ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JEAN-PHILIPPE ANKER
EWA DAMEK
CHOKRI YACOUB

Spherical analysis on harmonic *AN* groups

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 23, nº 4 (1996), p. 643-679
<http://www.numdam.org/item?id=ASNSP_1996_4_23_4_643_0>

Spherical Analysis on Harmonic AN Groups

JEAN-PHILIPPE ANKER - EWA DAMEK - CHOKRI YACOUB

0. – Introduction

Harmonic AN groups and analysis thereon have been studied by several authors in the past 10 years ([Bo], [BTV], [CDKR1], [CDKR2], [Dam1], [Dam2], [Dam3], [Dam4], [DR1], [DR2], [Ric2], [Sz2] ...). Recall that, as Riemannian manifolds, these solvable Lie groups include all symmetric spaces of noncompact type and rank one, namely the hyperbolic spaces $H^N(\mathbb{R})$, $H^N(\mathbb{C})$, $H^N(\mathbb{H})$ and $H^2(\mathbb{O})$ (the real hyperbolic spaces $H^N(\mathbb{R})$, which are somehow degenerate, are often disregarded), but that most of them are not symmetric, thus providing numerous counterexamples to the Lichnerowicz conjecture [DR1].

Despite the lack of symmetry, spherical analysis i.e. the analysis of radial functions on these spaces is quite similar to the hyperbolic space case. This was already apparent in the pioneer works [DR2], [Ric2] and will be made clear in this paper. First of all we shall emphasize that spherical analysis is again a particular case of the Jacobi function analysis, an elementary observation which is quite useful and seems to have been unnoticed. The main body of the paper will be devoted to establishing a series of analytical results, which illustrate the actual similarity with symmetric spaces: a Kunze-Stein phenomenon, sharp and simple criteria for the $L^p \to L^p$ and the weak $L^1 \to L^1$ boundedness of positive convolution kernels, the L^p behavior of (functions of) the Laplace-Beltrami operator, a detailed analysis of the heat kernel, the weak $L^1 \to L^1$ boundedness of both the heat maximal operator and the Riesz transform, ... The (modified) wave equation will be studied separately [AMPS].

Our paper is more precisely organized as follows. In Section 1 we recall the basic structure of harmonic AN groups and in Section 2 the basic spherical harmonic analysis thereon. In Section 3 we establish the analogues of the Herz and the Strömberg criteria for the $L^p \to L^p$ and the weak $L^1 \to L^1$ boundedness of positive radial convolution kernels, and get as first consequences a spherical Kunze-Stein phenomenon and the weak $L^1 \to L^1$ boundedness of the Hardy-Littlewood maximal function. In Section 5 we obtain an explicit formula for the heat kernel based on the inverse Abel transform, deduce from it optimal

Pervenuto alla Redazione il 2 gennaio 1995 e in forma definitiva il 11 ottobre 1996.

upper and lower bounds, both for the heat kernel and its gradient, and get as a consequence the weak $L^1 \to L^1$ boundedness of the heat maximal operator. We conclude in Section 6 with some further results and open problems.

This paper is the result of two independent researches. On one hand, J.-Ph. Anker and E. Damek worked out most of the material in Sections 2 to 5 during a(n enjoyable) stay of the first author in Wrocław in the Spring of 1993. On the other hand, C. Yacoub obtained parts of Sections 3, 5 and 6 in the first chapter of his thesis [Ya]. All three authors are grateful to N. Lohoué for suggesting to look at the weak $L^1 \to L^1$ boundedness of the heat maximal operator in this setting, which was the starting point of their study.

1. – Basic structure of $S = AN$

Harmonic AN groups form a class of solvable Lie groups, equipped with a left-invariant metric. More precisely, $S = A \ltimes N$ is a semi-direct product of $A \cong \mathbb{R}$ with a *Heisenberg type* Lie group N.

H-type Lie groups were introduced by Kaplan [Ka1] and studied later on by several authors (see [BTV], [CDKR1], [CDKR2], [DR2], [Ka2], [Ka3], [KR], [Kor], [Ric1], [Rie] ...). Recall briefly their structure. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra, equipped with an inner product $\langle\ ,\ \rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ (so that $[\mathfrak{v},\mathfrak{v}] \subset \mathfrak{z}$). Let $J_Z : \mathfrak{v} \to \mathfrak{v}$ be the linear map defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \qquad (X, Y \in \mathfrak{v};\ Z \in \mathfrak{z})\ . \tag{1.1}$$

Then $\mathfrak{n}$ is *of Heisenberg type* if the following equivalent conditions are satisfied:

(1.2) $\operatorname{ad} X$ is an isometry from $(\operatorname{Ker}\operatorname{ad} X)^{\perp}\ (\subseteq \mathfrak{v})$ onto $\mathfrak{z}$, for every unit vector $X \in \mathfrak{v}$;

(1.3) $J_Z^2 = -|Z|^2 I$ for every $Z \in \mathfrak{z}$.

Recall [Ka1] the possible dimensions $k = \dim \mathfrak{z}$ and $m = \dim \mathfrak{v}$:

$$k = \begin{cases} 8\kappa+1 \\ 8\kappa+2 \\ 8\kappa+3 \\ 8\kappa+4 \\ 8\kappa+5 \\ 8\kappa+6 \\ 8\kappa+7 \\ 8\kappa+8 \end{cases} \implies m = \begin{cases} 2^{4\kappa+1}\mu \\ 2^{4\kappa+2}\mu \\ 2^{4\kappa+2}\mu \\ 2^{4\kappa+3}\mu \\ 2^{4\kappa+3}\mu \\ 2^{4\kappa+3}\mu \\ 2^{4\kappa+3}\mu \\ 2^{4\kappa+4}\mu \end{cases} \tag{1.4}$$

where $\kappa \geq 0$ and $\mu > 0$ are arbitrary integers. In particular m is always even. (As was pointed out by A. Korányi, there is actually a simple and clear explanation for this fact: any J_Z corresponding to a unit vector $Z \in \mathfrak{z}$ induces a complex structure on $\mathfrak{v}$.)

The corresponding (connected and) simply connected Lie groups N are called *of Heisenberg type*. We shall identify them with their Lie algebra $\mathfrak{n}$ via the exponential map. Thus multiplication in $N \equiv \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ reads

$$(1.5)\qquad (X, Z)(X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']) .$$

Consider ([Bo], [BTV], [CDKR1], [CDKR2], [Dam1], [Dam2], [DR1], [DR2], [Sz2] ...) the semi-direct product $S = N \ltimes \mathbb{R}_+^*$ defined by

$$(1.6)\qquad (X, Z, a)(X', Z', a') = (X + a^{\frac{1}{2}} X', Z + aZ' + \frac{1}{2} a^{\frac{1}{2}} [X, X'], aa') .$$

S is a solvable (connected and) simply connected Lie group, with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ and Lie bracket

$$(1.7)\qquad [(X, Z, \ell), (X, Z', \ell')] = (\frac{1}{2}\ell X' - \frac{1}{2}\ell' X, \ell Z' - \ell' Z + [X, X'], 0) .$$

S is equipped with the left-invariant Riemannian metric induced by

$$(1.8)\qquad \langle (X, Z, \ell), (X', Z', \ell') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \ell\ell'$$

on $\mathfrak{s}$. The associated left-invariant (Riemann-Haar) measure on S is given by

$$(1.9)\qquad a^{-Q}\, dX\, dZ\, \frac{da}{a} ,$$

Here $Q = \frac{m}{2} + k$ is the *homogeneous dimension* of N.

Most Riemannian symmetric spaces G/K of noncompact type and rank one fit into this framework. According to the Iwasawa decomposition $G = NAK$, they can be realized indeed as $S = NA = AN$, with $A \cong \mathbb{R}$. N is abelian for real hyperbolic spaces $G/K = H^N(\mathbb{R})$ (which are therefore often disregarded) and of Heisenberg type in the other cases $G/K = H^N(\mathbb{C})$, $H^N(\mathbb{H})$, $H^2(\mathbb{O})$. Notice that these classical examples form only a very small subclass of harmonic AN groups, as can be seen by looking at the dimensions:

$$(1.10)\qquad \begin{array}{lcc} & k & m \\ [H^N(\mathbb{R}): & 0 & N-1\ ] \\ H^N(\mathbb{C}): & 1 & 2(N-1) \\ H^N(\mathbb{H}): & 3 & 4(N-1) \\ H^2\,(\mathbb{O}): & 7 & 8 \end{array}$$

It should be pointed out that different notations and normalizations are used in the setting of symmetric spaces on one hand and in the setting of AN groups on the other hand. In this paper we shall stick to the latter conventions.

Hyperbolic spaces $H^n(\mathbb{F})$ have a well known realization as the unit ball $B(\mathbb{F}^n)$ in $\mathbb{F}^n$. Our general groups S can be realized similarly as the unit ball

$$B(\mathfrak{s}) = \{(X, Z, \ell) \in \mathfrak{s} \mid |X|^2 + |Z|^2 + \ell^2 < 1\} \tag{1.11}$$

in $\mathfrak{s}$, via a *Cayley type transform*:

$$\begin{array}{cccc} C: & S & \longrightarrow & B(\mathfrak{s}) \\ & x = (X, Z, a) & \longmapsto & x' = (X', Z', \ell') \end{array} , \tag{1.12}$$

$$\text{where} \quad \begin{cases} X' = \left\{ \left(1 + a + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\}^{-1} \left\{1 + a + \frac{1}{4}|X|^2 - J_Z\right\} X, \\ Z' = \left\{ \left(1 + a + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\}^{-1} 2Z, \\ \ell' = \left\{ \left(1 + a + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\}^{-1} \left\{-1 + \left(a + \frac{1}{4}|X|^2\right)^2 + |Z|^2\right\}, \end{cases}$$

$$\text{and conversely} \quad \begin{cases} X = \{(1-\ell')^2 + |Z'|^2\}^{-1}\{1 - \ell' + J_{Z'}\}\, 2X', \\ Z = \{(1-\ell')^2 + |Z'|^2\}^{-1}\, 2Z', \\ a = \{(1-\ell')^2 + |Z'|^2\}^{-1}\{1 - \|x'\|^2\}. \end{cases}$$

Letting $a \overset{>}{\to} 0$ i.e. $\|x'\| \overset{<}{\to} 1$, this transform extends naturally to a *stereographic type projection* ∂C between the boundaries $\partial S \equiv N$ and $\partial B(\mathfrak{s}) \setminus \{(0,0,1)\}$. Notice that the Jacobian of ∂C is related to the *Poisson kernel*

$$P_a(X, Z) = 2^{k-1} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) a^Q \left\{ \left(a + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\}^{-Q} \tag{1.13}$$

on N ([Dam3], [Dam4], [DR2]). Precisely,

$$J(\partial C)(X, Z) = 2 \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} P_1(X, Z) = 2^k \left\{ \left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\}^{-Q}. \tag{1.14}$$

In the ball model $B(\mathfrak{s})$, the geodesics passing through the origin are the diameters, the geodesic distance to the origin is given by

$$r = d(x', 0) = \log \frac{1 + \|x'\|}{1 - \|x'\|} \quad \text{i.e.} \quad \rho = \|x'\| = \tanh \frac{r}{2}, \tag{1.15}$$

and the Riemannian volume writes

$$
\begin{aligned}
dV &= 2^n (1-\rho^2)^{-Q-1} \rho^{n-1} \, d\rho \, d\sigma \\
&= 2^{m+k} \left(\sinh \frac{r}{2}\right)^{m+k} \left(\cosh \frac{r}{2}\right)^{k} dr \, d\sigma \ ,
\end{aligned}
\tag{1.16}
$$

where $d\sigma$ denotes the surface measure on the unit sphere $\partial B(\mathfrak{s})$ in $\mathfrak{s}$ and $n = \dim S = m + k + 1$. In particular:

(1.17) The volume density in normal coordinates at the origin, and by translation at any point, is a purely radial function, which means that S is a *harmonic manifold* ([DR1], [Sz1]).

(1.18) The ball volume has the asymptotic behavior

$$
|B_r| \sim \frac{\pi^{\frac{n}{2}}}{Q\, 2^{k-1}\, \Gamma(\frac{n}{2})} \, e^{Qr} \ , \quad \text{as } r \to +\infty \ .
$$

Like all harmonic manifolds, S is an *Einstein manifold.* A lengthy computation yields the actual constant:

$$
\textit{Ricci curvature} = -\left(\frac{m}{4} + k\right) \times \textit{Riemannian metric} \ . \tag{1.19}
$$

The sectional curvature, as far as it is concerned, is nonpositive, with minimum $= -1$ [BTV] (see also [Bo], [Dam2]). Notice that it may vanish, contrarily to the hyperbolic space case.

Eventually let us mention an interesting inequality

$$
|\log a(x)| \leq r(x) \qquad (x \in S) \ , \tag{1.20}
$$

between the A-component $a(x)$ of x and the geodesic distance $r(x)$ from x to the origin, which follows from (1.12) and (1.15), and which is the analogue of a classical relation between the Iwasawa and the Cartan projections in the symmetric space case.

2. – Spherical harmonic analysis on S

Spherical harmonic analysis on S was developed in [DR2] and [Ric2], in close analogy with the classical (rank one) symmetric space case. We recall it in this section, place it in the general framework of *Jacobi analysis* [Koo] and introduce the *Abel transform.*

Harmonic analysis on symmetric spaces G/K relies on the commutativity of convolution on bi-K-invariant objects on G. A similar phenomenon occurs on general S, if one replaces bi-K-invariance by *radiality*. For functions on S, this notion has the obvious meaning. For distributions, invariant differential operators, ... radiality is defined by means of an *averaging operator* over spheres, which writes

$$f^{\natural}(x') = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{\partial B(s)} d\sigma\, f(\rho\sigma) \tag{2.1}$$

in the ball model and generalizes K-averages on rank one symmetric spaces G/K. The following fundamental properties were established in [DR2]:

(2.2) Convolution preserves radiality.

(2.3) Convolution is commutative on radial objects. In particular, the radial integrable functions on S form a commutative Banach algebra $L^1(S)^{\natural}$ under convolution.

(2.4) The algebra $\mathbb{D}(S)^{\natural}$ of invariant differential operators on S which are radial (i.e. which commute with the averaging operator) is a polynomial algebra with a single generator, the *Laplace-Beltrami operator* $\mathcal{L}$.

From there one can develop ([DR2], [Ric2]) a theory of spherical functions, spherical Fourier transform, ... quite similar to the classical (rank one) symmetric space case, for which we refer for instance to [An5], [Fa], [Hel]. Let us sum up:

(2.5) *Spherical functions* φ on S are characterized by the set of conditions

$$\begin{cases} \varphi \text{ is a radial eigenfunction of } \mathcal{L} \text{ (and thus automatically analytic)}, \\ \varphi(e) = 1\,. \end{cases}$$

(2.6) They are all obtained by a *Harish-Chandra type integral formula*:

$$\varphi_\lambda = \{\, a(\,.\,)^{\frac{Q}{2}-i\lambda} \,\}^{\natural} \qquad (\lambda \in \mathbb{C})\,.$$

Moreover $\mathcal{L}\varphi_\lambda = -\,(\frac{Q^2}{4} + \lambda^2)\,\varphi_\lambda$ and $\varphi_\lambda = \varphi_\mu \Longleftrightarrow \lambda = \pm\mu$.

(2.7) For $\operatorname{Re}(i\lambda) = -\operatorname{Im}\lambda > 0$, we have the following asymptotic behavior:

$$\varphi_\lambda(x) \sim \mathbf{c}(\lambda)\, e^{(i\lambda-\frac{Q}{2})r} \qquad \text{as } r = r(x) \to +\infty\,,$$

where $\mathbf{c}(\lambda) = \frac{\Gamma(m+k)}{\Gamma(\frac{m+k}{2})}\,\frac{\Gamma(i2\lambda)}{\Gamma(i2\lambda+\frac{m}{2})}\,\frac{\Gamma(i\lambda+\frac{m}{4})}{\Gamma(i\lambda+\frac{m}{4}+\frac{k}{2})} = \Gamma(\frac{n}{2})2^{Q-2i\lambda}\,\frac{\Gamma(i2\lambda)}{\Gamma(i\lambda+\frac{Q}{2})\,\Gamma(i\lambda+\frac{m}{4}+\frac{1}{2})}$.

(2.8) The *spherical Fourier transform* is defined by

$$\begin{aligned}\mathcal{H}f(\lambda) &= \int_S dx\, \varphi_\lambda(x)\, f(x)\\ &= \frac{2^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^{+\infty} dr \left(\sinh \frac{r}{2}\right)^{m+k} \left(\cosh \frac{r}{2}\right)^k \varphi_\lambda(r)\, f(r)\end{aligned}$$

for radial functions $f = f(x)$ on S, which we shall often identify with functions $f = f(r)$ of the geodesic distance to the origin $r = d(x,0) \in [0, +\infty)$.

(2.9) *Paley-Wiener theorem*:
$\mathcal{H}$ is an isomorphism from the space $C_c^\infty(S)^\natural$ of smooth radial functions with compact support on S onto the space $\mathcal{PW}(\mathbb{C})_{\text{even}}$ of even entire functions of exponential type.

(2.10) *Plancherel formula*:
We have the following inversion formula

$$f(x) = c_0 \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\, \varphi_\lambda(x)\, \mathcal{H}f(\lambda)\,.$$

where $c_0 = 2^{k-2} \pi^{-\frac{n}{2}-1}\, \Gamma(\frac{n}{2})$. Moreover $\mathcal{H}$ extends to an isometry from the space $L^2(S)^\natural$ of radial L^2 functions on S onto $L^2((0,+\infty), c_0|\mathbf{c}(\lambda)|^{-2} d\lambda)$.

Notice that the constant given in the inversion formula [Ric2; (4.1)] is slightly incorrect. The error is due to a discrepancy (already alluded to in Section 1) between standard normalizations in the harmonic AN group setting on one hand and in the symmetric space setting on the other hand. See also (5.61).

As for hyperbolic spaces, all this analysis fits actually into the general framework of Jacobi function analysis [Koo]. and this is the point of view we want to present now. The radial part (in geodesic polar coordinates) of the Laplace-Beltrami operator $\mathcal{L}$ on S writes

$$\text{rad}\,\mathcal{L} = \frac{\partial^2}{\partial r^2} + \left\{\frac{m+k}{2} \coth \frac{r}{2} + \frac{k}{2} \tanh \frac{r}{2}\right\} \frac{\partial}{\partial r}\,. \tag{2.11}$$

By substituting $t = \frac{r}{2}$, $4\,\text{rad}\,\mathcal{L}$ becomes the *Jacobi operator* [Koo; (2.9)]

$$\mathcal{L}_{\alpha,\beta} = \frac{\partial^2}{\partial t^2} + \{(2\alpha+1)\coth t + (2\beta+1)\tanh t\} \frac{\partial}{\partial t} \tag{2.12}$$

with indices $\alpha = \frac{m+k-1}{2}$ and $\beta = \frac{k-1}{2}$. Notice that we are in the ideal case $\alpha > \beta > -\frac{1}{2}$ for Jacobi analysis. A comparison of the preceding formulas with [Koo; §§ 2.1-2.2] yields the following identifications:

(2.13) Spherical functions on S are *Jacobi functions*:

$$\varphi_\lambda(r) = \phi_{2\lambda}^{(\alpha,\beta)}(\tfrac{r}{2})\ .$$

(2.14) The spherical Fourier transform coincides with the *Jacobi transform*:

$$\mathcal{H}f(\lambda) = 2^{2-k}\, \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\, \widehat{f(2\,.)}^{(\alpha,\beta)}(2\lambda)\ .$$

As a consequence, all above-mentioned results follow from the general theory developed in [Koo]. And much more actually.

For instance there is an analytic *Abel transform*, which can be interpreted again geometrically as an orbital integral, as in the symmetric space case, and which leads to an effective inversion formula for the spherical Fourier transform. Let us consider, for radial functions f on S, the integral transform

$$\mathcal{A}f(r) = a^{-\frac{Q}{2}} \int_N dn\, f(n,a)\ , \qquad \text{where } a = e^r\ . \tag{2.15}$$

It follows from the integral expression (2.6) that

$$\begin{aligned}\mathcal{H}f(\lambda) &= \int_S dx\, \varphi_\lambda(x)\, f(x) = \int_0^{+\infty} \frac{da}{a}\, a^{-Q} \int_N dn\, a^{\frac{Q}{2}-i\lambda}\, f(n,a)\\ &= \int_{-\infty}^{+\infty} dr\, e^{-i\lambda r}\, \mathcal{A}f(r)\end{aligned}$$

(2.16) i.e. $\mathcal{H} = \mathcal{F}\circ\mathcal{A}$,

where $\mathcal{F}$ denotes the classical Fourier transform on the real line. Hence

(2.17) Abel tranforms $\mathcal{A}f$ are even functions

and inverting effectively the spherical Fourier transform $\mathcal{H}$ or the Abel transform $\mathcal{A}$ amounts to the same problem.

Let us compute (2.15) explicitly. Write $n = (X, Z)$ as usual. (1.12) and (1.15) yield the expression

$$\begin{aligned}&\tanh^2 \frac{r'}{2} = 1 - \frac{4a}{(1 + a + \frac{1}{4}\,|X|^2)^2 + |Z|^2}\\ &\text{i.e.}\quad \cosh\frac{r'}{2} = \left\{\left(\cosh\frac{r}{2} + \frac{1}{8}\, e^{-\frac{r}{2}}\, |X|^2\right)^2 + \frac{1}{4}\, e^{-r}\, |Z|^2\right\}^{\frac{1}{2}}\end{aligned} \tag{2.18}$$

for the geodesic distance r' between $x = (X, Z, a)$ and the origin. Thus (2.15) becomes

$$\begin{aligned}(2.19)\quad & \mathcal{A}f(r) \\ &= e^{-\frac{Q}{2}r} \int_{\mathfrak{v}} dX \int_{\mathfrak{z}} dZ\, f\left(2 \operatorname{arg\,cosh} \left\{ \left(\cosh \frac{r}{2} + \frac{1}{8} e^{-\frac{r}{2}} |X|^2\right)^2 + \frac{1}{4} e^{-r} |Z|^2 \right\}^{\frac{1}{2}}\right) \\ &= 2^{\frac{3m}{2}+k} \int_{\mathfrak{v}} dX \int_{\mathfrak{z}} dZ\, f\left(2 \operatorname{arg\,cosh} \left\{ \left(\cosh \frac{r}{2} + |X|^2\right)^2 + |Z|^2 \right\}^{\frac{1}{2}}\right).\end{aligned}$$

Elementary integral calculus (introduce polar coordinates $X = \xi^{\frac{1}{2}} \frac{X}{|X|}$ in $\mathfrak{v}$, $Z = \zeta^{\frac{1}{2}} \frac{Z}{|Z|}$ in $\mathfrak{z}$ and substitute successively $\cosh \frac{s_1}{2} = \cosh \frac{r}{2} + \xi$, $\cosh^2 \frac{s_2}{2} = \cosh^2 \frac{s_1}{2} + \zeta$ i.e. $\cosh s_2 = \cosh s_1 + 2\zeta$) leads to the formula

$$(2.20)\qquad \mathcal{A} = 2^{\frac{3m+k}{2}} \pi^{\frac{m+k}{2}} \mathcal{W}^{\frac{1}{2}}_{\frac{m}{2}} \circ \mathcal{W}^{1}_{\frac{k}{2}}$$

expressing the Abel transform as a composition of two *Weyl type fractional integral transforms* [Koo; (5.61)], defined by

$$(2.21)\qquad \mathcal{W}^{\tau}_{\mu} f(r) = \frac{1}{\Gamma(\mu)} \int_r^{+\infty} d(\cosh \tau s)\, (\cosh \tau s - \cosh \tau r)^{\mu-1} f(s)$$

for $\tau > 0$, $\operatorname{Re} \mu > 0$.

In order to invert the Abel transform, we use the fact that $\{\mathcal{W}^{\tau}_{\mu}\}_{\mu\in\mathbb{C}}$ is (for fixed $\tau > 0$) a one-parameter group of transformations let say of $C_c^{\infty}(\mathbb{R})_{\text{even}}$, with

$$(2.22)\qquad \mathcal{W}^{\tau}_{-1} f(r) = -\frac{d}{d(\cosh \tau r)} f(r)\,.$$

Thus

$$(2.23)\qquad \mathcal{A}^{-1} = 2^{-\frac{3m+k}{2}} \pi^{-\frac{m+k}{2}} \mathcal{W}^{1}_{-\frac{k}{2}} \circ \mathcal{W}^{\frac{1}{2}}_{-\frac{m}{2}}\,.$$

Explicitly,

$$(2.24.1)\quad \mathcal{A}^{-1} f(r) = 2^{-\frac{3m+k}{2}} \pi^{-\frac{m+k}{2}} \left(-\frac{d}{d(\cosh r)}\right)^{\frac{k}{2}} \left(-\frac{d}{d(\cosh \frac{r}{2})}\right)^{\frac{m}{2}} f(r)$$

when k is even and

$$\begin{aligned}\mathcal{A}^{-1} f(r) = 2^{-\frac{3m+k}{2}} \pi^{-\frac{n}{2}} \int_r^{+\infty} \frac{d(\cosh s)}{\sqrt{\cosh s - \cosh r}} \left(-\frac{d}{d(\cosh s)}\right)^{\frac{k+1}{2}} \\ \cdot \left(-\frac{d}{d(\cosh \frac{s}{2})}\right)^{\frac{m}{2}} f(s)\end{aligned} \tag{2.24.2}$$

when k is odd (recall that m is always even).

The mapping properties in the setting of smooth functions with compact support can be summarized by the following commutative diagram

$$\begin{array}{ccc} & \mathcal{PW}(\mathbb{C})_{\text{even}} & \\ \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\ C_c^\infty(S)^\natural & \underset{\mathcal{A}}{\longrightarrow} & C_c^\infty(\mathbb{R})_{\text{even}} \end{array} \tag{2.25}$$

where each arrow is a (topological) isomorphism. This picture can be extended to larger function spaces. Inspired by the symmetric space case, let us introduce for instance the following radial *Schwartz type space* on S:

$$\begin{aligned}\mathcal{S}(S)^\natural = \Big\{ f \in C^\infty(S)^\natural \mid \left(\frac{d}{dr}\right)^M f(r) = O\Big((1+r)^{-N} e^{-\frac{Q}{2}r}\Big) \\ \text{for any nonnegative integers } M,\, N \Big\},\end{aligned} \tag{2.26}$$

which is nothing else but $(\cosh .)^{-\frac{Q}{2}} \mathcal{S}(\mathbb{R})_{\text{even}}$ (with the usual identification), $\mathcal{S}(\mathbb{R})$ denoting the classical Schwartz space on $\mathbb{R}$. Then we have again a commutative diagram

$$\begin{array}{ccc} & \mathcal{S}(\mathbb{R})_{\text{even}} & \\ \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\ \mathcal{S}(S)^\natural & \underset{\mathcal{A}}{\longrightarrow} & \mathcal{S}(\mathbb{R})_{\text{even}} \end{array} \tag{2.27}$$

where each arrow is a topological isomorphism [Koo; § 6]. Let us make two remarks:

(2.28) There is actually a whole range of radial L^p *Schwartz type spaces* $\mathcal{S}^p(S)^\natural$ $(0 < p \leq 2)$, which are defined by substituting $\frac{Q}{p}$ for $\frac{Q}{2}$ in (2.26). Their Abel transform is $(\cosh .)^{-(\frac{1}{p}-\frac{1}{2})Q} \mathcal{S}(\mathbb{R})_{\text{even}}$ and their Fourier transform, for $p < 2$, is the space of even holomorphic functions $h = h(\lambda)$ inside the strip $|\operatorname{Im} \lambda| < (\frac{1}{p} - \frac{1}{2})\, Q$, which satisfy

$$(2.29)\qquad \sup_{|\operatorname{Im}\lambda|<(\frac{1}{p}-\frac{1}{2})Q} (1+|\lambda|)^M \left|\left(\frac{d}{d\lambda}\right)^N h(\lambda)\right| < +\infty\,,$$

for any nonnegative integers M, N [Koo; Theorem 6.1].

(2.30) These Schwartz spaces can be defined more intrinsically by replacing the radial derivatives $\left(\frac{d}{dr}\right)^M$ in (2.26) by powers $\mathcal{L}^M$ of the Laplace-Beltrami operator, as in the symmetric space case (see for instance [An2; § 4, Remark (i)]).

For the sake of completeness, let us give the explicit expression of the *Plancherel measure*:

$$(2.31)\quad c_0\,|\mathbf{c}(\lambda)|^{-2} = \frac{1}{2^{n-1}\,\pi^{\frac{n}{2}}\,\Gamma(\frac{n}{2})}\times$$

$$\times\begin{cases} \prod\limits_{0<j\text{ integer}<\frac{m}{4}} \{\lambda^2+j^2\}^2\lambda^3\coth\pi\lambda\,, \\ \prod\limits_{0<j\text{ odd integer}<\frac{m}{2}} \left\{\lambda^2+\left(\frac{j}{2}\right)^2\right\}^2 \prod\limits_{\frac{m}{2}<j\text{ odd integer}<\frac{m}{2}+k} \left\{\lambda^2+\left(\frac{j}{2}\right)^2\right\}\lambda\tanh\pi\lambda\,, \\ \prod\limits_{0\le j\text{ integer}\le\frac{m}{2}} \left\{\lambda^2+\left(\frac{j}{2}\right)^2\right\} \prod\limits_{\frac{m}{2}<j\text{ even integer}<\frac{m}{2}+k} \left\{\lambda^2+\left(\frac{j}{2}\right)^2\right\}\,, \end{cases}$$

$$\text{depending whether}\quad \begin{cases} k=1 \text{ and } \dfrac{m}{2} \text{ is odd}\,, \\ k \text{ is odd and } \dfrac{m}{2} \text{ is even}\,, \\ k \text{ is even and } \dfrac{m}{2} \text{ is even}\,. \end{cases}$$

Notice that all possible cases are covered. See the list (1.4).

Eventually let us relate, as in the symmetric space case, the spherical functions φ_λ on S with a natural series of representations of S living on its boundary, namely the analytic continuation of the representations π_λ, unitarily induced from the characters $a \mapsto a^{i\lambda}$ ($\lambda \in \mathbb{R}$) of $A = \mathbb{R}_+^*$. These induced representations are realized on $L^2(N)$ by

$$(2.32)\qquad \{\pi_\lambda(x)\,f\,\}\,(n') = a^{-\frac{Q}{2}+i\lambda}\,f(a^{-1}n^{-1}n'a)\,,$$

where $x=(n,a)$ with $n\in N$ and $a\in A$. Via the stereographic projection, they can also be realized on $L^2(\partial B(\mathfrak{s}))$ by

$$\{\pi_\lambda(x)\,f\,\}(\sigma)=\left\{\frac{d(x^{-1}.\sigma)}{d\sigma}\right\}^{\frac12-i\frac{\lambda}{Q}} f(x^{-1}.\sigma)\,. \tag{2.33}$$

Both realizations admit obvious analytic continuation in $\lambda\in\mathbb{C}$. We want to show that

$$\varphi_\lambda(x)=\Big(\pi_\lambda(x)\,P_1^{\frac12-i\frac{\lambda}{Q}}\,,\,P_1^{\frac12-i\frac{\bar\lambda}{Q}}\Big)_{L^2(N)}=c^{-1}\,(\pi_\lambda(x)\,1\,,\,1)_{L^2(\partial B(\mathfrak{s}))}\,, \tag{2.34}$$

where P_1 is the Poisson kernel on N (see (1.13) or (1.14)) and $c=|\partial B(\mathfrak{s})|=2\,\frac{\pi^{\frac n2}}{\Gamma(\frac n2)}$. Contrary to the symmetric space case, this cannot be obtained here by an elementary substitution. Consider instead the *normalized λ-Poisson kernel* on S:

$$\mathcal{P}_\lambda(X,Z,a)=\frac{P_a(X,Z)^{\frac12-i\frac{\lambda}{Q}}}{P_1(0,0)^{\frac12-i\frac{\lambda}{Q}}}=a^{\frac{Q}{2}-i\lambda}\left\{\left(a+\frac14|X|^2\right)^2+|Z|^2\right\}^{-\frac{Q}{2}+i\lambda} \tag{2.35}$$

A direct computation as in [Dam3] yields $\mathcal{L}\mathcal{P}_\lambda=-\,(\frac{Q^2}{4}+\lambda^2)\,\mathcal{P}_\lambda$. Hence

$$\varphi_\lambda=(\mathcal{P}_\lambda)^\natural\,. \tag{2.36}$$

From now on, let us assume $\operatorname{Im}\lambda\geq 0$. Inspired by the determination of the *Martin boundary* of S in [DR2; § 7], we shall approximate $\mathcal{P}_\lambda(x)$ by ratios $\frac{G_\lambda(y^{-1}x)}{G_\lambda(y^{-1})}$ involving the *Green function* G_λ of $\mathcal{L}_\lambda=\mathcal{L}+\frac{Q^2}{4}+\lambda^2$. G_λ was determined in [DR2; Theorem 7.8] for $\lambda\in i\,[0,+\infty)$, but this restriction is not essential. Notice that, as in the hyperbolic space case ([An5], [Fa], [MW]), $G_\lambda=G_\lambda(r)$ is a multiple of the fundamental solution at infinity [Koo; (2.15)]

$$\begin{aligned}\Phi_\lambda(r)&=\Phi_{2\lambda}^{(\frac{m+k-1}{2},\frac{k-1}{2})}\left(\frac r2\right)\\&=\left(2\cosh\frac r2\right)^{-Q+i\,2\lambda}\\&\quad\cdot{}_2F_1\left(\frac Q2-i\lambda\,,\,\frac m4+\frac12-i\lambda\,;\,1-i\,2\lambda\,;\,\left(\cosh\frac r2\right)^{-2}\right)\\&\sim e^{(-\frac Q2+i\lambda)\,r}\qquad(r\to+\infty)\end{aligned} \tag{2.37}$$

of the Jacobi type equation $(\operatorname{rad}\mathcal{L}_\lambda)\,\Phi=0$. Precisely,

$$G_\lambda(r)=2^{k-2}\,\pi^{-\frac n2}\,\Gamma(\frac n2)\,(-i\lambda)^{-1}\,\mathbf{c}(-\lambda)^{-1}\,\Phi_\lambda(r)\,. \tag{2.38}$$

Now, let $y = (X, Z, a) \in S$ tend to $(X_0, Z_0, 0)$ in $N \times \mathbb{R}_+$. Then, as it was shown in the proof of [DR2; Theorem 7.11],

$$\tag{2.39} \begin{aligned} \frac{G_\lambda(y^{-1}x)}{G_\lambda(y^{-1})} = \frac{\Phi_\lambda(y^{-1}x)}{\Phi_\lambda(y^{-1})} &\sim e^{(-\frac{Q}{2}+i\lambda)\,(d(x,y)-d(e,y))} \\ &\to \frac{\mathcal{P}_\lambda(\,(X_0, Z_0, 1)^{-1}x\,)}{\mathcal{P}_\lambda(\,(X_0, Z_0, 1)^{\pm 1}\,)} \,. \end{aligned}$$

Moreover $\left| \frac{G_\lambda(y^{-1}x)}{G_\lambda(y^{\pm 1})} \right|$ is bounded above (and below) if x remains in a compact subset of S and $d(y,e) \to +\infty$.

We are now ready for the proof of (2.34). Take $y = (0, 0, a) \in S$ with $a \overset{>}{\to} 0$. By using (2.39) and dominated convergence, (2.36) becomes

$$\tag{2.40} \varphi_\lambda(x) = \lim_y \left\{ \frac{G_\lambda(y^{-1}\,.\,)}{G_\lambda(y^{\pm 1})} \right\}^\natural (x)\,.$$

Since G_λ is radial,

$$\tag{2.41} \{\,G_\lambda(y^{-1}\,.\,)\,\}^\natural = (\,\delta_y * G_\lambda\,)^\natural = (\delta_y)^\natural * G_\lambda\,,$$

where $(\delta_y)^\natural$ is the normalized surface measure on the geodesic sphere $\partial B(e, r)$ of radius $r = d(y, e)$ in S. Thus,

$$\tag{2.42} \begin{aligned} \varphi_\lambda(x) &= \lim_y \int_S d(\delta_y)^\natural(z)\, \frac{G_\lambda(z^{-1}x)}{G_\lambda(y^{\pm 1})} \\ &= \lim_{\rho \overset{<}{\to} 1} \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{\partial B(\mathfrak{s})} d\sigma\, \frac{G_\lambda(\{C^{-1}(\rho\sigma)\}^{-1}x)}{G_\lambda(\{C^{-1}(\rho\sigma)\}^{\pm 1})}\,. \end{aligned}$$

Using again (2.39) and dominated convergence, we obtain

$$\tag{2.43} \varphi_\lambda(x) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{\partial B(\mathfrak{s})} d\sigma\, \frac{\mathcal{P}_\lambda(\{n_\sigma\}^{-1}x)}{\mathcal{P}_\lambda(\{n_\sigma\}^{\pm 1})}\,,$$

where $n_\sigma = (\{\partial C\}^{-1}(\sigma), 1)$. This last expression is easily seen to be equivalent to (2.34).

Now that we have identified φ_λ with a matrix coefficient of π_λ, it becomes clear that

(2.44) φ_λ is a positive definite function on S when $\lambda \in \mathbb{R}$.

See (6.14) for further information.

3. – $L^p \to L^p$ and weak $L^1 \to L^1$ boundedness of positive radial convolution kernels

The problem consists in finding minimal conditions on a nonnegative locally integrable radial function $\kappa = \kappa(r)$ ensuring $L^p \to L^p$ boundedness, respectively weak $L^1 \to L^1$ boundedness of the corresponding right convolution operator $Tf = f * \kappa$, i.e.

$$\|Tf\|_p \le C_p \, \|f\|_p \quad \text{for all } f \in L^p(S)\,, \tag{3.1}$$

respectively

$$\sup_{t>0} |\{x \in S \mid |Tf(x)| > t\}| \le C_1 \frac{\|f\|_1}{t} \quad \text{for all } f \in L^1(S)\,. \tag{3.2}$$

We shall first establish the analogue of the *Herz criterion* for symmetric spaces ([Her; Theorem 7], [LRy; Anhang 3]), which takes care of (3.1). As we shall see later (Proposition 4.10.i), we can always reduce by duality to $1 \le p \le 2$.

(3.3) THEOREM. (3.1) *holds if and only if*

$$\mathcal{H}\kappa\left(i\left\{\frac{1}{p}-\frac{1}{2}\right\}Q\right) = \int_S dx\, \kappa(x)\, \varphi_{i(\frac{1}{p}-\frac{1}{2})Q}(x) \;<\; +\infty\,.$$

Moreover this quantity coincides with the $L^p \to L^p$ operator norm of T.

Notice that this integral condition reduces simply to $\kappa \in L^1(S)^\natural$ for $p = 1$, as should be expected, and that it can be made quite explicit in all cases, namely

$$\begin{cases} \left\{\int_0^1 dr\, r^{n-1} + \int_1^{+\infty} dr\, e^{\frac{Q}{p}r}\right\} \kappa(r) \;<\; +\infty & \text{if } 1 \le p < 2\,, \\ \left\{\int_0^1 dr\, r^{n-1} + \int_1^{+\infty} dr\, (1+r)\, e^{\frac{Q}{p}r}\right\} \kappa(r) \;<\; +\infty & \text{if } p = 2\,, \end{cases} \tag{3.4}$$

by introducing polar coordinates in S and using the basic behavior[†]

$$\varphi_{i(\frac{1}{p}-\frac{1}{2})Q}(r) \asymp \begin{cases} e^{-\frac{Q}{p'}r} & \text{if } 1 \le p < 2\,, \\ (1+r)\, e^{-\frac{Q}{2}r} & \text{if } p = 2\,. \end{cases} \tag{3.5}$$

[†] The symbol $\asymp$ between two positive expressions means for us that

$$C_1 \times \textit{second expression} \le \textit{first expression} \le C_2 \times \textit{second expression}$$

for some positive constants C_1 and C_2.

The *proof of Theorem 3.3* is different from the symmetric space case and actually simpler. Consider the right regular representation

$$\{R_p(x)f\}(y) = a(x)^{-\frac{Q}{p}} f(yx) \tag{3.6}$$

of S on $L^p(S)$. On one hand, given $f \in L^p(S)$ and $f' \in L^{p'}(S)$ with norms ≤ 1, we have

$$\int_S dy\, Tf(y)\, f'(y) = \int_S dx\, \kappa(x)\, a(x)^{\frac{Q}{p'}} \int_S dy\, \{R_p(x^{-1})f\}(y)\, f'(y)\,, \tag{3.7}$$

hence

$$\|T\|_{L^p \to L^p} \leq \int_S dx\, \kappa(x)\, a(x)^{\frac{Q}{p'}}\,. \tag{3.8}$$

On the other hand, since S is amenable, there exist $f_n \in L^p(S)$ and $f_n' \in L^{p'}(S)$ with norms ≤ 1 such that

$$\lim_{n\to+\infty} \int_S dy\, \{R_p(x^{-1})f_n\}(y)\, f_n'(y) = 1 \tag{3.9}$$

locally uniformly in $x \in S$. Hence,

$$\begin{aligned} \int_S dx\, \kappa(x) a(x)^{\frac{Q}{p'}} &\leq \sup_n \int_S dx\, \kappa(x) a(x)^{\frac{Q}{p'}} \int_S dy\, \{R_p(x^{-1})|f_n|\}(y)\, |f_n'(y)| \\ &= \sup_n \int_S dy\, \{T|f_n|\}(y)\, |f_n'(y)| \leq \|T\|_{L^p\to L^p}. \end{aligned} \tag{3.10}$$

In order to conclude, notice that

$$\int_S dx\, \kappa(x) a(x)^{\frac{Q}{p'}} = \int_S dx\, \kappa(x) \{a(\,.\,)^{\frac{Q}{p'}}\}^\natural(x) = \int_S dx\, \kappa(x)\, \varphi_{i(\frac{1}{p}-\frac{1}{2})Q}(x). \tag{3.11}$$

Theorem (3.3) implies the following *Kunze-Stein type phenomenon* for right convolution with radial functions. This is an amazing result at the first glance, since (any version of) the Kunze-Stein phenomenon is known to be false for general functions on S (see for instance [Co1], [CF], [Li], [Lo; pp. 414-415], ...).

(3.12) COROLLARY. *Given $1 \leq q < p \leq 2$, there is a positive constant C such that*

$$\|F * f\|_p \leq C\, \|F\|_p\, \|f\|_q\,,$$

for every $F \in L^p(S)$ and $f \in L^q(S)^\natural$.

Let us turn our attention to the weak type estimate (3.2) for $Tf = f * \kappa$. By taking for f an approximate unit in $L^1(S)$, such an estimate implies that κ

belongs to the *weak* L^1 *space* $L^{1,\infty}(S)^{\natural}$. Since the volume density behaves at infinity like (a positive constant times) e^{Qr}, a typical radial function which is $L^{1,\infty}$ at infinity is

$$\kappa(r) = e^{-Qr} \,. \tag{3.13}$$

The following theorem asserts that this type of decay at infinity is actually sufficient to imply weak $L^1 \to L^1$ boundedness. Both its statement and its proof are adapted from the analogous result for symmetric spaces [Str].

(3.14) THEOREM. *Let* κ *be a locally integrable radial function on* S *with the following behavior at infinity*

$$\kappa(r) = O\left(e^{-Qr}\right).$$

Then the associated right-convolution operator $Tf = f * \kappa$ *on* S *is of weak type* $L^1 \to L^1$.

PROOF. We can assume that $f \geq 0$ and $\kappa(r) = e^{-Qr}$. By definition,

$$Tf(x) = \int_S dy\, f(xy)\, \kappa(y^{\pm 1}) \,. \tag{3.15}$$

Write $y = (e^{\frac{r}{2}}X, e^r Z, e^r) = a_r n_{X,Z}$, where $a_r = (0,0,e^r)$ and $n_{X,Z} = (X, Z, 1)$. On one hand, the left-invariant Riemann-Haar measure on S in this decomposition is just $dy = dr\, dX\, dY$. On the other hand, (2.18) implies the following control of the geodesic distance r' between the point y and the origin:

$$e^{r'} \asymp e^r \left\{ \left(1 + \frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\} . \tag{3.16}$$

Thus

$$\begin{aligned} Tf(x) &\asymp \int_{+\infty}^{+\infty} dr\, e^{-Qr} \int_{\mathfrak{v}} dX \int_{\mathfrak{z}} dZ \left\{ \left(1+\frac{1}{4}|X|^2\right)^2 + |Z|^2 \right\}^{-Q} f(xa_r n_{X,Z}) \\ &\asymp \int_{+\infty}^{+\infty} dr\, e^{-Qr}\, T'f(xa_r) \,, \end{aligned} \tag{3.17}$$

where

$$T'f(x) = \int_N dn\, f(xn)\, P_1(n) \tag{3.18}$$

is the right-convolution operator on S associated with the Poisson kernel P_1 on N. Since T' is trivially bounded on $L^1(S)$, we are left with the weak $L^1 \to L^1$ boundedness of the operator

$$T''f(x) = \int_{-\infty}^{+\infty} dr\, e^{-Qr}\, f(xa_r) \,, \tag{3.19}$$

which is just the right-convolution on S with the function 1 on $A \equiv \mathbb{R}$. Writing this time $x = na_s$ (with $n \in N$ and $s \in \mathbb{R}$), we have

$$T''f(x) = e^{Qs} \int_{-\infty}^{+\infty} dr\, e^{-Qr} f(na_r)\,. \tag{3.20}$$

Hence

$$|\{x \in S \mid T''f(x) > t\}| \le \int_N dn \int_{s_0}^{+\infty} ds\, e^{-Qs} = \frac{1}{Q} \int_N dn\, e^{-Qs_0}\,, \tag{3.21}$$

where $e^{Qs_0} \int_{-\infty}^{+\infty} dr\, e^{-Qr} f(na_r) = t$. This brings us to the desired conclusion:

$$|\{x \in S \mid T''f(x) > t\}| \le \frac{1}{Qt} \int_S f\,. \tag{3.21}$$

(3.22) Corollary. *The* Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} dy\, |f(y)|$$

is of weak type $L^1 \to L^1$ on S.

Since $\mathcal{M}$ is trivially bounded on $L^\infty(S)$, it is also bounded on all intermediate spaces $L^p(S)$ $(1 < p < +\infty)$. For general Riemannian symmetric spaces of noncompact type, this L^p result was first obtained in [CS]. The weak $L^1 \to L^1$ boundedness, as far as it is concerned, was established in [Str].

Proof of Corollary (3.22). As usual, we consider separately the *local* Hardy-Littlewood maximal function

$$\mathcal{M}_0 f(x) = \sup_{0<r\le 1} \frac{1}{|B(x,r)|} \int_{B(x,r)} dy\, |f(y)| \tag{3.23}$$

and the *large scale* Hardy-Littlewood maximal function

$$\mathcal{M}_\infty f(x) = \sup_{r>1} \frac{1}{|B(x,r)|} \int_{B(x,r)} dy\, |f(y)| \tag{3.24}$$

On one hand, $\mathcal{M}_0$ can be handled as in the Euclidean case. On the other hand, $\mathcal{M}_\infty$ is dominated by the right-convolution operator $Tf = f * \kappa$ associated to the radial kernel

$$\kappa(r) = \left| B_{\max\{r,1\}} \right|^{-1} \asymp e^{-Qr}\,, \tag{3.25}$$

which is of weak type $L^1 \to L^1$ by Theorem (3.14).

4. – L^p radial multipliers

In the previous section, we have considered right convolution operators on $L^p(S)$ associated to radial functions which are essentially positive. This section is devoted to an actually wider class, consisting of all functions of the Laplace-Beltrami operator $\mathcal{L}$, which are at least bounded on $L^2(S)$.

Let us first make clear that, as in the symmetric space case, the spherical Plancherel formula (2.10) provides an explicit realization of the spectral decomposition of $\mathcal{L}$ on $L^2(S)$ and hence of all (spectrally defined) functions of $\mathcal{S}$. For every $h \in L^\infty(\mathbb{R})_{\text{even}}$ consider the right-convolution operator

$$T_h f = f * \mathcal{H}^{-1} h \tag{4.1}$$

let say from $C_c^\infty(S)$ to $C^\infty(S)$. The right-hand side of (4.1) should be understood in the following way:

$$f * \mathcal{H}^{-1} h(x) = c_0 \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} h(\lambda)\,(f * \varphi_\lambda)(x), \quad \text{with } f * \varphi_\lambda(x) = \mathcal{H}\{f(x\,.)^\natural\}(\lambda).$$

Notice that T_h is entirely determined by its action on radial functions, which corresponds simply to pointwise multiplication on the Fourier transform side:

$$\mathcal{H}(T_h f)(\lambda) = h(\lambda)\,\mathcal{H} f(\lambda)\,. \tag{4.2}$$

(4.3) Lemma *T_h is a bounded operator on $L^2(S)$, with operator norm equal to $\|h\|_{L^\infty}$.*

Proof. To be safe, we may assume that h belongs to the Paley-Wiener space $\mathcal{PW}(\mathbb{C})_{\text{even}}$. The general case follows by a routine completion argument. We have

$$\begin{aligned} \|T_h f\|_{L^2}^2 &= \big((T_h)^* T_h f, f\big)_{L^2} = \big(T_{|h|^2} f, f\big)_{L^2} = (f^* * T_{|h|^2} f)(e) \\ &= c_0 \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\, |h(\lambda)|^2 \int_S dx\, \varphi_\lambda(x)\,(f^* * f)(x)\,. \end{aligned} \tag{4.4}$$

Since φ_λ is positive definite when $\lambda \in \mathbb{R}$ (2.44),

$$\int_S dx\, \varphi_\lambda(x)\,(f^* * f)(x) \ge 0\,. \tag{4.5}$$

Hence

$$\|T_h f\|_{L^2}^2 \le c_0\, \|h\|_{L^\infty}^2 \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \int_S dx\, \varphi_\lambda(x)\,(f^* * f)(x)\,. \tag{4.6}$$

By (2.10) the right-hand side of (4.6) is equal to

$$\|h\|_{L^\infty}^2\,(f^* * f)^\natural(e) = \|h\|_{L^\infty}^2\,\|f\|_{L^2}^2\,. \tag{4.7}$$

Thus T_h is bounded on $L^2(S)$, with operator norm $\le \|h\|_{L^\infty}$. On the other hand, it is clear from (2.10) that T_h is bounded on $L^2(S)^\natural$ with operator norm $= \|h\|_{L^\infty}$.

It now follows easily from (2.6), (2.10) and Lemma (4.3) that the spectral decomposition of $\mathcal{L}$ on $L^2(S)$ is given by

$$E_{(-\frac{Q^2}{4}-\lambda_2^2,\,-\frac{Q^2}{4}-\lambda_1^2]}f = T_{\chi_{[\lambda_1,\lambda_2)}}f = c_0\int_{\lambda_1}^{\lambda_2}\frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\,(f*\varphi_\lambda)(x) \tag{4.8}$$

for $0 \le \lambda_1 < \lambda_2 \le +\infty$. Moreover all (spectrally defined) functions of $\mathcal{L}$ can be written

$$\widetilde{h}(\mathcal{L}) = h\Big(\sqrt{-\mathcal{L}-\frac{Q^2}{4}}\,\Big) = T_h \tag{4.9}$$

for some $h \in L^\infty(\mathbb{R})_{\text{even}}$.

The classical L^p *multiplier problem* consists in finding properties of h reflecting the boundedness of T_h on $L^p(S)$. As for symmetric spaces [CS], we have the following necessary conditions.

(4.10) PROPOSITION. (i) *Assume that T_h is bounded on $L^p(S)$, for some $1 \le p < +\infty$. Then T_h is bounded on the dual space $L^{p'}(S)$ and hence on $L^2(S)$.*
(ii) *Assume that T_h is bounded on $L^p(S)$, for some $1 \le p < 2$. Then $h = h(\lambda)$ extends to an even bounded holomorphic function in the strip $|\operatorname{Im}\lambda| < (\frac{1}{p}-\frac{1}{2})\,Q$ (continuous up to the boundary in the case $p = 1$).*

The *proof* is similar to the symmetric space case [CS]. Let us recall it. Since $h = \check{h}$, T_h coincide with the dual operator $(T_h)^t$, which implies immediately (i). For (ii) we use the fact that

$$\varphi_\lambda \text{ belongs to } L^{p'}(S)^\natural \text{ when } |\operatorname{Im}\lambda| < (\frac{1}{p}-\frac{1}{2})\,Q \tag{4.11}$$
$$\text{(respectively } |\operatorname{Im}\lambda| \le \frac{Q}{2} \text{ in the case } p = 1\text{)}.$$

This follows from the integral formula (2.6), which yields by convexity

$$|\varphi_\lambda(x)| \le \varphi_{\pm i\mu}(x) \quad \text{when } |\operatorname{Im}\lambda| \le \mu\,, \tag{4.12}$$

and from the asymptotic behavior (2.7). Our next step is the crucial formula

$$T\varphi_\lambda = h(\lambda)\,\varphi_\lambda\,, \tag{4.13}$$

which is easily established for $\lambda \in \mathbb{R}$:

$$(4.14)\quad \int_S (T_h\varphi_\lambda)\, f = \int_S \varphi_\lambda\, (T_h f) = \mathcal{H}(T_h f)(\lambda) = h(\lambda)\, \mathcal{H}f(\lambda) = h(\lambda) \int_S \varphi_\lambda\, f$$

holds indeed for every $f \in C_c^\infty(S)^\natural$. The analytic continuation of h follows now from (4.13), more precisely from

$$(4.15)\qquad h(\lambda) = \frac{\mathcal{H}(T_h f)(\lambda)}{\mathcal{H}f(\lambda)}$$

with suitable choice of test functions $f \in C_c^\infty(S)^\natural$. The uniform boundedness of h is obtained by taking the $L^{p'}$-norm in (4.13):

$$(4.16)\qquad |h(\lambda)|\, \|\varphi_\lambda\|_{L^{p'}} = \|T_h\varphi_\lambda\|_{L^{p'}} \leq \|T_h\|_{L^{p'}\to L^{p'}}\, \|\varphi_\lambda\|_{L^{p'}}\,,$$

and dividing by $\|\varphi_\lambda\|_{L^{p'}} \neq 0$. This concludes the proof of Proposition (4.10).

Reciprocally one can show, as in [An3], [ST], [Ta], ... , that T_h is bounded on $L^p(S)$ provided $h = h(\lambda)$ extends holomorphically to the strip $|\operatorname{Im}\lambda| < (\frac{1}{p} - \frac{1}{2})\, Q$ and satisfies there sufficiently many uniform symbol type conditions. Here is a sharp *Hörmander type* multiplier theorem, which was recently obtained in an actually wider context [AS].

(4.17) Theorem. *Let $1 < p < 2$. Assume that $h = h(\lambda)$ is an even bounded holomorphic function in the strip $|\operatorname{Im}\lambda| < (\frac{1}{p} - \frac{1}{2})\, Q$ whose boundary value $\widetilde{h}(\lambda) = \lim_{\mu \overset{<}{\to} (\frac{1}{p}-\frac{1}{2})Q} h(\lambda + i\mu)$ satisfies*

(i) *$\eta_0\, \widetilde{h}$ belongs to the* Besov space *$B^\alpha_{q,1}(\mathbb{R})$,*

(ii) *$\eta\, \widetilde{h}(t\,.)$ belongs to $B^\alpha_{q,1}(\mathbb{R})$, uniformly in $t \gg 0$,*

where $\eta_0 = \eta_0(\lambda)$, respectively $\eta = \eta(\lambda)$ is an even (smooth) bump function around $\lambda = 0$, respectively $\lambda = \pm 1$ and $\alpha = n\, (\frac{1}{p} - \frac{1}{2}) > \frac{n}{q} > 0$. Then T_h is a bounded operator on $L^p(S)$. Moreover $B^\alpha_{q,1}(\mathbb{R})$ can be replaced by $B^\alpha_{2,1}(\mathbb{R})$ when $p < 2\frac{n+1}{n+3}$.

As in the symmetric space case ([LRy], [Ta]), one gets as a consequence the following L^p spectral result.

(4.18) Corollary. *For $1 \leq p \leq +\infty$, the L^p spectrum of $\mathcal{L}$ consists of the parabolic region*

$$\left\{ -\frac{Q^2}{4} - \lambda^2 \mid |\operatorname{Im}\lambda| \leq |\frac{1}{p} - \frac{1}{2}|\, Q \right\}.$$

5. - The heat kernel

Consider the *heat equation*

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \mathcal{L}_x u(t,x) & (t>0,\ x\in S)\,, \\ \lim\limits_{t \overset{>}{\to} 0} u(t,x) = f(x)\,, \end{cases} \tag{5.1}$$

associated to the Laplace-Beltrami operator $\mathcal{L}$ on S. Its solution is given by the *heat semi-group*

$$u(t,x) = e^{t\mathcal{L}} f\,(x) = \int_S dy\, h_t(x,y)\, f(y)\,, \tag{5.2}$$

with corresponding *heat kernel* $h_t(x,y)$. Like all kernels of functions of $\mathcal{L}$, h_t is a radial right-convolution kernel on S:

$$h_t(x,y) = h_t\big(d(x,y)\big)\,. \tag{5.3}$$

Its analysis fits into the setting (2.27):

$$\mathcal{H}h_t(\lambda) = e^{-t\,(\frac{Q^2}{4}+\lambda^2)}\,. \tag{5.4}$$

By the inversion formula (2.10),

$$h_t(x) = c_0\, e^{-\frac{Q^2}{4}t} \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\, e^{-t\lambda^2}\, \varphi_\lambda(x)\,. \tag{5.5}$$

A better expression is obtained by applying first the inverse Euclidean Fourier transform $\mathcal{F}^{-1}$ and then the inverse Abel transform $\mathcal{A}^{-1}$:

$$\mathcal{A}h_t(r) = (\mathcal{F}^{-1}\circ\mathcal{H})\, h_t(r) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{Q^2}{4}t}\, e^{-\frac{r^2}{4t}}\,, \tag{5.6}$$

hence

$$h_t(r) = 2^{-m-\frac{k}{2}-1}\pi^{-\frac{n}{2}}t^{-\frac{1}{2}}\, e^{-\frac{Q^2}{4}t} \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{\frac{k}{2}} \left(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\right)^{\frac{m}{2}} e^{-\frac{r^2}{4t}} \tag{5.7}$$

by (2.24.1) when k is even and

$$\begin{aligned} h_t(r) &= 2^{-m-\frac{k}{2}-1}\pi^{-\frac{n+1}{2}}t^{-\frac{1}{2}}\, e^{-\frac{Q^2}{4}t} \\ &\times \int_r^{+\infty} \frac{ds\, \sinh s}{\sqrt{\cosh s - \cosh r}} \left(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{k+1}{2}} \left(-\frac{1}{\sinh\frac{s}{2}}\frac{\partial}{\partial s}\right)^{\frac{m}{2}} e^{-\frac{s^2}{4t}} \end{aligned} \tag{5.8}$$

by (2.24.2) when k is odd. Such formulas were previously obtained in [LRy] for all hyperbolic spaces.

Most of this section will be devoted to establishing the following sharp heat kernel estimate.

(5.9) THEOREM. *We have*

$$h_t(r) \asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{Q^2}{4}t-\frac{Q}{2}r-\frac{r^2}{4t}}$$

for $t>0$ *and* $r\geq 0$.

Let us make some bibliographical comments about Theorem (5.9). For general hyperbolic spaces $H^N(\mathbb{F})$ (and some higher rank symmetric spaces), the upper bound was given in [An1]. Simultaneously and independently, the upper and lower estimates were established in [DM] (see also [Dav; § 5.7]) for real hyperbolic spaces $H^N(\mathbb{R})$. The proof can be actually adapted, with some care, to the general hyperbolic space case ([GM; § 3], [Mu]). Here we shall extend it further to harmonic AN groups. Recently such upper and lower estimates were established by probabilistic methods for a much wider class of *radial Laplacians* [LRo]. We think however that our proof is not without interest and in any case simpler in our setting.

We shall first estimate expressions of the form

$$(5.10) \qquad \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^q \left(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\right)^p e^{-\frac{r^2}{4t}},$$

which constitute essentially the right-hand side of the even case (5.7) and appear also inside the integral in the odd case (5.8). In order to analyze (5.10), we shall borrow and improve some arguments from [CGGM; pp. 647-648]. Consider the *modified Jacobi operator*

$$(5.11)\quad \widetilde{\mathcal{L}}_{\alpha,\beta}=\frac{d^2}{dr^2}+\left\{\left(\alpha+\frac{1}{2}\right)\coth\frac{r}{2}+\left(\beta+\frac{1}{2}\right)\tanh\frac{r}{2}\right\}\frac{d}{dr}+\left(\frac{\alpha+\beta+1}{2}\right)^2.$$

Straightforward computations yield the following *shift* or *transmutation* formulas.

(5.12) LEMMA. *For any* $\alpha,\ \beta\in\mathbb{C}$,

(i) $\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)\circ\widetilde{\mathcal{L}}_{\alpha,-\frac{1}{2}}=\widetilde{\mathcal{L}}_{\alpha+1,-\frac{1}{2}}\circ\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)$,

(ii) $\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)\circ\widetilde{\mathcal{L}}_{\alpha,\beta}=\widetilde{\mathcal{L}}_{\alpha+1,\beta+1}\circ\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)$.

Set

$$(5.13)\qquad f_{p,q}(r)=\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)^q\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)^p(-r^2),$$

where p and q are nonnegative integers with $p+q>0$.

(5.14) COROLLARY. $f_{p,q}(r)=c\,\phi_0^{(p+q-\frac{1}{2},q-\frac{1}{2})}(\frac{r}{2})$ *for some constant* $c>0$.

PROOF. $f_{p,q}(r)$ and $\phi_0^{(p+q-\frac{1}{2},q-\frac{1}{2})}(\frac{r}{2})$ are even smooth functions on $\mathbb{R}$ satisfying the same differential equation:

$$\widetilde{\mathcal{L}}_{p+q-\frac{1}{2},q-\frac{1}{2}}f(r)=0\,. \tag{5.15}$$

For $f_{p,q}$ this follows by induction from Lemma (5.12) and from the trivial identity $\frac{d}{dr^3}r^2=0$. Equation (5.15) has a regular singularity at the origin with indices 0 and $1-2p-2q<0$. $\phi_0^{(p+q-\frac{1}{2},q-\frac{1}{2})}$ is known to be (stricly) positive. For instance

$$\begin{aligned}\phi_0^{(p+q-\frac{1}{2},q-\frac{1}{2})}(r)&=(\cosh r)^{-\alpha-\beta-1}\\&\quad\cdot{}_2F_1\left(\frac{\alpha+\beta+1}{2},\frac{\alpha-\beta+1}{2};\alpha+1;\tanh^2 r\right)\end{aligned} \tag{5.16}$$

and the *hypergeometric function* ${}_2F_1(a,b;c;z)=\sum_{j=0}^{+\infty}\frac{(a)_j\,(b)_j}{j!\,(c)_j}z^j$ is positive if $a\geq 0$, $b\geq 0$, $c>0$ and $0\leq z<1$. Hence $f_{p,q}(r)=c\,\phi_0^{(p+q-\frac{1}{2},q-\frac{1}{2})}(\frac{r}{2})$.

It remains to show that $c>0$. We shall assume $p>0$ and $p+q>1$ (the degenerate case $p=0$, $q>0$ is handled like $p>0$, $q=0$ and the case $f_{1,0}(r)=\frac{2r}{\sinh\frac{r}{2}}$ is trivial). Then $f_{p,q}(r)$ is equal to

$$2r\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)^q\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)^{p-1}\frac{1}{\sinh\frac{r}{2}} \tag{5.17.1}$$

plus a linear combination of products

$$\begin{aligned}&\left\{\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)^{q'}\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)^{p'}r\right\}\\&\times\left\{\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)^{q''}\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)^{p''}\frac{1}{\sinh\frac{r}{2}}\right\}\end{aligned} \tag{5.17.2}$$

with $p'+p''=p-1$, $q'+q''=q$ and $p'+q'>0$. By expanding

$$\frac{1}{\sinh\tau r}=\frac{2\,e^{-\tau r}}{1-e^{-2\tau r}}=2\sum_{j=0}^{+\infty}e^{-(2j+1)\tau r}\,, \tag{5.18}$$

one sees that

$$\left(-\frac{1}{\sinh r}\frac{d}{dr}\right)^{\widetilde{q}}\left(-\frac{1}{\sinh\frac{r}{2}}\frac{d}{dr}\right)^{\widetilde{p}}\frac{1}{\sinh\frac{r}{2}} \asymp e^{-(\frac{\widetilde{p}}{2}+\widetilde{q})r} \tag{5.19}$$

as $r \to +\infty$ and consequently

$$f_{p,q}(r) = c'\,r\,e^{-(\frac{p}{2}+q)r} + O\big(e^{-(\frac{p}{2}+q)r}\big) \tag{5.20}$$

for some constant $c' > 0$. In particular $f_{p,q}(r) > 0$ for $r \gg 0$ and therefore the constant c is positive.

(5.21) Corollary.

$$f_{p,q}(r) \asymp (1+r)\,e^{-(\frac{p}{2}+q)r} \qquad \textit{for } r \geq 0\,.$$

(5.22) Proposition. *Let p, q be nonnegative integers with $p+q>0$. Then*

$$\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{q}\left(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\right)^{p} e^{-\frac{r^2}{4t}} \asymp \frac{1+r}{t}\,(1+\frac{1+r}{t})^{p+q-1}\,e^{-(\frac{p}{2}+q)r-\frac{r^2}{4t}}$$

for $t > 0$ and $r \geq 0$.

Proof. We have

$$\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{q}\left(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\right)^{p} e^{-\frac{r^2}{4t}} = \sum_{j=1}^{p+q} a_j(r)\,t^{-j}\,e^{-\frac{r^2}{4t}}\,, \tag{5.23}$$

where $a_1 = \frac{1}{4}\,f_{p,q}$, $a_{p+q} = 4^{-p-q}\,f_{1,0}^{\,p}\,f_{0,1}^{\,q}$ and more generally a_j is a linear combination, with nonnegative coefficients, of products $f_{p_1,q_1}\cdots f_{p_j,q_j}$ with $p_1+\ldots+p_j = p$, $q_1+\ldots+q_j = q$. Therefore,

$$a_j(r) = O\big((1+r)^j\,e^{-(\frac{p}{2}+q)r}\big)\,. \tag{5.24}$$

Hence

$$\begin{aligned} &\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{q}\left(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\right)^{p} e^{-\frac{r^2}{4t}} \\ &\qquad \asymp \left\{\frac{1+r}{t} + \left(\frac{1+r}{t}\right)^{p+q}\right\} e^{-(\frac{p}{2}+q)r-\frac{r^2}{4t}} \\ &\qquad = \frac{1+r}{t}\left(1+\frac{1+r}{t}\right)^{p+q-1} e^{-(\frac{p}{2}+q)r-\frac{r^2}{4t}}\,, \end{aligned} \tag{5.25}$$

which finishes the proof of Proposition (5.22).

This settles the even case (5.7). The estimate (5.9) in the odd case (5.8) is obtained by combining (5.22) with the following Proposition, which handles the integral part.

(5.26) PROPOSITION. *Assume $p \geq 0$ and $q > \frac{1}{2}$. Then*

$$\int_r^{+\infty} \frac{ds\, \sinh s}{\sqrt{\cosh s - \cosh r}}\, (1+s) \left(1 + \frac{1+s}{t}\right)^p e^{-qs - \frac{s^2}{4t}} \asymp (1+r) \left(1 + \frac{1+r}{t}\right)^{p-\frac{1}{2}} e^{-(q-\frac{1}{2})r - \frac{r^2}{4t}} .$$

PROOF. After performing the change of variables $s = r + u$, the left-hand side of (5.26) becomes

$$(5.27) \qquad e^{-qr - \frac{r^2}{4t}} \int_0^{+\infty} \frac{du\, \sinh(r+u)}{\sqrt{\cosh(r+u) - \cosh r}} \cdot (1+r+u) \left(1 + \frac{1+r+u}{t}\right)^p e^{-qu - \frac{ru}{2t} - \frac{u^2}{4t}} .$$

Using the elementary estimates

$$(5.28) \qquad \sinh v \asymp \frac{v}{1+v}\, e^v \qquad \text{for } v \geq 0,$$

$$(5.29) \qquad \cosh v - \cosh(v-w) \asymp \sinh v\, \frac{w}{1+w} \qquad \text{for } v \geq w \geq 0,$$

$$(5.30) \qquad \sqrt{1+v} \asymp 1 + \sqrt{v} \qquad \text{for } v \geq 0,$$

the integral in (5.27) is easily seen to be equivalent to

$$(5.31) \qquad t^{-p}\, e^{\frac{r}{2}} \int_0^{+\infty} du\, \sqrt{1+u} \cdot \left(1 + \sqrt{\frac{r}{u}}\right) \sqrt{1+r+u}\, (1+t+r+u)^p\, e^{-(q-\frac{1}{2})u - \frac{ru}{2t} - \frac{u^2}{4t}} .$$

Call $I_1 = I_1(t,r)$ the last integral. It remains to show that

$$(5.32) \qquad I_1 \asymp \sqrt{t}\, (1+r)\, (1+t+r)^{p-\frac{1}{2}} .$$

To estimate I_1 from above, we use the elementary inequalities

$$(5.33) \qquad \sqrt{1+r+u} \leq \sqrt{1+r}\, \sqrt{1+u} ,$$

$$(5.34) \qquad (1+t+r+u)^p \leq (1+t+r)^p\, (1+u)^p ,$$

$$(5.35) \qquad (1+u)^p\, e^{-(q-\frac{1}{2})u} \leq C\, e^{-\varepsilon u} \qquad \text{for some } \varepsilon > 0 ,$$

and get

$$I_1 \le C\sqrt{1+r}\,(1+t+r)^p \int_0^{+\infty} du \left(1+\sqrt{\frac{r}{u}}\right) e^{-\varepsilon u-\frac{ru}{2t}-\frac{u^2}{4t}}\,. \tag{5.36}$$

Call $I_2 = I_2(t,r)$ the last integral. For $t \ge 1+r$, we have

$$I_2 \le \int_0^{+\infty} du\, e^{-\varepsilon u} + \sqrt{r}\int_0^{+\infty} \frac{du}{\sqrt{u}}\, e^{-\epsilon u} \le C\sqrt{1+r} \le C\,\frac{\sqrt{t}\sqrt{1+r}}{\sqrt{1+t+r}} \tag{5.37}$$

and, for $t \le 1+r$,

$$I_2 \le \int_0^{+\infty} du\, e^{-\frac{u^2}{4t}} + \sqrt{r}\int_0^{+\infty} \frac{du}{\sqrt{u}}\, e^{-\frac{ru}{2t}} \le C\sqrt{t} \le C\,\frac{\sqrt{t}\sqrt{1+r}}{\sqrt{1+t+r}}\,. \tag{5.38}$$

(5.36), (5.37) and (5.38) give the announced upper estimate (5.32) of I_1. For the lower estimate, we use

$$\sqrt{1+u} \ge 1 \tag{5.39}$$

$$\sqrt{1+r+u} \ge \sqrt{1+r} \tag{5.40}$$

$$(1+t+r+u)^p \ge (1+t+r)^p \tag{5.41}$$

and get

$$I_1 \ge \sqrt{1+r}\,(1+t+r)^p \int_0^{+\infty} du \left(1+\sqrt{\frac{r}{u}}\right) e^{-(q-\frac{1}{2})u-\frac{ru}{2t}-\frac{u^2}{4t}}\,. \tag{5.42}$$

Call $I_3 = I_3(t,r)$ the last integral. For $r \le \min\{1,\sqrt{t}\,\}$, we have

$$I_3 \ge \int_0^{+\infty} du\, e^{-(q-\frac{1}{2})u-\frac{ru}{2t}-\frac{u^2}{4t}} = e^{\frac{T^2}{t}} \int_0^{\infty} du\, e^{-\frac{(T+\frac{u}{2})^2}{t}}\,, \tag{5.43}$$

where $T = (q-\frac{1}{2})t + \frac{r}{2}$. Performing the change of variables $u' = \frac{T+\frac{u}{2}}{\sqrt{t}}$ and using

$$\int_{T'}^{+\infty} du'\, e^{-u'^2} \asymp \frac{1}{1+T'}\, e^{-T'^2} \qquad \text{for } T' \ge 0\,, \tag{5.44}$$

we obtain

$$I_3 \ge C\,\frac{\sqrt{t}}{1+\sqrt{t}+\frac{r}{\sqrt{t}}} \ge C\,\frac{\sqrt{t}\sqrt{1+r}}{\sqrt{1+t+r}}\,. \tag{5.45}$$

For $u \geq \min\{1, \sqrt{t}\}$, we have

$$I_3 \geq C \int_0^{\sqrt{t}} du \sqrt{\frac{r}{u}}\, e^{-(q-\frac{1}{2})u - \frac{ru}{2t}}\,. \tag{5.46}$$

Performing the change of variables $u'^2 = \frac{Tu}{t}$ and using this time

$$\int_0^{T'} du'\, e^{-u'^2} \asymp \frac{T'}{1+T'} \qquad \text{for } T' \geq 0, \tag{5.47}$$

we obtain

$$I_3 \geq C \frac{\sqrt{t}\sqrt{r}}{\sqrt{\sqrt{t}+t+r}}, \text{ which is } \geq C \frac{\sqrt{t}\sqrt{1+r}}{\sqrt{1+t+r}} \text{ if } r \geq 1 \text{ or } r \geq \sqrt{t}. \tag{5.48}$$

(5.42), (5.45) and (5.48) give the announced lower estimate (5.32) of I_1. And this concludes the proof of Theorem (5.9).

Since $|\operatorname{grad} h_t(r)| = -\frac{\partial}{\partial r} h_t(r) = \sinh r \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right) h_t(r)$, one obtains similarly the following gradient estimate.

(5.49) COROLLARY. *For all $t > 0$ and $r \geq 0$,*

$$|\operatorname{grad} h_t(r)| \asymp t^{-\frac{3}{2}}\, r \left(1 + \frac{1+r}{t}\right)^{\frac{n-1}{2}} e^{-\frac{Q^2}{4}t - \frac{Q}{2}r - \frac{r^2}{4t}}\,.$$

The heat kernel upper estimate in Theorem (5.9) implies the following refined result.

(5.50) THEOREM. *The* heat maximal operator

$$\mathcal{T} f(x) = \sup_{t>0} |e^{t\mathcal{L}} f(x)|$$

is of weak type $L^1 \to L^1$.

Since $\mathcal{T}$ is trivially bounded on $L^\infty(S)$, it is also bounded on all intermediate spaces $L^p(S)$ $(1 < p < +\infty)$. The weak $L^1 \to L^1$ boundedness of the heat maximal operator was established in [LZ] for hyperbolic spaces and some higher rank Riemannian symmetric spaces of noncompact type, in [An4; Corollary 3.2] for general Riemannian symmetric spaces of noncompact type, in [CGGM] for groups with polynomial growth and for a distinguished Laplacian on some Iwasawa AN groups, in [AJ] for general Iwasawa AN groups, ...

PROOF OF THEOREM (5.50). As usual, we estimate separately the *small time* maximal operator

$$\mathcal{T}_0 f(x) = \sup_{0<t\leq 1} |f * h_t(x)| \tag{5.51}$$

and the *large time* maximal operator

$$\mathcal{T}_\infty f(x) = \sup_{t>1} |\, f * h_t(x) \,| \tag{5.52}$$

Consider first $\mathcal{T}_0$. The purely local maximal operator

$$\mathcal{T}_0^0 f(x) = \sup_{0<t\leq 1} |\, f * (\chi h_t)(x) \,| \tag{5.53}$$

where χ denotes the characteristic function of the unit ball in S, can be handled as in the Euclidean case. For instance, it is dominated by the local Hardy-Littlewood maximal function $\mathcal{M}_0$ (3.23). The remaining part of $\mathcal{T}_0$ is controled by an integrable convolution kernel, thanks to the Gaussian factor $e^{-\frac{r^2}{4t}}$ in (5.9):

$$\begin{aligned} &\sup_{0<t\leq 1} |\, f * \{(1-\chi)h_t\} \,| \leq |f| * \{(1-\chi) \sup_{0<t\leq 1} h_t\}\,, \\ &\text{with} \quad \sup_{0<t\leq 1} h_t(r) = O\big(e^{-\frac{r^2}{4}}\big) \quad \text{for } r \gg 0\,. \end{aligned} \tag{5.54}$$

$\mathcal{T}_\infty$ is controled similarly by a radial convolution kernel, which satisfies the decay condition of Theorem (3.14):

$$\begin{aligned} &\sup_{t>1} |\, f * h_t \,| \leq |f| * \{\sup_{t>1} h_t\} \\ &\text{with} \quad \sup_{t>1} h_t(r) = O\big((1+r)^{-\frac{1}{2}}\, e^{-Qr}\big) \quad \text{for } r \geq 0\,. \end{aligned} \tag{5.55}$$

The last estimate follows from (5.9) by analyzing the function $t \mapsto t^{-p} e^{-\frac{Q^2}{4}t - \frac{r^2}{4t}}$, which has a maximum at $t(r) = \frac{r}{Q} + O(1)$ (as $r \to +\infty$). Hence $\mathcal{T}_\infty$ is of weak type $L^1 \to L^1$. And this finishes the proof of Theorem (5.50).

Eventually let us check the actual values of the various constants in the inversion formulas

$$f(x) = c_0 \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\, \varphi_\lambda(x)\, \mathcal{H}f(\lambda) \tag{2.10}$$

and

$$\mathcal{A}^{-1} f(r) = \begin{cases} c_1 \left(-\dfrac{d}{d(\cosh r)}\right)^{\frac{k}{2}} \left(-\dfrac{d}{d\,(\cosh \frac{r}{2})}\right)^{\frac{m}{2}} f(r) \\ \text{when } k \text{ is even}, \\ c_2 \displaystyle\int_r^{+\infty} \frac{d(\cosh s)}{\sqrt{\cosh s - \cosh r}} \left(-\frac{d}{d(\cosh s)}\right)^{\frac{k+1}{2}} \\ \qquad \cdot \left(-\dfrac{d}{d(\cosh \frac{s}{2})}\right)^{\frac{m}{2}} f(s) \\ \text{when } k \text{ is odd}, \end{cases} \tag{2.24}$$

by using the classical heat kernel behavior:

$$h_t(e) \sim (4\pi t)^{-\frac{n}{2}}, \qquad \text{as } t \to 0. \tag{5.56}$$

For $f = h_t$ and $x = e$, these formulas read respectively

$$h_t(e) = c_0\, e^{-\frac{Q^2}{4}t} \int_0^{+\infty} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\, e^{-t\lambda^2}, \tag{5.57}$$

$$\begin{aligned} h_t(e) &= c_1\, 2^{\frac{m}{2}-1}\, \pi^{-\frac{1}{2}}\, t^{-\frac{1}{2}}\, e^{-\frac{Q^2}{4}t} \\ &\quad\times \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{\frac{k}{2}} \left(-\frac{1}{\sinh \frac{r}{2}}\frac{\partial}{\partial r}\right)^{\frac{m}{2}} \Big|_{r=0} e^{-\frac{r^2}{4t}}, \end{aligned} \tag{5.58}$$

$$\begin{aligned} h_t(e) &= c_2\, 2^{\frac{m-1}{2}}\, \pi^{-\frac{1}{2}}\, t^{-\frac{1}{2}}\, e^{-\frac{Q^2}{4}t} \\ &\quad\times \int_0^{+\infty} ds \cosh\frac{s}{2} \left(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{k+1}{2}} \left(-\frac{1}{\sinh \frac{s}{2}}\frac{\partial}{\partial s}\right)^{\frac{m}{2}} e^{-\frac{s^2}{4t}}. \end{aligned} \tag{5.59}$$

For small time asymptotics, the factor $e^{-\frac{Q^2}{4}t}$ can be disregarded.

Consider first (5.57). Using either the explicit expression (2.7) of the $\mathbf{c}$-function and Stirling's formula or the explicit computation (2.31) of the Plancherel measure, we see that

$$|\mathbf{c}(\lambda)|^{-2} \sim 2^{2-2Q}\,\pi\,\Gamma\left(\frac{n}{2}\right)^{-2} \lambda^{n-1} \qquad \text{for } \lambda \to +\infty. \tag{5.60}$$

After performing the change of variables $\lambda' = \sqrt{t}\,\lambda$ and letting $t \to 0$, we obtain

$$h_t(e) \sim c_0\, 2^{2-2Q}\,\pi\,\Gamma\left(\frac{n}{2}\right)^{-2} \int_0^{+\infty} d\lambda'\, \lambda'^{n-1}\, e^{-\lambda'^2}. \tag{5.61}$$

Since the last integral is equal to $\frac{1}{2}\Gamma(\frac{n}{2})$, (5.56) implies

$$c_0 = 2^{k-2}\,\pi^{-\frac{n}{2}-1}\,\Gamma\left(\frac{n}{2}\right), \tag{5.62}$$

which was the value given in (2.10).

Look next at (5.58). By (5.23) the expression under consideration has an finite expansion in negative powers of t, with leading term

$$c_1\, 2^{-Q-1}\pi^{-\frac{1}{2}} \left(\frac{2r}{\sinh\frac{r}{2}}\Big|_{r=0}\right)^{\frac{m}{2}} \left(\frac{2r}{\sinh r}\Big|_{r=0}\right)^{\frac{k}{2}} t^{-\frac{n}{2}} = c_1\, 2^{\frac{m-k}{2}-1}\pi^{-\frac{1}{2}} t^{-\frac{n}{2}}. \tag{5.63}$$

(5.56) implies

$$c_1 = 2^{-\frac{3m+k}{2}} \pi^{-\frac{m+k}{2}} , \tag{5.64}$$

which was the value found in (2.24.1).

For the last case (5.59), we keep inside the integral the leading term in t, which was just identified:

$$h_t(e) \sim c_2\, 2^{\frac{m-k}{2}-1} \pi^{-\frac{1}{2}} t^{-\frac{n+1}{2}} \int_0^{+\infty} ds\, \cosh\frac{s}{2} \left(\frac{\frac{s}{2}}{\sinh\frac{s}{2}}\right)^{\frac{m}{2}} \left(\frac{s}{\sinh s}\right)^{\frac{k+1}{2}} e^{-\frac{s^2}{4t}} . \tag{5.65}$$

After performing the change of variables $s' = \frac{s}{2\sqrt{t}}$ and letting $t \to 0$, we obtain

$$h_t(e) \sim c_2\, 2^{\frac{m-k}{2}} \pi^{-\frac{1}{2}} t^{-\frac{n}{2}} \int_0^{+\infty} ds'\, e^{-s'^2} = c_2\, 2^{\frac{m-k}{2}-1} t^{-\frac{n}{2}} . \tag{5.66}$$

(5.56) implies

$$c_2 = 2^{-\frac{3m+k}{2}} \pi^{-\frac{n}{2}} , \tag{5.67}$$

which was the value found in (2.24.2).

6. – Further results and open problems

More generally, all analytical results established for symmetric spaces in [An4] and [CGM] remain valid in our setting. Let us be more specific.

(6.1) The *Riesz transform* is defined by

$$Rf = \operatorname{grad} (-\mathcal{L})^{-\frac{1}{2}} f \tag{6.2}$$

for smooth functions $f \in C_c^\infty(S)$ and in general as a singular integral operator

$$Rf(x) = \text{P.V.} \int_S dy\, f(y)\, (d_{y^{-1}x} L_y)\, \operatorname{grad} k(y^{-1}x) , \tag{6.3}$$

where κ is the (distributional) kernel of $(-\mathcal{L})^{-\frac{1}{2}}$ and $L_y(z) = yz$ denotes left translation in S. Its $L^p \to L^p$ mapping properties are stated below. Everything relies upon the following kernel estimates.

(6.4) Lemma. (i) $\operatorname{grad} \kappa(x) = -\pi^{-\frac{n+1}{2}} \Gamma(\frac{n+1}{2}) r^{-n} \frac{\partial}{\partial r} + O(r^{1-n})$ *as* $r = r(x) \overset{>}{\to} 0$.

(ii) *The second derivatives of* $\kappa(x)$ *are* $O(r^{-n-1})$, *as* $r = r(x) \overset{>}{\to} 0$.

(iii) $|\operatorname{grad}\kappa(x)| \asymp r^{-\frac{1}{2}} e^{-Qr}$, *as* $r = r(x) \to +\infty$.

(6.5) THEOREM. (i) *Let* $1 < p < +\infty$ *and* $f \in L^p(S)$. *Then*

$$R_\varepsilon f(x) = \int_{d(x,y)\geq\varepsilon} f(y)\, (d_{y^{-1}x} L_y)\, \operatorname{grad} k(y^{-1}x)$$

converges in L^p *norm and almost everywhere, as* $\varepsilon \overset{>}{\to} 0$. *The operator* $f \mapsto Rf$ *so defined is* $L^p \to L^p$ *bounded.*

(ii) *Let* $f \in L^1(S)$. *Then* $\lim_{\varepsilon \overset{>}{\to} 0} R_\varepsilon f(x) = Rf(x)$ *exists almost everywhere and defines an operator of weak type* $L^1 \to L^1$.

(6.6) The full $L^p \to L^q$ behavior of the *heat semigroup* $e^{t\mathcal{L}}$ $(0 < t < +\infty)$ and of the *Bessel-Green-Riesz potentials* $\{(\Lambda - \frac{Q^2}{4})I - \mathcal{L}\}^{-z}$ $(0 \leq \Lambda < +\infty$, $\operatorname{Re} z \geq 0)$ can be determined as in [An4], [CGM]. Notice that Vretare's recurrence formula for spherical functions, which plays an important role in [CGM], writes here [Vr]:

$$\begin{aligned}(\cosh r)\varphi_\lambda(r) &= \frac{(Q-i2\lambda)(\frac{m}{2}+1-i2\lambda)}{2(-i2\lambda)(1-i2\lambda)}\,\varphi_{\lambda+i}(r)\\ &+ \frac{(Q+i2\lambda)(\frac{m}{2}+1+i2\lambda)}{2(i2\lambda)(1+i2\lambda)}\,\varphi_{\lambda-i}(r) + \frac{(Q-1)\frac{m}{2}}{1+4\lambda^2}\,\varphi_\lambda(r)\,.\end{aligned} \tag{6.7}$$

(6.8) The so-called *Poisson semi-group* $e^{-t(-\mathcal{L})^{\frac{1}{2}}}$ $(0 < t < +\infty)$ is the most basic object in Paley-Littlewood-Stein theory. The subordination formula

$$e^{-t(-\mathcal{L})^{\frac{1}{2}}} = t\int_0^{+\infty} \frac{du}{u}\,(4\pi u)^{-\frac{1}{2}}\, e^{-\frac{t^2}{4u}}\, e^{u\mathcal{L}}\,, \tag{6.9}$$

which writes

$$p_t(x) = t\int_0^{+\infty} \frac{du}{u}\,(4\pi u)^{-\frac{1}{2}}\, e^{-\frac{t^2}{4u}}\, h_u(x) \tag{6.10}$$

at the kernel level, allows reduction to the heat equation. This yields for instance the following sharp estimates.

(6.11) THEOREM. (i) $p_t(r) = \pi^{-\frac{n+1}{2}}\,\Gamma(\frac{n+1}{2})\, t\,(t^2+r^2)^{-\frac{n+1}{2}} + O(t\,(t^2+r^2)^{-\frac{n-1}{2}})$ *for* $t > 0$ *and* $r \geq 0$ *small,*

(ii) $p_t(r) \asymp t\,(1+r)\,(t+r)^{-\frac{5}{2}}\, e^{-\frac{Q}{2}r - \frac{Q}{2}\sqrt{t^2+r^2}}$ *otherwise.*

Derivatives of $p_t(x)$ can be estimated above similarly. All this leads clearly to the same results as in [An4; § 6].

As a conclusion, let us mention some open problems and naive questions, which are worth studying in our opinion.

(6.12) *Problem*:
What is the right Lichnerowicz conjecture in the noncompact case? More precisely, are there harmonic manifolds with (stricly) negative Ricci curvature beside these AN groups (and the real hyperbolic spaces $H^N(\mathbb{R})$)? This problem is of course well recognized among specialists of harmonic spaces ([BTV], [Sz2]).

(6.13) *Question*:
Are there similar classes of solvable Lie groups, generalizing Iwasawa AN groups in higher rank?

(6.14) *Problem*:
Determine the set of all positive definite spherical functions on $S = AN$. As usual there are elementary necessary conditions: If φ_λ is positive definite, then φ_λ is bounded and $\varphi_\lambda(x) = \overline{\varphi_\lambda(x^{-1})}$, which amounts respectively to $|\operatorname{Im}\lambda| \leq \frac{Q}{2}$ and $\lambda \in \mathbb{R} \cup i\mathbb{R}$, and consequently $\lambda \in \mathbb{R} \cup [-i\frac{Q}{2}, i\frac{Q}{2}]$. In analogy with the hyperbolic space case, the answer to Problem (6.14) should be:
(6.15)

$$\varphi_\lambda \text{ is positive definite} \Longleftrightarrow \lambda \in \mathbb{R} \cup \left[-i\left(\frac{m}{4}+\frac{1}{2}\right), i\left(\frac{m}{4}+\frac{1}{2}\right)\right] \cup \left\{\pm i\frac{Q}{2}\right\}.$$

The easy case $\lambda \in \mathbb{R}$ was settled positively at the end of Section 2. For imaginary λ's, we tried unsuccessfully to follow the approach [FK], but got stuck in the intricate formula for the distance between two generic points in the ball model (1.11). B. Di Blasio [Di2] (see also [Di1; § 4.1]) was more successfull in her attempt to use reduction to the complex hyperbolic space $H^{\frac{m}{2}+1}(\mathbb{C})$ and was able to establish this way the implication $\Longleftarrow$ in (6.15).

(6.16) *Question*:
So far spherical analysis on harmonic AN groups has proved to be quite similar to the hyperbolic space case. (Here we have of course the results in mind, not the proofs.) It would be interesting to exhibit analytical phenomena reflecting symmetry or assymmetry among these groups. For this it is likely that one should go beyond radial function analysis and consider more general functions, sections of vector bundles, ... on $S = AN$.

7. – Later information

Since this article circulated as a preprint in 1994, further progress has been made. Here are some works we are aware of.

(7.1) The radial L^p Schwartz spaces $\mathcal{S}^p(S)^\natural$ (see § 2) are characterized in [Di3] (see also [Di1; § 2.1]) by a better set of intrinsic conditions, namely

$$\sup_{x\in S}\,(1+r(x))^N\,e^{\frac{Q}{2}r(x)}\,|Df(x)|\;<\;+\infty\;,$$

where D ranges over the left-invariant differential operators on S and N over the nonnegative integers.

(7.2) Formula (2.34) is established in [Di4] (see also [Di1; § 4.2]) by reduction to the complex hyperbolic space $H^{\frac{m}{2}+1}(\mathbb{C})$.

(7.3) The Fourier transform of general functions on $S=AN$, which is defined by means of the series (2.32) of representations π_λ, is investigated in [ACD].

(7.4) Another L^p radial multiplier theorem, different from (4.17), is established in [As].

(7.5) The characterization (6.15) of positive-definite spherical functions has now been achieved in [DZ] by following the approach of M. G. Cowling in the hyperbolic space case ([Co2], [Co3], [CH]).

REFERENCES

[An1] J.-Ph. Anker, *Le noyau de la chaleur sur les espaces symétriques* $U(p,q)\,/\,U(p)\times U(q)$, In: "Harmonic Analysis, Luxembourg 1987", P. Eymard - J.-P. Pier (eds.), Lecture Notes in Mathematics 1359, Springer, 1988, pp. 60-82

[An2] J.-Ph. Anker, *The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi and Varadarajan*, J. Funct. Anal. **96** (1991), 331-349.

[An3] J.-Ph. Anker, *L_p Fourier multipliers on Riemannian symmetric spaces of the noncompact type*, Ann. of Math. **132** (1990), 597-628

[An4] J.-PH. ANKER, *Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces*, Duke Math. J. **65** (1992), 257-297.

[An5] J.-PH. ANKER, *Analyse sphérique sur les espaces hyperboliques*, Cours D.E.A. 1992-1993, Université Nancy I.

[AJ] J.-PH. ANKER - L. JI, *Heat kernel and Green function estimates on noncompact symmetric spaces*, in preparation.

[AMPS] J.-PH. ANKER - P. MARTINOT - E. PEDON - A. SETTI, *The wave equation on harmonic AN groups*, in preparation.

[AS] J.-PH. ANKER - A. SEEGER, *L^p estimates for functions of the Laplacian on noncompact manifolds*, in preparation.

[As] F. ASTENGO, *Asymptotic expansions for spherical functions and multipliers on solvable extensions of H-type groups*, J. Lie Theory **5** (1995), 147-164.

[ACD] F. ASTENGO - R. CAMPORESI - B. DI BLASIO, *The Helgason Fourier transform on a class of nonsymmetric harmonic spaces*, preprint (1996).

[Bo] J. BOGGINO, *Generalized Heisenberg groups and solvmanifolds naturally associated*, Rend. Sem. Mat. Univ. Politec. Torino **43** (1985), 529-547.

[BTV] J. BERNDT - F. TRICERRI - L. VANHECKE, Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Mathematics 1598, Springer, 1995.

[CS] J.-L. CLERC - E.M. STEIN, *L^p-multipliers for noncompact symmetric spaces*, Proc. Nat. Acad. Sci. USA **71** (1974), 3911-3912.

[Co1] M.G. COWLING, *The Kunze-Stein phenomenon*, Ann. of Math. **107** (1978), 209-234.

[Co2] M.G. COWLING, *Unitary and uniformly bounded representations of some simple Lie groups*, In: "Harmonic analysis and group representations, C.I.M.E. 1980", Liguori, Napoli, 1982, pp. 49-128.

[Co3] M.G. COWLING, *Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups)*, In: "Topics in modern harmonic analysis (Proc. Sem. Torino Milano, May-June 1982)" vol. **I**, Ist. Naz. Alta Mat. F. Severi, Roma, 1983, pp. 81-123.

[CDKR1] M. G. COWLING - A. H. DOOLEY - A. KORÁNYI - F. RICCI, *H-type groups and Iwasawa decompositions*, Adv. Math. **87** (1991), 1-41.

[CDKR2] M. G. COWLING - A. H. DOOLEY - A. KORÁNYI - F. RICCI, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal. (to appear).

[CF] M. G. COWLING - J. J. F. FOURNIER, *Inclusions and non-inclusions of spaces of convolution operators*, Trans. Amer. Math. Soc. **221** (1976), 59-95.

[CGGM] M.G. COWLING - G.I. GAUDRY - S. GIULINI - G. MAUCERI, *Weak type (1,1) estimates for heat kernel maximal functions on Lie groups*, Trans. Amer. Math. Soc. **323** (1991), 637-649.

[CGM] M.G. COWLING - S. GIULINI - S. MEDA, *L^p-L^q estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces I*, Duke Math. J. **72** (1993), 109-150.

[CH] M.G. COWLING - U. HAAGERUP, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. math. **96** (1989), 507-549.

[Dam1] E. DAMEK, *Geometry of a semi-direct extension of a Heisenberg type nilpotent group*, Collect. Math. **53** (1987), 255-268.

[Dam2] E. DAMEK, *Curvature of a semi-direct extension of a Heisenberg type nilpotent group*, Collect. Math. **53** (1987), 249-253.

[Dam3] E. DAMEK, *A Poisson kernel on Heisenberg type nilpotent groups*, Collect. Math. **53** (1987), 239-247.

[Dam4] E. DAMEK, *Harmonic functions on a semidirect extension of type H nilpotent groups*, Trans. Amer. Math. Soc. **290** (1985), 375-384.

[DR1] E. DAMEK - F. RICCI, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), 139-142.

[DR2] E. DAMEK - F. RICCI, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. **2** (1992), 213-248.

[Dav] E.B. DAVIES, Heat kernels and spectral theory, Cambridge Univ. Press, 1989.

[DM] E.B. DAVIES - N. MANDOUVALOS, *Heat kernels bounds on hyperbolic space and Kleinian groups*, Proc. London Math. Soc. **57** (1988), 182-208.

[Di1] B. DI BLASIO, Analisi di Fourier sferica su estensioni risolubili di gruppi di Heisenberg generalizzati, Tesi di Dottorato, Politecnico di Torino (1996).

[Di2] B. DI BLASIO, *Positive definite spherical functions on harmonic spaces NA*, Boll. Unione Mat. Ital. (to appear).

[Di3] B. DI BLASIO, *Paley-Wiener type theorems on harmonic extensions of H-type groups*, Monatsh. Math. (to appear).

[Di4] B. DI BLASIO, *An extension of the theory of Gelfand pairs to radial functions on Lie groups*, Boll. Unione Mat. Ital. (to appear).

[DZ] A.H. DOOLEY - G. ZHANG, *Spherical functions on H-type groups*, preprint (1995).

[Fa] J. FARAUT, *Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques*, In: "Analyse Harmonique", J.-L. Clerc *et al.* (eds.), C.I.M.P.A., Nice 1983, pp. 315-446.

[FK] M. FLENSTED-JENSEN - T. H. KOORNWINDER, *Positive definite spherical functions on a non-compact, rank one symmetric space*, In: "Analyse Harmonique sur les Groupes de Lie II, Séminaire Nancy-Strabourg 1976-78", P. Eymard *et al.* (eds.), Lecture Notes in Mathematics 739, Springer, 1979, pp. 249-282.

[GM] S. GIULINI - G. MAUCERI, *Analysis of a distinguished Laplacian on solvable Lie groups*, Math. Nachr. **163** (1993), 151-162.

[Hel] S. HELGASON, Groups and Geometric Analysis, Academic Press, 1984.

[Her] C. HERZ, *Problems of extrapolation and spectral synthesis on groups*, In: "Conference on harmonic analysis, College Park 1971", D. Gulick - R. L. Lipsman (eds.), Lecture Notes in Mathematics 266, Springer, 1972, pp. 157-166.

[Ka1] A. KAPLAN, *Fundamental solutions for a class of hypoelliptic PDE generated by compositions of quadratic forms*, Trans. Amer. Math. Soc. **258** (1980), 147-153.

[Ka2] A. KAPLAN, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Ded. **11** (1981), 127-136.

[Ka3] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35-42.

[KR] A. Kaplan - F. Ricci, Harmonic analysis on groups of Heisenberg type, Lecture Notes in Mathematics 992, Springer, 1983, pp. 416-435.

[Koo] T. H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, In: "Special functions: Group theoretical aspects and applications", R. A. Askey *et al.* (eds.), Reidel, 1984, pp. 1-85.

[Kor] A. Korányi, *Geometric properties of Heisenberg-type groups*, Adv. Math. **56** (1985), 28-38.

[Li] R.L. Lipsman, *An indicator diagram II. The L^p convolution theorem for connected unimodular groups*, Duke. Math. J. **37** (1970), 459-466.

[Lo] N. Lohoué, *Inégalités de Sobolev pour les sous Laplaciens de certains groupes unimodulaires*, Geom. Funct. Anal. **2** (1992), 394-420.

[LRy] N. Lohoué - Th. Rychener, *Die Resolvente von Δ auf symmetrischen Räumen vom nichtkompakten Typ*, Comm. Math. Helv. **57** (1982), 445-468.

[LZ] N. Lohoué - Zhu Fu Liu, *Estimation faible de certaines fonctions maximales*, C. R. Acad. Sci. Paris Sér. I **302** (1986), 303-305.

[LRo] G. Lorang - B. Roynette, *Etude d'une fonctionnelle liée au pont de Bessel*, Ann. Inst. Henri Poincaré Section B (*Prob. Stat.*) **32** (1996), 107-133.

[MW] R.J. Miatello - N.R. Wallach, *The resolvent of the Laplacian on locally symmetric spaces*, J. Diff. Geom. **36** (1992), 663-698.

[Mu] A. Munch, Comportement exact du noyau de la chaleur sur les espaces hyperboliques, Mémoire D.E.A., Université Nancy I 1993.

[Ric1] F. Ricci, *Commutative algebras of invariant functions on groups of Heisenberg type*, J. London Math. Soc. **32** (1985), 265-271.

[Ric2] F. Ricci, *The spherical transform on harmonic extensions of H-type groups*, Rend. Sem. Mat. Univ. Politec. Torino **50** (1992), 381-392.

[Rie] C. Riehm, *The automorphism group of a composition of quadratic forms*, Trans. Amer. Math. Soc. **269** (1982), 403-414.

[ST] R.J. Stanton - P.A. Tomas, *Expansions for spherical functions on noncompact symmetric spaces*, Acta Math. **140** (1978), 251-276.

[Ste] E.M. Stein, *Analytic continuation of group representations*, Adv. Math. **4** (1970), 172-207.

[Str] J.-O. Strömberg, *Weak type L^1 estimates for maximal functions on non-compact symmetric spaces*, Ann. of Math. **114** (1981), 115-126.

[Sz1] Z.I. Szabó, *The Lichnerowicz conjecture on harmonic manifolds*, J. Differential Geom. **31** (1990), 1-28.

[Sz2] Z.I. Szabó, *Spectral theory for operator families on Riemannian manifolds*, In: "Differential geometry, UCLA 1990", R. Greene - S. T. Yau (eds.) Proc. Symp. Pure Math. 54, 3, 1993, pp. 615-665.

[Ta] M.E. Taylor, *L^p estimates on functions of the Laplace operator*, Duke Math. J. **58** (1989), 773-793.

[Vr] L. Vretare, *On L^p Fourier multipliers on certain symmetric spaces*, Math. Scand. **37** (1975), 111-121.

[Ya] C. Yacoub, Analyse radiale sur des variétés harmoniques AN et d'opérateurs différentiels généralisant l'opérateur de Jacobi, Thèse, Université Paris-Sud / Orsay, 1994.

Université Henri Poincaré (Nancy I)
Institut Elie Cartan
(Laboratoire Commun de Mathématiques
UHP-CNRS-INRIA)
B.P. 239,
F-54506 Vandoeuvre-lès-Nancy Cedex, France
E-mail: anker@iecn.u-nancy.fr

Uniwersytet Wrocławski
Instytut Matematyczny
Plac Grunwaldzki 2/4
PL-50384 Wrocław, Polska
E-mail: edamek@math.uni.wroc.pl

Faculté des Sciences
Département de Mathématiques
500 Monastir, Tunisie