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Dirichlet operators on $\mathbb{R}^n$


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Elliptic Regularity and Essential Self-adjointness of Dirichlet Operators on $\mathbb{R}^n$

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One of the classical problems in mathematical physics is the problem of essential self-adjointness for Dirichlet operators

$$L := \Delta + \beta \cdot \nabla,$$

with domain $C_0^\infty(\mathbb{R}^n)$ (:= all infinitely differentiable functions on $\mathbb{R}^n$ with compact support) on $L^2(\mathbb{R}^n, \mu)$, where $\mu$ is a measure on $\mathbb{R}^n$ with density $\rho := \varphi^2$, with $\varphi \in H_{loc}^{2,1}(\mathbb{R}^n)$ and $\beta := \nabla \rho / \rho$. (By definition $\beta(x) = 0$ if $\rho(x) = 0$).

The results obtained in [1], [8], [9], [11], [14], [25] have been important steps in the investigation of this problem. One motivation to study this problem is that the operator $-L$ is unitary equivalent to the Schrödinger operator $H := -\Delta + V$, $V := \Delta \varphi / \varphi$, considered on $L^2(\mathbb{R}^n, dx)$ (see, e.g., [1], [5]) where $dx$ denotes Lebesgue measure on $\mathbb{R}^n$. The corresponding isomorphism $L^2(\mathbb{R}^n, \mu) \rightarrow L^2(\mathbb{R}^n, dx)$ is given by $f \mapsto \varphi \cdot f$. Conversely, if $H = -\Delta + V$ is a Schrödinger operator on $L^2(\mathbb{R}^n, dx)$ with lower bounded spectrum $\sigma(H)$ whose minimum is an eigenvalue $E$, then the isomorphism above holds for the potential $V - E$ (and $\varphi :=$ the ground state). Since this unitary equivalence only holds for sufficiently regular $\varphi$, Dirichlet operators are also sometimes called generalized Schrödinger operators. We emphasize that under the above isomorphism in general domains change drastically. Hence known results on the essential self-adjointness of $H$ with domain $C_0^\infty(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n, dx)$ do not apply. On the contrary in many cases the essential self-adjointness of Dirichlet operators implies this property for Schrödinger operators (see e.g. [16, pp. 217, 218]).

There are basically two different types of sufficient conditions known for the essential self-adjointness of Dirichlet operators: global and local. A typical global condition obtained in [14] is: $|\beta| \in L^4(\mathbb{R}^n, \mu)$ (provided $\rho > 0$ a.e.). The best local condition obtained so far has been found in [25] where $\rho$ has been required to be locally Lipschitzian and strictly positive if $n \geq 2$ (and

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with even weaker conditions if $n = 1$, cf. Remark 2 below). In particular, this means that $\beta$ is locally bounded. One of our main results in this paper (cf. Theorem 7 below) says that $L$ is essentially self-adjoint provided that $\rho$ is merely locally bounded and \textit{locally uniformly positive} (cf. below) and $|\beta| \in L^\gamma_{\text{loc}}(\mathbb{R}^n, \mu)$ for some $\gamma > n$ (which as we shall show below, is equivalent to $|\beta| \in L^\gamma_{\text{loc}}(\mathbb{R}^n, dx)$; cf. Corollary 8). The proof of Theorem 7 is based on an elliptic regularity result (which is the main result of this paper) giving $H^\gamma_{\text{loc}}$-regularity of distributional solutions of the elliptic equation $L^* F = 0$, where $\rho := \Delta f + \langle B, \nabla f \rangle + cf$. This result is formulated as Theorem 1 below. As a consequence one gets $H^\gamma_{\text{loc}}$-regularity of invariant measures for diffusion processes with drifts satisfying certain mild local integrability conditions (which extends a result from [3], [4]). Finally, we note that for the above mentioned special applications to Schrödinger operators $H = -\Delta + V$, of course, one still needs corresponding information about the ground state $\varphi$ to ensure that $|\beta| = 2|\nabla \varphi/\varphi| \in L^\gamma_{\text{loc}}(\mathbb{R}^n, \mu)$.

Throughout this paper, $\Omega$ is a (fixed) open subset of $\mathbb{R}^n$, and for $r \in (-\infty, \infty)$ and $p \geq 1$, $H^p_{\text{loc}}(\Omega)$ denotes the class of (generalized) functions $u$ on $\Omega$, such that $(1-\Delta)^{r/2} u \in L^p(\mathbb{R}^n, dx)$ for every $\psi \in C_0^\infty(\Omega)$. These spaces coincide with the usual Sobolev spaces for integer $r \geq 1$. All properties of these spaces which are needed below can be found, for instance, in [23]. If $\nu$ is a signed measure, then by definition $\int f \, d\nu = \int f \chi \, d|\nu|$, where $\chi := \nu/|\nu|$, and $L^p(\Omega, \nu) := L^p(\Omega, |\nu|)$. If, in addition, $\nu \ll dx$, then we write $\nu$ instead of $d\nu/dx$. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$ and $|\cdot|$ the corresponding norm.

**THEOREM 1.** Let $n \geq 2$ and let $\mu, \nu$ be (signed) Radon measures on $\Omega$. Let $B = (B^i) : \Omega \to \mathbb{R}^n$, $c : \Omega \to \mathbb{R}$ be maps such that $|B|, c \in L^1_{\text{loc}}(\Omega, \mu)$. Assume that

1. $\int L\varphi(x) \mu(dx) = \int \varphi(x) v(dx) \quad \forall \varphi \in C_0^\infty(\Omega)$,

where

\begin{equation}
L\varphi(x) := \Delta \varphi(x) + \langle B(x), \nabla \varphi(x) \rangle + c(x) \varphi(x).
\end{equation}

Then:

(i) $\mu \in H^p_{\text{loc}}(\Omega, \mu)$ for any $p \geq 1$ and $\varepsilon > 0$. Here $1 - n(p - 1)/p > 0$ if $p \in [1, n-1]$ and, in particular, $\mu$ admits a density $F \in L^p_{\text{loc}}(\Omega, dx)$ for any $p \in [1, n/(n-1)]$.

(ii) If $|B| \in L^\gamma_{\text{loc}}(\Omega, \mu)$, $c \in L^{\gamma/2}_{\text{loc}}(\Omega, \mu)$ and $\nu \in L^n(\Omega, dx)$ where $n \geq \gamma > 1$, then $F := d\mu/dx \in H^{p,1}_{\text{loc}}(\Omega)$ for any $p \in [1, n/(n-\gamma + 1))$. In particular, $F \in L^p_{\text{loc}}(\Omega, dx)$ for any $p \in [1, n/(n-\gamma))$, where (here and below) $n/(n-\gamma) := \infty$ if $\gamma = n$.

(iii) If $\gamma > n$ and either.
REMARK 2 (i) There is a similar regularity result for \( n = 1 \) (whose proof is easier and, in fact, quite elementary). Therefore, Theorem 7 and Corollary 8 below also hold in this case. However, our conditions there for \( n = 1 \) are then obviously equivalent to: \( \phi(= \sqrt{\rho}) \in H_{\text{loc}}^{2,1}(\mathbb{R}) \) and (the continuous version of) \( \rho \) is strictly positive. But under these conditions in the special case \( n = 1 \) both results are already contained in [25]. So, we state and prove our results only for \( n \geq 2 \).

(ii) Note that since \( c \in L_{\text{loc}}^1(\Omega, \mu) \) and \( \nu \) is a Radon measure, the assumptions on \( c, \nu \) in Theorem 1 (ii) are automatically fulfilled, if \( \gamma \leq 2 \), provided \( \nu \ll dx \).

To prove Theorem 1 we use the following lemma.

**LEMMA 3.** (i) For any \( r \in (-\infty, \infty) \) and \( p > 1 \), if \( \Delta u \in H_{\text{loc}}^{p,r}(\Omega) \), then \( u \in H_{\text{loc}}^{p,r+2}(\Omega) \); also if \( u \in H_{\text{loc}}^{p,r}(\Omega) \), then \( u_{,i} \in H_{\text{loc}}^{p,r-1}(\Omega) \), \( 1 \leq i \leq n \).

(ii) We have \( H_{\text{loc}}^{p,1}(\Omega) \subset L_{\text{loc}}^{np/(n-p)}(\Omega, dx) \) and \( L_{\text{loc}}^{p}(\Omega, dx) \subset H_{\text{loc}}^{n(p/(n-p)-1)}(\Omega) \) whenever \( 1 < p < n \), and \( H_{\text{loc}}^{p,1}(\Omega) \subset C_{\text{loc}}^{1-n/p}(\Omega) \) if \( p > n \), so that in the latter case elements of \( H_{\text{loc}}^{p,1}(\Omega) \) are locally bounded. Also for \( q > p > 1 \), \( L_{\text{loc}}^{p}(\Omega, dx) \subset H_{\text{loc}}^{q,n/q-n/p}(\Omega) \).

(iii) If \( \mu \) is a Radon measure on \( \Omega \), then \( \mu \in H_{\text{loc}}^{p,-m}(\Omega) \) whenever \( p > 1 \) and \( m > n(1 - 1/p) \).

**PROOF.** Assertion (i) is well-known. Specifically, its first statement is a well-known elliptic regularity result and the second statement follows from the boundedness of Riesz’s transforms. Assertion (ii) is just the Sobolev imbedding theorems (see [23]). Assertion (iii) follows from these imbedding theorems since, for regular sub-domains \( U \) of \( \Omega \), \( H^{q,m}(U) \subset C(\bar{U}) \) if \( qm > n \) whence by duality the space \( H^{q/(q-1),-m}(U) = [H^{q,m}(U)]^* \) contains all finite measures on \( U \).  

**PROOF OF THEOREM 1.** (i): We have that in the sense of distributions

\[
\Delta \mu = (B^i \mu)_{,i} - c \mu + \nu
\]

on \( \Omega \). Here by Lemma 3 (iii), the right-hand side belongs to \( H_{\text{loc}}^{p,-m-1}(\Omega) \) if \( m > n(1 - 1/p) \). By Lemma 3 (i) we conclude \( \mu \in H_{\text{loc}}^{p,-m+1}(\Omega) \), which leads to the result after substituting \( m = n(1 - 1/p) + \varepsilon \).
Before we prove (ii), (iii) we need some preparations. Fix a $p_1 > 1$ and assume that $F := \frac{d\mu}{dx} \in L^{p_1}_{\text{loc}}(\Omega, dx)$. (Such $p_1$ exists by (i).) Define

$$r := r(p_1) := \frac{\gamma p_1}{\gamma - 1 + p_1}$$

and observe that owing to the inequalities $1 < \gamma$ and $p_1 > 1$, we have $1 < r < \gamma$. Next, starting with the formula

$$|B F|^r = (|B||F|^{1/\gamma})^r |F|^{r-1/\gamma}$$

and using Hölder’s inequality (with $s = \frac{\gamma}{r} (> 1)$ and $t := \frac{s}{s-1} = \frac{\gamma}{\gamma - r}$) and the assumptions $|B||F|^{1/\gamma} \in L^r_{\text{loc}}(\Omega, dx)$ and $F \in L^{p_1}_{\text{loc}}(\Omega, dx)$, we get that $B^i F \in L^r_{\text{loc}}(\Omega, dx)$. By Lemma 3 (i)

$$B^i F \in H^{r,0}_{\text{loc}}(\Omega), \quad (B^i F)_x \in H^{r-1}_{\text{loc}}(\Omega).$$

(ii): Set

$$q := q(p_1) := \frac{\gamma p_1}{\gamma - 2 + 2 p_1} + 1,$$

and note that $q > 1 \leftrightarrow \gamma > 2 \leftrightarrow q < \frac{\gamma}{r}$, in particular, $q < \gamma$ in any case. Hence repeating the above argument with $c, \gamma/2, q$ replacing $|B|, \gamma, r$, respectively we obtain that

$$c F \in L^q_{\text{loc}}(\Omega, dx)$$

Fix $p_1 > 1$ such that $F := \frac{d\mu}{dx} \in L^{p_1}_{\text{loc}}(\Omega, dx)$ and let $r, q$ be as in (4), (6), correspondingly. Since $\gamma \leq n$ we have that $q < n$, which by (7) and Lemma 3 (ii) resp. (iii) yields $c F \in H^{nq/(n-q) - 1}_{\text{loc}}(\Omega)$ if $q > 1$ resp. $c F \in H^{n-1}_{\text{loc}}(\Omega)$ for any $s \in (1, n/(n-1))$ if $q = 1$.

It turns out that if $p_1 < n/(n-\gamma)$, then

$$c F \in H^{r-1}_{\text{loc}}(\Omega).$$

Indeed, if $q > 1$, then (8) follows from the fact that if $p_1 \in (1, n/(n-\gamma))$ the inequality $r \leq nq/(n-q)$ holds. If $q = 1$, then $\gamma \leq 2$ and (8) follows from the fact that $r < n/(n-\gamma + 1) \leq n/(n-1)$ for $p_1 < n/(n-\gamma)$.

Finally by Lemma 3 (ii) we have $\nu \in H^{n/(n-\gamma + 1) - 1}_{\text{loc}}(\Omega)$ if $\gamma > 2$ and $\nu \in H^{r-1}_{\text{loc}}(\Omega)$ for any $s \in (1, n/(n-1))$ if $\gamma \leq 2$. In the same way as above, $\nu \in H^{r-1}_{\text{loc}}(\Omega)$ whenever $1 < p_1 < n/(n-\gamma)$. This along with (5) and (8) shows that the right-hand side of (3) is now in $H^{r-1}_{\text{loc}}(\Omega)$. By Lemma 3 (i) we have

$$\mu \in H^{r,1}_{\text{loc}}(\Omega)$$
and by Lemma 3 (ii) \( F \in L^{p_1}_{\text{loc}}(\Omega, dx) \), where

\[
p_2 := \frac{n r}{n - r} = \frac{n y p_1}{n y - n + (n - y) p_1} =: f(p_1).
\]

Thus we get

\[
p_1 \in \left(1, \frac{n}{n - y}\right) \quad \text{and} \quad F \in L^{p_1}_{\text{loc}}(\Omega, dx) \implies F \in L^{f(p_1)}_{\text{loc}}(\Omega, dx).
\]

One can easily check that \( p_2 = f(p_1) > p_1 \) if \( p_1 < n/(n - y) \), and that the only positive solution of the equation \( q = f(q) \) is \( q = n/(n - y) \). Therefore, by taking \( p_1 \) from \((1, n/(n - 1))\), which is possible by (i), and by defining \( p_{k+1} = f(p_k) \) we get an increasing sequence of \( p_k \uparrow n/(n - y) \), which implies that \( F \in L^{p_k}_{\text{loc}}(\Omega, dx) \) for any \( p < n/(n - y) \).

But as \( p_k \not\to n/(n - y) \), \( r(p_k) \) (defined according to (4)) increasingly converges to

\[
\frac{\gamma n/(n - y)}{\gamma - 1 + n/(n - y)} = \frac{n}{n - y + 1}.
\]

By (9) this proves (ii).

(iii): First we consider case (b) in which \( |B| \in L^{q}_{\text{loc}}(\Omega, \mu), \quad c \in L^{n y/(n + y)}_{\text{loc}}(\Omega, \mu), \quad v \in L^{n y/(n + y)}_{\text{loc}}(\Omega, dx) \). By the last assertion in (ii) we have \( F \in L^{p_1}_{\text{loc}}(\Omega, dx) \) for any (finite) \( p_1 > 1 \). Let \( r := r(p_1) \) be defined as in (4). Then \( 1 < r < \gamma \) and (5) holds. Set

\[
q := q(p_1) := \frac{n y}{n + \gamma p_1}.
\]

Hence repeating the arguments that led to (5) with \( c, n y/(n + y), q \) replacing \( |B|, \gamma, r \) respectively we obtain \( c F \in L^{q}_{\text{loc}}(\Omega, dx) \), thus \( c F \in H^{n q/(n - q) - 1}_{\text{loc}}(\Omega) \) by Lemma 3 (ii). Observe that when \( p_1 \to \infty \), we have \( r \uparrow \gamma, \quad q \uparrow n y/(n + \gamma), \) and \( n q/(n - q) \uparrow \gamma \). Therefore, combining this with our assumption that \( v \in L^{n y/(n + y)}_{\text{loc}}(\Omega, dx) \) which by Lemma 3 (ii) is contained in \( H^{\gamma - 1}_{\text{loc}}(\Omega) \), by taking \( p_1 \) large enough, we see that the right-hand side in (3) is in \( H^{\gamma - 1}_{\text{loc}}(\Omega) \) for any \( \varepsilon \in (0, \gamma - 1) \). By Lemma 3 (ii) we conclude \( F \in H^{\gamma - 1}_{\text{loc}}(\Omega) \) and since \( \gamma > n \), the function \( F \) is locally bounded. Now we see that above we can take \( p_1 = \infty \) and therefore the right-hand side of (3) is in \( H^{\gamma - 1}_{\text{loc}}(\Omega) \), which by Lemma 3 (i) gives us the desired result.

In the remaining case (a) we take \( p_1 > \gamma/(\gamma - 1) \) and assume that \( F \in L^{p_1}_{\text{loc}}(\Omega, dx) \). Then instead of (4) and (10) we define

\[
r := r(p_1) := \frac{\gamma p_1}{\gamma + p_1}, \quad q := q(p_1) := \frac{n y}{n + y p_1} + p_1 \vee 1.
\]
and observe that owing to $p_1 > \gamma / (\gamma - 1)$ we have $r > 1$, which (because $p_1^{-1} + \gamma^{-1} = r^{-1}$) allows us to apply Hölder's inequality starting with $|BF| = |B|^r|F|^r$ to conclude that (5) holds. Since $c \in L^1_{\text{loc}}(\Omega, \mu)$, resp. $\frac{n\gamma}{n + \gamma} > 1$ and \( \left( \frac{n\gamma}{n + \gamma} \right)^{-1} + p_1^{-1} = q^{-1} \), we also have that $cF \in L^q_{\text{loc}}(\Omega, dx)$. Obviously, $q < n$. As in part (ii) this yields that $cF \in H^{q/(n-q),-1}_{\text{loc}}(\Omega)$ if $q > 1$ and $cF \in H^{r,-1}_{\text{loc}}(\Omega)$ for any $s \in (1, n/(n - 1))$ if $q = 1$. We claim that (8) holds (with $r = r(p_1)$ as in (11) for all $p_1 > \gamma / (\gamma - 1)$, $p_1 \neq n\gamma / (n\gamma - n - \gamma)$).

Indeed, if $q > 1$, then $nq/(n-q) = r$. If $q = 1$, then $p_1 \leq n\gamma / (n\gamma - n - \gamma)$. But since $p_1 \neq n\gamma / (n\gamma - n - \gamma)$, we have $p_1 < n\gamma / (n\gamma - n - \gamma)$, which is equivalent to the inequality $r < n/(n - 1)$.

Thus, since $v \in L^{n\gamma/(n+\gamma)}_{\text{loc}}(\Omega, dx) \subset H^{\gamma,-1}_{\text{loc}}(\Omega) \subset H^{r,-1}_{\text{loc}}(\Omega)$ (because $r < \gamma$), it follows by Lemma 2 (i) that:

\[
(12) \quad \left( p_1 > \frac{\gamma}{\gamma - 1} \text{ and } p_1 \neq \frac{n\gamma}{n\gamma - n - \gamma} \right) \Rightarrow F \in H^{r,-1}_{\text{loc}}(\Omega).
\]

Provided $r < n$ the latter in turn by Lemma 3 (ii) implies that $F \in L^p_{\text{loc}}(\Omega, dx)$. Summarizing we have thus shown:

\[
(13) \quad \left( p_1 > \frac{\gamma}{\gamma - 1} \text{ and } p_1 \neq \frac{n\gamma}{n\gamma - n - \gamma} \right) \Rightarrow F \in L^p_{\text{loc}}(\Omega, dx),
\]

where

\[
p_2 := \frac{nr}{n - r} = \frac{n\gamma p_1}{n\gamma - (\gamma - n)p_1} > \frac{n\gamma}{n\gamma - (\gamma - n)} p_1.
\]

Also notice that $\gamma / (\gamma - 1) < n/(n - 1) < \frac{n\gamma}{\gamma n - n - \gamma}$ so that by (i) we can take a $p_1$ to start with. Then starting with $p_1$ close enough to $n/(n - 1)$, by iterating (13) we always increase $p$ by a certain factor $> 1$. While doing so we can obviously choose the first $p$ so that the iterated $p$'s will be never equal to $n\gamma / (n\gamma - n - \gamma)$ and the corresponding $r$'s will not coincide with $n$. Then after several steps we shall come to the situation where $r > n$, and then we conclude from (12) that $F$ is locally bounded (one cannot keep iterating (13) infinitely having the restriction $r < n$). As in case (b) one can now easily complete the proof.

**Remark 4** (i) For sufficiently regular $F$ with no zeros operators of the type considered above become special cases of operators $L = \sum_{i,j} \partial_i (a_{ij} \partial_j) + q$. Additional information (including further references) about the essential self-adjointness of such operators, however, considered on $L^2(R^n, dx)$ can be found in [8], [15].

(ii) In a forthcoming paper the parabolic case will be studied. It is, however, immediate from Theorem 1 that if $t \mapsto \mu_t$ is differentiable such that $\frac{\partial}{\partial t} \mu_t$ is a
Radon measure, then for fixed \( t \) the densities \( F_t \) of \( \mu_t \) w.r.t. \( dx \) exist and all respective assertions in Theorem 1 hold for \( F_t \).

(iii) Note that the only property of the operator \( L_0 := \Delta \) used above was the one mentioned in Lemma 3 (i), i.e., that \( u \in H^{n,r+2}_{\text{loc}}(\Omega) \) provided \( L_0 u \in H^{n,r}_{\text{loc}}(\Omega) \). It is known (see, e.g., [21, p. 270]) that this holds for arbitrary non-degenerate second order elliptic operators with smooth coefficients. Therefore, Theorem 1 remains valid if we replace \( \Delta \) by any non-degenerate second order elliptic operator \( L_0 \) with smooth coefficients. Moreover, as a thorough inspection of the proof of Theorem 4.2.4 in [22] shows, one can relax the assumption about the smoothness of the coefficients of \( L_0 \) here even more. Note, in particular, that Theorem 1 extends to elliptic second order operators on smooth Riemannian manifolds with non-degenerate smooth second order parts.

(iv) It should be noted that the elliptic equations discussed here cannot be reduced to those considered e.g. in [10], [13], [18], [24]. There are two major differences. The first is that the solutions considered there by definition are supposed to be in \( H^{n+1}_{\text{loc}}(\mathbb{R}^n) \). Secondly, our integrability conditions for \( B \) are w.r.t. a measure \( \mu \) which is a solution of our equation. For this reason, \( B \) need not be locally Lebesgue integrable; e.g. if \( \mu \) is given by the density \( x^2 \exp(-x^2) \) on \( \mathbb{R} \), then it solves our elliptic equation with \( B(x) = \beta(x) = -2x + 2/x \). Of course, Theorem 1 (iii) shows that under sufficient integrability conditions our solutions become solutions also in the sense of the above mentioned references. However, in general we get a wider class of solutions. Note also that in our setting due to the weak assumptions on \( B \) the elliptic regularity does not imply that solutions belong to the second Sobolev class \( H^{2,1}_{\text{loc}} \) (e.g. any \( \mu = \rho dx \) with \( \rho \in H^{1,1}_{\text{loc}} \) satisfies (1) with \( B := \nabla \rho / \rho \), \( c := 0 \), \( v := 0 \)).

The next example shows that assertion (iii) of Theorem 1 fails if \( n + \varepsilon \) is replaced by \( n - \varepsilon \). (Then \( F \) does not even need to be in \( H^{2,1}_{\text{loc}}(\Omega) \).)

**Example 5.** Let \( n > 3 \) and

\[
L^* F(x) = \Delta F(x) + \alpha(x^1|x|^{-2}F)_{x^1}(x) - F(x),
\]

where \( \alpha = n - 3 \). Then the function \( F(x) = (e^r - e^{-r}) r^{-(n-2)} \), \( r = |x| \), is locally \( dx \)-integrable and \( L^* F = 0 \) in the sense of distributions, but \( F \) is not in \( H^{2,1}_{\text{loc}}(\mathbb{R}^n) \). Here \( B(x) = -\alpha x \|x\|^{-2} = \nabla(|x|^{-\alpha})/|x|^{-\alpha} \) and \( |B| \in L^{n-\varepsilon}_{\text{loc}}(\mathbb{R}^n, dx) \) for all \( \varepsilon > 0 \). In a similar way, if there is no """"F"""" in the equation above, then the function \( F(x) = r^{-(n-3)} \) has the same properties.

**Proof.** Observe that \( F_{x^i}, F_{x^i x^j} \) are locally \( dx \)-integrable. Therefore, the equation \( L^* F = 0 \) follows easily from the equation on \((0, \infty)\)

\[
f'' + \frac{(n-1+\alpha)}{r} f' + \frac{n-2}{r^2} f = f = 0,
\]

which is satisfied for the function \( f(r) = (e^r - e^{-r}) r^{-(n-2)} \). It remains to note that \( F, \nabla F \) and \( \Delta F \) are locally \( dx \)-integrable, since \( f(r)r^{n-1}, f'(r)r^{n-1}, \)
Local boundedness of $f''(r)r^{n-1}$ and non-square-integrability of $\nabla F$ at the origin.

**Remark 6.** Applying the regularity result in Theorem 1 (ii) above to the case $c = 0 = \nu$ we get, in particular, the existence of a density in $H_{\text{loc}}^{0,1}(\mathbb{R}^n)$, for $p \in \left[1, \frac{n}{n-2}\right)$, for any invariant measure $\mu$ of a diffusion $\xi_t$ driven by the stochastic differential equation $d\xi_t = dw_t + B(\xi_t)dt$, where the drift $B$ is assumed to be in $L_{\text{loc}}^{1+\delta}(\mathbb{R}^n, \mu)$. This is true for any interpretation of a solution which implies (1) for invariant measures. Thus, we get an improvement of a part of a theorem in [3], [4] (see also [2] for the case of a non-constant second order part). In [3], [4] under the a priori assumption that $\mu$ is a probability measure and assuming that $|B|$ is globally in $L^2(\mathbb{R}^n, \mu)$, it was shown that $\mu$ admits a density in $H^{1,1}(\mathbb{R}^n)$. (We would like to mention that under these stronger conditions the latter result can also be deduced from [6]).

We say that a measurable function $f$ on $\mathbb{R}^n$ is **locally uniformly positive** if $\inf_U f > 0$ for every ball $U \subset \mathbb{R}^n$.

**Theorem 7.** Let $n \geq 2$ and let $\mu$ be a measure on $\mathbb{R}^n$ with density $\rho := \phi^2$, $\phi \in H_{\text{loc}}^{2,1}(\mathbb{R}^n)$, which is locally uniformly positive. Assume that $|\beta| \in L_{\text{loc}}^{\gamma}(\mathbb{R}^n, \mu)$, where $\beta := \nabla \rho / \rho$ and $\gamma > n$. Then the operator

$$L\psi = \Delta \psi + \langle \nabla \psi, \beta \rangle$$

with domain $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint on $L^2(\mathbb{R}^n, \mu)$.

**Proof.** First we note that since $\mu$ satisfies (1) with $B := \beta$, $c := 0$, $\nu := 0$, it follows by Theorem 1 (iii), part (b), that $\rho$ is continuous, hence locally bounded. Assume that there is a function $g \in L^2(\mathbb{R}^n, \mu)$ such that

$$\int (L - 1)\xi(x)g(x)\mu(dx) = 0 \quad \forall \xi \in C_0^\infty(\mathbb{R}^n). \quad (14)$$

Recall that by definition $\beta = 0$ on the set $\{\rho = 0\}$ (which is reasonable since $\nabla \rho = 0$ $dx$-a.e. on $\{\rho = 0\}$). Clearly, $|\beta| \in L_{\text{loc}}^{\gamma}(\mathbb{R}^n, dx)$. Consequently, by Theorem 1 (iii), Part (a), $F \in H_{\text{loc}}^{0,1}(\mathbb{R}^n)$. In particular, $F$ is continuous and locally bounded. Then $g = F/\rho \in H_{\text{loc}}^{0,1}(\mathbb{R}^n) \cap L_{\text{loc}}^{\infty}(\mathbb{R}^n)$, $g|\beta| \in L_{\text{loc}}^{\gamma}(\mathbb{R}^n, dx)$. Therefore, we can integrate by parts in equality (14) which yields for every $\xi \in C_0^\infty(\mathbb{R}^n)$

$$0 = \int \langle \nabla \xi, \nabla g \rangle d\mu - \int \langle \nabla \xi, \beta \rangle g d\mu + \int \langle \nabla \xi, \beta \rangle g d\mu - \int \xi g d\mu \quad (15)$$

$$= -\int \langle \nabla \xi, \nabla g \rangle d\mu - \int \xi g d\mu.$$
Now let \( \psi \in C_0^\infty(\mathbb{R}^n) \) and \( \varphi \in H^{2,1}_{\text{loc}}(\mathbb{R}^n) \). Then by the product rule

\[
(\nabla \varphi, \nabla (\psi \varphi)) = (\nabla (\psi \varphi), \nabla \varphi) - \varphi (\nabla \psi, \nabla \varphi) + g(\nabla \varphi, \nabla \psi) .
\]

Since equality (15) extends to all \( \zeta \) in \( H^{2,1}(\mathbb{R}^n) \) with compact support, we can apply (15) to \( \zeta := \psi \varphi \) and use (16) to obtain

\[
\int (\nabla \varphi, \nabla (\psi \varphi)) \, d\mu + \int \varphi \psi g \, d\mu
\]

\[
= \int (\nabla (\psi \varphi), \nabla \varphi) \, d\mu - \int \varphi (\nabla \psi, \nabla \varphi) \, d\mu
\]

\[
+ \int g(\nabla \varphi, \nabla \psi) \, d\mu + \int \varphi \psi g \, d\mu
\]

\[
= \int \varphi (\nabla \psi, \nabla g) + \int g(\nabla \varphi, \nabla \psi) \, d\mu .
\]

Taking \( \varphi := \psi g \), one gets

\[
\int (\nabla (\psi g), \nabla (\psi g)) \, d\mu + \int (\psi g)^2 \, d\mu
\]

\[
= - \int \psi g (\nabla \psi, \nabla g) \, d\mu + \int g(\nabla (\psi g), \nabla \psi) \, d\mu
\]

\[
= \int g^2(\nabla \psi, \nabla \psi) \, d\mu .
\]

Hence, we get

\[
(17) \quad \int (\psi g)^2 \, d\mu \leq \int g^2|\nabla \psi|^2 \, d\mu .
\]

Taking a sequence \( \psi_k \in C_0^\infty(\mathbb{R}^n), k \in \mathbb{N} \), such that \( 0 \leq \psi_k \leq 1 \), \( \psi_k(x) = 1 \) if \( |x| \leq k \), \( \psi_k(x) = 0 \) if \( |x| \geq k + 1 \), and \( \sup_k |\nabla \psi_k| = M < \infty \), we get by Lebesgue's dominated convergence theorem that the left hand side of (17) tends to \( \|g\|_2^2 \), while the right hand side tends to zero. Thus, \( g = 0 \). By a standard result (see, e.g., [12]) this implies the essential self-adjointness of \( (L, C_0^\infty(\mathbb{R}^n)) \) on \( L^2(\mathbb{R}^n, \mu) \).

**Corollary 8.** The assertion of the previous theorem holds true if \( \mu \) is a measure on \( \mathbb{R}^n \) with density \( \rho := \varphi^2, \varphi \in H^{2,1}_{\text{loc}}(\mathbb{R}^n) \), and \( |\beta| \in L^\gamma_{\text{loc}}(\mathbb{R}^n, dx) \), where \( \beta := \nabla \rho/\rho \) and \( \gamma > n \).

**Proof.** Note that \( \rho \) admits a continuous strictly positive modification. Indeed, if \( f_n := \log(\rho + \frac{1}{n}) \), \( n \in \mathbb{N} \), then \( f_n \xrightarrow{n \to \infty} \log \rho \) in \( L^1_{\text{loc}}(\mathbb{R}^n, dx) \), which easily follows from the fact that \( \log \rho \in L^1_{\text{loc}}(\mathbb{R}^n, dx) \). The latter in turn follows from [3, Lemma 6.4]. Consequently by the Poincaré inequality, the sequence \( (f_n)_{n \in \mathbb{N}} \) is bounded in \( H^{\gamma,1}(U) \) for every open ball \( U \subset \mathbb{R}^n \). By the compactness of the embedding \( H^{\gamma,1}(U) \hookrightarrow C(U) \), a subsequence of the sequence of the continuous modifications of \( (f_n)_{n \in \mathbb{N}} \) converges locally uniformly to \( \log \rho \). Whence \( \rho \) is continuous and strictly positive. In particular, \( |\nabla \rho/\rho| \in L^\gamma_{\text{loc}}(\mathbb{R}^n, \mu) \).
Remark 9. If \( \mu = \rho \, dx \) with \( \rho = \varphi^2 \) and \( \varphi \in H^{2,1}_{\text{loc}}(\mathbb{R}^n) \), the so-called Markov uniqueness (i.e., the uniqueness of a Markovian semigroup on \( L^2(\mathbb{R}^n, \mu) \) with generator given by \( Lf = \Delta f + (\nabla f, \beta) \) on \( C^\infty_0(\mathbb{R}^n) \)) always holds with \( \beta := \nabla \rho / \rho \) (see [16], [17]). However, in general Markov uniqueness is weaker than the essential self-adjointness of \( (L, C^\infty_0(\mathbb{R}^n)) \) on \( L^2(\mathbb{R}^n, \mu) \). (see [7]). Optimal (local or global) conditions for the essential self-adjointness remain unknown except for the one-dimensional case investigated in [25] and [7]. In fact, recently in [7] a complete characterization of the essential self-adjointness for Dirichlet operators has been given in the case \( n = 1 \).

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