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Elliptic Regularity and Essential Self-adjointness of Dirichlet Operators on $\mathbb{R}^n$

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One of the classical problems in mathematical physics is the problem of essential self-adjointness for *Dirichlet operators*

$$L := \Delta + \beta \cdot \nabla,$$

with domain $C_0^\infty(\mathbb{R}^n)$ (:= all infinitely differentiable functions on $\mathbb{R}^n$ with compact support) on $L^2(\mathbb{R}^n, \mu)$, where μ is a measure on $\mathbb{R}^n$ with density $\rho := \varphi^2$, with $\varphi \in H_{\text{loc}}^{2,1}(\mathbb{R}^n)$ and $\beta := \nabla\rho/\rho$. (By definition $\beta(x) = 0$ if $\rho(x) = 0$). The results obtained in [1], [8], [9], [11], [14], [25] have been important steps in the investigation of this problem. One motivation to study this problem is that the operator $-L$ is unitary equivalent to the Schrödinger operator $H := -\Delta + V$, $V := \Delta\varphi/\varphi$, considered on $L^2(\mathbb{R}^n, dx)$ (see, e.g., [1], [5]) where dx denotes Lebesgue measure on $\mathbb{R}^n$. The corresponding isomorphism $L^2(\mathbb{R}^n, \mu) \to L^2(\mathbb{R}^n, dx)$ is given by $f \mapsto \varphi \cdot f$. Conversely, if $H = -\Delta + V$ is a Schrödinger operator on $L^2(\mathbb{R}^n, dx)$ with lower bounded spectrum $\sigma(H)$ whose minimum is an eigenvalue E, then the isomorphism above holds for the potential $V - E$ (and $\varphi :=$ the ground state). Since this unitary equivalence only holds for sufficiently regular φ, Dirichlet operators are also sometimes called *generalized Schrödinger* operators. We emphasize that under the above isomorphism in general domains change drastically. Hence known results on the essential self-adjointness of H with domain $C_0^\infty(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n, dx)$ do not apply. On the contrary in many cases the essential self-adjointness of Dirichlet operators implies this property for Schrödinger operators (see e.g. [16, pp. 217, 218]).

There are basically two different types of sufficient conditions known for the essential self-adjointness of Dirichlet operators: global and local. A typical global condition obtained in [14] is: $|\beta| \in L^4(\mathbb{R}^n, \mu)$ (provided $\rho > 0$ a.e.). The best local condition obtained so far has been found in [25] where ρ has been required to be locally Lipschitzian and strictly positive if $n \geq 2$ (and

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with even weaker conditions if $n = 1$, cf. Remark 2 below). In particular, this means that β is locally bounded. One of our main results in this paper (cf. Theorem 7 below) says that L is essentially self-adjoint provided that ρ is merely locally bounded and *locally uniformly positive* (cf. below) and $|\beta| \in L^{\gamma}_{\rm loc}(\mathbb{R}^n, \mu)$ for some $\gamma > n$ (which as we shall show below, is equivalent to $|\beta| \in L^{\gamma}_{\rm loc}(\mathbb{R}^n, dx)$; cf. Corollary 8). The proof of Theorem 7 is based on an elliptic regularity result (which is the first main result of this paper) giving $H^{\gamma,1}_{\rm loc}$-regularity of distributional solutions of the elliptic equation $L^*F = 0$, where $Lf := \Delta f + \langle B, \nabla f\rangle + cf$. This result is formulated as Theorem 1 below. As a consequence one gets $H^{\gamma,1}_{\rm loc}$-regularity of invariant measures for diffusion processes with drifts satisfying certain mild local integrability conditions (which extends a result from [3], [4]). Finally, we note that for the above mentioned special applications to Schrödinger operators $H = -\Delta + V$, of course, one still needs corresponding information about the ground state φ to ensure that $|\beta| = 2|\nabla\varphi/\varphi| \in L^{\gamma}_{\rm loc}(\mathbb{R}^n; \mu)$.

Throughout this paper, Ω is a (fixed) open subset of $\mathbb{R}^n$, and for $r \in (-\infty, \infty)$ and $p \geq 1$, $H^{p,r}_{\rm loc}(\Omega)$ denotes the class of (generalized) functions u on Ω, such that $(1-\Delta)^{r/2}\psi u \in L^p(\mathbb{R}^n, dx)$ for every $\psi \in C_0^\infty(\Omega)$. These spaces coincide with the usual Sobolev spaces for integer $r \geq 1$. All properties of these spaces which are needed below can be found, for instance, in [23]. If ν is a signed measure, then by definition $\int f\,d\nu = \int f\chi\,d|\nu|$, where $\chi := d\nu/d|\nu|$, and $L^p(\Omega, \nu) := L^p(\Omega, |\nu|)$. If, in addition, $\nu \ll dx$, then we write ν instead of $\frac{d\nu}{dx}$. Furthermore, $\langle\ ,\ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$ and $|\cdot|$ the corresponding norm.

THEOREM 1. *Let $n \geq 2$ and let μ, ν be (signed) Radon measures on Ω. Let $B = (B^i) : \Omega \to \mathbb{R}^n$, $c : \Omega \to \mathbb{R}$ be maps such that $|B|,\ c \in L^1_{\rm loc}(\Omega, \mu)$. Assume that*

$$\int L\varphi(x)\,\mu(dx) = \int \varphi(x)\nu(dx) \quad \forall\varphi \in C_0^\infty(\Omega), \tag{1}$$

where

$$L\varphi(x) := \Delta\varphi(x) + \langle B(x), \nabla\varphi(x)\rangle + c(x)\varphi(x). \tag{2}$$

Then:

(i) *$\mu \in H^{p,1-n(p-1)/p-\varepsilon}_{\rm loc}(\Omega)$ for any $p \geq 1$ and $\varepsilon > 0$. Here $1 - n(p-1)/p > 0$ if $p \in [1, \frac{n}{n-1})$ and, in particular, μ admits a density $F \in L^p_{\rm loc}(\Omega, dx)$ for any $p \in [1, \frac{n}{n-1})$.*

(ii) *If $|B| \in L^{\gamma}_{\rm loc}(\Omega, \mu)$, $c \in L^{\gamma/2}_{\rm loc}(\Omega, \mu)$ and $\nu \in L^{n/(n-\gamma+2)}_{\rm loc}(\Omega, dx)$ where $n \geq \gamma > 1$, then $F := \frac{d\mu}{dx} \in H^{p,1}_{\rm loc}(\Omega)$ for any $p \in [1, n/(n-\gamma+1))$. In particular, $F \in L^p_{\rm loc}(\Omega, dx)$ for any $p \in [1, n/(n-\gamma))$, where (here and below) $\frac{n}{n-\gamma} := \infty$, if $\gamma = n$.*

(iii) *If $\gamma > n$ and either*

(a) $|B| \in L^{\gamma}_{\mathrm{loc}}(\Omega, dx)$ *and* $c, \nu \in L^{\gamma n/(n+\gamma)}_{\mathrm{loc}}(\Omega, dx)$,
or
(b) $|B| \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mu)$, $c \in L^{\gamma n/(n+\gamma)}_{\mathrm{loc}}(\Omega, \mu)$, *and* $\nu \in L^{\gamma n/(n+\gamma)}_{\mathrm{loc}}(\Omega, dx)$,
then μ *admits a density* $F \in H^{\gamma,1}_{\mathrm{loc}}(\Omega)$, *and, in particular,* $F \in C^{1-n/\gamma}_{\mathrm{loc}}(\Omega)$.

REMARK 2 (i) There is a similar regularity result for $n = 1$ (whose proof is easier and, in fact, quite elementary). Therefore, Theorem 7 and Corollary 8 below also hold in this case. However, our conditions there for $n = 1$ are then obviously equivalent to: $\varphi(= \sqrt{\rho}) \in H^{2,1}_{\mathrm{loc}}(\mathbb{R})$ and (the continuous version of) ρ is strictly positive. But under these conditions in the special case $n = 1$ both results are already contained in [25]. So, we state and prove our results only for $n \geq 2$.

(ii) Note that since $c \in L^1_{\mathrm{loc}}(\Omega, \mu)$ and ν is a Radon measure, the assumptions on c, ν in Theorem 1 (ii) are automatically fulfilled, if $\gamma \leq 2$, provided $\nu \ll dx$.

To prove Theorem 1 we use the following lemma.

LEMMA 3. (i) *For any* $r \in (-\infty, \infty)$ *and* $p > 1$, *if* $\Delta u \in H^{p,r}_{\mathrm{loc}}(\Omega)$, *then* $u \in H^{p,r+2}_{\mathrm{loc}}(\Omega)$; *also if* $u \in H^{p,r}_{\mathrm{loc}}(\Omega)$, *then* $u_{x^i} \in H^{p,r-1}_{\mathrm{loc}}(\Omega)$, $1 \leq i \leq n$.

(ii) *We have* $H^{p,1}_{\mathrm{loc}}(\Omega) \subset L^{np/(n-p)}_{\mathrm{loc}}(\Omega, dx)$ *and* $L^{p}_{\mathrm{loc}}(\Omega, dx) \subset H^{np/(n-p),-1}_{\mathrm{loc}}(\Omega)$ *whenever* $1 < p < n$, *and* $H^{p,1}_{\mathrm{loc}}(\Omega) \subset C^{1-n/p}_{\mathrm{loc}}(\Omega)$ *if* $p > n$, *so that in the latter case elements of* $H^{p,1}_{\mathrm{loc}}(\Omega)$ *are locally bounded. Also for* $q > p > 1$, $L^{p}_{\mathrm{loc}}(\Omega, dx) \subset H^{q,n/q-n/p}_{\mathrm{loc}}(\Omega)$.

(iii) *If* μ *is a Radon measure on* Ω, *then* $\mu \in H^{p,-m}_{\mathrm{loc}}(\Omega)$ *whenever* $p > 1$ *and* $m > n(1 - 1/p)$.

PROOF. Assertion (i) is well-known. Specifically, its first statement is a well-known elliptic regularity result and the second statement follows from the boundedness of Riesz's transforms. Assertion (ii) is just the Sobolev imbedding theorems (see [23]). Assertion (iii) follows from these imbedding theorems since, for regular sub-domains U of Ω, $H^{q,m}(U) \subset C(\overline{U})$ if $qm > n$ whence by duality the space $H^{q/(q-1),-m}(U) = [H^{q,m}(U)]^*$ contains all finite measures on U. □

PROOF OF THEOREM 1. (i): We have that in the sense of distributions

$$\Delta\mu = (B^i\mu)_{x^i} - c\mu + \nu \tag{3}$$

on Ω. Here by Lemma 3 (iii), the right-hand side belongs to $H^{p,-m-1}_{\mathrm{loc}}(\Omega)$ if $m > n(1 - 1/p)$. By Lemma 3 (i) we conclude $\mu \in H^{p,-m+1}_{\mathrm{loc}}(\Omega)$, which leads to the result after substituting $m = n(1 - 1/p) + \varepsilon$.

Before we prove (ii), (iii) we need some preparations. Fix a $p_1 > 1$ and assume that $F := \frac{d\mu}{dx} \in L^{p_1}_{\text{loc}}(\Omega, dx)$. (Such p_1 exists by (i).) Define

$$r := r(p_1) := \frac{\gamma p_1}{\gamma - 1 + p_1} \tag{4}$$

and observe that owing to the inequalities $1 < \gamma$ and $p_1 > 1$, we have $1 < r < \gamma$. Next, starting with the formula

$$|BF|^r = (|B||F|^{1/\gamma})^r |F|^{r-r/\gamma}$$

and using Hölder's inequality (with $s = \frac{\gamma}{r} (> 1)$ and $t := \frac{s}{s-1} = \frac{\gamma}{\gamma - r}$) and the assumptions $|B||F|^{1/\gamma} \in L^{\gamma}_{\text{loc}}(\Omega, dx)$ and $F \in L^{p_1}_{\text{loc}}(\Omega, dx)$, we get that $B^i F \in L^r_{\text{loc}}(\Omega, dx)$. By Lemma 3 (i)

$$B^i F \in H^{r,0}_{\text{loc}}(\Omega), \quad (B^i F)_{x^i} \in H^{r,-1}_{\text{loc}}(\Omega). \tag{5}$$

(ii): Set

$$q := q(p_1) := \frac{\gamma p_1}{\gamma - 2 + 2p_1} \vee 1, \tag{6}$$

and note that $q > 1 \Leftrightarrow \gamma > 2 \Leftrightarrow q < \frac{\gamma}{2}$, in particular, $q < \gamma$ in any case. Hence repeating the above argument with c, $\gamma/2$, q replacing $|B|$, γ, r, respectively we obtain that

$$cF \in L^q_{\text{loc}}(\Omega, dx) \tag{7}$$

Fix $p_1 > 1$ such that $F := \frac{d\mu}{dx} \in L^{p_1}_{\text{loc}}(\Omega, dx)$ and let r, q be as in (4), (6), correspondingly. Since $\gamma \leq n$ we have that $q < n$, which by (7) and Lemma 3 (ii) resp. (iii) yields $cF \in H^{nq/(n-q),-1}_{\text{loc}}(\Omega)$ if $q > 1$ resp. $cF \in H^{s,-1}_{\text{loc}}(\Omega)$ for any $s \in (1, n/(n-1))$ if $q = 1$.

It turns out that if $p_1 < n/(n-\gamma)$, then

$$cF \in H^{r,-1}_{\text{loc}}(\Omega). \tag{8}$$

Indeed, if $q > 1$, then (8) follows from the fact that if $p_1 \in (1, n/(n-\gamma))$ the inequality $r \leq nq/(n-q)$ holds. If $q = 1$, then $\gamma \leq 2$ and (8) follows from the fact that $r < n/(n-\gamma+1) \leq n/(n-1)$ for $p_1 < n/(n-\gamma)$.

Finally by Lemma 3 (ii) we have $\nu \in H^{n/(n-\gamma+1),-1}_{\text{loc}}(\Omega)$ if $\gamma > 2$ and $\nu \in H^{s,-1}_{\text{loc}}(\Omega)$ for any $s \in (1, n/(n-1))$ if $\gamma \leq 2$. In the same way as above, $\nu \in H^{r,-1}_{\text{loc}}(\Omega)$ whenever $1 < p_1 < n/(n-\gamma)$. This along with (5) and (8) shows that the right-hand side of (3) is now in $H^{r,-1}_{\text{loc}}(\Omega)$. By Lemma 3 (i) we have

$$\mu \in H^{r,1}_{\text{loc}}(\Omega) \tag{9}$$

and by Lemma 3 (ii) $F \in L^{p_2}_{\mathrm{loc}}(\Omega, dx)$, where

$$p_2 := \frac{nr}{n-r} = \frac{n\gamma p_1}{n\gamma - n + (n-\gamma)p_1} =: f(p_1).$$

Thus we get

$$p_1 \in \left(1, \frac{n}{n-\gamma}\right) \text{ and } F \in L^{p_1}_{\mathrm{loc}}(\Omega, dx) \Longrightarrow F \in L^{f(p_1)}_{\mathrm{loc}}(\Omega, dx).$$

One can easily check that $p_2 = f(p_1) > p_1$ if $p_1 < n/(n-\gamma)$, and that the only positive solution of the equation $q = f(q)$ is $q = n/(n-\gamma)$. Therefore, by taking p_1 from $(1, n/(n-1))$, which is possible by (i), and by defining $p_{k+1} = f(p_k)$ we get an increasing sequence of $p_k \uparrow n/(n-\gamma)$, which implies that $F \in L^p_{\mathrm{loc}}(\Omega, dx)$ for any $p < n/(n-\gamma)$.

But as $p_k \nearrow n/(n-\gamma)$, $r(p_k)$ (defined according to (4)) increasingly converges to

$$\frac{\gamma n/(n-\gamma)}{\gamma - 1 + n/(n-\gamma)} = \frac{n}{n-\gamma+1}\ .$$

By (9) this proves (ii).

(iii): First we consider case (b) in which $|B| \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mu)$, $c \in L^{n\gamma/(n+\gamma)}_{\mathrm{loc}}(\Omega, \mu)$, $\nu \in L^{n\gamma/(n+\gamma)}_{\mathrm{loc}}(\Omega, dx)$. By the last assertion in (ii) we have $F \in L^{p_1}_{\mathrm{loc}}(\Omega, dx)$ for any (finite) $p_1 > 1$. Let $r := r(p_1)$ be defined as in (4). Then $1 < r < \gamma$ and (5) holds. Set

$$q := q(p_1) := \frac{\frac{n\gamma}{n+\gamma} p_1}{\frac{n\gamma}{n+\gamma} - 1 + p_1}. \tag{10}$$

$2 \le n < \gamma$, implies $\frac{n\gamma}{n+\gamma} > 1$. Therefore, (since $p_1 > 1$) it follows that $1 < q < \frac{n\gamma}{n+\gamma}$. Hence repeating the arguments that led to (5) with c, $\frac{n\gamma}{n+\gamma}$, q replacing $|B|$, γ, r respectively we obtain $cF \in L^q_{\mathrm{loc}}(\Omega, dx)$, thus $cF \in H^{nq/(n-q),-1}(\Omega)$ by Lemma 3 (ii). Observe that when $p_1 \to \infty$, we have $r \uparrow \gamma$, $q \uparrow n\gamma/(n+\gamma)$, and $nq/(n-q) \uparrow \gamma$. Therefore, combining this with our assumption that $\nu \in L^{n\gamma/(n+\gamma)}_{\mathrm{loc}}(\Omega, dx)$ which by Lemma 3 (ii) is contained in $H^{\gamma,-1}_{\mathrm{loc}}(\Omega)$, by taking p_1 large enough, we see that the right-hand side in (3) is in $H^{\gamma-\varepsilon,-1}_{\mathrm{loc}}(\Omega)$ for any $\varepsilon \in (0, \gamma-1)$. By Lemma 3 (ii) we conclude $F \in H^{\gamma-\varepsilon,1}_{\mathrm{loc}}(\Omega)$ and since $\gamma > n$, the function F is locally bounded. Now we see that above we can take $p_1 = \infty$ and therefore the right-hand side of (3) is in $H^{\gamma,-1}_{\mathrm{loc}}(\Omega)$, which by Lemma 3 (i) gives us the desired result.

In the remaining case (a) we take $p_1 > \gamma/(\gamma-1)$ and assume that $F \in L^{p_1}_{\mathrm{loc}}(\Omega, dx)$. Then instead of (4) and (10) we define

$$r := r(p_1) := \frac{\gamma p_1}{\gamma + p_1}, \quad q := q(p_1) := \frac{\frac{n\gamma}{n+\gamma} p_1}{\frac{n\gamma}{n+\gamma} + p_1} \vee 1 \tag{11}$$

and observe that owing to $p_1 > \gamma/(\gamma-1)$ we have $r > 1$, which (because $p_1^{-1}+\gamma^{-1}=r^{-1}$) allows us to apply Hölder's inequality starting with $|BF|^r = |B|^r|F|^r$ to conclude that (5) holds. Since $c \in L^1_{\mathrm{loc}}(\Omega,\mu)$, resp. $\frac{n\gamma}{n+\gamma} > 1$ and $\left(\frac{n\gamma}{n+\gamma}\right)^{-1} + p_1^{-1} = q^{-1}$, we also have that $cF \in L^q_{\mathrm{loc}}(\Omega, dx)$. Obviously, $q < n$. As in part (ii) this yields that $cF \in H^{nq/(n-q),-1}_{\mathrm{loc}}(\Omega)$ if $q > 1$ and $cF \in H^{s,-1}_{\mathrm{loc}}(\Omega)$ for any $s \in (1, n/(n-1))$ if $q = 1$. We claim that (8) holds (with $r = r(p_1)$ as in (11) for all $p_1 > \gamma/(\gamma-1)$, $p_1 \neq n\gamma/(n\gamma - n - \gamma)$.

Indeed, if $q > 1$, then $nq/(n-q) = r$. If $q = 1$, then $p_1 \le n\gamma/(n\gamma - n - \gamma)$. But since $p_1 \neq n\gamma/(n\gamma-n-\gamma)$, we have $p_1 < n\gamma/(n\gamma - n - \gamma)$, which is equivalent to the inequality $r < n/(n-1)$.

Thus, since $\nu \in L^{n\gamma/(n+\gamma)}_{\mathrm{loc}}(\Omega, dx) \subset H^{\gamma,-1}_{\mathrm{loc}}(\Omega) \subset H^{r,-1}_{\mathrm{loc}}(\Omega)$ (because $r < \gamma$), it follows by Lemma 2 (i) that:

$$
(12)\qquad \begin{pmatrix} p_1 > \dfrac{\gamma}{\gamma-1} \text{ and } p_1 \neq \dfrac{n\gamma}{n\gamma - n - \gamma} \\ \text{and } F \in L^{p_1}_{\mathrm{loc}}(\Omega, dx) \end{pmatrix} \Longrightarrow F \in H^{r,1}_{\mathrm{loc}}(\Omega)\ .
$$

Provided $r < n$ the latter in turn by Lemma 3 (ii) implies that $F \in L^{p_2}_{\mathrm{loc}}(\Omega, dx)$. Summarizing we have thus shown:

$$
(13)\qquad \begin{pmatrix} p_1 > \dfrac{\gamma}{\gamma-1} & \text{and } p_1 \neq \dfrac{n\gamma}{n\gamma-n-\gamma} \\ \text{and } r := \dfrac{\gamma p_1}{\gamma + p_1} < n & \text{and } F \in L^{p_1}_{\mathrm{loc}}(\Omega, dx) \end{pmatrix} \Longrightarrow F \in L^{p_2}_{\mathrm{loc}}(\Omega, dx),
$$

where

$$
p_2 := \frac{nr}{n-r} = \frac{n\gamma p_1}{n\gamma - (\gamma - n)p_1} > \frac{n\gamma}{n\gamma - (\gamma - n)} p_1.
$$

Also notice that $\gamma/(\gamma-1) < n/(n-1) < \frac{n\gamma}{\gamma n - n - \gamma}$ so that by (i) we can take a p_1 to start with. Then starting with p_1 close enough to $n/(n-1)$, by iterating (13) we always increase p by a certain factor > 1. While doing so we can obviously choose the first p so that the iterated p's will be never equal to $n\gamma/(n\gamma - n - \gamma)$ and the corresponding r's will not coincide with n. Then after several steps we shall come to the situation where $r > n$, and then we conclude from (12) that F is locally bounded (one cannot keep iterating (13) infinitely having the restriction $r < n$). As in case (b) one can now easily complete the proof. □

REMARK 4 (i) For sufficiently regular F with no zeros operators of the type considered above become special cases of operators $L = \sum_{i,j} \partial_i(a_{ij}\partial_j) + q$. Additional information (including further references) about the essential self-adjointness of such operators, however, considered on $L^2(R^n, dx)$ can be found in [8], [15].

(ii) In a forthcoming paper the parabolic case will be studied. It is, however, immediate from Theorem 1 that if $t \mapsto \mu_t$ is differentiable such that $\frac{\partial}{\partial t}\mu_t$ is a

Radon measure, then for fixed t the densities F_t of μ_t w.r.t. dx exist and all respective assertions in Theorem 1 hold for F_t.

(iii) Note that the only property of the operator $L_0 := \Delta$ used above was the one mentioned in Lemma 3 (i), i.e., that $u \in H_{\text{loc}}^{p,r+2}(\Omega)$ provided $L_0 u \in H_{\text{loc}}^{p,r}(\Omega)$. It is known (see, e.g., [21, p. 270]) that this holds for arbitrary non-degenerate second order elliptic operators with smooth coefficients. Therefore, Theorem 1 remains valid if we replace Δ by any non-degenerate second order elliptic operator L_0 with smooth coefficients. Moreover, as a thorough inspection of the proof of Theorem 4.2.4 in [22] shows, one can relax the assumption about the smoothness of the coefficients of L_0 here even more. Note, in particular, that Theorem 1 extends to elliptic second order operators on smooth Riemannian manifolds with non-degenerate smooth second order parts.

(iv) It should be noted that the elliptic equations discussed here cannot be reduced to those considered e.g. in [10], [13], [18], [24]. There are two major differences. The first is that the solutions considered there by definition are supposed to be in $H_{\text{loc}}^{\gamma,1}(\mathbb{R}^n)$. Secondly, our integrability conditions for B are w.r.t. a measure μ which is a solution of our equation. For this reason, B need not be locally Lebesgue integrable; e.g. if μ is given by the density $x^2 \exp(-x^2)$ on $\mathbb{R}^1$, then it solves our elliptic equation with $B(x) = \beta(x) = -2x + 2/x$. Of course, Theorem 1 (iii) shows that under sufficient integrability conditions our solutions become solutions also in the sense of the above mentioned references. However, in general we get a wider class of solutions. Note also that in our setting due to the weak assumptions on B the elliptic regularity does not imply that solutions belong to the second Sobolev class $H_{\text{loc}}^{\gamma,2}$ (e.g. any $\mu = \rho dx$ with $\rho \in H_{\text{loc}}^{1,1}$ satisfies (1) with $B := \nabla\rho/\rho$, $c := 0$, $\nu := 0$).

The next example shows that assertion (iii) of Theorem 1 fails if $n + \varepsilon$ is replaced by $n - \varepsilon$. (Then F does not even need to be in $H_{\text{loc}}^{2,1}(\Omega)$.)

EXAMPLE 5. Let $n > 3$ and

$$L^*F(x) = \Delta F(x) + \alpha(x^i|x|^{-2}F)_{x^i}(x) - F(x),$$

where $\alpha = n - 3$. Then the function $F(x) = (e^r - e^{-r})r^{-(n-2)}$, $r = |x|$, is locally dx-integrable and $L^*F = 0$ in the sense of distributions, but F is not in $H_{\text{loc}}^{2,1}(\mathbb{R}^n)$. Here $B(x) = -\alpha x\|x\|^{-2} = \nabla(|x|^{-\alpha})/|x|^{-\alpha}$ and $|B| \in L_{\text{loc}}^{n-\varepsilon}(\mathbb{R}^n, dx)$ for all $\varepsilon > 0$. In a similar way, if there is no "$-F$" in the equation above, then the function $F(x) = r^{-(n-3)}$ has the same properties.

PROOF. Observe that $F_{x^i}, F_{x^i x^j}$ are locally dx-integrable. Therefore, the equation $L^*F = 0$ follows easily from the equation on $(0, \infty)$

$$f'' + \frac{(n-1+\alpha)}{r}f' + \alpha\frac{n-2}{r^2}f - f = 0,$$

which is satisfied for the function $f(r) = (e^r - e^{-r})r^{-(n-2)}$. It remains to note that F, ∇F and ΔF are locally dx-integrable, since $f(r)r^{n-1}$, $f'(r)r^{n-1}$,

$f''(r)r^{n-1}$ are locally bounded, but ∇F is not dx-square-integrable at the origin. (If $n \geq 6$, then also F is not dx-square-integrable at the origin). In the case without "$-F$" in the equation similar (but even simpler) arguments can be used to show that $F(x) = r^{-(n-3)}$ has the same properties. □

REMARK 6. Applying the regularity result in Theorem 1 (ii) above to the case $c = 0 = \nu$ we get, in particular, the existence of a density in $H^{p,1}_{\mathrm{loc}}(\mathbb{R}^n)$, for $p \in [1, \frac{n}{n-\varepsilon})$, for any invariant measure μ of a diffusion ξ_t driven by the stochastic differential equation $d\xi_t = dw_t + B(\xi_t)dt$, where the drift B is assumed to be in $L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n, \mu)$. This is true for any interpretation of a solution which implies (1) for invariant measures. Thus, we get an improvement of a part of a theorem in [3], [4] (see also [2] for the case of a non-constant second order part). In [3], [4] under the a priori assumption that μ is a *probability* measure and assuming that $|B|$ is *globally* in $L^2(\mathbb{R}^n, \mu)$, it was shown that μ admits a density in $H^{1,1}(\mathbb{R}^n)$. (We would like to mention that under these stronger conditions the latter result can also be deduced from [6]).

We say that a measurable function f on $\mathbb{R}^n$ is *locally uniformly positive* if $\operatorname{essinf}_U f > 0$ for every ball $U \subset \mathbb{R}^n$.

THEOREM 7. *Let $n \geq 2$ and let μ be a measure on $\mathbb{R}^n$ with density $\rho := \varphi^2$, $\varphi \in H^{2,1}_{\mathrm{loc}}(\mathbb{R}^n)$, which is locally uniformly positive. Assume that $|\beta| \in L^{\gamma}_{\mathrm{loc}}(\mathbb{R}^n, \mu)$, where $\beta := \nabla\rho/\rho$ and $\gamma > n$. Then the operator*

$$L\psi = \Delta\psi + \langle \nabla\psi, \beta\rangle$$

with domain $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint on $L^2(\mathbb{R}^n, \mu)$.

PROOF. First we note that since μ satisfies (1) with $B := \beta$, $c :\equiv 0$, $\nu :\equiv 0$, it follows by Theorem 1 (iii), part (b), that ρ is continuous, hence locally bounded. Assume that there is a function $g \in L^2(\mathbb{R}^n, \mu)$ such that

$$\int (L-1)\zeta(x)g(x)\,\mu(dx) = 0 \qquad \forall \zeta \in C_0^\infty(\mathbb{R}^n). \tag{14}$$

Recall that by definition $\beta = 0$ on the set $\{\rho = 0\}$ (which is reasonable since $\nabla\rho = 0$ dx-a.e. on $\{\rho = 0\}$). Clearly, $|\beta| \in L^{\gamma}_{\mathrm{loc}}(\mathbb{R}^n, dx)$. Consequently, by Theorem 1 (iii), Part (a), $F \in H^{\gamma,1}_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, F is continuous and locally bounded. Then $g = F/\rho \in H^{\gamma,1}_{\mathrm{loc}}(\mathbb{R}^n) \bigcap L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, $g|\beta| \in L^{\gamma}_{\mathrm{loc}}(\mathbb{R}^n, dx)$. Therefore, we can integrate by parts in equality (14) which yields for every $\zeta \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} 0 &= -\int \langle\nabla\zeta, \nabla g\rangle\, d\mu - \int \langle\nabla\zeta, \beta\rangle g\, d\mu + \int \langle\nabla\zeta, \beta\rangle g\, d\mu - \int \zeta g\, d\mu \\ &= -\int \langle\nabla\zeta, \nabla g\rangle\, d\mu - \int \zeta g\, d\mu. \end{aligned} \tag{15}$$

Now let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \in H_{\mathrm{loc}}^{2,1}(\mathbb{R}^n)$. Then by the product rule

$$\langle \nabla\varphi, \nabla(\psi g)\rangle = \langle \nabla(\psi\varphi), \nabla g\rangle - \varphi\langle \nabla\psi, \nabla g\rangle + g\langle \nabla\varphi, \nabla\psi\rangle \ . \tag{16}$$

Since equality (15) extends to all ζ in $H^{2,1}(\mathbb{R}^n)$ with compact support, we can apply (15) to $\zeta := \psi\varphi$ and use (16) to obtain

$$\begin{aligned}
&\int \langle \nabla\varphi, \nabla(\psi g)\rangle \, d\mu + \int \varphi\psi g \, d\mu \\
&\underset{(16)}{=} \int \langle \nabla(\psi\varphi), \nabla g\rangle \, d\mu - \int \varphi\langle \nabla\psi, \nabla g\rangle \, d\mu \\
&\qquad\qquad + \int g\langle \nabla\varphi, \nabla\psi\rangle \, d\mu + \int \varphi\psi g \, d\mu \\
&\underset{(15)}{=} -\int \varphi\langle \nabla\psi, \nabla g\rangle + \int g\langle \nabla\varphi, \nabla\psi\rangle \, d\mu.
\end{aligned}$$

Taking $\varphi := \psi g$, one gets

$$\begin{aligned}
&\int \langle \nabla(\psi g), \nabla(\psi g)\rangle \, d\mu + \int (\psi g)^2 \, d\mu \\
&\qquad = -\int \psi g\langle \nabla\psi, \nabla g\rangle \, d\mu + \int g\langle \nabla(\psi g), \nabla\psi\rangle \, d\mu \\
&\qquad = \int g^2\langle \nabla\psi, \nabla\psi\rangle \, d\mu.
\end{aligned}$$

Hence, we get

$$\int (\psi g)^2 \, d\mu \leq \int g^2 |\nabla\psi|^2 \, d\mu. \tag{17}$$

Taking a sequence $\psi_k \in C_0^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$, such that $0 \leq \psi_k \leq 1$, $\psi_k(x) = 1$ if $|x| \leq k$, $\psi_k(x) = 0$ if $|x| \geq k+1$, and $\sup_k |\nabla\psi_k| = M < \infty$, we get by Lebesgue's dominated convergence theorem that the left hand side of (17) tends to $\|g\|_2^2$, while the right hand side tends to zero. Thus, $g = 0$. By a standard result (see, e.g., [12]) this implies the essential self-adjointness of $(L, C_0^\infty(\mathbb{R}^n))$ on $L^2(\mathbb{R}^n, \mu)$. □

COROLLARY 8. *The assertion of the previous theorem holds true if μ is a measure on $\mathbb{R}^n$ with density $\rho := \varphi^2$, $\varphi \in H_{\mathrm{loc}}^{2,1}(\mathbb{R}^n)$, and $|\beta| \in L_{\mathrm{loc}}^\gamma(\mathbb{R}^n, dx)$, where $\beta := \nabla\rho/\rho$ and $\gamma > n$.*

PROOF. Note that ρ admits a continuous strictly positive modification. Indeed, if $f_n := \log(\rho + \frac{1}{n})$, $n \in \mathbb{N}$, then $f_n \underset{n\to\infty}{\longrightarrow} \log\rho$ in $L_{\mathrm{loc}}^1(\mathbb{R}^n, dx)$, which easily follows from the fact that $\log\rho \in L_{\mathrm{loc}}^1(\mathbb{R}^n, dx)$. The latter in turn follows from [3, Lemma 6.4]. Consequently by the Poincaré inequality, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $H^{\gamma,1}(U)$ for every open ball $U \subset \mathbb{R}^n$. By the compactness of the embedding $H^{\gamma,1}(U) \to C(U)$, a subsequence of the sequence of the continuous modifications of $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to $\log\rho$. Whence ρ is continuous and strictly positive. In particular, $|\nabla\rho/\rho| \in L_{\mathrm{loc}}^\gamma(\mathbb{R}^n, \mu)$. □

REMARK 9. If $\mu = \rho\, dx$ with $\rho = \varphi^2$ and $\varphi \in H^{2,1}_{loc}(\mathbb{R}^n)$, the so-called *Markov uniqueness* (i.e, the uniqueness of a Markovian semigroup on $L^2(\mathbb{R}^n, \mu)$ with generator given by $Lf = \Delta f + \langle \nabla f, \beta \rangle$ on $C_0^\infty(\mathbb{R}^n)$) always holds with $\beta := \nabla\rho/\rho$ (see [16], [17]). However, in general Markov uniqueness is weaker than the essential self-adjointness of $(L, C_0^\infty(\mathbb{R}^n))$ on $L^2(\mathbb{R}^n, \mu)$. (see [7]). Optimal (local or global) conditions for the essential self-adjointness remain unknown except for the one-dimensional case investigated in [25] and [7]. In fact, recently in [7] a complete characterization of the essential self-adjointness for Dirichlet operators has been given in the case $n = 1$.

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