GIUSEPPE DA PRATO

Poincaré inequality for some measures in Hilbert spaces and application to spectral gap for transition semigroups


<http://www.numdam.org/item?id=ASNSP_1997_4_25_3-4_419_0>
1. Introduction and setting of the problem

Let $H$ be a separable Hilbert space (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), and let $\nu$ be a Borel measure on $H$. This paper is devoted to prove, under suitable assumptions on $\nu$, an estimate of this kind (Poincaré inequality):

$$\int_H |\varphi(x) - \int_H \varphi(y) \nu(dy)|^2 \nu(dx) \leq C \int_H |D\varphi(x)|^2 \nu(dx),$$

where $C$ is a suitable positive constant.

Estimate (1.1) can be used to study the spectral gap for a transition semigroup corresponding to a differential stochastic equation:

$$\begin{cases}
dX(t) = (AX(t) + F(X(t)))dt + Q^{1/2}dW(t), & t \geq 0, \\
X(0) = x,
\end{cases}$$

where $A : D(A) \subset H \to H$ and $Q : H \to H$, are linear operators, $F : H \to H$ is nonlinear, and $W(t)$, $t \geq 0$ is an $H$-valued cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [5].

Assume that problem (1.2) has unique solution $X(t, x)$, then the corresponding transition semigroup $P_t$, $t \geq 0$, is defined by

$$P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in B_b(H),$$

where $B_b(H)$ is the Banach space of all bounded and Borel functions from $H$ into $\mathbb{R}$. We want to prove, under suitable assumptions, an estimate

$$\int_H \left| P_t \varphi(x) - \int_H \varphi(y) \nu(dy) \right|^2 \nu(dx) \leq C e^{-\alpha t} \int_H |\varphi(x)|^2 \nu(dx),$$

Partially supported by the Italian National Project MURST Equazioni di Evoluzione e Applicazioni Fisico-Matematiche.
for all \( \varphi \in L^2(H, \nu) \), where \( \nu \) is an invariant measure for the semigroup, and \( C, \omega \) are positive constants.

Estimate (1.4) implies that the spectrum \( \sigma(\mathcal{L}) \) of the infinitesimal generator \( \mathcal{L} \) of \( P_t \) in \( L^2(H, \nu) \) has the following property

\[
\sigma(\mathcal{L}) \setminus \{0\} \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda < \omega \}.
\]

This spectral gap property is important in the applications, it has been studied in the literature, mainly when the semigroup \( P_t \) is symmetric, see [8], [9], [5].

The content of the paper is the following. In Section 2 we prove a Poincaré inequality when \( \nu = \mu_R \) is a Gaussian measure of mean 0 and covariance operator \( R \in \mathcal{L}_+^1(H) \), the space of all nonnegative, symmetric, linear operators from \( H \) into \( H \) of trace class. In this case estimate (1.1) is a natural generalization of a well known result when \( H \) is finite dimensional. Then we consider in Section 3 the case when \( \nu \) is absolutely continuous with respect to a Gaussian measure \( \mu_R \). Finally section Section 4 is devoted to the spectral gap property.

2. – Poincaré inequality for Gaussian measures

We are given a Gaussian measure \( \mu_R \), on \( H \) with mean 0 and covariance operator \( R \in \mathcal{L}_+^1(H) \). We denote by \( \{e_k\} \) a complete orthonormal system in \( H \) consisting of eigenvectors of \( R \) and by \( \{\lambda_k\} \) the corresponding sequence of eigenvalues:

\[
\text{Re}e_k = \lambda_k e_k, \quad k \in \mathbb{N}.
\]

We shall assume that sequence \( \{\lambda_k\} \) is nonincreasing and that \( \lambda_k > 0 \) for all \( k \in \mathbb{N} \). For any \( k \in \mathbb{N} \) we shall denote by \( D_k \) the derivative in the direction of \( e_k \), and we shall set \( x_k = \langle x, e_k \rangle \) for any \( x \in H \). It is well known that \( D_k \) is a closable operator on \( L^2(H, \mu) \), see e.g. [7]. The Sobolev space \( W^{1,2}(H, \mu_R) \) is the Hilbert space of all \( \varphi \in L^2(H, \mu_R) \cap \text{dom}(D_k), \quad k \in \mathbb{N} \), such that

\[
\|\varphi\|_{W^{1,2}(H,\mu_R)} := \int_H |\varphi(x)|^2 \mu_R(\text{d}x) + \sum_{k=1}^{\infty} \int_H |D_k\varphi(x)|^2 \mu_R(\text{d}x) < +\infty.
\]

We denote by \( \mathcal{E}(H) \) the linear space spanned by all exponential functions \( \psi(x) = e^{\langle h, x \rangle}, \quad x \in H \). Obviously

\[
\mathcal{E}(H) \subset C^\infty(H) \cap L^2(H, \mu_R).
\]

and \( \mathcal{E}(H) \) is dense in \( L^2(H, \mu_R) \).

We denote by \( T_t, \quad t \geq 0 \), the Ornstein-Uhlenbeck semigroup:

\[
T_t \varphi(x) = \int_H \varphi(e^{-t/2}x + y) \mu_{(1-e^{-t})R}(\text{d}y), \quad t \geq 0, \quad \varphi \in L^2(H, \mu_R).
\]
It is well known that $T_t$, $t \geq 0$, is a strongly continuous semigroup of contractions on $L^2(H, \mu_R)$ having as unique invariant measure $\mu_R$:

\begin{equation}
\int_H T_t \varphi(x) \mu_R(dx) = \int_H \varphi(x) \mu_R(dx), \quad t \geq 0, \quad \varphi \in L^2(H, \mu_R).
\end{equation}

We denote by $\mathcal{L}$ the infinitesimal generator of $T_t$, $t \geq 0$. $\mathcal{L}$ is defined as the closure of the linear operator $\mathcal{L}_0$:

\begin{equation}
\mathcal{L}_0 \varphi(x) = \frac{1}{2} \text{Tr}[RD^2 \varphi(x)] - \frac{1}{2} \langle x, D\varphi(x) \rangle, \quad \varphi \in \mathcal{E}(H), \quad x \in H.
\end{equation}

We recall also that, for any $\varphi \in D(\mathcal{L})$ we have, see [1], [6],

\begin{equation}
\int_H \mathcal{L} \varphi(x) \varphi(x) \mu_R(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \mu_R(dx).
\end{equation}

Now we prove the result

**Theorem 2.1.** The following estimate holds

\begin{equation}
\int_H |\varphi(x) - \overline{\varphi}|^2 \mu_R(dx) \leq \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx), \quad \varphi \in W^{1,2}(H, \mu_R),
\end{equation}

where

\begin{equation}
\overline{\varphi} = \int_H \varphi(x) \mu_R(dx).
\end{equation}

**Proof.** For any $\varphi \in D(\mathcal{L})$ we have, in view of (2.4)

\begin{equation}
\frac{d}{dt} \int_H |T_t \varphi(x)|^2 \mu(dx) = 2 \int_H \mathcal{L} T_t \varphi(x) T_t \varphi(x) \mu(dx) = -\int_H |R^{1/2} D T_t \varphi(x)|^2 \mu(dx).
\end{equation}

To estimate $|R^{1/2} D T_t \varphi(x)|^2$ note that, in view of (2.1),

\begin{equation}
\langle Q^{1/2} D T_t \varphi(x), h \rangle = e^{-t/2} \int_H \langle D\varphi(e^{-t/2} x + y), h \rangle \mu_R(1-e^{-t})(dy),
\end{equation}

for all $h \in H$. It follows, using Hölder’s inequality

\begin{equation}
|\langle R^{1/2} D T_t \varphi(x), h \rangle|^2 \leq e^{-t/2} h^2 T_t (|R^{1/2} D \varphi|^2)(x), \quad h \in H.
\end{equation}

Therefore, due to the arbitrariness of $h$,

\begin{equation}
|R^{1/2} D T_t \varphi(x)|^2 \leq e^{-t} T_t (|R^{1/2} D \varphi|^2)(x).
\end{equation}
By integrating on $H$ with respect to $\mu_R$, and taking into account the invariance of $\mu_R$, we have
\[ \int_H |R^{1/2}DT_t \varphi(x)|^2 \mu_R(dx) \leq e^{-t} \int_H |R^{1/2}D\varphi(x)|^2 \mu_R(dx). \]

Now, comparing with (2.7) we find
\[ \frac{d}{dt} \int_H |T_t \varphi(x)|^2 \mu_R(dx) \geq -e^{-t} \int_H |R^{1/2}D\varphi(x)|^2 \mu_R(dx). \]

Integrating in $t$ find
\[ \int_H |T_t \varphi(x)|^2 \mu_R(dx) \geq \int_H |\varphi(x)|^2 \mu_R(dx) - (1 - e^{-t}) \int_H |R^{1/2}D\varphi(x)|^2 \mu_R(dx). \]

Finally, letting $t$ tend to $+\infty$, and using the fact that, as easily checked,
\[ \lim_{t \to +\infty} P_t \varphi(x) = \overline{\varphi}, \text{ x a.e. in } H, \]
we get
\[ (\overline{\varphi})^2 \geq \int_H |\varphi(x)|^2 \mu_R(dx) - \int_H |R^{1/2}D\varphi(x)|^2 \mu_R(dx), \]
that is equivalent to (2.5).

3. – Poincaré inequality for non Gaussian measures

Here we are given, besides a Gaussian measure $\mu = \mu_R$, with $R \in \mathcal{L}_+^1(H)$ and $\ker R = \{0\}$, a function $U : H \to \mathbb{R}$, such that

HYPOTHESIS 1.
(i) $U$ is convex and of class $C^2$.
(ii) $DU$ is Lipschitz continuous.

We set
\[ \alpha(x) = ke^{-2U(x)}, \quad x \in H, \]
where $k$ is chosen such that
\[ \int_H \alpha(x) \mu(dx) = 1. \]

Finally we consider the Borel probability measure on $H$
\[ v(dx) = \alpha(x) \mu(dx). \]
We are going to prove a Poincaré estimate for measure $\nu$. We notice that assumptions on $\alpha$ could be considerably weakened. It will be enough to assume convexity of $U$ (that implies dissipativity of $-DU$), and some additional properties similar to [5]. But we prefer to make Hypothesis 1 for the sake of simplicity.

It is useful to introduce a differential stochastic equation having $\nu$ as invariant measure:

\[
\begin{cases}
    dZ = (AZ - DU(Z))dt + dW(t) \\
    Z(0) = x \in H,
\end{cases}
\]

where $A$ is the negative self-adjoint operator in $H$ defined as

\[ A = -\frac{1}{2} R^{-1} , \]

and $W$ is a cylindrical $H$-valued Wiener process in some probability space $(\Omega, \mathcal{F}, P)$.

Problem (3.2) has a unique solution $Z(t, x)$, and measure $\nu$ is invariant, see [5]. The corresponding transition semigroup is defined in $L^2(H, \nu)$ by

\[
N_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \; \varphi \in L^2(H, \nu), \; t \geq 0.
\]

Its infinitesimal generator $N$ is defined by, see [4]

\[
D(N) = \left\{ \varphi \in W^{2,2}(H; \nu) \cap W_A^{1,2}(H; \nu) : \int_H \langle D^2U(x)D\varphi(x), D\varphi(x) \rangle \nu(dx) < +\infty \right\},
\]

where $W_A^{1,2}(H; \nu)$ is the linear space of all $\varphi \in W^{1,2}(H, \nu)$ such that $\langle AD\varphi, D\varphi \rangle \in L^2(H, \nu)$.

Finally in [5] it is proved that $\nu$ is strongly mixing

\[
\lim_{t \to \infty} N_t \varphi(x) = \int_H \varphi(y) \nu(dy), \; \varphi \in L^2(H, \nu).
\]

We can now prove

**Theorem 3.1.** The following estimate holds

\[
\int_H |\varphi(x) - \bar{\varphi}|^2 \nu(dx) \leq \frac{1}{R} \int_H |D\varphi(x)|^2 \nu(dx), \; \varphi \in W^{1,2}(H, \nu),
\]

where

\[
\bar{\varphi} = \int_H \varphi(x) \nu(dx).
\]
PROOF. For any $\varphi \in D(\mathcal{N})$ we have, see [4],

$$
\frac{d}{dt} \int_H |N_t\varphi(x)|^2 \nu(dx) = 2 \int_H N_t N^* \varphi(x) N_t \varphi(x) \nu(dx)
$$
(3.8)

$$
= -\int_H |D N_t \varphi(x)|^2 \nu(dx).
$$

We want now to estimate $|D N_t \varphi(x)|^2$. To this purpose we note that $X(t, x)$ is differentiable with respect to $x$ and

$$
\|X^*_t(t, x)\| \leq e^{-\frac{\|\mathcal{N}\|^t}{\|\mathcal{N}\|}}, \quad t \geq 0.
$$

(3.9)

It follows

$$
D N_t \varphi(x) = \mathbb{E}[X^*_t(t, x) D \varphi(X(t, x))].
$$

(3.10)

Now by (3.10) and the Hölder’s estimate, it follows,

$$
|D N_t \varphi(x)|^2 \leq e^{-\frac{1}{\|\mathcal{N}\|^t} N_t |D \varphi|^2(x)}.
$$

(3.11)

By integrating on $H$ with respect to $\nu$, and taking into account the invariance of $\nu$, we have

$$
\int_H |D N_t \varphi(x)|^2 \nu(dx) \leq e^{-\frac{1}{\|\mathcal{N}\|^t}} \int_H |D \varphi(x)|^2 \nu(dx).
$$

By substituting in (3.8) we find

$$
\frac{d}{dt} \int_H |N_t \varphi(x)|^2 \nu(dx) \geq -e^{-\frac{1}{\|\mathcal{N}\|^t}} \int_H |D \varphi(x)|^2 \nu(dx).
$$

Integrating in $t$ we have

$$
\int_H |N_t \varphi(x)|^2 \nu(dx) \geq \int_H |\varphi(x)|^2 \nu(dx) - \|R\|(1 - e^{-\frac{1}{\|\mathcal{N}\|^t}}) \int_H |D \varphi(x)|^2 \nu(dx).
$$

Finally, letting $t$ tend to $+\infty$, and using (3.5) we get

$$
(\bar{\varphi})^2 \geq \int_H |\varphi(x)|^2 \nu(dx) - \|R\| \int_H |D \varphi(x)|^2 \nu(dx),
$$

and the conclusion follows. \qed
4. – Spectral gap

4.1. – Gaussian case

We are here concerned with the Ornstein-Uhlenbeck process $X(\cdot, x)$ solution of the following differential stochastic equation

\[
\begin{aligned}
&dX(t) = AX(t)dt + Q^{1/2}dW(t), \ t \geq 0, \\
&X(0) = x,
\end{aligned}
\]

under the following assumptions.

HYPOTHESIS 2.

(i) $A$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$ on $H$.

(ii) $Q$ is bounded, symmetric, and nonnegative.

(iii) For all $t > 0$ the operator $e^{tA}Qe^{tA^*}$ is of trace class and its kernel is equal to $\{0\}$. Moreover

\[
\int_0^{+\infty} \text{Tr}[e^{tA}Qe^{tA^*}]dt < +\infty.
\]

If Hypothesis 2 holds the linear operator

\[
Q_\infty x = \int_0^{+\infty} e^{tA}Qe^{tA^*}x \, dt, \ x \in H,
\]

is well defined and it is of trace-class. Moreover problem (4.1) has a unique mild solution given by, see [5]

\[
X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}dW(s).
\]

The corresponding transition semigroup $P_t, t \geq 0$, is defined by

\[
P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))] = \int_H \varphi(e^{tA}x + y)\mu_{Q_t}(dy), \ \varphi \in \mathcal{B}_b(H),
\]

where

\[
Q_t x = \int_0^t e^{sA}Qe^{sA^*}x \, ds, \ x \in H.
\]

Finally the measure $\mu_{Q_\infty}$ is invariant, and so the semigroup $P_t, t \geq 0$, can be uniquely extended to a strongly continuous semigroup of contractions on $L^2(H, \mu)$, that we still denote by $P_t, t \geq 0$. Its infinitesimal generator will be denoted by $\mathcal{L}$. 

THEOREM 4.1. Assume, besides Hypothesis 2 that

\[ Q^{1/2}(H) \subset Q^{1/2}_\infty(H). \]

Then for any \( \varphi \in W^{1,2}(H, \mu) \) we have

\[ \int_H |P_t \varphi(x) - \bar{\varphi}|^2 \, \mu(dx) \leq e^{-t/2} \frac{1}{\|Q^{-1/2}Q^{-1/2}_\infty\|} \int_H |\varphi(x)|^2 \, \mu(dx), \]

where

\[ \bar{\varphi} = \int_H \varphi(y) \mu(dy) \]

PROOF. By the Poincaré inequality (2.5), with \( R = Q_\infty \), it follows

\[ \int_H |\varphi(x) - \bar{\varphi}|^2 \, \mu(dx) \leq \|Q^{-1/2}Q^{-1/2}_\infty\| \int_H |Q^{1/2}D\varphi(x)|^2 \, \mu(dx). \]

We also recall that, for any \( \varphi \in D(\mathcal{L}) \) we have, see [1], [6],

\[ \int_H \mathcal{L}\varphi(x)\varphi(x) \, \mu(dx) = -\frac{1}{2} \int_H |Q^{1/2}D\varphi(x)|^2 \, \mu(dx). \]

This implies

\[ \int_H \mathcal{L}\varphi(x)\varphi(x) \, \mu(dx) \leq \frac{1}{2\|Q^{-1/2}Q^{-1/2}_\infty\|} \int_H |\varphi(x) - \bar{\varphi}|^2 \, \mu(dx). \]

Let now consider the space

\[ Y = \left\{ \varphi \in L^2(H, \mu) : \bar{\varphi} = 0 \right\}. \]

\( Y \) is obviously an invariant subspace of \( P_t, \ t \geq 0 \); denote by \( \mathcal{L}_Y \) the part of \( \mathcal{L} \) in \( Y \). By (4.6) it follows

\[ \int_H \mathcal{L}_Y\varphi(x)\varphi(x) \, \mu(dx) \leq \frac{1}{2\|Q^{-1/2}Q^{-1/2}_\infty\|} \int_H |\varphi(x)|^2 \, \mu(dx), \quad \varphi \in D(\mathcal{L}_Y). \]

It is easy to check that this inequality yields (4.5). \( \square \)
Another condition implying the spectral gap property holds when the semigroup \( P_t, t \geq 0 \), is strong Feller.

**Hypothesis 3.** For any \( t > 0 \) we have

\[
e^{tA}(H) \subset Q_t^{1/2}(H).
\]

When Hypothesis 3 is fulfilled we set

\[
\Gamma(t) = Q_t^{1/2}Q_t^{-1/2}e^{tA}, \ t > 0.
\]

We recall that \( \|\Gamma(t)\| \) is nonincreasing in \( t \) and \( \lim_{t \to 0} \|\Gamma(t)\| = +\infty \). Moreover for any \( \varphi \in L^2(H, \mu) \) and any \( t > 0 \), one has \( P_t\varphi \in W^{1,2}(H, \mu) \) and the following estimate holds, see [5],

\[
(4.8) \quad \int_H |DP_t\varphi(x)|^2 \mu(dx) \leq \|\Gamma(t)\|^2 \int_H |\varphi(x)|^2 \mu(dx).
\]

**Theorem 4.2.** Assume, besides Hypotheses 2 and 3, that there exist \( M, \omega > 0 \) such that

\[
\|Q_\infty^{1/2}e^{tA}\| \leq Me^{-\omega t}, \ t \geq 0.
\]

The there exists \( M_1 > 0 \) such that the following estimate holds

\[
(4.9) \quad \int_H |P_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) \leq M_1 e^{-\omega t} \int_H |\varphi(x)|^2 \mu(dx).
\]

**Proof.** Replacing in (2.5) \( \varphi \) with \( P_t\varphi \), and taking into account that \( P_t\varphi = \overline{\varphi} \) by the invariance of \( \mu \), we have

\[
\int_H |P_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) \leq \int_H |Q_\infty^{1/2}DP_t\varphi(x)|^2 \mu(dx), \ \varphi \in W^{1,2}(H, \mu).
\]

Since

\[
DP_t\varphi(x) = e^{tA^*}P_tD\varphi(x),
\]

it follows

\[
\int_H |P_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) \leq \|Q_\infty^{1/2}e^{tA^*}\|^2 \int_H |DP_t\varphi(x)|^2 \mu(dx)
\]

\[
\leq M^2 e^{-2\omega t} \int_H |D\varphi(x)|^2 \mu(dx).
\]

By replacing \( \varphi \) with \( P_t\varphi \), and taking into account (4.8), we find

\[
\int_H |P_{t+1}\varphi(x) - \overline{\varphi}|^2 \mu(dx) \leq M^2 e^{-2\omega t} \int_H |DP_t\varphi(x)|^2 \mu(dx)
\]

\[
\leq M^2 e^{-2\omega t} \|\Gamma(t)\|^2 \int_H |\varphi(x)|^2 \mu(dx).
\]

By replacing \( t + 1 \) with \( t \) the conclusion follows. \( \square \)
4.2. – Non Gaussian case

We are here concerned with the solution \( X(\cdot, x) \) of the following differential stochastic equation

\[
\begin{align*}
&\begin{cases}
\frac{dX(t)}{dt} = (AX(t) + F(X))dt + dW(t), \ t \geq 0, \\
X(0) = x,
\end{cases}
\end{align*}
\]

under the following assumptions.

**Hypothesis 4.**

(i) \( A \) is the infinitesimal generator of a strongly continuous semigroup \( e^{tA} \) on \( H \) and there exists \( \omega > 0 \) such that \( \| e^{tA} \| \leq e^{-\omega t}, \ t \geq 0 \).

(ii) For all \( t > 0 \) the operator \( e^{tA}e^{tA^*} \) is of trace class, and \( \int_0^\infty \text{Tr}[e^{tA}e^{tA^*}]dt < +\infty \).

(iii) \( F : H \to H \) is uniformly continuous and bounded together with its Fréchet derivative.

If Hypothesis 4 holds the linear operator

\[ Q_\infty x = \int_0^{+\infty} e^{tA}e^{tA^*}x \, dt, \ x \in H, \]

is well defined and it is of trace-class. Moreover problem (4.10) has a unique mild solution, see [5]. The corresponding transition semigroup \( P_t, t \geq 0, \) is defined by as before by

\[ P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \ \varphi \in B_b(H). \]

We set \( \mu = \mu_{Q_\infty} \), and denote by \( \mathcal{E}_A(H) \) the vector space generated by all functions of the form

\[ \varphi(x) = e^{(h, x)}, \ h \in D(A^*). \]

We denote by \( \mathcal{L} \) the infinitesimal generator of \( P_t, t \geq 0. \) \( \mathcal{L} \) is defined as the closure of the linear operator \( \mathcal{L}_0: \)

\[
\mathcal{L} \varphi(x) = \frac{1}{2} \text{Tr}[D^2 \varphi(x)] + \langle x, A^* D\varphi(x) \rangle + \langle F(x), D\varphi(x) \rangle, \ \varphi \in \mathcal{E}_A(H), \ x \in H.
\]

We need an integration by parts formula.

**Lemma 4.3.** Assume that Hypotheses 1 and 4 hold. Let \( \alpha \) be defined by (3.1), and let \( \varphi, \psi \in \mathcal{E}_A(H). \) Then the following identity holds.

\[
\begin{align*}
\int_H \left[ D_k \varphi(x) \psi(x) + \varphi(x) D_k \psi(x) \right] v(dx) & = \int_H \left( \frac{x_k}{\lambda_k} - D_k \log \alpha(x) \right) \varphi(x) \psi(x) v(dx).
\end{align*}
\]
PROOF. Denote by $J$ the left hand side of (4.13). Taking into account a well known result on Gaussian measures, we have

$$J = \int_H [D_k \varphi(x) \psi(x) \alpha(x) + \varphi(x) D_k \psi(x) \alpha(x)] \mu(dx)$$

$$= \int_H [-\varphi(x) D_k (\psi(x) \alpha(x)) \psi(x) D_k \psi(x) \alpha(x)] \mu(dx)$$

$$+ \int_H \frac{x_k}{\lambda_k} \alpha(x) \varphi(x) \psi(x) \mu(dx)$$

$$= \int_H \left( \frac{x_k}{\lambda_k} - D_k \alpha(x) \right) \varphi(x) \psi(x) \mu(dx).$$

The conclusion follows. \(\square\)

**Proposition 4.4.** Assume that Hypotheses 1 and 4 hold. Let $\alpha$ be defined by (3.1) and $L$ by (4.12). Then for any $\varphi, \psi \in \mathcal{E}(H)$ we have

$$\int_H \mathcal{L} \varphi(x) \psi(x) \nu(dx) = \int_H (AQ_\infty D \varphi(x), D \psi(x)) \nu(dx)$$

$$+ \int_H (AQ_\infty D \log \alpha(x) + F(x), D \psi(x)) \psi(x) \nu(dx),$$

and

$$\int_H \mathcal{L} \varphi(x) \varphi(x) \nu(dx) = -\frac{1}{2} \int_H |D \varphi(x)|^2 \nu(dx)$$

$$+ \int_H (AQ_\infty D \log \alpha(x) + F(x), D \varphi(x)) \psi(x) \nu(dx).$$

Notice that

$$Q_\infty(H) \subset D(A),$$

see [3], so that $AQ_\infty$ is a well defined bounded operator.

**Proof.** We first compute the integral

$$J = \int_H (Ax, D \psi(x)) \psi(x) \nu(dx).$$

We denote by $\{e_k\}$ a complete orthonormal system in $H$ consisting of eigenvectors of $Q_\infty$ and by $\{\lambda_k\}$ the corresponding sequence of eigenvalues:

$$Q_\infty e_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

We assume for simplicity that $\{e_k\} \subset D(A)$, this extra assumption can be easily removed by approximating $A$ with its Yosida approximations. We have

$$\langle Ax, D \varphi(x) \rangle = \sum_{h,k=1}^\infty a_{h,k} x_k D_h \varphi(x),$$
where \( \alpha_{h,k} = \langle A e_k, e_h \rangle \), and \( x_k = \langle x, e_k \rangle \). We proceed here as in [6]. By integration by parts formula (4.11) we have

\[
\int_H x_k D_h \varphi(x) \psi(x) v(dx) = \int_H \lambda_k D_h D_k \varphi(x) \psi(x) v(dx) + \int_H \lambda_k D_h \varphi(x) D_k \psi(x) v(dx) + \int_H \lambda_k D_k \log \alpha(x) D_h \varphi(x) \psi(x) v(dx).
\]

It follows

\[
J = \int_H \text{Tr}[AQ_{\infty}D^2 \varphi(x)] \psi(x) v(dx) + \int_H (AQ_{\infty}D \varphi(x), D \varphi(x)) v(dx) + \int_H (AQ_{\infty}D \log \alpha(x), D \varphi(x)) \psi(x) v(dx).
\]

Now, taking into account (4.12), a simple computation yields (4.14). Finally (4.15) follows as in [6], recalling the Lyapunov equation

\[
AQ + QA^* + Q_{\infty} = 0.
\]

**Theorem 4.5.** Assume that Hypotheses 1 and 4 hold. Assume in addition that \( \alpha \), defined by (3.1), can be chosen such that

\[
F(x) = -AQ_{\infty}D \log \alpha(x), \quad x \in H.
\]

Then \( v \) is an invariant measure for \( P_t, t \geq 0 \), and for all \( \varphi \in L^2(H, \mu) \) we have

\[
\int_H |P_t \varphi(x) - \overline{\varphi}|^2 v(dx) \leq e^{-\frac{1}{Q_{\infty}} t} \int_H |\varphi(x)|^2 v(dx),
\]

where

\[
\overline{\varphi} = \int_H \varphi(y) v(dy)
\]

**Proof.** First notice that if (4.16) holds, then setting \( \psi(x) = 1, \ x \in H \), we have by (4.14)

\[
\int_H \mathcal{L} \varphi(x) v(dx) = 0, \ \varphi \in D(\mathcal{L}).
\]

This implies that \( v \) is invariant for \( P_t, t \geq 0 \). Now by (4.15) it follows

\[
\int_H \mathcal{L} \varphi(x) \varphi(x) v(dx) = -\frac{1}{2} \int_H |D \varphi(x)|^2 v(dx), \ \varphi \in D(\mathcal{L}).
\]

Consequently, by (3.6) we have

\[
\int_H \mathcal{L} \varphi(x) \varphi(x) v(dx) \leq -\frac{1}{2 \|Q_{\infty}\|} \int_H |\varphi(x) - \overline{\varphi}| v(dx).
\]

Arguing as in the proof of Theorem 4.1, we arrive at (4.17). \( \square \)
REFERENCES


Scuola Normale Superiore di Pisa
Piazza dei Cavalieri, 7
56100 Pisa, Italy