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**Existence of regular solutions to the steady Navier-Stokes equations in bounded six-dimensional domains**

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# Existence of Regular Solutions to the Steady Navier-Stokes Equations in Bounded Six-Dimensional Domains

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## 1. – Introduction

In this paper we establish the existence of a weak solution to the steady Navier-Stokes equations in a bounded six-dimensional domain $\Omega$, which additionally satisfies

$$\sup_{\Omega_0}\left(\frac{\mathbf{u}^2}{2}+p\right)\le c \tag{M}$$

for all compact subdomains $\Omega_0 \subseteq\subseteq \Omega$. Consequently, from former results of the authors [5], [9] follows the existence of a regular solution.

In a series of papers Frehse, Růžička [2]-[6], [9] have studied the regularity of solutions of the steady Navier-Stokes equations(1)

$$\left.\begin{aligned} -\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p&=\mathbf{f}\\ \operatorname{div}\mathbf{u}&=0\end{aligned}\right\}\quad \text{in } \Omega\,. \tag{1.1}$$

In the case of a bounded domain $\Omega\subseteq\mathbb{R}^N$ with Dirichlet boundary conditions we proved the existence of a regular solution only for $N=5$. On the other hand in the space periodic situation the existence of regular solutions was established for $5\le N\le 15$ (cf. Struwe [10], who studied the case $\Omega=\mathbb{R}^5$).

The existence of a regular solution for a bounded five-dimensional domain is based on a general result (cf. [5]), which states that every "maximum solution", i.e. inequality (M) is satisfied, is regular and on the construction of such a "maximum solution" (cf. [2]). The idea, which worked for $N=5$, at the first sight can not be carried over to higher-dimensional situations. Here we show, using a "dimensional reduction", that also for $N=6$ a "maximum solution" can be constructed.

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(1) Here we normalized the viscosity $\nu$ to one, but all arguments work also for arbitrary $\nu>0$.

## 2. – Maximum property for the head pressure $\frac{\mathbf{u}^2}{2} + p$

Let $\Omega \subseteq \mathbb{R}^6$ be a bounded smooth domain and let $\mathbf{f} \in L^\infty(\Omega)$ be given. We want to prove the existence of a weak solution $\mathbf{u}$, $p$ to the steady Navier-Stokes equations

$$\begin{aligned} -\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned} \quad \text{in } \Omega \tag{2.1}$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega\,,$$

which additionally satisfies for all $\Omega_0 \subseteq\subseteq \Omega$

$$\sup_{\Omega_0}\left(\frac{\mathbf{u}^2}{2} + p\right) \le c\,. \tag{2.2}$$

As in [2] we use the following approximation of (2.1) for $\varepsilon > 0$

$$\begin{aligned} -\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \varepsilon|\mathbf{u}|^2\mathbf{u} + \nabla\mathbf{p} &= \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned} \quad \text{in } \Omega \tag{2.3}$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega\,.$$

Here and in the sequel we will drop the dependence of solutions of the considered equations on various parameters, but we will clearly indicate this dependence in estimates. One easily gets:

LEMMA 2.4. *Let $\mathbf{f} \in L^\infty(\Omega)$. Then, for all $\varepsilon > 0$, there exists a weak solution $\mathbf{u} = \mathbf{u}^\varepsilon$, $p = p^\varepsilon$ to* (2.3) *satisfying for all $\varphi \in C_0^\infty(\Omega)$*

$$\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial \varphi_i}{\partial x_j} + u_j\frac{\partial u_i}{\partial x_j}\varphi_i + \varepsilon|\mathbf{u}|^2 u_i\,\varphi_i\,dx + \int_\Omega \frac{\partial p}{\partial x_i}\varphi_i\,dx = \int_\Omega f_i\,\varphi_i\,dx\,, \tag{2.5}$$

*such that*

$$\begin{aligned} \|\mathbf{u}\|_{1,2} &\le K\,, \\ \varepsilon\|\mathbf{u}\|_{0,4}^4 &\le K\,, \end{aligned} \tag{2.6}$$

$$\|\varepsilon\mathbf{u}|\mathbf{u}|^2\|_{0,4/3} \le K\,\varepsilon^{1/4}\,, \tag{2.7}$$

$$\begin{aligned} \|p\|_{1,6/5} &\le K\,, \\ \|p\|_{1,4/3} &\le c(\varepsilon)\,, \end{aligned} \tag{2.6}$$

*where the constant $K$ is independent of $\varepsilon$.*

One possibility to get a maximum property is to use the so-called duality method. Let us therefore consider the Green-type function $G$ solving

$$
\begin{aligned}
-\Delta G - \mathbf{v}\cdot\nabla G &= \delta_h(x_0) \quad \text{in } \Omega \\
G &= 0 \qquad\quad \text{on } \partial\Omega .
\end{aligned} \tag{2.8}
$$

Here $\mathbf{v} = \mathbf{v}^k \in \mathcal{V} = \{\mathbf{v} \in C_0^\infty(\Omega), \operatorname{div} \mathbf{v} = 0\}$ is an approximation of the solution $\mathbf{u} = \mathbf{u}^\varepsilon$ of (2.3), such that $\mathbf{v}^k \to \mathbf{u}$ in $W_0^{1,2}(\Omega)$ and $\delta_h(x_0)$, $0 < h < \operatorname{dist}(x_0, \partial\Omega)$, is a smooth non-negative approximation of the Dirac distribution satisfying

$$
\operatorname{supp}\, \delta_h(x_0) \subseteq B_h(x_0), \qquad \int_\Omega \delta_h(x_0)dx = 1 . \tag{2.9}
$$

In the same way as in [2], [9] we get:

LEMMA 2.10. *For all $\varepsilon > 0$, $h > 0$ and $k \in \mathbb{N}$ there exists a solution $G = G^k_{h,\varepsilon} \in C^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ to (2.8) satisfying*

$$
\int_\Omega \nabla G \nabla \varphi\, dx - \int_\Omega \mathbf{v}\cdot\nabla G\, \varphi\, dx = \int_\Omega \delta_h(x_0)\varphi\, dx \qquad \forall \varphi \in W_0^{1,2}(\Omega) , \tag{2.11}
$$

*such that*

$$
\begin{aligned}
G &\geq 0 , \\
\|G\|_{0,\infty} &\leq c(h) ,
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
\|\nabla^2 G\|_{0,2}^2 &\leq c(h)(1 + \|\nabla \mathbf{v}\|_{0,2}^2) , \\
\|\nabla G\|_{0,4}^4 &\leq c(h)(1 + \|\nabla \mathbf{v}\|_{0,2}^2) ,
\end{aligned} \tag{2.13}
$$

*where the constant $c(h)$ is independent of $\varepsilon$ and $k$.*

The weak formulation of the pressure equation for the approximative system (2.3) reads

$$
\int_\Omega \nabla p \nabla \psi\, dx = \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\psi\, dx - \varepsilon \int_\Omega |\mathbf{u}|^2 \mathbf{u}\cdot\nabla\psi\, dx + \int_\Omega \mathbf{f}\cdot\nabla\psi\, dx . \tag{2.14}
$$

for all $\psi \in L^\infty(\Omega) \cap W_0^{1,4}(\Omega)$. From $(2.12)_2$ and $(2.13)_2$ follows that $\psi = G\zeta^2$, $0 \leq \zeta \in C_0^\infty(\Omega)$ is an admissible test function in (2.14). On the other hand $\varphi = \mathbf{u}G\zeta^2$ is due to Lemma 2.4 an admissible test function in (2.5). Thus we get, denoting $\nabla\mathbf{u} \circ \nabla\mathbf{u} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$,

$$
\begin{aligned}
&\int_\Omega \nabla\left(\frac{\mathbf{u}^2}{2} + p\right)\nabla(G\zeta^2) + \mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2} + p\right)G\zeta^2\, dx + \varepsilon\int_\Omega |\mathbf{u}|^4 G\zeta^2\, dx \\
&= \int_\Omega (\nabla\mathbf{u}\circ\nabla\mathbf{u} - |\nabla\mathbf{u}|^2)G\zeta^2\, dx - \varepsilon\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla(G\zeta^2)\, dx \\
&\quad + \int_\Omega \mathbf{f}\cdot\nabla(G\zeta^2) + \mathbf{f}\cdot\mathbf{u}G\zeta^2\, dx .
\end{aligned} \tag{2.15}
$$

Using now $\nabla\mathbf{u}\circ\nabla\mathbf{u}-|\nabla\mathbf{u}|^2\le 0$ and $(2.12)_1$ we obtain

$$\begin{aligned}&\int_\Omega \nabla\Big(\frac{\mathbf{u}^2}{2}+p\Big)\nabla(G\zeta^2)+\mathbf{u}\cdot\nabla\Big(\frac{\mathbf{u}^2}{2}+p\Big)G\zeta^2\,dx\\ &\le -\varepsilon\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla(G\zeta^2)dx+\int_\Omega \mathbf{f}\cdot\nabla(G\zeta^2)+\mathbf{f}\cdot\mathbf{u}G\zeta^2\,dx\,.\end{aligned}\tag{2.16}$$

Because all estimates in Lemma 2.10 are independent of $k$ we can justify the limiting process $k\to\infty$. From (2.11) and (2.16) we thus get

$$\begin{aligned}&\int_\Omega \delta_h(x_0)\Big(\frac{\mathbf{u}^2}{2}+p\Big)\zeta^2\,dx\\ &\le\int_\Omega \mathbf{u}\cdot\nabla\zeta^2\Big(\frac{\mathbf{u}^2}{2}+p\Big)G\,dx-2\int_\Omega G\nabla\Big(\frac{\mathbf{u}^2}{2}+p\Big)\nabla\zeta^2\,dx\\ &\quad-\int_\Omega G\Big(\frac{\mathbf{u}^2}{2}+p\Big)\Delta\zeta^2dx-\varepsilon\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla(G\zeta^2)dx\\ &\quad+\int_\Omega \mathbf{f}\cdot\nabla(G\zeta^2)+\mathbf{f}\cdot\mathbf{u}G\zeta^2\,dx\,,\end{aligned}\tag{2.17}$$

where $\mathbf{u}=\mathbf{u}^\varepsilon$ and $G=G_{h,\varepsilon}$.

Let us now analyse the limiting processes $\varepsilon\to 0$ and then $h\to 0$ in inequality (2.17). There are two main difficulties. Namely, in the first integral at the right-hand side appears the term $|\mathbf{u}|^3$ and thus we need an $L^\infty$-estimate of $G$ independent of $\varepsilon$ and $h$, but only away from the singularity $x_0$ (due to $\nabla\zeta^2$). Further, if we would use in the last integral on the right-hand side only the information $\mathbf{u}\in W^{1,2}(\Omega)\hookrightarrow L^3(\Omega)$, we would need $G\in L^{3/2}(\Omega)$ independent on $\varepsilon$ and $h$ near the singularity $x_0$, which is even more than holds for the Laplace operator. Therefore we also need additional information on $\mathbf{u}$.

We will deal with these two problems in the next two sections. Namely, we will prove:

Proposition 2.18. *Let $G=G_{h,\varepsilon}$ be the solution of* (2.8) (*now with* $\mathbf{u}=\mathbf{u}^\varepsilon$ *solving* (2.3) *instead of* $\mathbf{v}$). *Then:*

(i)

$$\begin{aligned}&\int_\Omega |G|^q\,dx\le K \qquad \forall q\in[1,3/2)\,,\\ &\int_\Omega |\nabla G|^s\,dx\le K \qquad \forall s\in[1,6/5)\,,\end{aligned}\tag{2.19}$$

*where $K$ is independent of $\varepsilon$ and $h$.*

(ii) *Let $B_{2R}\subseteq\Omega$ be a ball such that $B_h(x_0)\cap B_R=\emptyset$ for $0<h\le h_0$. Then we have*

$$\|G\|_{0,\infty,B_R}\le c(R)\,,\tag{2.20}$$

*where $c(R)$ is independent of $\varepsilon$ and $h$.*

PROPOSITION 2.21. *The solution* $\mathbf{u}$ *of* (2.1), *obtained as the limit as* $\varepsilon \to 0$ *of the solutions* $\mathbf{u}^\varepsilon$ *of* (2.3), *satisfies for* $q \in [1,4)$

$$\|\mathbf{u}\|_{0,q,\mathrm{loc}} \le K\,, \tag{2.22}$$

*where* $K = K(q)$ *is independent on* $\varepsilon$ *and* $h$.

Based on Proposition 2.18, Proposition 2.21 and Lemma 2.4 we can handle the limiting processes $\varepsilon \to 0$ and then $h \to 0$ in (2.17) and obtain that the right-hand side of (2.17) remains bounded. Indeed, we have for some small $\delta > 0$

$$\begin{aligned}\Big(\frac{\mathbf{u}^2}{2} + p\Big)(x_0) \le c(\zeta)(&\|\nabla \mathbf{u}\|_{0,2}^3 \|G\|_{0,\infty,\Omega\setminus B_{h_0}(x_0)} + \|\nabla \mathbf{u}\|_{0,2}^2 \|G\|_{0,\infty,\Omega\setminus B_{h_0}(x_0)} \\ &+ \|\mathbf{f}\|_{0,\infty}\|G\|_{1,6/5-\delta} + \|\mathbf{f}\|_{0,\infty}\|\mathbf{u}\|_{0,4-\delta}\|G\|_{0,3/2-\delta})\end{aligned}$$

and therefore we proved:

THEOREM 2.23. *Let* $\mathbf{f} \in L^\infty(\Omega)$. *Then the weak solution of* (2.1), *obtained as the limit as* $\varepsilon \to 0$ *of the solutions* $\mathbf{u}^\varepsilon$ *of* (2.3), *satisfies for all compact subdomains* $\Omega_0 \subseteq\subseteq \Omega$

$$\sup_{\Omega_0} \Big(\frac{\mathbf{u}^2}{2} + p\Big) \le c(\mathbf{f}, \Omega_0)\,. \tag{2.24}$$

In a former paper Frehse, Růžička [5, Theorem 1.8] it is shown that every weak solution of (2.1) satisfying (2.24) is regular. Thus we proved:

THEOREM 2.25. *Let* $\mathbf{f} \in L^\infty(\Omega)$ *and let* $\mathbf{u}$, $p$ *be the solution of* (2.1) *constructed before. Then* $\mathbf{u}$, $p$ *is regular, i.e. for all* $q \in (1,\infty)$.

$$\begin{aligned}&\mathbf{u} \in W^{2,q}_{\mathrm{loc}}(\Omega)\,, \\ &p \in W^{1,q}_{\mathrm{loc}}(\Omega)\,.\end{aligned} \tag{2.26}$$

## 3. – Properties of the Green-type function $G$

In this section we study the properties of $G$, which are independent of $h$ and $\varepsilon$. In the same way as for the Laplace operator one can show (see e.g. [2]):

LEMMA 3.1. *Let* $G = G^k_{h,\varepsilon}$ *be the solution of* (2.8). *Then we have*

$$\begin{aligned}&\int_\Omega |G|^q\,dx \le K \qquad \forall q \in [1,3/2)\,, \\ &\int_\Omega |\nabla G|^s\,dx \le K \qquad \forall s \in [1,6/5)\,,\end{aligned} \tag{3.2}$$

*where* $K$ *is independent of* $\varepsilon$, $h$ *and* $k$.

This proves Proposition 2.18 (i). In order to prove Proposition 2.18 (ii) we will use a method, which we would call "dimensional reduction". This method however is different from the dimensional reduction used e.g. in [1].

Let us fix some $R > 0$ and a ball $B_{2R}$, such that $B_{2R} \cap \operatorname{supp} \delta_h(x_0) = \emptyset$, and let further $R < r < s < \rho < 2R$. We now multiply (2.8) by $\chi_s |G - G_R|^{l-2}(G - G_R)$, where $l \in \mathbb{R}_+$ and $\chi_s$ is the characteristic function of the ball $B_s$. The constant $G_R$ will be specified later on. After integration over $\Omega$ and partial integration we arrive at (note that due our assumptions the integral involving $\delta_h(x_0)$ is zero)

$$
\begin{aligned}
(l-1) \int_{B_s} & |\nabla G|^2 |G - G_R|^{l-2} \, dx \\
&= \int_{\partial B_s} \nu \cdot \nabla G |G - G_R|^{l-2}(G - G_R) dS \\
&\quad + \frac{1}{l} \int_{\partial B_s} \nu \cdot \mathbf{v} |G - G_R|^l \, dS \\
&\equiv I_1 + I_2 \,.
\end{aligned} \tag{3.3}
$$

Using Hölder's and Young's inequalities the integrals $I_1$, $I_2$ can be estimated as follows:

$$
\begin{aligned}
|I_1| &\leq \int_{\partial B_s} |\nabla G| \, |G - G_R|^{l-1} \, dS \\
&\leq c \left( \int_{\partial B_s} |\nabla G|^2 \, |G - G_R|^{\frac{20}{21} l - 2} \, dS \right)^{\frac{21}{20}} \\
&\quad + c \left( \int_{\partial B_s} |G - G_R|^{\frac{22}{21} l} \, dS \right)^{\frac{21}{22}} \\
&\leq c \left( \int_{\partial B_s} |\nabla G|^2 \, |G - G_R|^{\frac{20}{21} l - 2} \, dS \right)^{\frac{21}{20}} \\
&\quad + c \left( \int_{\partial B_s} |G - G_R|^{\frac{10}{7} l} \, dS \right)^{\frac{7}{10}}
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
|I_2| &\leq \frac{1}{l} \int_{\partial B_s} |\mathbf{v}| \, |G - G_R|^l \, dS \\
&\leq \frac{1}{l} \left( \int_{\partial B_s} |\mathbf{v}|^{\frac{10}{3}} \, dS \right)^{\frac{3}{10}} \left( \int_{\partial B_s} |G - G_R|^{\frac{10}{7} l} \, dS \right)^{\frac{7}{10}} \\
&\leq \frac{c}{l} \left( \int_{\partial B_s} \frac{|\mathbf{v}|^2}{\mu(\partial B_s)} + |\nabla \mathbf{v}|^2 \, dS \right)^{\frac{1}{2}} \left( \int_{\partial B_s} |G - G_R|^{\frac{10}{7} l} \, dS \right)^{\frac{7}{10}},
\end{aligned} \tag{3.5}
$$

where we also used the Sobolev embedding in the last line and where $\mu$ denotes the five-dimensional surface measure. Indeed, from the Sobolev embedding theorem in dimension 5 we get

$$\|\mathbf{v}-\bar{\mathbf{v}}\|_{0,10/3,\partial B_s} \le K\|\nabla\mathbf{v}\|_{0,2,\partial B_s}\,,$$

where $\bar{\mathbf{v}} = \frac{1}{|\partial B_s|}\int_{\partial B_s}\mathbf{v}\,dS$. Consequently, we obtain

$$\begin{aligned}\left(\int_{\partial B_s}|\mathbf{v}|^{\frac{10}{3}}\,dS\right)^{\frac{3}{10}} &\le c\left(|\bar{\mathbf{v}}|^2+\int_{\partial B_s}|\nabla\mathbf{v}|^2\,dS\right)^{\frac{1}{2}}\\ &\le\left(\int_{\partial B_s}\frac{|\mathbf{v}|^2}{\mu(\partial B_s)}+|\nabla\mathbf{v}|^2\,dS\right)^{\frac{1}{2}}\,.\end{aligned}\tag{3.6}$$

From (3.3)-(3.6) we get

$$\begin{aligned}&(l-1)\int_{B_s}|\nabla G|^2\,|G-G_R|^{l-2}\,dx\\ &\le c\left(\int_{\partial B_s}|\nabla G|^2\,|G-G_R|^{\frac{20}{21}l-2}\,dS\right)^{\frac{21}{20}}\\ &\quad+c\left(\int_{\partial B_s}|G-G_R|^{\frac{10}{7}l}\,dS\right)^{\frac{7}{10}}\\ &\quad+\frac{c}{l}\left(\int_{\partial B_s}\frac{|\mathbf{v}|^2}{\mu(\partial B_s)}+|\nabla\mathbf{v}|^2\,dS\right)^{\frac{1}{2}}\left(\int_{\partial B_s}|G-G_R|^{\frac{10}{7}l}\,dS\right)^{\frac{7}{10}}\,.\end{aligned}\tag{3.7}$$

We use now the following lemma, which will be proved in the appendix (cf. Lemma 5.1):

LEMMA 3.8. *Let $0<r\le s\le\rho$ and let $g_i\in L^1(B_\rho\backslash B_r)$, $i=1,2,3$. Then there exists a set $E\subset[r,\rho]$ with $|E|\ge\frac{1}{4}(\rho-r)$, such that for all $s\in E$ and $i=1,2,3$*

$$\int_{\partial B_s}|g_i|\,dS\le\frac{4}{\rho-r}\int_{B_\rho\backslash B_r}|g_i|\,dx\,.\tag{3.9}$$

For

$$\begin{aligned}g_1&=|\nabla G|^2\,|G-G_R|^{\frac{20}{21}l-2}\\ g_2&=\frac{|\mathbf{v}|^2}{\mu(B_1)R^5}+|\nabla\mathbf{v}|^2\\ g_3&=|G-G_R|^{\frac{10}{7}l}\end{aligned}\tag{3.10}$$

we obtain from (3.7)

$$
\begin{aligned}
&(l-1)\int_{B_s} |\nabla G|^2\,|G-G_R|^{l-2}\,dx \\
&\le c\left(\frac{4}{\rho-r}\int_{B_\rho\setminus B_r} |\nabla G|^2\,|G-G_R|^{\frac{20}{21}l-2}\,dx\right)^{\frac{21}{20}} \\
&+c\left(\frac{4}{\rho-r}\int_{B_\rho\setminus B_r} |G-G_R|^{\frac{10}{7}l}\,dx\right)^{\frac{7}{10}} \qquad (3.11)\\
&+\frac{c}{l}\left(\frac{4}{\rho-r}\int_{B_\rho\setminus B_r} \frac{|\mathbf{v}|^2}{\mu(B_1)R^5}+|\nabla\mathbf{v}|^2\,dx\right)^{\frac{1}{2}} \\
&\cdot\left(\frac{4}{\rho-r}\int_{B_\rho\setminus B_r} |G-G_R|^{\frac{10}{7}l}\,dx\right)^{\frac{7}{10}}.
\end{aligned}
$$

The last inequality holds for all $s \in E$, where the set $E \subset [r,\rho]$ of course depends on $\varepsilon, h, k, l$ and $R$. The left-hand side of (3.11) is estimated from below by

$$(l-1)\int_{B_r} |\nabla G|^2\,|G-G_R|^{l-2}\,dx\,.$$

Furthermore, $\mathbf{v}=\mathbf{v}^k$ is a smooth approximation in $W^{1,2}(\Omega)$ of $\mathbf{u}=\mathbf{u}^\varepsilon$. Due to estimate $(2.6)_1$ we have $\|\mathbf{v}\|_{1,2} \le K$, where $K$ is independent of $\varepsilon, h, k$ and $R$. Using this at the right-hand side of (3.11) we get for $\rho - r < 4$, $R < 1$

$$
\begin{aligned}
&(l-1)\int_{B_r} |\nabla G|^2\,|G-G_R|^{l-2}\,dx \\
&\le \frac{K}{(\rho-r)^{21/20}}\left(\int_{B_\rho\setminus B_r} |\nabla G|^2\,|G-G_R|^{\frac{20}{21}l-2}\,dx\right)^{\frac{21}{20}} \\
&+K\left(\frac{1}{(\rho-r)^{7/10}}+\frac{1}{l\,R^{5/2}}\,\frac{1}{(\rho-r)^{6/5}}\right)\left(\int_{B_\rho\setminus B_r} |G-G_R|^{\frac{10}{7}l}\,dx\right)^{\frac{7}{10}} \qquad (3.12)\\
&\le \frac{K}{(\rho-r)^{21/20}}\left(\int_{B_\rho} |\nabla G|^2\,|G-G_R|^{\frac{20}{21}l-2}\,dx\right)^{\frac{21}{20}} \\
&+\frac{K}{l(\rho-r)^{6/5}}\left(\int_{B_\rho} |G-G_R|^{\frac{10}{7}l}\,dx\right)^{\frac{7}{10}},
\end{aligned}
$$

where $K$ is independent of $\varepsilon, h, k, \rho, r$ and $l$.

Let us now specify the constant $G_R$ such that

$$\mu(\{x \in B_R; G - G_R \geq 0\}) \geq \frac{1}{2}\mu(B_R) \tag{3.13}$$
$$\mu(\{x \in B_R; G - G_R \leq 0\}) \geq \frac{1}{2}\mu(B_R)$$

Applying now the Sobolev inequality (cf. [7, p. 81]) for $(G - G_R)_+$ and $(G - G_R)_-$ we get

$$\begin{aligned} &\left(\int_{B_\rho} |G - G_R|^{\frac{10}{7}l}\,dx\right)^{\frac{7}{10}} \\ &\leq K\left(\int_{B_\rho} |\nabla|G - G_R|^{\frac{10}{21}l}|^2\,dx\right)^{\frac{21}{20}} \\ &\leq K l^{\frac{21}{10}}\left(\int_{B_\rho} |\nabla G|^2\,|G - G_R|^{\frac{20}{21}l-2}\,dx\right)^{\frac{21}{20}} \end{aligned} \tag{3.14}$$

where $k$ is independent of $\varepsilon, h, k, R$, and $l$. Thus we arrived at

$$\begin{aligned} &\int_{B_r} |\nabla G|^2\,|G - G_R|^{l-2}\,dx \\ &\leq K\frac{l^{11/10}}{(\rho - r)^{6/5}}\left(\int_{B_\rho} |\nabla G|^2\,|G - G_R|^{\frac{20}{21}l-2}\,dx\right)^{\frac{21}{20}}, \end{aligned} \tag{3.15}$$

which holds for all $R < r < \rho < 2R$, and where the constant is independent of $\varepsilon, h, k, \rho, r$ and $l$. But inequality (3.15) is nothing else than the starting point for the Moser iteration technique (cf. (4.22), (4.23)). If we put

$$\begin{aligned} &l_0 = \frac{21}{10}, \qquad l_{i+1} = \frac{21}{20}l_i = \left(\frac{21}{20}\right)^i l_0 \\ &R_0 = 2R, \qquad R_i = \left(2 - \frac{6}{\pi^2}\sum_{k=1}^{i}\frac{1}{k^2}\right)R \end{aligned} \tag{3.16}$$

and use Lemma 5.8, which shows that the starting point of the iteration is finite, we proved:

LEMMA 3.17. *Let $B_R$ be a ball such that $B_{2R} \cap \operatorname{supp}\delta_h(x_0) = \emptyset$ for $0 < h < h_0$. Then we have for the solution $G = G^k_{h,\varepsilon}$ of (2.8)*

$$\|G\|_{0,\infty,B_R} \leq K(R)\,, \tag{3.18}$$

*where the constant $K(R)$ is independent of $\varepsilon, h, k$.*

But (3.18) is independent of $k$ and thus the limiting process $k \to \infty$ is possible. Hence we proved Proposition 2.18 (ii).

## 4. – Regularity of u

In this section we will establish the higher integrability of the solution $\mathbf{u}$ of (2.1), which is needed to prove the full regularity as stated in Theorem 2.23. We will use a similar Green-type function as in Sections 2 and 3. Therefore the treatment will be brief and we discuss only the additional new features.

Let us consider $H$ solving

$$\begin{aligned} -\Delta H - \mathbf{v}\cdot\nabla H &= \frac{\chi_+}{|x-x_0|^\alpha} * \omega_\rho \quad \text{in } \Omega\,, \\ H &= 0 \qquad\qquad\qquad \text{on } \partial\Omega\,. \end{aligned} \tag{4.1}$$

Here $x_0 \in \Omega_0 \subseteq\subseteq \Omega$ is an arbitrary point, $\mathbf{v} = \mathbf{v}^k \in \mathcal{V}$ is the same approximation of $\mathbf{u} = \mathbf{u}^\varepsilon$ as in Section 2, $\chi_+$ is the characteristic function of the set $\{x;\, \frac{\mathbf{u}^2}{2}(x) + p(x) > 0\}$, $\alpha \in [4,6)$ is fixed but arbitrary and $\omega_\rho$ is the usual mollification kernel.

The right-hand side of $(4.1)_1$ is non-negative and belongs to the space $L^{q/\alpha}(\Omega)$, $q \in [4,6)$ independent of $\varepsilon > 0$ and $\rho > 0$. In the same way as in Lemma 2.10 we obtain:

LEMMA 4.2. *For all $\varepsilon > 0$, $\rho > 0$ and $k \in \mathbb{N}$ there exists a solution $H = H^k_{\rho,\varepsilon} \in C^\infty(\Omega) \cap W^{2,q/\alpha} \cap W^{1,q/\alpha}_0(\Omega)$ to* (4.1) *satisfying*

$$\int_\Omega \nabla H\, \nabla\varphi\, dx - \int_\Omega \mathbf{v}\cdot\nabla H\, \varphi\, dx = \int_\Omega \frac{\chi_+}{|x-x_0|^\alpha} * \omega_\rho\, \varphi\, dx \qquad \forall \varphi \in C^\infty_0(\Omega)\,, \tag{4.3}$$

*such that*

$$\begin{aligned} &H \geq 0\,, \\ &\|H\|_{0,\infty} \leq c(\rho)\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \|\nabla^2 H\|^2_{0,2,\mathrm{loc}} &\leq c(\rho)(1 + \|\nabla\mathbf{v}\|^2_{0,2})\,, \\ \|\nabla H\|^4_{0,4,\mathrm{loc}} &\leq c(\rho)(1 + \|\nabla\mathbf{v}\|^2_{0,2})\,, \end{aligned} \tag{4.5}$$

*where the constant $c(\rho)$ is independent of $\varepsilon$ and $k$.*

LEMMA 4.6. *Let $H = H^k_{\rho,\varepsilon}$ be the solution of* (4.1). *Then we have*

$$\begin{aligned} &\int_\Omega |H|^q\, dx \leq K \quad \forall q \in \left[1, \frac{6}{\alpha-2}\right)\,, \\ &\int_\Omega |\nabla H|^s\, dx \leq K \quad \forall s \in [1, 6/5)\,, \end{aligned} \tag{4.7}$$

*where $K$ is independent of $\varepsilon$, $\rho$ and $k$.*

PROOF. In the same way as in Lemma 3.1 we get $(4.7)_2$ and $(4.7)_1$ for $q \in [1, 3/2)$. In order to prove $(4.7)_1$ completely we use $\varphi = \frac{H^r}{(1+H^m)^{1/m}}$, $r > 1$, $m > 0$[(2)] in (4.3) and obtain (note that the convective term vanishes)

$$r \int_\Omega |\nabla H|^2 \frac{H^{r-1}}{(1+H^m)^{1+1/m}} dx + (r-1) \int_\Omega |\nabla H|^2 \frac{H^{r+m-1}}{(1+H^m)^{1+1/m}} dx$$
$$= \int_\Omega \frac{\chi_+}{|x-x_0|^\alpha} * \omega_\rho \frac{H^r}{(1+H^m)^{1/m}} dx$$

and consequently $(q < 6, r > 1)$

$$\int_\Omega |\nabla |H|^{\frac{r+1}{2}}|^2 \frac{1}{(1+H^m)^{1+1/m}} dx$$
$$\leq c \left\| \frac{\chi_+}{|x-x_0|^\alpha} * \omega_\rho \right\|_{q/\alpha} \left\| \frac{H^r}{(1+H^m)^{1+1/m}} \right\|_{\frac{q}{q-\alpha}} \tag{4.8}$$
$$\leq c \|H\|^{r-1}_{(r-1)\frac{q}{q-\alpha}} .$$

Further, we have for $3 > \gamma \geq 1$ (using (4.8))

$$\|H\|^{\frac{r+1}{2}}_{\frac{r+1}{2}\gamma} \leq \|\nabla |H|^{\frac{r+1}{2}}\|_{\frac{6\gamma}{6+\gamma}}$$
$$\leq \left( \int_\Omega \left( \frac{|\nabla |H|^{\frac{r+1}{2}}|^2}{(1+H^m)^{1+1/m}} \right)^{\frac{3\gamma}{6+\gamma}} (1+H^m)^{\frac{m+1}{m}\frac{3\gamma}{6+\gamma}} dx \right)^{\frac{6\gamma}{6+\gamma}} \tag{4.9}$$
$$\leq c \|H\|^{\frac{r-1}{2}}_{(r-1)\frac{q}{q-\alpha}} \left( 1 + \|H\|^{\frac{m+1}{2}}_{\frac{m+1}{2}\frac{3\gamma}{3-\gamma}} \right) .$$

Setting now

$$\frac{r+1}{2}\gamma = \frac{m+1}{2}\frac{3\gamma}{3-\gamma} = (r-1)\frac{q}{q-\alpha} \tag{4.10}$$

we get restrictions on $\gamma$ and $r$ in terms of $\alpha$, namely

$$\gamma \in \left[1, \frac{3r}{1+r}\right), \qquad r \in \left(1, \frac{4}{\alpha-2}\right) . \tag{4.11}$$

Finally we get

$$\|H\|_{\frac{r+1}{2}\gamma} \leq c ,$$

where $\frac{r+1}{2}\gamma \in \left[1, \frac{6}{\alpha-2}\right)$, which is $(4.7)_1$. □

(2) The factor $\frac{1}{(1+H^m)^{1/m}}$ is a normalization which changes the polynomial growth of the test function only slightly.

We shall also need a statement for $H$ similar to that one in Proposition 2.18 (ii). Let us first state a Moser iteration lemma, which will be proved in the appendix (cf. Lemma 5.14).

LEMMA 4.12. *Let $\sigma > 1$, $l_0 > 1$, $s, t, \alpha > 0$ be given and let us denote for all $n \in \mathbb{N}$*

$$A_{l_{n+1}}^{l_n} \leq c l_n^s A_{l_n}^{l_n} + c l_n^t A_{l_n}^{l_n-\alpha}. \tag{4.13}$$

*Then we have*

$$A_\infty = \overline{\lim_{n\to\infty}} A_{l_n} < \infty. \tag{4.14}$$

PROPOSITION 4.15. *Let $H = H^k_{\rho,\varepsilon}$ be a solution of* (4.1), *let $R > 0$ be arbitrary but fixed and let $0 < \rho < R$. Let $y_0$ be such that* $\operatorname{dist}(x_0, y_0) > 4R$. *Then we have*

$$\|H\|_{0,\infty,B_R(y_0)} \leq c(R), \tag{4.16}$$

*where the constant $c(R)$ is independent of $\rho$, $\varepsilon$ and $k$.*

PROOF. The proof follows the lines of that one of Proposition 2.18 (ii). Let $y_0$ and $R$ be given and let $R < r < s < \rho < 2R$. We multiply (4.1) by $\chi_s|H - H_R|^{l-2}(H - H_R)$, where $l \in \mathbb{R}_+$ and $\chi_s$ is the characteristic function of the ball $B_s(y_0)$. The constant $H_R$ will be specified later on. We get

$$\begin{aligned}
(l-1)\int_{B_s} &|\nabla H|^2\,|H - H_R|^{l-2}\,dx \\
&= \int_{\partial B_s} \nu\cdot\nabla H|H - H_R|^{l-2}(H - H_R)\,dS \\
&\quad + \frac{1}{l}\int_{\partial B_s} \nu\cdot\mathbf{v}|H - H_R|^l\,dS \\
&\quad + \int_{B_s} \frac{\chi_+}{|x - x_0|^\alpha} * \omega_\rho |H - H_R|^{l-2}(H - H_R)\,dx \\
&\equiv I_1 + I_2 + I_3.
\end{aligned} \tag{4.17}$$

The left-hand side of (4.17) and the integrals $I_1$ and $I_2$ will be treated in the same way as in Section 3. Let us therefore discuss $I_3$. We have (note that $0 < \rho < R$)

$$\begin{aligned}
|I_3| &\leq \int_{B_s} \frac{1}{R^\alpha}\int_{B_\rho(x)} \omega_\rho(x-y)dy|H - H_R|^{l-1}\,dx \\
&\leq \frac{1}{R^\alpha}\int_{B_s} |H - H_R|^{l-1}\,dx \\
&\leq \frac{1}{R^{\alpha-9/5}}\left(\int_{B_s} |H - H_R|^{\frac{10}{7}l}\,dx\right)^{\frac{7}{10l}(l-1)}.
\end{aligned} \tag{4.18}$$

If we specify the constant $H_R$ similar as in (3.13) we get (see also (3.14))

$$|I_3| \le K \frac{l^{21/10}}{R^{\alpha-9/5}} \left( \int_{B_\rho} |\nabla H|^2 \, |H - H_R|^{\frac{20}{21}l-2} \, dx \right)^{\frac{21}{20l}(l-1)} \tag{4.19}$$

Alltogether from (4.17), (4.19) and a similar treatment of $I_1$ and $I_2$ as in Section 3 we obtain

$$\begin{aligned} &\int_{B_r} |\nabla H|^2 \, |H - H_R|^{l-2} \, dx \\ &\le K \frac{l^{11/10}}{(\rho - r)^{6/5}} \left( \int_{B_\rho} |\nabla H|^2 \, |H - H_R|^{\frac{20}{21}l-2} \, dx \right)^{21/20} \\ &\quad + K \frac{l^{21/10}}{R^{\alpha-9/5}} \left( \int_{B_\rho} |\nabla H|^2 \, |H - H_R|^{\frac{20}{21}l-2} \, dx \right)^{\frac{21}{20l}(l-1)} \end{aligned} \tag{4.20}$$

If we denote

$$\begin{aligned} &l_0 = \frac{21}{10}, \quad l_{i+1} = \frac{21}{20} l_i = \left(\frac{21}{20}\right)^i l_0 \\ &R_0 = 2R, \quad R_i = \left( 2 - \frac{6}{\pi^2} \sum_{k=1}^{i} \frac{1}{k^2} \right) R \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} A_{l_i} &= \left( \int_{B_{R_i}(y_0)} |\nabla H|^2 \, |H - H_R|^{l_i-2} \, dx \right)^{1/l_i} \\ A_{l_0} &= \left( \int_{B_{2R}(y_0)} |\nabla H|^2 \, dx \right)^{1/l_0} \end{aligned} \tag{4.22}$$

we get

$$A_{l_{i+1}}^{l_i} \le K \frac{l_i^{7/2}}{R^2} A_{l_i}^{l_i} + K \frac{l_i^{5/2}}{R^2} A_{l_i}^{l_i - \frac{20}{21}} \tag{4.23}$$

and thus be Lemma 4.12, Lemma 5.12 and

$$A_{l_i} \ge \|H - H_R\|_{\frac{3}{2}l_i, B_{R-i}(y_0)}$$

we get (4.16). □

Based on Lemma 4.2 and Lemma 2.4 we can use $H\zeta^2$ as a test function in the head pressure equation and we obtain (cf. (2.15))

$$\begin{aligned}(4.24)\qquad &\int_\Omega \nabla\left(\frac{\mathbf{u}^2}{2}+p\right)\nabla(H\zeta^2)+\mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2}+p\right)H\zeta^2\,dx\\ &\le -\varepsilon\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla(H\zeta^2)dx+\int_\Omega \mathbf{f}\cdot\nabla(H\zeta^2)+\mathbf{f}\cdot\mathbf{u}\,H\zeta^2\,dx\,.\end{aligned}$$

With the appropriate changes we now proceed in the same way as in Section 2. Concerning the limiting process $\varepsilon\to 0$, let us only mention that the right-hand side of (4.1) converges strongly in $L^{q/\alpha}(\Omega)$ as $\varepsilon\to 0$. We arrive at

$$\begin{aligned}&\int_\Omega \frac{\chi_+}{|x-x_0|^\alpha}*\omega_\rho\left(\frac{\mathbf{u}^2}{2}+p\right)\zeta^2\,dx\\ &\le \lim_{\varepsilon\to 0}\int_\Omega \mathbf{u}^\varepsilon\cdot\nabla\zeta^2\left(\frac{|\mathbf{u}^\varepsilon|^2}{2}+p^\varepsilon\right)H_{\rho,\varepsilon}\,dx\\ (4.25)\qquad &\quad -2\int_\Omega H\nabla\left(\frac{\mathbf{u}^2}{2}+p\right)\nabla\zeta^2\,dx-\int_\Omega H\left(\frac{\mathbf{u}^2}{2}+p\right)\Delta\zeta^2\,dx\\ &\quad -\varepsilon\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla(H\zeta^2)dx+\int_\Omega \mathbf{f}\cdot\nabla(H\zeta^2)+\mathbf{f}\cdot\mathbf{u}H\zeta^2\,dx\,,\end{aligned}$$

where $\mathbf{u}$ is a solution to (2.1) and $H=H_\rho$ solves (4.1) with $\mathbf{v}$ replaced by $\mathbf{u}$. Now Proposition 4.15 ensures that the first term on the right-hand side of (4.25) remains bounded also as $\rho\to 0$. We can handle the last term at the right-hand side of (4.25), as $\rho\to 0$, due to Lemma 4.6, especially $(4.7)_1$. Hence we have

PROPOSITION 4.26. *Let $\alpha\in[4,6)$ and let $x_0\in\Omega_0\subset\subset\Omega$ be given. Then the weak solution $\mathbf{u}$, $p$ of* (2.1) *constructed before satisfies*

$$(4.27)\qquad \int_\Omega \frac{\zeta^2}{|x-x_0|^\alpha}\left(\frac{\mathbf{u}^2}{2}+p\right)_+dx\le K\,,$$

*where the constant $K=K(\alpha)$ is independent of $x_0\in\Omega_0$.*

In particular for $\alpha=4$ we can proceed as in Frehse, Růžička [5, Theorem 2.1, Theorem 2.11] (cf. [9]) in order to prove Proposition 2.21.

## 5. – Appendix

LEMMA 5.1. *Let $0 < r \le s \le \rho$ and let $g_i \in L^1(B_\rho \backslash B_r)$, $i = 1, 2, 3$. Then there exists a set $E \subset [r, \rho]$ with $\mu(E) \ge \frac{1}{4}(\rho - r)$, such that for all $s \in E$ and $i = 1, 2, 3$*

$$\int_{\partial B_s} |g_i| dS \le \frac{4}{\rho - r} \int_{B_\rho \backslash B_r} |g_i|\, dx\,. \tag{5.2}$$

PROOF. Let us denote

$$G_i = \{t \in [r, \rho]; \int_{\partial B_t} |g_i| dS \ge \frac{4}{\rho - r} \int_{B_\rho \backslash B_r} |g_i| dx\} \tag{5.3}$$

and $E_i = [r, \rho] \backslash G_i$. Then we have

$$\int_{G_i} \int_{\partial B_t} |g_i| ds\, dt \ge \frac{4}{\rho - r} \mu(G_i) \int_{B_\rho \backslash B_r} |g_i|\, dx\,. \tag{5.4}$$

On the other hand we have

$$\int_{G_i} \int_{\partial B_t} |g_i| ds\, dt \le \int_{B_\rho \backslash B_r} |g_i| dx\,, \tag{5.5}$$

and hence

$$\mu(G_i) \le \frac{\rho - r}{4}, \qquad \mu(E_i) \ge \frac{3}{4}(\rho - r)\,. \tag{5.6}$$

This immediately implies

$$\mu\left(\bigcap_{i=1}^{3} E_i\right) \ge \frac{1}{4}(\rho - r)\,. \tag{5.7}$$

Indeed, we have $E_1 \cap E_2 = E_1 \cap F_{12}$, where $F_{12} = E_2 \cap ([r, \rho] \backslash E_1)$. This implies

$$\mu(F_{12}) \le \mu([r, \rho] \backslash E_1) \le \frac{\rho - r}{4}\,.$$

Further we have

$$\mu(E_1 \cap F_{12}) = \mu(E_1) - \mu(F_{12}) \ge \frac{1}{2}(\rho - r)\,.$$

The same argument, with $E_1$ replaced by $E_1 \cap E_2$ and $E_2$ replaced by $E_3$ yields

$$\mu(E_1 \cap E_2 \cap E_3) \ge \frac{1}{4}(\rho - r)\,,$$

which is the assertion of the lemma if we put $E = \bigcap_{i=1}^{3} E_i$. □

LEMMA 5.8. *Let $G = G^k_{h,\varepsilon}$ be the solution of* (2.7) *and let $B_{2R}$ be as in Section 3. Then*

$$\int_{B_R} |\nabla G|^2 dx \leq c(R)\,, \tag{5.9}$$

*where the constant $c(R)$ is independent of $k$, $h$ and $\varepsilon$.*

PROOF. We just have to modify the procedure from Section 3 a little. Let us multiply (2.7) by $\chi_s(G+1)^{t-1}$, $t > 1$. Thus we get for $s \in E$ (cf. Lemma 3.8)

$$\begin{aligned}
&(t-1)\int_{B_s} |\nabla G|^2 (G+1)^{t-2}\, dx \\
&= \int_{\partial B_s} \nu \cdot \nabla G (G+1)^{t-1} dS + \frac{1}{t}\int_{\partial B_s} \nu \cdot \mathbf{v}(G+1)^t\, dS \\
&\leq \frac{c}{\rho - r}\left(\int_{B_\rho \setminus B_r} |\nabla G|^q dx\right)^{\frac{1}{q}} \left(\int_{B_\rho \setminus B_r} |G+1|^{(t-1)\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \\
&\quad + \frac{c}{t(\rho - r)}\left(\int_{B_\rho \setminus B_r} \frac{|\mathbf{v}|^2}{\mu(B_1)R^5} + |\nabla \mathbf{v}|^2 dx\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{B_\rho \setminus B_r} |G+1|^{\frac{10}{7}t} dx\right)^{\frac{7}{10}}.
\end{aligned} \tag{5.10}$$

Using (3.2) we see that the right-hand side is finite if $q < 6/5$ and $t < 21/20$. Thus we get from the Sobolev embedding theorem

$$\begin{aligned}
\|G+1\|_{\alpha, B_s} &\leq c\|\nabla G\|_{\frac{6\alpha}{6+\alpha}, B_s} + \|\overline{G+1}\|_{\alpha, B_s} \\
&\leq \left(\int_{B_s} \left(\frac{|\nabla G|^2}{(G+1)^{2-t}}\right)^{\frac{3\alpha}{6+\alpha}} (G+1)^{(2-t)\frac{3\alpha}{6+\alpha}} dx\right)^{\frac{6+\alpha}{6\alpha}} \\
&\quad + cs^{\frac{6}{\alpha}-6}\|G+1\|_{1, B_s} \\
&\leq c(R)\left(\int_{B_s} (G+1)^{(2-t)\frac{3\alpha}{6-2\alpha}} dx\right)^{\frac{6-2\alpha}{6\alpha}} + c(R)\,.
\end{aligned}$$

Choosing $\alpha = \frac{3}{2}t$ we get

$$G \in W^{1, \frac{6t}{4+t}}(B_r)\,. \tag{5.11}$$

Using now the new local information (5.11) instead of (3.2) we can repeat the procedure and thus we obtain

$$G \in W^{1, \frac{6t_i}{4+t_i}}(B_{R_i})\,,$$

where

$$t_{i+1} < \frac{21}{20} t_i\,, \qquad q_i < \frac{6t_i}{4+t_i}$$

and $R_i$ are as in (3.16). The lemma follows immediately. □

LEMMA 5.12. *Let $H = H^k_{\rho,\varepsilon}$ be the solution of* (4.1) *and let $B_{2R}$ be as in Section 4. Then*

$$\int_{B_R} |\nabla H|^2 dx \le c(R)\,, \tag{5.13}$$

*where the constant $c(R)$ is independent of $k$, $\rho$ and $\varepsilon$.*

PROOF. The proof follows the lines of he previous lemma and that one Proposition 4.15. Due to the right-hand side of (4.1) we get one additional term and thus we have

$$\begin{aligned}
&\int_{B_s} |\nabla G|^2 (H+1)^{t-2} dx \\
&\le \frac{c}{\rho - r}\left(\int_{B_\rho \setminus B_r} |\nabla H|^q \, dx\right)^{\frac{1}{q}} \left(\int_{B_\rho \setminus B_r} |H+1|^{(t-1)\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \\
&\quad + \frac{c}{t(\rho - r)}\left(\int_{B_\rho \setminus B_r} \frac{|\mathbf{v}|^2}{\mu(B_1)R^5} + |\nabla \mathbf{v}|^2 dx\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{B_\rho \setminus B_r} |H+1|^{\frac{10}{7}t} dx\right)^{\frac{7}{10}} \\
&\quad + \frac{c}{R^{\alpha - 9/5}}\left(\int_{B_\rho \setminus B_r} |H+1|^{\frac{10}{7}t} dx\right)^{\frac{7}{10t}(t-1)} .
\end{aligned} \tag{5.13}$$

Using Lemma (4.7) we see that the right-hand side is finite if $q < 6/5$ and $t < 21/20$. We conclude the proof in the same way as in the previous lemma. □

LEMMA 5.14. *Let $\sigma > 1$, $l_0 > 1$, $s, t, \alpha > 0$ be given and let us denote for all $n \in \mathbb{N}$*

$$A^{l_n}_{l_{n+1}} \le c l_n^s A^{l_n}_{l_n} + c l_n^t A^{l_n - \alpha}_{l_n} . \tag{5.15}$$

*Then we have*

$$A_\infty = \overline{\lim_{n \to \infty}} A_{l_n} < \infty . \tag{5.16}$$

PROOF. There are two possibilities. Either

$$\exists\, c_0\, \exists\, l_n \to \infty : \forall\, n \qquad A^{l_n}_{l_n} \le c_0^{l_n} , \tag{5.17}$$

which gives immediately (5.16), or

$$\begin{gathered} \forall\, c_0\, \exists\, j_0 \text{ (minimal) } : \ \forall\, i \ge j_0 \\ A^{l_i}_{l_i} \ge c_0^{l_i} . \end{gathered} \tag{5.18}$$

This together with (5.15) implies

$$A_{l_{i+1}}^{l_i} \leq c l_i^{s+t} A_{l_i}^{l_i} \left(1 + \frac{1}{c_0^\alpha}\right) \qquad \forall\, i \geq j_0$$

and hence

$$A_{l_{i+1}} \leq c \prod_{i=j_o}^{\infty} (1 + c_0^{-\alpha})^{1/l_i} \prod_{i=j_0}^{\infty} l_i^{\frac{s+t}{l_i}} A_{l_{j_0}} \qquad \forall\, i \geq j_0\,. \tag{5.19}$$

Now either $j_0 = 0$ and we get immediately (5.16) (cf. [7]) or $j_0 > 0$. But then we have $A_{l_{j_0}-1} \leq c_0$ and thus we obtain

$$A_{l_{j_0}}^{l_{j_0-1}} \leq l_{j_0-1}^{s+t} A_{l_{j_0-1}}^{l_{j_0-1}} \left(1 + \frac{1}{c_0^\alpha}\right)$$

and consequently

$$A_{l_{j_0}} \leq c(1 + c_0^{-\alpha})^{1/l_{j_0-1}} l_{j_0-1}^{\frac{s+t}{l_{j_0-1}}}\,.$$

This together with (5.19) implies

$$A_{l_{i+1}} \leq c \prod_{i=j_0-1}^{\infty} (1 + c_0^{-\alpha})^{1/l_i} \prod_{i=j_0-1}^{\infty} l_i^{\frac{s+t}{l_i}}\,. \tag{5.20}$$

Inequality (5.20) again immediately gives (5.16). □

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