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On Veech’s Conjecture for Harmonic Functions

W. HANSEN - N. NADIRASHVILI

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday

0. - Introduction

Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$. For every $x \in \mathbb{R}^d$ and $r > 0$ let $B(x, r) = \{ y \in \mathbb{R}^d : |y - x| < r \}$ and $\lambda_{B(x,r)} = \lambda(B(x, r))^{-1} B(x, r) \lambda$. A function $r > 0$ on a domain $U$ in $\mathbb{R}^d$ is called admissible provided $B(x, r(x)) \subset U$ for every $x \in U$. Given an admissible function $r$ on $U$, let us say that a Lebesgue measurable real function $f$ on $U$ is $r$-median if

$$f(x) = \lambda_{B(x,r(x))}(f)$$

for every $x \in U$. In [HN1, HN2, HN5] we proved the following converse to the mean value theorem for harmonic functions (for the case $U = \mathbb{R}^d$ see [HN4]):

**Theorem 0.1.** Let $r$ be an admissible function on a proper subdomain $U$ of $\mathbb{R}^d$. Let $f$ be an $r$-median function on $U$ which is bounded by some harmonic function on $U$ and suppose that $f$ is continuous or that $r$ is locally bounded away from zero. Then $f$ is harmonic.

Simple counterexamples reveal that the boundedness condition for $f$ cannot be completely dropped. However, under additional assumptions on $r$ boundedness from one side is sufficient. Work of J.R. Baxter [Ba2], A. Cornea and J. Veselý [CV], W.A. Veech [Ve3], led to the following (for a detailed account of the history see [NV]):

**Theorem 0.2.** Let $U$ be a Green domain in $\mathbb{R}^d$, let $\rho, r : U \to ]0, \infty[$ and $\alpha > 0$ be such that, for all $x, y \in U$:

$$\rho(x) \leq \text{dist}(x, \partial U), \quad |\rho(x) - \rho(y)| \leq |x - y|, \quad \alpha \rho(x) < r(x) < (1 - \alpha) \rho(x).$$

Then any $r$-median function $f \geq 0$ on $U$ is harmonic.

About twenty years ago W.A. Veech ([Ve3]) formulated the:

**Conjecture 0.3.** Let $U$ be a bounded domain in $\mathbb{R}^d$ and let $r$ be an admissible function on $U$ which is locally bounded away from zero, i.e., such that $\inf r(K) > 0$ for any compact subset $K$ of $U$. Then every $r$-median function $f \geq 0$ on $U$ is harmonic.

Previous work by F. Huckemann [Hu] shows that this is true if $d = 1$.

In this paper we shall see, however, that the conjecture fails already for open balls in any $\mathbb{R}^d$, $d \geq 2$. In fact, even a weakened version of the conjecture where $r$ and $f$ are assumed to be continuous (or $C^\infty$) is wrong. As in [HN3] our counterexample will be based on properties of the random walk given by the transition kernel $P : (x, A) \mapsto \lambda_{B(x,r(x))}(A)$.

1. - A measurable counterexample

Let $U$ denote the open unit ball in $\mathbb{R}^d$, $d \geq 2$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\Omega = U^{\mathbb{N}_0}$, $X_i(\omega) = \omega_i$ and let $\mathcal{M}$ be the $\sigma$-algebra on $\Omega$ generated by $X_i$, $i \in \mathbb{N}_0$. As usual $\theta_j$, $j \in \mathbb{N}_0$, will be the canonical shift $\theta_j : \Omega \to \Omega$ defined by $(\theta_j \omega)_i = \omega_{i+j}$, i.e., $X_i \circ \theta_j = X_{i+j}$. Given a Markov kernel $P$ on $U$ and $x \in U$, let $P^x$ denote the probability measure on $(\Omega, \mathcal{M})$ such that $(\Omega, \mathcal{M}, P^x)$ is the random walk starting at $x$ having transition kernel $P$, i.e., for all $n \in \mathbb{N}_0$ and Borel subsets $A_0, A_1, \ldots, A_n$ of $U$

$$P^x[X_0 \in A_0, X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n]$$

$$(*)$$

$$= \varepsilon_x(A_0) \int_{A_1} P(x, dx_1) \int_{A_2} P(x_1, dx_2) \ldots \int_{A_n} P(x_{n-1}, dx_n).$$

For every Borel subset $A$ of $U$ let

$$T_A := \inf \{i \in \mathbb{N}_0 : X_i \in A\}$$

(where $\inf \emptyset = \infty$). A Borel function $v \geq 0$ on $U$ is called $P$-supermedian if $Pv \leq v$. We recall that the function $x \mapsto P^x[T_A < \infty]$ is the smallest $P$-supermedian function $v$ on $U$ such that $v \geq 1$ on $A$ ([DM], [Re]). Moreover, we shall use the (strong) Markov property for random walks (cf. [DM], [Re]) and we shall exploit the following simple fact which is intuitively clear and can easily be derived formally from $(*)$: If $P$ and $Q$ are two Markov kernels on $U$ such that $P(x, \cdot) = Q(x, \cdot)$ for every $x$ in the complement of a Borel set $B$ then

$$P^x[X_0 \in A_0, \ldots, X_n \in A_n, n \leq T_B] = Q^x[X_0 \in A_0, \ldots, X_n \in A_n, n \leq T_B]$$
for all \( x \in U, \ n \in \mathbb{N}_0, \) and Borel sets \( A_0, \ldots, A_n \) in \( U. \) In particular, 
\[ P^x[T_B = n] = Q^x[T_B = n] \] for every \( n \geq 0. \)

Now let us choose points \( x_n = (\xi_n, 0, \ldots, 0) \in \mathbb{R}^d \) and radii \( 0 < \rho_n < 1, \ n \in \mathbb{N}_0, \) such that \( \xi_0 = 0, \ \xi_n < \xi_{n+1}, \ \lim_{n \to \infty} \xi_n = 1, \)

\[ \{ x_{n+1} \} = B(x_n, \rho_n) \cap \{ x_n : j \neq n \}, \quad x_{n+1} \notin B(x_n, \rho_n/2). \]

More precisely, fix \( 0 < \alpha < 1/4, \) define

\[ \alpha_n := (1 - \alpha)^n \alpha, \quad n = -1, 0, 1, 2, \ldots, \]

and, for every \( n \in \mathbb{N}_0, \)

\[ \xi_n := \sum_{j=0}^{n-1} \alpha_j = 1 - (1 - \alpha)^n, \quad x_n := (\xi_n, 0, \ldots, 0) \in \mathbb{R}^d, \]

\[ \rho_n := \frac{\alpha_{n-1} + \alpha_n}{2} = \left( 1 - \frac{\alpha}{2} \right) \alpha_{n-1}. \]

Then, for every \( n \in \mathbb{N}_0, \)

\[ \alpha_n - \frac{\rho_n}{2} = (1 - \alpha)\alpha_{n-1} - \frac{\rho_n}{2} > (1 - \alpha)\rho_n - \frac{\rho_n}{2} > \alpha \rho_n, \]

\[ \alpha_n + \alpha_{n+1} = (1 - \alpha + (1 - \alpha)^2)\alpha_{n-1} > (2 - 3\alpha)\alpha_{n-1} > \alpha_{n-1}, \]

\[ \alpha_{n-1} - \rho_n = \left( 1 - \frac{\alpha}{2} \right) \rho_n > \frac{\alpha}{2} \rho_n, \]

\[ \rho_n - \alpha_n = \left( 1 - \frac{1 - \alpha}{1 - \alpha} \right) \rho_n > \frac{\alpha}{2} \rho_n. \]

Choosing real numbers \( c_n > 0 \) such that \( c_n \leq \inf(\alpha/5, 2^{-n})\rho_n \) and defining

\[ C_n := B(x_n, c_n), \quad n = 0, 1, 2, \ldots, \]

we hence obtain that, for every \( x \in C_0, \)

\[ B(x, \rho_0/2) \cap C_1 = B(x, \rho_0) \cap C_2 = \emptyset, \quad C_1 \subset B(x, \rho_0) \]

and, for every \( x \in C_n, \ n \geq 1, \)

\[ B(x, \rho_n/2) \cap C_{n+1} = B(x, \rho_n) \cap (C_{n-1} \cup C_{n+2}) = \emptyset, \quad C_{n+1} \subset B(x, \rho_n). \]
Moreover, for every $n \in \mathbb{N}_0$ and every $x \in C_n$,

$$\rho_n \leq 2\alpha(1 - |x|)$$

since $2\alpha(1 - |x|) \geq 2\alpha(1 - \xi_n - c_n) = 2\alpha_n - 2\alpha c_n \geq \alpha_{n-1} - 2\alpha c_n \geq \rho_{n-1} - 2\alpha c_n \geq \rho_n$.

Take $0 < b_0 < c_0/2$, $a_0 = b_0/2$, and define

$$A_0 = B(0, a_0), \quad B_0 = B(0, b_0).$$

In order to understand the idea of our counterexample let us assume for a moment that we have chosen real numbers $0 < a_n < c_n$, $n \in \mathbb{N}$. Then we take $A_n = B(x_n, a_n)$ and we may define a measurable function $f \geq 0$ on $U$ by

$$f = \begin{cases} 
\prod_{j=1}^{n} \frac{\rho_{j-1}^d - a_{j-1}^d}{a_j^d} & \text{on } A_n, n \geq 0, \\
0 & \text{on } U \setminus \bigcup_{n=0}^{\infty} A_n.
\end{cases}$$

Obviously, $f$ is not harmonic, but it is $r$-median if we define

$$r(x) = \begin{cases} 
\rho_n, & x \in A_n, n \geq 0, \\
\inf \left( \alpha(1 - |x|), \text{dist} \left( x, \bigcup_{n=0}^{\infty} A_n \right) \right), & x \in U \setminus \bigcup_{n=0}^{\infty} A_n.
\end{cases}$$

Of course, $r$ is not locally bounded away from zero. We may, however, modify this construction and obtain an $r$-median function for which $r$ is locally bounded away from zero. To that end we shall arrange that the random walk given by the kernel

$$P : (x, A) \mapsto \lambda_{B(x, r(x))}(A)$$

has almost no chance to get to $A_{n+1}$ except to go to $A_n$ first and then to hit $A_{n+1}$ at the next step.

Let us see how this can be achieved. We choose a continuous function $0 < \rho \leq \rho_0$ on $U \setminus \{x_n : n \in \mathbb{N}\}$ such that $\rho = \rho_0$ on $A_0$ and

$$\rho(x) \leq \inf \left( \alpha(1 - |x|), \frac{1}{3} \inf_{n \in \mathbb{N}} |x - x_n| \right)$$

for every $x \in U \setminus (B_0 \cup \{x_n : n \in \mathbb{N}\})$, take

$$C'_n = B(x_n, c_n/2), \quad C' = \bigcup_{n=1}^{\infty} C'_n,$$
and consider the Markov kernel $Q$ given by
\[
Q(x, \omega) = \begin{cases} 
\lambda_{B(x, \rho(x))}, & x \in U \setminus C', \\
\varepsilon_x, & x \in C'. 
\end{cases}
\]
Using the associated random walk we define a function $g$ on $U$ by
\[
g(x) = Q^x[T_{A_0} < \infty], \quad x \in U.
\]
Obviously,
\[
\liminf_{|x| \to 0} g(x) \geq \lim_{|z| \to \rho_0} Q^z[X_1 \in A_0] = \left( \frac{a_0}{\rho_0} \right)^d > 0.
\]
Since $\rho$ is continuous and $Qg = g$ on $U \setminus A_0$, we know that $g$ is continuous on $U \setminus (A_0 \cup C')$ and that \( \{ y \in U \setminus (A_0 \cup C') : g(y) = 0 \} \) is an open set in $U \setminus (A_0 \cup C')$. Therefore $g > 0$ on $U \setminus C'$ and, for every $n \in \mathbb{N}$,
\[
\beta_n := \inf \left\{ \frac{c}{2} \leq |y - x_n| \leq c_n \right\} > 0.
\]
Now let $n \in \mathbb{N}_0$ and suppose that real numbers $a_j \in [0, c_j/4]$, $j = 0, \ldots, n$, have already been chosen. Define
\[
\gamma_n := \prod_{j=1}^{n} \left( \frac{\rho_{j-1}}{a_j} \right)^d
\]
(in particular, $\gamma_0 = 1$). Then there exists $b_{n+1} \in [0, c_{n+1}/2]$ such that $B_{n+1} := B(x_{n+1}, b_{n+1})$ satisfies
\[
R^B_{n+1} := \inf \{ s : s \geq 0 \text{ superharmonic on } U, s \geq 1 \text{ on } B_{n+1} \} \leq 2^{-(n+2d)} \beta_n / \gamma_n \text{ on } U \setminus C'_{n+1}.
\]
Take
\[
a_{n+1} = \frac{b_{n+1}}{2}, \quad A_{n+1} = B(x_{n+1}, a_{n+1}).
\]
By Harnack’s inequality there exists $c \in \mathbb{R}^+$ such that for every harmonic function $h \geq 0$ on $U$
\[
h(x_2) \leq ch(x_1)
\]
and we clearly may assume that $b_2$ and hence $a_2$ are chosen so small that
\[
\gamma_2 > (e^4 \gamma_1 + 1)c.
\]
Having obtained the sequences of balls $(A_n)$ and $(B_n)$ we define $\tau : U \to [0, 1[$ by
\[
\tau(x) := \begin{cases} 
\rho(x), & x \in U \setminus \bigcup_{n=1}^{\infty} B_n, \\
\rho_n, & x \in A_n, n \in \mathbb{N}, \\
\rho_n / 2, & x \in B_n \setminus A_n, n \in \mathbb{N}.
\end{cases}
\]
Then \( r(x) \leq 2\alpha(1 - |x|) \) for every \( x \in U \) and \( \inf r(K) > 0 \) for every compact \( K \) in \( U \).

Let \( P \) denote the corresponding Markov kernel, i.e.,

\[
P(x, \cdot) = \lambda_{B(x, r(x))}, \quad x \in U.
\]

Using the associated random walk we define measurable functions \( f_n \) on \( U \), \( n \in \mathbb{N}_0 \), by

\[
f_n(x) := \gamma_n P^{x}[T_{A_n} < \infty], \quad x \in U.
\]

Obviously, \( 0 \leq f_n \leq \gamma_n \). The sequence \( (f_n) \) is increasing since

\[
f_{n+1}(x) \geq \gamma_{n+1} P^{x}[T_{A_n} < \infty, X_{T_{A_n+1}} \in A_{n+1}]
\]

\[
= \gamma_{n+1} \int_{[T_{A_n} < \infty]} P^{x_{T_{A_n}}}[X_{1} \in A_{n+1}] dP^{x}
\]

\[
= \gamma_{n+1} \left( \frac{a_{n+1}}{\rho_{n}} \right)^d P^{x}[T_{A_n} < \infty] = \gamma_{n} P^{x}[T_{A_n} < \infty] = f_n(x)
\]

for every \( x \in U \). Let

\[
f := \lim_{n \to \infty} f_n.
\]

Since obviously \( Pf_n(x) = f_n(x) \) for every \( n \in \mathbb{N}_0 \) and every \( x \in U \setminus A_n \), we conclude that \( Pf = f \).

We intend to show that, for every \( n \in \mathbb{N} \),

\[
(*) \quad f_{n+1} \leq (1 + 2^{2^{-n}})f_n \quad \text{on } U \setminus C_{n+1}.
\]

Since \( \prod_{n=1}^{\infty} (1 + 2^{-n}) \leq \exp \left( \sum_{n=1}^{\infty} 2^{-n} \right) = e^4 \), we then obtain that, for every \( n \in \mathbb{N} \),

\[
f \leq e^4 f_n \quad \text{on } U \setminus \bigcup_{j=n+1}^{\infty} C_j.
\]

In particular, \( f \) turns out to be locally bounded on \( U \) and

\[
f(x_2) \geq f_2(x_2) = \gamma_2 > ce^4 \gamma_1 = ce^4 f_1(x_1) \geq cf(x_1),
\]

so \( f \) is not harmonic. Thus (\( *) \) will yield that \( f \) is a counterexample to Veech's conjecture.

In order to prove (\( *) \) let us first establish a general lemma (for the purpose of this section it will be sufficient to take \( E = \emptyset \), i.e., \( T_E = \infty \)):

**Lemma.**\( 1.1. \) Let \( P \) be a Markov kernel on \( U \), let \( A, B, C, A_0, C_0, E \) be Borel subsets of \( U \) such that \( A \subset B \subset C \subset U \setminus C_0 \), \( A_0 \subset C_0 \) and let
0 < \delta \leq \epsilon < 1/6 such that \( P(y, A) = 0 \) for every \( y \in U \setminus (A_0 \cup B \cup E) \), \( P(y, C_0) \leq \epsilon \) for every \( y \in A_0 \), \( P^y[T_{A_0} < \infty] \leq \epsilon \) for every \( y \in U \setminus C_0 \), \( P(y, C) \leq \epsilon \) for every \( y \in B \), and \( P^y[T_B < T_E] \leq \delta P^y[T_{A_0} < \infty] \) for every \( y \in U \setminus C \). Then for every \( x \in U \setminus C \)

\[
P^x[T_A < T_E] \leq (1 + 3\epsilon) \left( \sup_{y \in A_0} P(y, A) + \delta \sup_{y \in B \setminus E} P(y, A) \right) P^x[T_{A_0} < \infty].
\]

**Proof.** Fix \( x \in U \setminus C \) and let

\[ S = \inf \{ j \in \mathbb{N} : X_j \in A_0 \}. \]

For every \( y \in A_0 \),

\[
P^y[X_1 \not\in C_0, S < \infty] = P^y[X_1 \not\in C_0, T_{A_0} \circ \theta_1 < \infty] = \int_{\{X_1 \not\in C_0\}} P^x[T_{A_0} < \infty] dP^y \leq \epsilon,
\]

hence

\[
P^y[S < \infty] \leq P^y[X_1 \in C_0] + P^y[X_1 \not\in C_0, S < \infty] \leq 2\epsilon.
\]

We define an increasing sequence \((S_m)\) of stopping times by

\[ S_1 := T_{A_0}, \quad S_{m+1} := S_m + S \circ \theta_{S_m}. \]

Clearly, for every \( \omega \in \Omega \),

\[ \{ k \in \mathbb{N} : X_k(\omega) \in A_0 \} = \{ S_m(\omega) : m \in \mathbb{N}, S_m(\omega) < \infty \}. \]

For every \( m \in \mathbb{N} \),

\[
P^x[S_m < \infty, X_{S_{m+1}} \in A] = \int_{[S_m < \infty]} P^{X_{S_m}}[X_1 \in A] dP^x \leq \sup_{y \in A_0} P(y, A) P^x[S_m < \infty].
\]

Moreover,

\[
P^x[S_{m+1} < \infty] = P^x[S_m < \infty, S \circ \theta_{S_m} < \infty] = \int_{[S_m < \infty]} P^{X_{S_m}}[S < \infty] dP^x \leq 2\epsilon P^x[S_m < \infty],
\]

hence by induction

\[
P^x[S_{m+1} < \infty] \leq (2\epsilon)^m P^x[S_1 < \infty].
\]
Therefore
\[ \sum_{m=1}^{\infty} P^x[S_m < \infty, X_{S_{m+1}} \in A] \leq P^x[T_{A_0} < \infty] \sup_{y \in A_0} P(y, A) \sum_{k=0}^{\infty} (2\varepsilon)^k \]
where \( \sum_{k=0}^{\infty} (2\varepsilon)^k = (1 - 2\varepsilon)^{-1} \leq 1 + 3\varepsilon \) since \( \varepsilon \leq \frac{1}{6} \).

Define similarly a sequence \((T_m)\) by
\[ T := \inf \{ j \in \mathbb{N} : X_j \in B \}, \quad T_1 = T_B, \quad T_{m+1} = T_m + T \circ \theta_{T_m}. \]

We know that \( P(y, C) \leq \varepsilon \) for every \( y \in B \) and \( \delta P^x[T_B < T_E] \leq \delta \leq \varepsilon \) for every \( y \in U \setminus C \). Arguing in a similar way as for the sequence \((S_m)\) we hence obtain that
\[ \sum_{m=1}^{\infty} P^x[T_m < T_E, X_{T_{m+1}} \in A] \leq (1 + 3\varepsilon) P^x[T_B < T_E] \sup_{y \in B \setminus E} P(y, A) \]
\[ \leq (1 + 3\varepsilon) \delta P^x[T_{A_0} < \infty] \sup_{y \in B \setminus E} P(y, A). \]

To finish the proof it suffices to note that, for every \( k \in \mathbb{N}_0 \),
\[ P^x[X_k \not\in A_0 \cup B, X_{k+1} \in A, k < T_E] = \int_{\{X_k \not\in A_0 \cup B, k < T_E\}} P^x[X_1 \in A] dP^x = 0 \]
since \( P(y, A) = 0 \) for every \( y \in U \setminus (A_0 \cup B \cup E) \), and hence
\[ P^x[T_A < T_E] = P^x[1 \leq T_A < T_E] \]
\[ \leq \sum_{m=1}^{\infty} (P^x[S_m < \infty, X_{S_{m+1}} \in A] + P^x[T_m < T_E, X_{T_{m+1}} \in A]). \]

\[ \square \]

**Proposition 1.2.** For every \( n \in \mathbb{N} \),
\[ f_{n+1} \leq (1 + 2^{2^{-n}}) f_n \quad \text{on} \quad U \setminus C_{n+1}. \]

**Proof.** We know by construction of \( P \) that
\[ P(y, A_{n+1}) = 0 \quad \text{for every} \quad y \in U \setminus (A_n \cup B_{n+1}), \]
\[ P(y, C_n) = \left( \frac{c_n}{\rho_n} \right)^d \leq 2^{-nd} \leq 2^{-(n+1)} \quad \text{for every} \quad y \in A_n, \]
\[ P(y, C_{n+1}) \leq \left( \frac{2c_{n+1}}{\rho_{n+1}} \right)^d \leq 2^{-nd} \leq 2^{-(n+1)} \quad \text{for every} \quad y \in B_{n+1}. \]
In order to get the necessary estimates for $P^y[T_{A_n} < \infty]$ and $P^y[T_{B_{n+1}} < \infty]$ we note that every superharmonic function $s > 0$ on $U$ is $P$-supermedian and hence for every Borel subset $D$ of $U$

$$P^D[T_D < \infty] = \inf \{ s : s \text{ } P\text{-supermedian, } s \geq 1 \text{ on } D \} \leq R_1^D.$$ 

In particular,

$$P^R[T_{A_n} < \infty] \leq R_{n+1}^A \leq R_1^{B_n} \leq 2^{-(n+1)} \text{ on } U \setminus C_n$$

and

$$P^R[T_{B_{n+1}} < \infty] \leq R_1^{B_{n+1}} \leq 2^{-(n+2d)} \beta_{n+1} / \gamma_n \text{ on } U \setminus C_{n+1}'.$$

On the other hand, the random walk associated with $Q$ is obtained from the random walk associated with $P$ by stopping on $C'$. Hence clearly

$$P^R[T_{A_0} < \infty] \geq Q^R[T_{A_0} < \infty] = g.$$ 

Moreover, for every $y \in U$,

$$P^y[T_{A_n} < \infty] \geq P^y[T_{A_0} < \infty, X_{T_{A_0}+1} \in A_1, \ldots, X_{T_{A_n}+1} \in A_n]$$

$$= \int_{[T_{A_0} < \infty]} P^{X_{T_{A_0}}}[X_1 \in A_1, \ldots, X_n \in A_n] dP^y$$

$$= \left( \frac{a_1}{\rho_0} \right)^d \cdots \left( \frac{a_n}{\rho_{n-1}} \right)^d P^y[T_{A_0} < \infty]$$

$$= \gamma_n^{-1} P^y[T_{A_0} < \infty].$$

Thus, for every $y \in C_{n+1} \setminus C_{n+1}'$,

$$P^y[T_{A_n} < \infty] \geq \gamma_n^{-1} g(y) \geq \beta_{n+1} / \gamma_n$$

$$\geq 2^{n+2d} R_1^{B_{n+1}}(y) \geq 2^{n+2d} P^y[T_{B_{n+1}} < \infty].$$

In addition, for every $y \in A_n$,

$$P^y[T_{A_n} < \infty] = 1 \geq 2^{n+2d} R_1^{B_{n+1}}(y) \geq 2^{n+2d} P^y[T_{B_{n+1}} < \infty].$$

Let

$$R := T_{A_n \cup C_{n+1}} = \inf (T_{A_n}, T_{C_{n+1}})$$

and fix $x \in U \setminus C_{n+1}$. Since $P(y, C_{n+1}') = 0$ for every $y \in U \setminus (A_n \cup C_{n+1})$ we know that

$$X_R \in A_n \cup (C_{n+1} \setminus C_{n+1}') \text{ } P^x\text{-a.s. on } [R < \infty].$$
(Indeed,

\[ P^x[R < \infty, X_R \in C_{n+1}'] = P^x[1 \leq R < \infty, X_R \in C_{n+1}'] \]

\[ \leq \sum_{k=0}^{\infty} P^x[X_k \in C(A_n \cup C_{n+1}), X_{k+1} \in C_{n+1}'] \]

\[ = \sum_{k=0}^{\infty} \int_{\{ X_k \in C(A_n \cup C_{n+1}) \}} P^x[X_1 \in C_{n+1}'] dP^x = 0. \]

Obviously, \( T_{A_n} = R + T_{A_n} \circ \theta_R \) and \( T_{C_{n+1}} = R + T_{C_{n+1}} \circ \theta_R \), hence the strong Markov property implies that

\[ P^x[T_{A_n} < \infty] = \int_{[R<\infty]} P^{X_n}[T_{A_n} < \infty] dP^x \]

and

\[ P^x[T_{B_{n+1}} < \infty] = \int_{[R<\infty]} P^{X_n}[T_{B_{n+1}} < \infty] dP^x. \]

Since

\[ P^y[T_{A_n} < \infty] \geq 2^{n+v_2} P^y[T_{B_{n+1}} < \infty] \]

for every \( y \in A_n \cup (C_{n+1} \setminus C_{n+1}') \) we conclude that

\[ P^x[T_{A_n} < \infty] \geq 2^{n+v_2} P^x[T_{B_{n+1}} < \infty]. \]

Furthermore, \( P(y, A_{n+1}) = \left( \frac{a_{n+1}}{\rho_n} \right)^d \) for every \( y \in A_n \), whereas, for every \( y \in B_{n+1} \),

\[ P(y, A_{n+1}) \leq \left( \frac{2a_{n+1}}{\rho_{n+1}} \right)^d \leq 2^d \left( \frac{a_{n+1}}{\rho_n} \right)^d. \]

Therefore by our Lemma

\[ f_{n+1}(x) = \gamma_{n+1} P^x[T_{A_{n+1}} < \infty] \]

\[ \leq (1 + 3 \cdot 2^{-v_2}) (1 + 2^{-v_2}) \gamma_{n+1} \left( \frac{a_{n+1}}{\rho_n} \right)^d P^x[T_{A_n} < \infty] \leq (1 + 2^{-v_2}) f_n(x). \]

Thus we have proven the following result which shows that Veech’s conjecture is wrong:

**Theorem 1.3.** For every \( 0 < \alpha \leq 1 \) there exist Borel functions \( r, f > 0 \) on \( U \) such that \( r \leq \alpha \text{dist}(\cdot, \partial U) \), \( \inf r(K) > 0 \) for every compact subset \( K \) of \( U \), \( f \) is locally bounded and \( r \)-median, but not harmonic on \( U \).
2. - A continuous counterexample

Having constructed a measurable counterexample to Veech’s conjecture the question arises if perhaps a weakened version is true where \( r \) and \( f \) are supposed to be continuous. In this section we shall see that this is not the case:

**THEOREM 2.1.** Given \( 0 < \alpha \leq 1 \), there exist continuous strictly positive functions \( r \) and \( f \) on \( U \) such that \( r \leq \alpha \text{dist}(\cdot, \partial U) \), \( f \) is \( r \)-median, but not harmonic.

In order to get this result it suffices to modify the measurable counterexample removing the discontinuities at \( \partial A_n \cup \partial B_n, \ n \in \mathbb{N} \). To that end we shall use a general property of random walks given by means having a locally bounded density with respect to the Lebesgue measure (cf. a similar argument in [HN3]):

**LEMMA 2.2.** Let \( P \) be the transition kernel of a random walk given by an admissible function \( r \) on \( U \) which is locally bounded away from zero, let \( K \) be a compact subset of \( U \), fix \( x \in U \) and \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that \( P^\alpha [0 < T_A < \infty] < \epsilon \) for every Borel subset \( A \) of \( K \) satisfying \( \lambda_U(A) < \delta \).

**PROOF.** Define \( q(y) = 1 - |y|, \ y \in U \), and let \( \gamma := \inf q(K) = \text{dist}(K, \partial U) \). Since \( q \) is a continuous potential on \( U \), we know by Lemma 1 in [HN5] that \( \lim_{m \to \infty} P^m q = 0 \). So there exists \( m \in \mathbb{N} \) such that

\[
P^m q(x) < \frac{\epsilon^2}{9} \gamma.
\]

Then

\[
P^x \left[ q(X_m) \geq \frac{\epsilon}{3} \gamma \right] \leq \left( \frac{\epsilon}{3} \gamma \right)^{-1} P^m q(x) < \frac{\epsilon}{3}.
\]

Moreover, for every \( y \in \left\{ q < \frac{\epsilon}{3} \gamma \right\} \),

\[
P^y [T_{\{q \geq \gamma\}} < \infty] \leq R_{\{q \geq \gamma\}}(y) \leq \frac{1}{\gamma} q(y) < \frac{\epsilon}{3}.
\]

Therefore

\[
P^x [X_i \in K \text{ for some } i \geq m] \leq P^x [q(X_i) \geq \gamma \text{ for some } i \geq m]
\]

\[
\leq P^x \left[ q(X_m) \geq \frac{\epsilon}{3} \gamma \right] + P^x \left[ q(X_m) < \frac{\epsilon}{3} \gamma, q(X_i) \geq \gamma \text{ for some } i \geq m \right]
\]

\[
\leq \frac{\epsilon}{3} + \int_{\left[ q(X_m) < \frac{\epsilon}{3} \gamma \right]} P^x_m \left[ T_{\{q \geq \gamma\}} < \infty \right] dP^x < \frac{2}{3} \epsilon.
\]

Let

\[
\eta := \inf \{ r(y) : q(y) \geq \gamma/2 \}.
\]
If \( y \in U \) such that \( B(y, r(y)) \cap K \neq \emptyset \) then \( q(y) > \gamma/2 \) and hence \( P(y, \cdot) = \lambda_{B(y,r(y))} \leq \eta^{-d} \lambda_U \). This implies that, for every \( i \in \mathbb{N} \),

\[
P_{X_i | K} \leq \eta^{-d} \lambda_U.
\]

Take

\[
\delta := \frac{\eta^d}{3m} \varepsilon
\]

and let \( A \) be a Borel subset of \( K \) such that \( \lambda_U(A) < \delta \). Then

\[
P^\varepsilon[0 < T_A < \infty] \leq \sum_{i=1}^{m} P^\varepsilon[X_i \in A] + P^\varepsilon[X_i \in K \text{ for some } i \geq m]
\]

\[
\leq m \cdot \eta^{-d} \delta + \frac{2}{3} \varepsilon = \varepsilon.
\]

Let us now return to the situation considered in the previous section. Defining

\[
\varepsilon_n := 2^{-(n+1)} \gamma_{n+1}^{-1}
\]

we know that for every \( n \in \mathbb{N} \),

\[
\sum_{j=n}^{\infty} \varepsilon_j \leq \gamma_{n+1}^{-1} \sum_{j=n}^{\infty} 2^{-(j+1)} = 2^{-n} \gamma_{n+1}^{-1}.
\]

Let \( V = U \setminus \bigcup_{j=1}^{n-1} (B_j \setminus A_j) \), fix \( n \in \mathbb{N} \), and suppose that we have already defined a continuous function \( \tilde{r} \) on \( V \cup \bigcup_{j=1}^{n-1} (B_j \setminus A_j) \) such that \( \tilde{r} = r \) on \( V \) and \( 0 < \tilde{r} \leq \rho_j \) on \( B_j \setminus A_j \), \( j = 0, 1, \ldots, n-1 \). Let \( P_n \) denote the transition kernel on \( U \) given by \( \tilde{r} \) on \( V \cup \bigcup_{j=1}^{n-1} (B_j \setminus A_j) \) and by \( r \) on \( \bigcup_{j=n}^{\infty} (B_j \setminus A_j) \), i.e.,

\[
P_n(x, \cdot) = \begin{cases} 
\lambda_{B(x, \tilde{r}(x))}, & x \in U \setminus \bigcup_{j=n}^{\infty} (B_j \setminus A_j), \\
\lambda_{B(x,r(x))}, & x \in \bigcup_{j=n}^{\infty} (B_j \setminus A_j).
\end{cases}
\]

By Lemma 2.2 we know that, for each \( x \in \overline{B}_{n-1} \cup (\overline{C}_n \setminus C_n') \), there exist \( b'_n \), \( b''_n \in [a_n, b_n] \) such that the set

\[
E_n = \{ y \in B_n \setminus A_n : |y - x_n| < b'_n \text{ or } |y - x_n| > b''_n \}
\]
satisfies

\[ e_n(x) := P_n^x[T_{E_n} < \infty] < \epsilon_n. \]

By the strong Markov property

\[ P_n e_n = e_n \quad \text{on } U \setminus B_n. \]

Since \( \tilde{r} \) is continuous on \( \overline{B}_{n-1} \cup (C_n \setminus C'_n) \) we conclude that \( e_n \) is continuous on \( \overline{B}_{n-1} \cup (C_n \setminus C'_n) \). So a simple compactness argument shows that we may choose \( b'_n, b''_n \) such that \( e_n < \epsilon_n \) on \( \overline{B}_{n-1} \cup (C_n \setminus C'_n) \). In fact,

\[ e_n < \epsilon_n \quad \text{on } U \setminus C'_n \]

since \( T_{\overline{B}_{n-1} \cup (C_n \setminus C'_n)} \leq T_{B_n} \leq T_{E_n} P_n^x \)-a.s. for every \( x \in U \setminus C_n \).

We now extend \( \tilde{r} \) to a continuous function on \( V \cup \bigcup_{j=1}^n (B_j \setminus A_j) \) such that

\[ \tilde{r} = r = \frac{\rho_n}{2} \quad \text{on } (B_n \setminus A_n) \setminus E_n, \quad 0 < \tilde{r} \leq \rho_n \quad \text{on } E_n. \]

By induction we obtain an admissible function \( \hat{r} \) on \( U \) such that

\[ \hat{r} = r \quad \text{on } U \setminus \bigcup_{n=1}^\infty E_n. \]

Define the Markov kernel \( \hat{P} \) on \( U \) by

\[ \hat{P}(x, \cdot) = \lambda_B(x, \tilde{r}(x)), \quad x \in U. \]

Using the corresponding random walk we obtain functions \( \tilde{f}_n \) on \( U, \, n \in \mathbb{N}_0 \), by

\[ \tilde{f}_n(x) := \gamma_n \hat{P}^x[T_{A_n} < \infty], \quad x \in U. \]

Repeating the argument we used for the sequence \( (f_n) \) we obtain that the sequence \( (\tilde{f}_n) \) is increasing (note that \( \tilde{r} = \rho_n \) on \( A_n, \, n \in \mathbb{N}_0 \)) and that

\[ \tilde{f} := \sup \tilde{f}_n \]

satisfies

\[ \hat{P} \tilde{f} = \tilde{f}. \]

**Proposition 2.3.** For every \( n \in \mathbb{N} \),

\[ \tilde{f}_{n+1} \leq (1 + 2^{-n}) \tilde{f}_n + 2^{-n} \quad \text{on } U \setminus C_{n+1}. \]
PROOF. Fix $n \in \mathbb{N}$ and let

$$F_n = \bigcup_{j=1}^{\infty} E_j.$$ 

By construction of $\tilde{P}$ we know that

$$\tilde{P}(y, A_{n+1}) = 0 \quad \text{for every } x \in U \setminus (A_n \cup B_{n+1} \cup F_n).$$

Moreover,

$$\tilde{P}(y, C_n) = \left( \frac{c_n}{\rho_n} \right)^d \leq 2^{-(n+1)} \quad \text{for every } y \in A_n,$$

$$\tilde{P}(y, C_{n+1}) \leq \left( \frac{2c_{n+1}}{\rho_{n+1}} \right)^d \leq 2^{-(n+1)} \quad \text{for every } y \in B_{n+1} \setminus F_n.$$

As in the proof of Proposition 1.2 we get that

$$\tilde{P}[T_{A_n} < \infty] \leq 2^{-(n+1)} \quad \text{on } U \setminus C_n,$$

$$\tilde{P}[T_{B_{n+1}} < \infty] \leq 2^{-(n+2d)} \beta_{n+1}/\gamma_n \quad \text{on } U \setminus C_{n+1},$$

and

$$\tilde{P}[T_{A_n} < T_{F_n}] \geq 2^{n+2d} \tilde{P}[T_{B_{n+1}} < \infty] \geq \gamma_n^{-1} \tilde{P}[T_{A_0} < T_{F_n}] \geq \gamma_n^{-1} g \geq 2^{n+2d} \tilde{P}[T_{B_{n+1}} < T_{F_n}] \quad \text{on } A_n \cup (C_{n+1} \setminus C'_{n+1}).$$

As before let $R := T_{A_n} \wedge C_{n+1}$ and fix $x \in U \setminus C_{n+1}$. Since $\tilde{P}(y, C_{n+1}) = 0$ for every $y \in U \setminus (A_n \cup C_{n+1} \cup F_n)$, we know that

$$X_R \in A_n \cup (C_{n+1} \setminus C'_{n+1}) \quad \tilde{P}^x\text{-a.s. on } [R < T_{F_n}].$$

By the strong Markov property

$$\tilde{P}^x[T_{A_n} < T_{F_n}] = \int_{[R < T_{F_n}]} \tilde{P}^{X_R}[T_{A_n} < T_{F_n}] d\tilde{P}^x$$

and

$$\tilde{P}^x[T_{B_{n+1}} < T_{F_n}] = \int_{[R < T_{F_n}]} \tilde{P}^{X_R}[T_{B_{n+1}} < T_{F_n}] d\tilde{P}^x,$$

hence

$$\tilde{P}^x[T_{A_n} < T_{F_n}] \geq 2^{n+2d} \tilde{P}^x[T_{B_{n+1}} < T_{F_n}].$$
Therefore Lemma 1.1 now yields
\[ \gamma_{n+1} \tilde{P}^x[T_{A_{n+1}} < T_{F_n}] \]
\[ \leq (1 + 3 \cdot 2^{-{(n+1)}})(1 + 2^{-n})\gamma_{n+1} \left( \frac{a_{n+1}}{\rho_n} \right)^d \tilde{P}^x[T_{A_n} < \infty] \]
\[ \leq (1 + 2^{-n})\tilde{f}_n(x). \]

On the other hand by definition
\[ \tilde{P}(y, \cdot) = P_j(y, \cdot) \quad \text{for every } y \in U \setminus F_n \text{ and } j \geq n, \]
hence
\[ \tilde{P}^x[T_{F_n} < \infty] = P_n^x[T_{F_n} < \infty] \]
\[ = \sum_{j=n}^{\infty} P_n^x[T_{E_j} < \infty, T_{E_n} = T_{E_{n+1}} = \ldots = T_{E_{j-1}} = \infty] \]
\[ = \sum_{j=n}^{\infty} P_j^x[T_{E_j} < \infty, T_{E_n} = T_{E_{n+1}} = \ldots = T_{E_{j-1}} = \infty] \]
\[ \leq \sum_{j=n}^{\infty} \epsilon_j \leq 2^{-n}\gamma_{n+1}^{-1}. \]

Finally, \([T_{A_{n+1}} < \infty] \subset [T_{A_{n+1}} < T_{F_n}] \cup [T_{F_n} < \infty], \) therefore
\[ \tilde{f}_{n+1}(x) = \gamma_{n+1} \tilde{P}^x[T_{A_{n+1}} < \infty] \leq (1 + 2^{-n})\tilde{f}_n(x) + 2^{-n}. \]

PROOF OF THEOREM 2.1. By Proposition 2.3, for every \( n \in \mathbb{N}, \)
\[ \tilde{f} \leq e^4 \tilde{f}_n + 1 \quad \text{on } U \setminus \bigcup_{j=n+1}^{\infty} C_j. \]

Thus \( \tilde{f} \) is locally bounded. Since \( \tilde{P}\tilde{f} = \tilde{f}, \) since \( \tilde{r} \) is continuous and \( \tilde{r} \leq \frac{1}{2} \text{dist}(\cdot, \partial U), \) we conclude that \( \tilde{f} \) is a continuous \( \tilde{r} \)-median function. Because of
\[ \tilde{f}(x_2) \geq \tilde{f}_2(x_2) = \gamma_2 > c (e^4\gamma_1 + 1) = c \left(e^4\tilde{f}_1(x_1) + 1\right) \geq c\tilde{f}(x_1) \]
the function \( \tilde{f} \) is not harmonic.

REMARKS 2.4. 1. For every \( z \in \partial U, \ z \neq (1,0,\ldots,0), \)
\[ \lim_{z \to \infty} \tilde{f}(z) = 0. \]
Indeed, the proof of (2.3) shows that

\[ \tilde{f} \leq e^4 \tilde{f}_1 + \sum_{m=1}^{\infty} m e_{m+1} \gamma_{m+1}^{\epsilon_m} \quad \text{on } U \setminus \bigcup_{j=2}^{\infty} C_j. \]

The functions $\tilde{f}_1$ and $e_m$, $m \in \mathbb{N}$, tend to zero at $\partial U$ (since $\tilde{f}_1 \leq \gamma_{1} R_{1}^{A_1}$ and $e_j \leq R_{1}^{B_j}$) and, for every $m \in \mathbb{N}$,

\[ m e_{m+1} \gamma_{m+1}^{\epsilon_m} < m e_{m+1} \gamma_{m+1}^{\epsilon_m} = m 2^{-(m+1)} \quad \text{on } U \setminus \bigcup_{j=2}^{\infty} C_j. \]

So our claim follows from the fact that $\sum_{m=1}^{\infty} m 2^{-(m+1)} < \infty$ and the point $(1,0,\ldots,0)$ is the only limit point of the set $\bigcup_{j=2}^{\infty} C_j$ contained in the boundary $\partial U$.

2. We could have arranged without difficulty that $\tilde{f}$ is a $C^\infty$-function and then $\tilde{P} \tilde{f} = \tilde{f}$ implies that $\tilde{f}$ is a $C^\infty$-function as well.

3. Every $\tilde{f}$-median function on $U$ which is bounded by some harmonic function on $U$ is harmonic, hence every extremal positive harmonic function on $U$ is an extremal $\tilde{f}$-median function. So the euclidean boundary of $U$ is a proper subset of the Martin boundary for the random walk given by $\tilde{f}$. (A close inspection should reveal that the function $f$ we constructed is an extremal $\tilde{f}$-median function and that it is up to constant multiples the only extremal positive $\tilde{f}$-median function which is not harmonic).

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