSITION 3.1. *Assume that* $m(t)$ *is a non-negative function in* $C^1([0,T])$. *Let* $\tilde{m}(t)=m(t)+\varepsilon e^{-\gamma t}$, $\lambda(t)=e^{-2\gamma t}$, $\rho(t)=\rho_1 e^{-\gamma t}$, *and* $v(t)\in\bigcap_{j=0}^{2} C^{2-j}([0,T);H^{j+s})$. *Then there are* $\varepsilon>0$ *and* $\gamma>0$ *such that if* $v(t)$ *satisfies* (3.1) *we have*

$$E(t)\le e^{\int_0^t p(\tau)d\tau}E(0)+\int_0^t e^{\int_\tau^t p(\sigma)d\sigma}\|g(\tau)\|_s d\tau \tag{3.10}$$

for $t\in[0,T)$, *where*

$$p(t)=c\gamma e^{\gamma t}+m(t)^2\,\frac{\rho_1^2}{\gamma}\,e^{\gamma t}+\frac{|m_t(t)|}{\tilde{m}(t)}+\frac{m(t)^2e^{3\gamma t}}{\gamma}. \tag{3.11}$$

PROOF. It suffices to prove that the terms in the right-hand side of (3.9) except for the first one and the second one are negative, if $\varepsilon>0$ and $\gamma>0$ are suitably chosen. In fact the third term is negative because of $\rho_t(t)<0$. The fourth term is

$$\begin{aligned}&\lambda(t)\left\{\rho_t(t)+\frac{\lambda(t)}{|\rho_t(t)|}\right\}+c\,\frac{|\tilde{m}(t)-m(t)|^2}{|\rho_t(t)|}+cm(t)^2\,\frac{\rho(t)^4}{|\rho_t(t)|}\\&=-\frac{\rho_1\gamma}{2}\,e^{-3\gamma t}+c\varepsilon^2\,\frac{e^{-\gamma t}}{\rho_1\gamma}+c\,\frac{\rho_1^3}{\gamma}\,m(t)^2e^{-3\gamma t}<0\end{aligned} \tag{3.13}$$

if we take

$$\gamma^2>\frac{3}{2\rho_1^2}+c\rho_1^2m(t)^2,\quad \varepsilon=\rho^{-\gamma T}. \tag{3.14}$$

Moreover we have the fifth term

$$\begin{aligned}&\tilde{m}(t)\rho_t(t)+cm(t)^2\,\frac{\rho(t)^2}{|\rho_t(t)|}\\&\le-\rho_1\gamma m(t)e^{-\gamma t}+c\,\frac{\rho_1}{\gamma}\,m(t)^2e^{-\gamma t}<0\end{aligned} \tag{3.15}$$

if we take $\gamma>0$ such that

$$\gamma^2>c\max_{0\le t\le T}m(t). \tag{3.16}$$

Therefore, choosing $\varepsilon > 0$ and $\gamma > 0$ such that (3.14) and (3.16) are valid, we obtain (3.11) from (3.9). □

For $m(t) \in L^1([0,T])$ and $\varepsilon > 0$, we define

$$\tilde{m}(t) = \int_0^T \chi_\varepsilon(t-\tau)m(\tau)d\tau + \varepsilon \tag{3.17}$$

where $\chi_\varepsilon(t) = \frac{1}{\varepsilon}\chi\left(\frac{t}{\varepsilon}\right)$ and $\chi(t) \in C_0^\infty((0,1))$ satisfying that $\chi(t) \geq 0$ and $\int_0^1 \chi(t)dt = 1$.

PROPOSITION 3.2. *Assume that $m(t)$ is a non-negative function in $C^1([0,T)) \cap L^1([0,T])$. Let $\tilde{m}(t)$ be a function defined by* (3.17) *and $v(t) \in \bigcap_{j=0}^{2} C^{2-j}(0,T); H^{s+j})$. Then there are $\rho(t)$ and $\lambda(t)$ in $C^1([0,T))$ with $\rho_t(t) \in L^1([0,T])$ and $\varepsilon > 0$ such that if $v(t)$ satisfies* (3.1) *we have*

$$E(t) \leq e^{\int_0^t p(\tau)d\tau} E(0) + \int_0^t e^{\int_\tau^t p(\sigma)d\sigma} \|g(\tau)\|_s d\tau \tag{3.18}$$

for $t \in [0,T)$, where $E(t) = E(t,s,\tilde{m}(t),\rho(t))$ is defined by (3.3) *and*

$$p(t) = \frac{c|\rho_t(t)|}{\lambda(t)} + \frac{m(t)^2}{|\rho_t(t)|}(\rho(t)^2+1) + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)}, \tag{3.19}$$

where c depends only on s and A.

PROOF. If we choose $\rho(t)$ and $\varepsilon > 0$ suitably, we can prove that the terms in the right-hand side of (3.9) except for the first one and the second one are negative. We can take $\rho(t)$ with $\rho_t(t) < 0$ such that the first terms of (3.13) and (3.15) are negative respectively. In fact, it suffices to find a function $\rho(t)$ satisfying

$$\begin{cases} \rho_t(t) \leq -c\left\{\dfrac{|\tilde{m}(t)-m(t)|}{\sqrt{\lambda(t)}} + \dfrac{m(t)\rho(t)^2}{\sqrt{\lambda(t)}} + \dfrac{m(t)\rho(t)}{\sqrt{\tilde{m}(t)}} + \sqrt{\lambda(t)}\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \in (0,T)) \\ \rho(0) = \rho_1. \end{cases} \tag{3.20}$$

Put

$$\rho(t) = \rho_1 e^{-ct} - \int_0^t \frac{|\tilde{m}(\tau) - m(\tau)|}{\sqrt{\rho_1}} d\tau, \tag{3.21}$$

$$\lambda(t) = \rho_1^2 e^{-2c\left\{t + \int_0^t m(\tau)(1+1/\tilde{m}(\tau))d\tau\right\}}.$$

Here we take $\varepsilon > 0$ sufficiently small such that $\rho(t) > 0$ for $t \in [0, T)$. Since $\rho(t) \le \sqrt{\lambda(t)}$ and $\lambda(t) \le \lambda(0)$, we can see easily that $\rho(t)$ defined by (3.21) satisfies (3.20). Hence, we obtain (3.18) from (3.9) defining $p(t)$ by (3.19). □

4. - Existence of solutions for the linear problem

In this section, we consider the following linear Cauchy problem:

$$\begin{cases} \partial_t^2 u(t) + m(t)Au(t) = f(t), \ t \in (0, T) \\ u(0) = u_0, \\ \partial_t u(0) = u_1. \end{cases} \tag{4.1}$$

Following the idea of the proof of the theorem in [5], we shall prove that the Cauchy problem (4.1) has a unique solution.

THEOREM 4.1. *Assume that* (1.2) *and* (1.3) *are valid. Let* $0 < \rho_1 < \rho_0/\sqrt{n}$, $s \in \mathbb{R}$ *and* $m(t) \in C^0([0, T])$. *Then there is* $\gamma > 0$ *such that for any* $u_0 \in H^{s+2}_{\rho_1}$, $u_1 \in H^{s+2}_{\rho_1}$ *and* $e^{\Lambda(t)} f(t) \in C^0([0, t]; H^{s+1})$, (4.1) *has a unique solution* $u(t)$ *satisfying* $e^{\Lambda(t)} u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j})$, *where* $\Lambda(t) = \rho_1 e^{-\gamma t}\langle D\rangle$. *Moreover if* $m(t) \in C^1([0, T])$, *the solution* $u(t)$ *satisfies*

$$\begin{aligned} &\{\|e^{\Lambda(t)}\partial_t u(t)\|_s^2 + e^{-2\gamma t}\|e^{\Lambda(t)} u(t)\|_{s+1}^2\}^{1/2} \\ &\le e^{\int_0^t p(\sigma)d\sigma} \Big[\{\|e^{\rho_1\langle D\rangle} u_1\|_s^2 + (m(0)+\varepsilon)(Ae^{\rho_1\langle D\rangle}\langle D\rangle^s u_0, e^{\rho_1\langle D\rangle}\langle D\rangle^s u_0)_{L^2} \\ &\quad + \|e^{\rho_1\langle D\rangle} u_0\|_{s+1}^2\}^{\frac{1}{2}} + \int_0^t \|e^{\Lambda(\sigma)} f(\sigma)\|_s d\sigma\Big], \end{aligned} \tag{4.2}$$

for $t \in [0, T]$, *where* $p(t)$, γ *and* ε *are given by Proposition* 3.1.

PROOF. Put $v(t) = e^{\Lambda(t)} u(t)$. If $v(t)$ is a solution of (3.1), it is evident that $u(t)$ satisfies (4.1). So it suffices to prove that problem (3.1) has a solution.

Now we put

$$V_1(t) = \langle D\rangle v(t),$$
$$V_2(t) = (\partial_t - \Lambda_t)v(t),$$
$$V(t) = {}^t(V_1(t), V_2(t)).$$

Then if $v(t)$ is a solution of (3.1), $V(t)$ satisfies

$$\begin{cases} \dfrac{d}{dt}V(t) = P(t)V(t) + F(t), & t > 0 \\ V(t) = V_0, \end{cases} \tag{4.3}$$

where $F(t) = {}^t(0, g(t))$, $V_0 = {}^t(v_0, v_1)$ and

$$P(t) = \begin{pmatrix} \Lambda_t & \langle D\rangle \\ m(t)A_\Lambda\langle D\rangle^{-1} & \Lambda_t \end{pmatrix}. \tag{4.4}$$

Conversely, it is evident that if $V(t)$ is a solution of (4.3), then $v(t) = \langle D\rangle^{-1}V_1(t)$ becomes a solution of (3.1). It follows from (4.4) and (ii) of Proposition 2.3 that $P(t)$ is a pseudo-differential operator of order 1 with symbol satisfying

$$\det(\lambda I - p(t; x, \xi)) = (\lambda + \gamma\rho(t)\langle\xi\rangle)^2 - \gamma^2\rho(t)\langle\xi\rangle$$
$$+m(t)\{a(x, \xi) + \rho(t)a_1(x, \xi) + \rho(t)^2a_2(a, \xi) + r(x, \xi)\}.$$

Since $m(t) \geq 0$, $a(x, \xi) \geq 0$ and $r \in S^1$ there are $\gamma_0 > 0$ and $R_0 > 0$ such that $\det(\lambda - p(t, x, \xi)) \neq 0$ for $\Re\lambda \geq -2^{-1}\gamma e^{-\gamma T}\langle\xi\rangle$, $\gamma \geq \gamma_0 \sup_{0\leq t\leq T} m(t)$ and $|\xi| \geq R_0\gamma^{-1}e^{2\gamma T}$. Therefore it follows from Proposition 2.4 that there exists a unique solution $V(t)$ of (4.3) and consequently $v(t) = \langle D\rangle^{-1}V_1(t)$ satisfies (3.1) and belongs to $\bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j})$. Put $u(t) = e^{-\Lambda(t)}v(t)$. Then $u(t)$ satisfies (4.1) and $e^{\Lambda(t)}u(t)$ is in $\bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j})$. If $m(t)$ is in $C^1([0, T])$, it follows from Proposition 3.1 that $v(t)$ satisfies (3.10) so $u(t)$ satisfies (4.2). In particular, if $u_0 = u_1 \equiv 0$, $f(t) \equiv 0$ and $e^{\Lambda(t;\gamma)}u(t) \subset \bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j})$ for some $\gamma > 0$, $u(t)$ identically vanishes. This implies the uniqueness of the the solution of (4.1). Note that $v(t)$ may depend on γ but $u(t) = e^{-\Lambda(t)}v(t)$ does not depend on γ. In fact, $\tilde{u}(t) = u(t; \gamma) - u(t; \gamma')$ satisfies (4.1) with $u_0 = u_1 = f(t) \equiv 0$ and $e^{\Lambda(t;\overline{\gamma})}\tilde{u}(t) \in \bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j})$, where $\overline{\gamma} = \max(\gamma, \overline{\gamma})$. Therefore we have $\tilde{u}(t) \equiv 0$ from the uniqueness of solution of (4.1), and consequently $u(t; \gamma) = u(t; \gamma')$. □

Finally we remark that it follows from (4.2) that $u(t)$, the solution of (4.1), satisfies

$$
(4.5)\qquad
\begin{aligned}
&\|\partial_t u(t)\|_s + e^{-\gamma t}\|u(t)\|_{s+1}\\
&\le \|e^{\Lambda(t)}\partial_t u(t)\|_s + e^{-\gamma t}\|e^{\Lambda(t)}u(t)\|_{s+1}\\
&\le c e^{\int_0^t p(\sigma)d\sigma}\left\{\|e^{\rho_1\langle D\rangle}u_1\|_s + \|e^{\rho_1\langle D\rangle}u_0\|_{s+1} + \int_0^t \|e^{\Lambda(\sigma)}f(\sigma)\|_s d\sigma\right\}
\end{aligned}
$$

for $t \in [0, T]$, where the positive constant c is independent of γ.

5. - Local existence of solutions of the nonlinear problem

Let $0 \le \tau < T_1 < \infty$. For $T \in (\tau, T_1]$ we consider the Cauchy problem

$$
(5.1)\qquad
\begin{cases}
\partial_t^1 u(t) + M((Au(t), u(t)))Au(t) = f(t), \quad \tau < t < T\\
u(\tau) = u_0,\\
u(\tau) = u_1.
\end{cases}
$$

THEOREM 5.1. *Assume that the conditions* (1.2), (1.3) *and* (1.4) *are valid. Let* $s \in \mathbb{R}$ *and* $0 < \rho_2 < \rho_0/\sqrt{n}$. *Then for any* $u_0 \in H^2_{\rho_2}$, $u_1 \in H^1_{\rho_2}$ *and* $e^{\Lambda(t)}f(t) \in C^0([\tau, T_1]; H^1)$ *where* $\Lambda(t) = \rho_2 e^{-\gamma(t-\tau)}\langle D\rangle$, *there are* $T \in (\tau, T_1]$ *and* $\gamma_0 > 0$ *such that the Cauchy problem* (4.1) *has a unique solution* $u(t)$ *satisfying* $e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([\tau, T]; H^j)$ *for any* $\gamma \ge \gamma_0$.

PROOF. We may assume $\tau = 0$ without loss of generality. We shall prove the existence of solutions of (5.1) by the principle of *contraction mapping*. For $T > 0$ and $s \in \mathbb{R}$, we introduce a space of functions

$$
X_T^s = C^0([0, T]; H^{s+1}) \cap C^1([0, T]; H^s)
$$

equipped with its *norm* $\|\cdot\|_{X_T^s}$ as

$$
(5.2)\qquad \|w\|_{X_T^s} = \sup_{0\le t\le T}\left\{\frac{1}{2}(\|\partial_t u(t)\|_s^2 + \|u(t)\|_{s+1}^2)\right\}^{1/2}
$$

for every $w \in X_T^s$. We now define two functions

$$
(5.3)\qquad
\begin{aligned}
&m(t) = m(t; w) = M(\eta(t; w)),\\
&\eta(t; w) = \sum_{i,j=1}^{n}(a_{ij}D_j w(t), D_i w(t))_{L_2},
\end{aligned}
$$

for each $w \in X_T^1$. Note that $m(t) \in C^1([0,T])$ if $w \in X_T^1$, and it satisfies

$$\sup_{0 \le t \le T} \{m(t) + |m'(t)|\} \le K(\|w\|_{X_T^1}), \tag{5.4}$$

where K is a positive and continuous function defined in $[0, \infty)$.

Let us consider the Cauchy problem (4.1) with $m(t) = m(t; w)$. Then it follows from Theorem 4.1 that there exists a unique solution $u(t)$ of (4.1) satisfying that $e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T]; H^j)$, where $\Lambda(t) = \rho_1 e^{-\gamma t}\langle D\rangle$. So the correspondence with each $w \in X_T^1$ to $u \in X_T^1$ defines a map

$$\Psi : X_T^1 \ni w \mapsto u \in X_T^1$$

such that

$$\Psi(w) = u;\ \ \partial_t^2 u + m(t; w)Au = f,\ \ u(0) = u_0,\ \ \partial_t u(0) = u_1.$$

We shall prove that Ψ is a contraction mapping if T is sufficiently small. For $k > 0$, let us define a set

$$B_T(k) = \left\{ e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T]; H^j); \|u\|_{X_T^1} \le k \right\}.$$

Then we can prove that for every $k \gg 1$ there is a real number $T = T(k) > 0$ such that $\Psi(w) \in B_T(k)$ as long as $w \in B_T(k)$. Actually, we can gain an estimate

$$\|\Psi(w)\|_{X_T^1} \le c e^{\int_0^T (p(t)+\gamma)dt} \quad \text{for } w \in B_T(k), \tag{5.5}$$

which is deduced from the estimate (4.5) with $s = 1$ and the fact that

$$\Lambda(t, \xi) = \rho_2 e^{-\gamma t}\langle \xi \rangle \le \rho_2 \langle \xi \rangle.$$

Note that the constant c appearing in (5.5) is independent of T, k and w. Since $p(t)$ is determined by (3.1) and (5.3), we can find a function $\overline{p}(t, k) \in C^0([0,T] \times [0, \infty))$, by virtue of (5.4), such that

$$p(t) + \gamma \le \overline{p}(t, k) \quad t \in (0, T)$$

if $w \in B_T(k)$. Since the constant c in (5.5) is independent of k and the function $\overline{p}(t, k)$ is continuous in (t, k), we can find $T = T(k) > 0$ such that

$$c e^{\int_0^T \overline{p}(t,k)dt} = k$$

for every $k > c$. Hence (5.5) implies that $\Psi(w)$ belongs to $B_T(k)$ provided $w \in B_T(k)$.

Next we shall prove Ψ is Lipschitz continuous in X_T^0, that is, with sufficiently small $T > 0$ we have the inequality

$$\|\Psi(w) - \Psi(w')\|_{X_T^0} \leq \frac{1}{2}\|w - w'\|_{X_T^0} \tag{5.6}$$

for any w, $w' \in B_T(k)$. Since the difference $\Psi(w) - \Psi(w')$ satisfies

$$\begin{aligned}
&(\partial_t^2 + m(t;w)A)(\Psi(w) - \Psi(w')) = (m(t;w') - m(t;w))A\Psi(w'), \ t > 0;\\
&(\Psi(w) - \Psi(w'))(0) = 0,\\
&\partial_t(\Psi(w) - \Psi(w'))(0) = 0,
\end{aligned}$$

we obtain, by virtue of (4.5) with $s = 0$

$$\begin{aligned}
\|\Psi(w) - \Psi(w')\|_{X_T^0} &\leq ce^{\int_0^T p(\sigma)d\sigma}\\
&\times \int_0^T |m(\sigma;w) - m(\sigma;w')|\|e^{\Lambda(\sigma)}A\Psi(w')\|_{L^2}d\sigma.
\end{aligned} \tag{5.7}$$

On the other hand, an application of Proposition 2.3 to A and the estimate (4.5) with $s = 1$ yield

$$\begin{aligned}
\|e^{\Lambda(\sigma)}A\Psi(w')(\sigma)\|_{L^2} &\leq c\|e^{\Lambda(\sigma)}\Psi(w')(\sigma)\|_2\\
&\leq c_1 e^{\int_0^\sigma \overline{p}(\tau,k)d\tau} \leq C_1(k)
\end{aligned}$$

for $w' \in B_T(k)$. Moreover, taking into account (5.4) we gain

$$\begin{aligned}
|m(\sigma;w) - m(\sigma;w')| &\leq \|M((Aw,w)) - M((Aw',w'))|\\
&\leq C_2(k)\|w - w'\|_{X_T^0}.
\end{aligned}$$

Hence, from (5.7) we have $C_3(k) > 0$ satisfying

$$\|\Psi(w) - \Psi(w')\|_{X_T^0} \leq C_3(k)T\|w - w'\|_{X_T^0}$$

for w, $w' \in B_T(k)$, which proves assertion (4.6) if $T \leq (2C_3(k))^{-1}$.

Thus once we choose $T = \min\{T(k), (2C_3(k))^{-1}\}$, we can find the solution u of (5.1) with the initial plane $\tau = 0$ which belongs to $B_T(k)$. □

6. - Existence of time global solutions for the nonlinear problem

In this section we shall prove our *main theorem*. According to D'Ancona and Spagnolo [3], we introduce the following energy,

$$e(t)^2 = \frac{1}{2}\{\|\partial_t u(t) + u(t)\|^2 + \|u(t)\|^2 + F(\eta(t))\} \tag{6.1}$$

where $F(\eta) = \int_0^\eta M(\lambda)d\lambda$, $\eta(t) = ((Au(t), u(t))_{L^2}$ and $\|\cdot\|$ stands for a norm of $L^2(\mathbb{R}^n)$.

PROPOSITION 6.1 ([3]). *Assume that $M(\eta)$ is a non-negative continuous function in $[0,\infty)$ and $f(t) \in C^0([0,T]; L^2)$. If $u(t)$ is a solution of the Cauchy problem of* (1.1) *in* $(0,T)$ *such that* $u \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j)$, *then we have the energy inequality*

$$e(t)^2 + \int_0^t e^{\frac{5}{2}(t-\tau)} M(\eta(\tau))\eta(\tau)d\tau \le e^{\frac{5}{2}t}e(0)^2 + \int_0^t e^{\frac{5}{2}(t-\tau)}\|f(\tau)\|^2 d\tau \tag{6.2}$$

for $t \in [0,T)$.

PROOF. Differentiating (6.1), we get from (1.1)

$$\begin{aligned}\frac{d}{dt}(e(t)^2) &= \Re(f(t) + \partial_t u(t), \partial_t u(t) + u(t)) - M(\eta(t))\eta(t)\\ &\le \frac{1}{2}\|f(t)\|^2 + \frac{5}{2}e(t)^2 - M(\eta(t))\eta(t)\end{aligned}$$

for $t \in [0,T)$, which yields (6.2). □

PROPOSITION 6.2 ([3]). *If* (6.2) *holds and* $T < \infty$, *then* $M(\eta(t)) \in L^1([0,T])$.

PROOF. From (6.2), it is evident that $M(\eta(t))\eta(t) \in L^1([0,T])$. On the other hand

$$\begin{aligned}\int_0^t M(\eta(\tau))d\tau &= \int_{[0,t]\cap\{\tau;\eta(\tau)>1\}} M(\eta(\tau))d\tau + \int_{[0,t]\cap\{\tau;\eta(\tau)\le 1\}} M(\eta(\tau))d\tau\\ &\le \int_0^t M(\eta(\tau))\eta(t)d\tau + t\sup_{0\le\eta\le 1} M(\eta)\end{aligned}$$

for all $t \in [0,T)$, which implies that $M(\eta(t)) \in L^1([0,T])$, □

Now we can prove our main theorem. Let $\Lambda(t,\gamma) = \rho_1 e^{-\gamma t\langle D\rangle}$, and let T^* be a real number defined by

$$T^* = \max\left\{ T > 0;\ \text{there exist } \gamma > 0 \text{ and a solution } u(t) \text{ satisfying (1.1)} \right.$$
$$\left. \text{in } (0,T) \text{ such that } e^{\Lambda(t,\gamma)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T);H^j) \right\}.$$

Theorem 4.1 ensures that $T^* > 0$. We claim that $T^* = \infty$. Suppose that $T^* < \infty$. Then it follows from Proposition 6.2 that $m(t) = M((Au(t),u(t))$ is in $L^1([0,T^*])$. Hence, Proposition 3.2 and the fact that $m(t) \in C^1([0,T^*))\cap L^1([0,T^*])$ yield that $v(t) = e^{\Lambda(t)}u(t)$ which satisfies (3.18) with $s = 0,1$ and $T = T^*$, where $\Lambda(t) = \rho(t)\langle D\rangle$ and $\rho(t)$ is what is introduced in (3.21). Let us take $\gamma > 0$ such that $\rho_1 e^{-\gamma t} \leq \rho(t)$ for $t \in [0,T^*)$. Then the definition of T^* and (3.18) imply $e^{\Lambda(t,\gamma)}u(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T^*];H^j)$, where $\Lambda(t,\gamma) = \rho_1 e^{-\gamma t}\langle D\rangle$. Hence we have the limits $u(T^*-0)$ and $\partial_t u(T^*-0)$ which satisfy $e^{\Lambda(T^*,\gamma)}u(T^*-0) \in H^2$ and $e^{\Lambda(T^*,\gamma)}\partial_t u(T^*0) \in H^1$. Therefore, applying Theorem 5.1 with $\rho_2 = \rho_1 e^{-\gamma T^*}$, we have a solution $\tilde{u}(t)$ of the Cauchy problem (5.1) in $(T^*,T)(T > T^*)$ with initial data $\tilde{u}(T^*) = u(T^*-0)$ and $\partial_t\tilde{u}(T^*) = \partial_t u(T^*-0)$, which satisfies

$$\exp(\rho_2 e^{-\gamma(t-T^*)}\langle D\rangle)\tilde{u}(t) \in \bigcap_{j=0}^{2} C^{2-j}([T^*,T];H^j).$$

Then $\Lambda(t,\gamma) = \rho_2 e^{-\gamma(T-T^*)}\langle D\rangle$ implies that $e^{\Lambda(t,\gamma)}\tilde{u}(t) \in \bigcap_{j=0}^{2} C^{2-j}([T^*,T];H^j)$. Now let us define

$$w(t) = \begin{cases} u(t), & t \in (0,T^*) \\ \tilde{u}(t), & t \in [T^*,T). \end{cases}$$

Then $w(t)$ has to satisfy (1.1) in $(0,T)$ and $e^{\Lambda(t,\gamma)}w(t) \in \bigcap_{j=0}^{2} C^{2-j}([0,T);H^j)$. This contradicts the definition of T^*. Thus, we have proved that $T^* = \infty$. Since $M(\eta)$ is of class C^1, we can prove easily the uniqueness of the solution of (1.1). □

REFERENCES

[1] L. AROSIO - S. SPAGNOLO, *Global solution of the Cauchy problem for a nonlinear hyperbolic equation*, Pitman Reseach Notes in Math. **109** (1984), 1-26.

[2] S. BERNSTEIN, *Sur une classe d'équations fonctionnelles aux dérivées partielles*. Izv. Akad. Nauk SSSR, Ser. Mat. **4** (1940), 17-26.

[3] P. D'ANCONA - S. SPAGNOLO, *Global solvability for the degenerate Kirchhoff equation with real analytic data*. Invent. Math. **108** (1992), 247-246.

[4] C. FEFFERMAN - D.H. PHONG, *On positivity of pseudo-differential operators*. Proc. Nat. Acad. Sci. USA **75** (1978), 4677-4674.