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Compactness of Conformal Metrics with Positive Gaussian Curvature in $\mathbb{R}^2$

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Abstract. In this paper we consider the compactness of a sequence of solutions u_n of

$$\Delta u + K(x)e^{2u} = 0 \quad \text{in } \mathbb{R}^2, \tag{0.1}$$

where $K(x)$ is positive in $\mathbb{R}^2$ and decays like $|x|^{-b}$ at ∞ for some $b > 0$. Assuming that the limit of the total curvature of u_n satisfies

$$2 - b \neq \lim_{n \longrightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx < 2, \tag{0.2}$$

we prove that u_n must be bounded in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. We also construct a specific $K(x) = K(|x|)$ to show that the total curvature of any solution u of equation (0.1) with this $K(|x|)$ must satisfy

$$(2 - b) < \frac{1}{2\pi} \int_{R^2} K(x)e^{2u}dx < 2. \tag{0.3}$$

This appears to be in contrast with the statement of Theorem A[1] in [A]. In this respect, we show that for any K which decays like $|x|^{-b}$ for $0 < b < 2$, there exists $\alpha_0(K) > \frac{2-b}{2}$ such that the total curvature of any solution u of (0.1) must satisfy

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} Ke^{2u}dx \geq \alpha_0(K) > \frac{2-b}{2}.$$

1. – Introduction

In this paper, we consider the entire solution of the equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where Δ is the Laplacian operator of $\mathbb{R}^2$ and $K(x)$ is a given function in $\mathbb{R}^2$. Equation (1.1) arises in the problem of finding a Riemannian metric which is

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conformal to the flat metric of $\mathbb{R}^2$ and realizes the given function $K(x)$ as its Gaussian curvature. We refer the reader to [CN1] for a brief description of the background and the history of this problem.

In case K is nonpostive on $\mathbb{R}^2$, a fairly complete understanding of the the solution set of (1.1) was achieved in [CN1], [CN2]. To state the results in [CN2], we introduce α_1 as

$$\alpha_1 = \sup\left\{\alpha \in \mathbb{R} \mid \int_{\mathbb{R}^2} \Big|K(x)|(1+|x|^2)^\alpha dx < +\infty\right\}. \tag{1.2}$$

Then the main result in [CN2] is

THEOREM A. *Suppose that $K \leq 0$ in $\mathbb{R}^2$ and that*

$$|x|^{-m} \leq |K(x)| \leq |x|^m \tag{1.3}$$

for $|x|$ large and some positive constant m. Then we have:

(I) *If $\alpha_1 \leq 0$, then (1.1) possesses no entire solution in $\mathbb{R}^2$.*

(II) *If $\alpha_1 > 0$, then the following conclusions hold:*

(i) *For each $\alpha \in (0, \alpha_1)$, (1.1) possesses a unique solution u_α such that*

$$u_\alpha(x) = \alpha \log|x| + O(1) \quad at\, \infty. \tag{1.4}$$

(ii) *The function $U(x)$ given by*

$$U(x) \equiv \sup\{u(x) \mid u \quad \text{is an entire solution of } (1.1) \text{ in } \mathbb{R}^2\}$$

is well-defined everywhere in $\mathbb{R}^2$ and is a solution of (1.1) in $\mathbb{R}^2$. Moreover, $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$.

(iii) *Let u be an arbitrary solution of (1.1) in $\mathbb{R}^2$. Then either $u \equiv U$ or $u \equiv u_\alpha$ for some $\alpha \in (0, \alpha_1)$.*

(iv) *If $0 < \alpha < \beta < \alpha_1$, then $u_\alpha(x) < u_\beta(x) < U(x)$ for all $x \in \mathbb{R}^2$. Furthermore, for any given $\varepsilon > 0$, there exists a constant $R = R(\varepsilon)$ such that for $|x| > R$,*

$$(\alpha_1 - \varepsilon)\log|x| - C \leq U(x) \leq \alpha_1 \log|x| + C.$$

In this paper, K is always assumed locally bounded and positive in $\mathbb{R}^2$. A solution u means $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$ and satisfies (1.1) in the distributional sense. For the case $K(x)$ is positive in $\mathbb{R}^2$, it is not expected that results similar to Theorem A should hold. However, for some special $K(x)$ as stated in Theorem 1.1 below, we have the following result in the spirit of Theorem A.

THEOREM 1.1. *Let* $K(x) \equiv 1$ *for* $|x| \le 1$ *and* $K(x) \equiv |x|^{-b}$ *for* $|x| \ge 1$ *for some constant* $b > 0$. *Then the following statements hold:*

(i) *For every* α *satisfying* $-2 < \alpha < \min\{0, b-2\}$, (1.1) *possesses a unique* C^2 *radial solution* $u_\alpha(r)$ *satisfying* (1.4).
(ii) *Let* u *be an arbitrary solution of* (1.1) *satisfying* (1.4) *for some* α, *then* α *satisfies* $-2 < \alpha < \min\{0, b-2\}$ *and* $u(x) \equiv u_\alpha(x)$ *where* $u_\alpha(x)$ *is the solution in* (i) *above.*

REMARK 1.2. On the constrast to the case $K \le 0$, the family of solution $u_\alpha(x)$ in Theorem 1.1 does not have the monotone property in α as the case in Theorem A. In fact, by the concrete construction of solutions in the proof of Theorem 1.1, it can be seen that $u_\alpha(r)$ and $u_\beta(r)$ exactly intersects once for $\alpha \ne \beta$. We hope that it will be useful in a future study.

Although Theorem 1.1 are only concerned with some specific $K(x)$, it still provides an interesting example to the situaton when $K(x)$ is positive in $\mathbb{R}^2$. In [A], Aviles proved the following theorem, (See Theorem A^1 in [A]).

THEOREM B. *Assume* $K(x) > 0$ *in* $\mathbb{R}^2$ *and* $\lim_{|x| \longrightarrow +\infty} K(x)|x|^b = 1$ *for some positive constant* $b > 0$. *Then, for any* α *satisfying*

$$-2 < \alpha < \min\left(0, \frac{b-2}{2}\right), \tag{1.5}$$

there exists a solution u *of* (1.1) *satisfying*

$$u(x) = \alpha \log |x| + O(1) \quad \text{at } \infty .$$

Let $K(x)$ be the specific funtion given in Theorem 1.1 with $0 < b < 2$. Then Theorem 1.1 contradicts to the result of Theorem B. In fact, Theorem 1.1 is not an isolated case to show that Theorem B does not hold. For a general $K(x)$, set

$$\alpha_0 = \sup\{\alpha \mid \text{there is an entire solution } u \text{ of (1.1) such that } u(x) = \alpha \log|x| + O(1) \quad \text{at } \infty\}. \tag{1.6}$$

Our main result is

THEOREM 1.2. *Suppose that* $K(x)$ *is positive and locally bounded in* $\mathbb{R}^2$ *and satisfies*

$$B|x|^{-b} \le K(x) \le A|x|^{-b} \tag{1.7}$$

for $|x| \ge 1$ *and for positive constants* A, B *and* $0 < b < 2$. *Then* $\alpha_0 < -\frac{2-b}{2}$, *where* α_0 *is given in* (1.6).

Obviously, Theorem 1.2 implies that Theorem B does not hold in general. We note that the real number α_1 in (1.2) is $-\frac{2-b}{2}$ if $K(x)$ satisfies (1.7). Theorem 1.2 provides a major contrast to Theorem A for the case $K(x) \leq 0$. We would like to remark that solutions possessing the asymptotic behavior (1.4) have a geometric meaning. Following conventional notations, *a solution $u(x)$ of (1.1) is said to have a finite total curvature if $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$, and the quantity $\frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx$ is called the total curvature of u.* Assume $K(x)$ satisfies (1.7). A consequence of our previous results in [CLn] is that a solution u has a finite total curvature if and only if u possesses the asymptotic behavior (1.4), or more precisely, $\lim_{|x|\longrightarrow+\infty} u(x)/\log|x|$ exists, and the identity

$$\frac{-1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx = \lim_{|x|\longrightarrow+\infty} \frac{u(x)}{\log|x|}$$

are always true. Please see Lemma 2.1 in Section 2. Thus, it is interesting to know what is the possible range of α or equivalently, the possible range of the total curvature of solutions. In [M], McOwen proved that if $0 < K(x) \leq C|x|^{-b}$ at ∞, then for every $\alpha \in (-2, (b-2)^-)$ where $(b-2)^- = \min(0, b-2)$, there exists a solution of (1.1) satisfying (1.4). Together with Theorem 1.1, we see that the result of McOwen is the best possible for a general K which decays like $|x|^{-b}$ at ∞.

THEOREM 1.3. *Suppose $K(x)$ is a positive continuous function in $\mathbb{R}^2$ and satisfies $\lim_{|x|\longrightarrow+\infty} K(x)|x|^b = 1$ for some $0 \leq b < 2$. Assume u_n is a sequence of solutions of (1.1) such that*

$$2 - b \neq \lim_{n\longrightarrow+\infty} \frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx < 2 \tag{1.8}$$

Then u_n is bounded in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p > 1$. Furthermore if u_n converges to u in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$, then

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx = \int_{R^2} K(x)e^{2u(x)}dx\,. \tag{1.9}$$

COROLLARY 1.4. *Suppose K satisfies the assumption of Theorem 1.3 and u_n is a sequence of solutions of (1.1). If $|u_n(0)| \longrightarrow +\infty$ as $n \longrightarrow +\infty$ and $\frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx \leq 2 - \varepsilon_0$ for some $\varepsilon_0 > 0$, then we always have*

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx = 2\pi(2-b)\,. \tag{1.10}$$

COROLLARY 1.5. *Suppose K satisfies the assumption of Theorem 1.3 and $\alpha_0(K)$ is defined in (1.6). If $\alpha_0(K) > -(2-b)$, then $\alpha_0(K)$ is achieved, i.e. there exists a solution u of (1.1) with*

$$-\alpha_0(K) = \frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx\,. \tag{1.11}$$

REMARK 1.6. When $K(x)$ decays like $|x|^{-b}$ for $b \geq 2$, and u_n is a sequence of solutions of (1.4) satisfying

$$0 < \varepsilon_0 \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx \leq 2 - \varepsilon_0$$

for some $\varepsilon_0 > 0$, then u_n is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. The proof is easy, and will be omitted.

The paper is organized as follows. In Section 2, we will give a proof of Theorem 1.1. Both Theorem 1.2 and Theorem 1.3 will be proved in Section 3.

2. – Proof of Theorem 1.1

Let K be positive in $\mathbb{R}^2$ and satisfy

$$|x|^{-m} \leq K(x) \leq |x|^m \tag{2.1}$$

for $|x|$ large, where m is a positive constant. A solution u of (1.1) is said to have a finite total curvature if $Ke^{2u} \in L^1(\mathbb{R}^2)$, and the quantity $\frac{1}{2\pi}\int Ke^{2u}dx$ is called the *total curvature* of u. Theorem 1.1 in [CLn] says that if u is a solution of (1.1) with a finite total curvature, then $\lim_{|x|\longrightarrow+\infty} \frac{u(x)}{\log|x|}$ exists and

$$\lim_{|x|\longrightarrow+\infty} \frac{u(x)}{\log|x|} = -\frac{1}{2\pi}\int_{\mathbb{R}^2} Ke^{2u}dx\,. \tag{2.2}$$

Conversely, it is easy to see that if $\lim_{|x|\longrightarrow+\infty} \frac{u(x)}{\log|x|}$ exists, then $Ke^{2u} \in L^1(\mathbb{R}^2)$ and (2.2) holds. Hence, we have

LEMMA 2.1. *Suppose K satisfies* (2.1). *Then $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$ if and only if $\lim_{|x|\longrightarrow+\infty} \frac{u(x)}{\log|x|}$ exists. Moreover,* (2.2) *always holds.*

REMARK 2.2. In fact, Theorem 1.1 in [CLn] also shows that for a solution u of (1.1) having a finite total curvature α, there exists a constant C such that

$$\alpha \log|x| - C \leq u(x) \tag{2.3}$$

holds. Hence, if $C_2|x|^{-b} \leq K(x) \leq C_1|x|^{-b}$ for large $|x|$, then $\alpha < -\frac{(2-b)^+}{2}$ where $(2-b)^+ = \max\{2-b, 0\}$.

PROOF OF THEOREM 1.1. Let

$$u_\alpha(r) = \frac{1}{2}\log(4B_1) - \log[1 + B_1 r^2], \quad r \in [0, 1] \tag{2.4}$$

and

$$(2.5)\qquad u_\alpha(r) = \frac{1}{2}\log(4A_2^2B_2) + \left(A_2 - 1 + \frac{b}{2}\right)\log r - \log[1 + B_2 r^{2A_2}],$$
$$r \in [1, \infty)\,,$$

where $B_1 > 0$ is a constant and $\alpha = -A_2 - 1 + \frac{b}{2}$. Then it is not very difficult to verify that u_α is a C^2-solution of (1.1) provided that

$$(2.6)\qquad A_2 = \left\{ \frac{4B_1 + \left[B_1\left(1+\frac{b}{2}\right) - \left(1-\frac{b}{2}\right)\right]^2}{(1+B_1)^2} \right\}^{\frac{1}{2}},$$

$$(2.7)\qquad B_2 = \frac{A_2(1+B_1) + \left[B_1\left(1+\frac{b}{2}\right) - \left(1-\frac{b}{2}\right)\right]}{A_2(1+B_1) - \left[B_1\left(1+\frac{b}{2}\right) - \left(1-\frac{b}{2}\right)\right]}\,.$$

Since $u_\alpha(0) = \frac{1}{2}\log(4B_1)$, we see that $B_1 > 0$ exhausts all radial solutions. It is easy to see that u_α satisfies (1.4) with $\alpha = -A_2 - 1 + \frac{b}{2}$. Now A_2 is a monotonic function of B_1 satisfying

$$\lim_{B_1 \longrightarrow 0^+} A_2(B_1) = \left|\frac{b}{2} - 1\right| \quad \text{and} \quad \lim_{B_1 \longrightarrow \infty} A_2(B_1) = \frac{b}{2} + 1\,.$$

Hence α satisfies $-2 < \alpha < \min\{0, b-2\}$. This proves (i).

Now suppose that u be an arbitrary solution of (1.1) with finite total curvature. Since $K(x) = K(|x|)$ is nonincreasing in r and $K(r) \geq e^{-r^\beta}$ for any $0 < \beta < 1$, then from Theorem 1.7 in [CLn], we conclude that u must be a radial function. Hence $u \equiv u_\alpha$ for some α in the range $-2 < \alpha < \min\{0, b-2\}$, where u_α is defined in (2.4) and (2.5). This proves (ii). □

3. – Proofs of compactness theorems

In this section, we begin with a proof of Theorem 1.2. First, we need the following result which was proved in [BM].

THEOREM 3.1 (Theorem 3 in [BM]). *Assume u_n is a sequence of solutions of*

$$(3.1)\qquad \Delta u_n + K_n e^{2u_n} = 0 \quad \text{in } \Omega$$

satisfying

$$0 \le K_n \le C_1 \quad \textit{in } \Omega\,, \tag{3.2}$$

and

$$\|e^{2u_n}\|_{L^1(\Omega)} \le C_2 \tag{3.3}$$

for two constants C_1 and C_2. Then either u_n is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$ or there exists a subsequence of u_n (still denoted by u_n) such that either $u_n \longrightarrow -\infty$ uniformly on any compact sets of Ω or the blow-up set S is a set of finite number of points, $u_n \longrightarrow -\infty$ uniformly on any compact set of $\Omega \backslash S$, and $K_n e^{2u_n}$ converges to $\sum_i \alpha_i \delta_{p_i}$ with $\alpha_i \ge 2\pi$ and $S = \cup_i \{p_i\}$.

REMARK 3.2. When either K_n is uniformly convergent or converges to a positive constant then Theorem 3.1 can be improved to have $\alpha_i \ge 4\pi$.

PROOF OF THEOREM 1.2. Suppose $\alpha_0 = -(\frac{2-b}{2})$. Since $\alpha_0 = -\frac{2-b}{2}$ can not be achived by some sloution of (1.1) by Remark 2.2, there exists a sequence of solutions of u_n such that the total curvature

$$\lim_{n \longrightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx = \frac{2-b}{2} < 1\,. \tag{3.4}$$

Since K has a lower positive bound in any compact set of $\mathbb{R}^2$, by Theorem 3.1, we have either u_n is uniformly bounded in any compact set or u_n is uniformly convergent to $-\infty$ in any compact set of $\mathbb{R}^2$.

STEP 1. We claim that $u_n \longrightarrow -\infty$ uniformly in any compact set of $\mathbb{R}^2$. Suppose u_n is uniformly bounded in any compact set of $\mathbb{R}^2$. By the elliptic estimates, we may assume $u_n \longrightarrow u$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p > 1$. In particular, u satisfies (1.1) and the total curvature

$$\frac{1}{2\pi} \int K(x) e^{2u(x)} dx \le \lim_{n \longrightarrow +\infty} \frac{1}{2\pi} \int K(x) e^{2u_n(x)} dx = \frac{2-b}{2}\,,$$

which yields a contradiction by Remark 2.2. Hence, by Theorem 3.1, we have $u_n \longrightarrow -\infty$ uniformly in any compact set of $\mathbb{R}^2$.

STEP 2. We claim there exists a constant $C > 0$ such that

$$K(x) e^{2u_n(x)} \le C|x|^{-2} \quad \text{for} \quad x \in \mathbb{R}^2\,. \tag{3.5}$$

To prove the claim, we assume there exists $x_n \in \mathbb{R}^2$ such that $u_n(x_n) + \frac{(2-b)}{2} \log |x_n| \longrightarrow +\infty$. By Step 1, we have $|x_n| \longrightarrow +\infty$ as $n \longrightarrow +\infty$. Set

$$v_n(y) = u_n(x_n + |x_n| y) + \frac{2-b}{2} \log |x_n|\,.$$

Then v_n satisfies

$$\Delta v_n + K_n(y)e^{2v_n(y)} = 0 \quad \text{in} \quad |y| < \frac{1}{2}, \tag{3.6}$$

where $K_n(y) = |x_n|^b K_n(x_n + |x_n|y)$. By the assumption on $K, 0 < \bar{C}_1 \leq K_n(y) \leq \bar{C}_2$ for $|y| < \frac{1}{2}$, and

$$\int_{|y|<\frac{1}{2}} K_n(y)e^{2v_n(y)}dy \leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx < 2\pi .$$

By Theorem 3.1, we conclude that $v_n(0) \leq C$ for some constant C, which yields a contradiction to the assumption.

STEP 3. There exists a positive constant C such that $|\nabla u_n(x)| \leq C|x|^{-1}$ and $|u_n(x) - u_n(y)| \leq C$ for $|x| = |y|$.

In [CLn], we have proved that u_n has the following representation

$$u_n(x) = u_n(0) + \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|y|}{|x-y|}K(y)e^{2u_n(y)}dy . \tag{3.7}$$

Thus, we have

$$\begin{aligned} |\nabla u_n(x)| &\leq \frac{1}{2\pi}\int_{\mathbb{R}^2} |x-y|^{-1}K(y)e^{2u_n(y)}dy \\ &= \frac{1}{2\pi}\int_{|y-x|\leq\frac{|x|}{2}} |x-y|^{-1}K(y)e^{2u_n(y)}dy \\ &\quad + \frac{1}{2\pi}\int_{|y-x|\geq\frac{|x|}{2}} |x-y|^{-1}K(y)e^{2u_n(y)}dy \end{aligned}$$

By Step 2, the first integral can be estimated by

$$\frac{1}{2\pi}\int_{|y-x|\leq\frac{|x|}{2}} |x-y|^{-1}K(y)e^{2u_n(y)}dy \leq C_1|x|^{-2}\int_{|y-x|\leq\frac{|x|}{2}} |x-y|^{-1}dy = C_2|x|^{-1} .$$

For the second integral, we have

$$\frac{1}{2\pi}\int_{|y-x|\geq\frac{|x|}{2}} |x-y|^{-1}K(y)e^{2u_n(y)}dy \leq \frac{1}{\pi|x|}\int_{\mathbb{R}^2} K(y)e^{2u_n(y)}dy .$$

Combined these two estimates together, we have

$$|\nabla u_n(x)| \leq C_3|x|^{-1} .$$

Set $w_n(x) = e^{2u_n(x)}$. Then $w_n(x)$ satisfies

$$\Delta w_n(x) + 4(Ke^{2u_n} + |\nabla u_n|^2)w_n = 0 . \tag{3.8}$$

Since $K(x)e^{2u_n(x)} + |\nabla u_n|^2 \le C_4|x|^{-2}$ for some constant C_4, by Harnack inequality, for any $a \ge 1$, there exists a positive constant $C_5 = C_5(a)$ such that

$$\sup_{a^{-1}r\le|x|\le ar} w_n(x) \le C_5 \inf_{a^{-1}r\le|x|\le ar} w_n(x). \tag{3.9}$$

Hence, Step 3 is proved.

STEP 4. For any $\varepsilon>0$ there exists $R=R(\varepsilon)>0$ such that $|u_n(x)-u_n(y)|\le\varepsilon$ for $|x|=|y|\ge R_\varepsilon$ and large n.

Step 4 will be proved by contradiction. Suppose there exist a positive number $\varepsilon_0>0$ and x_n, y_n with $r_n=|x_n|=|\bar{x}_n| \longrightarrow +\infty$ such that $u_n(\bar{x}_n) - u_n(x_n) \ge \varepsilon_0$. Let

$$v_n(y) = u_n(r_n y) - u_n(x_n).$$

Then v_n satisfies

$$\Delta v_n + K_n(y)e^{2v_n} = 0,$$

where $K_n(y) = e^{2u_n(x_n)}K(r_n y)r_n^2$. By Step 2,

$$K_n(y) \le C_1 e^{2u_n(x_n)} r_n^{2-b}|y|^{-b} \le C_2|y|^{-b}. \tag{3.10}$$

For $|y|\ge 1$, we have

$$K_n(y) \ge C_3 e^{2u_n(x_n)} r_n^{2-b}|y|^{-b}. \tag{3.11}$$

By Step 3 and the Harnack inequality (3.9), $v_n(y)$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. By the elliptic estimates, we may assume $v_n(y) \longrightarrow v_0(y)$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p>1$. Suppose there exists a subsequence of x_n (still denoted by x_n) such that $\lim_{n\longrightarrow+\infty} e^{2u_n(x_n)}r_n^{2-b} = S > 0$, then by (3.10) and (3.11), we may assume $K_n(y) \longrightarrow K_0(y)$ weakly in $L^\infty_{\text{loc}}(\mathbb{R}^2\backslash\{0\})$, where $K_0(y)$ satisfies

$$C_1|y|^{-b} \le K_0(y) \le C_2|y|^{-b}$$

for some positive constants C_1 and C_2, and $v_0(y)$ satisfies

$$\Delta v_0(y) + K_0(y)e^{2v_0(y)} = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\{0\}.$$

For any $0<r_0<r_1$, we have

$$\begin{aligned}\int_{r_0\le|y|\le r_1} K_n(y)e^{2v_0(y)}dy &= \lim_{n\longrightarrow+\infty}\int_{r_0\le|y|\le r_1} K_n(y)e^{2v_n(y)}dy\\ &= \int_{r_0 r_n\le|y|\le r_1 r_n} K(x)e^{2u_n(x)}dx\\ &\le \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx\\ &\longrightarrow \left(\frac{2-b}{2}\right)2\pi\end{aligned} \tag{3.12}$$

Thus, the total curvature

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} K_0(y)e^{2v_0(y)}dy \le \frac{2-b}{2}.$$

Applying Corollary 1.4 in [CLn], $v_0(y)$ in fact satisfies

$$\Delta v_0(y) + K_0(y)e^{2v_0(y)} = 2\pi\beta\delta(0) \quad \text{in} \quad \mathbb{R}^2 \tag{3.13}$$

for some $\beta \in \mathbb{R}$, where $\delta(0)$ is the Dirac measure at the origin, and the function $v_1(y) = v_0(y) - \beta \log|y|$ satisfies

$$\Delta v_1(y) + K_0(y)|y|^{2\beta}e^{2v_1(y)}dy = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{3.14}$$

It is easy to see that

$$\begin{aligned} o(1) + \beta &= \frac{1}{2\pi}\int_{|y|=r} \frac{\partial v_0}{\partial \nu}(y)d\sigma \\ &= \lim_{n\longrightarrow+\infty} \frac{1}{2\pi}\int_{|y|=r} \frac{\partial v_n}{\partial \nu}(y)d\sigma \\ &= \lim_{n\longrightarrow+\infty} \frac{-1}{2\pi}\int_{|y|\le r} K_n(y)e^{2v_n(y)}dy \\ &= \lim_{n\longrightarrow+\infty} \frac{-1}{2\pi}\int_{|x|\le r_n r} K(x)e^{2u_n(x)}dx, \end{aligned}$$

where o(1) denotes o(1) $\longrightarrow 0$ as $r \longrightarrow 0$. Thus, putting (3.12) and the above together, we have

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}^2} K_0(y) \mid y \mid^{2\beta} e^{2v_1(y)}dy &= \frac{1}{2\pi}\int_{\mathbb{R}^2} K_0(y)e^{2v_0(y)}dy \\ &\le \frac{2-b}{2} + \beta = \frac{2-b+2\beta}{2}. \end{aligned}$$

Obviously, $2 - b + 2\beta > 0$. Since $K_0(y)|y|^{2\beta} \sim |y|^{-b+2\beta}$ at ∞, by Remark 2.2, there exists no entire solution (3.14) with the total curvature equal to $\frac{2-b+2\beta}{2}$. Thus, it yields a contradiction. Hence we have proved $\lim_{n\longrightarrow+\infty} e^{2u_n(x_n)} r_n^{2-b} = 0$.

Since $e^{2u_n(x_n)}r_n^{2-b} \longrightarrow 0$ as $n \longrightarrow +\infty$, then $v_0(y)$ is harmonic in $\mathbb{R}^2\backslash\{0\}$. By Step 3,

$$|\nabla v_n(y)| = r_n|\nabla u_n(r_n y)| \le C|y|^{-1}.$$

By Liouville's theorem, we have

$$v_0(y) = \alpha_0 \log|y| + C,$$

where both α_0 and C are constant. Since $v_0(y)$ is radially symmetric, it obviously yields a contradiction to the assumption. Hence, Step 4 is proved.

STEP 5. Set

$$F_n(r) = \int_{B_r} K(x) e^{2u_n(x)} dx \,, \tag{3.15}$$

and

$$\bar{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u_n(x) ds \,. \tag{3.16}$$

Define $\bar{K}_n(r)$ by

$$\bar{K}_n(r) = (2\pi r)^{-1} e^{-2\bar{u}_n(r)} \int_{|x|=r} K(x) e^{2u_n(x)} ds \,.$$

Differentiating (3.15) and (3.16) with respect to r, we have

$$F_n'(r) = (2\pi r) \bar{K}_n(r) e^{2\bar{u}_n(r)} \tag{3.17}$$

$$\bar{u}_n'(r) = \frac{-1}{2\pi r} \int_{B_r} K(x) e^{2u_n(x)} dx = \frac{-F_n(r)}{2\pi r} \,. \tag{3.18}$$

Thus, we have

$$\begin{aligned} \left(\frac{r^{1-b} F_n'(r)}{\bar{K}_n(r)} \right)' &= (2\pi r^{2-b} e^{2\bar{u}_n(r)})' \\ &= 2\pi [(2-b) r^{1-b} e^{2\bar{u}_n(r)} + 2r^{2-b} e^{2\bar{u}_n(r)} \bar{u}_n'(r)] \\ &= \frac{(2-b) F_n'}{r^b \bar{K}_n} - \frac{F_n F_n'(r)}{\pi r^b \bar{K}_n} \\ &= \frac{-F_n'(r)}{r^b \bar{K}_n} \left[\frac{F_n(r)}{\pi} - (2-b) \right] . \end{aligned} \tag{3.19}$$

Since $F_n(\infty) > \pi(2-b)$, set r_n to satisfy $F_n(r_n) = \pi(2-b)$. Obviously, $\lim_{n \longrightarrow +\infty} r_n = +\infty$. For any $\varepsilon > 0$, by Step 4, there exists $R = R(\varepsilon) > 0$ such that

$$A e^{2\varepsilon} r^{-b} \le \bar{K}_n(r) \le e^{-2\varepsilon} B r^{-b} \quad \text{for} \quad r \ge R_\varepsilon \,.$$

Hence,

$$\left(\frac{r^{1-b} F_n'(r)}{\bar{K}_n(r)} \right)' \ge \begin{cases} \dfrac{e^{2\varepsilon}}{B} \left((2-b) - \dfrac{F_n(r)}{\pi} \right) F_n'(r) & \text{for} \quad R \le r \le r_n \,, \\ \dfrac{e^{-2\varepsilon}}{A} \left((2-b) - \dfrac{F_n(r)}{\pi} \right) F_n'(r) & \text{for} \quad r \ge r_n \,. \end{cases}$$

Since $\lim_{r\longrightarrow+\infty}\frac{rF_n'(r)}{r^b\bar{K}_n(r)}=0$ for any n, we have

$$
\begin{aligned}
-\frac{r^{1-b}F_n'(r)}{\bar{K}_n(r)}\Big|_{r=R} \geq & \frac{e^{-2\varepsilon}}{B}\int_R^{r_n}\left[(2-b)-\frac{F_n(r)}{\pi}\right]F_n'(r)dr \\
& +\frac{e^{2\varepsilon}}{A}\int_{r_n}^{\infty}\left((2-b)-\frac{F_n(r)}{\pi}\right)F_n'(r)dr \\
= & -e^{-2\varepsilon}B^{-1}\left[(2-b)F_n(r)-\frac{F_n^2(r)}{2\pi}\right]\Big|_{r=R} \\
& +\left(\frac{e^{2\varepsilon}}{B}-\frac{e^{-2\varepsilon}}{A}\right)\frac{\pi(2-b)^2}{2}+e^{-2\varepsilon}A^{-1}\left((2-b)F_n(\infty)-\frac{F_n^2(\infty)}{2\pi}\right).
\end{aligned}
$$

By Step 1, we note that the boundary term at R tends to 0 as $n\longrightarrow+\infty$. By letting $n\longrightarrow+\infty$ first and then $\varepsilon\longrightarrow 0$ the above yields

$$
\begin{aligned}
0 \geq & \left(\frac{1}{B}-\frac{1}{A}\right)\frac{\pi(2-b)^2}{2}+\frac{1}{A}\left[(2-b)\lim_{n\longrightarrow+\infty}F_n(\infty)-\frac{\lim_{n\longrightarrow+\infty}F_n^2(\infty)}{2\pi}\right] \\
= & \frac{\pi(2-b)^2}{2B},
\end{aligned}
$$

a contradiction, where $\lim_{n\longrightarrow+\infty}F_n(\infty)=(2-b)\pi$ is used. Therefore, the proof of Theroem 1.2 is completely finished. □

PROOF OF THEOREM 1.3. Suppose u_n is a sequence of solution of (1.1) and satisfies the assumption of Theorem 1.3. By Remark 3.2, we may assume that either u_n is uniformly bounded in any compact set or u_n uniformly converges to $-\infty$ in any compact set of $\mathbb{R}^2$. By the the same reasoning of Step 1 and Step 2 of Theorem 1.2, there exists a constant $C>0$ such that inequalities

$$K(x)e^{2u_n(x)}\leq C|x|^{-2}, \tag{3.20}$$

$$|\nabla u_n(x)|\leq C|x|^{-1}, \tag{3.21}$$

$$|u_n(x)-u_n(y)|\leq C \quad \text{whenever} \quad |x|=|y| \tag{3.22}$$

hold.

First, we want to prove u_n is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. Suppose the claim is not true. As before, we want to prove the asymptotic symmetry of u_n, i.e. for any $\varepsilon>0$, there exists $R=R(\varepsilon)>0$ such that for $|y|=|x|\geq R$, $|u_n(x)-u_n(y)|\leq\varepsilon$. Assume the conclusion is not true. Then there exists $r_n\longrightarrow+\infty$ such that $u_n(\bar{x}_n)\geq u_n(x_n)+\varepsilon_0$ with $|\bar{x}_n|=|x_n|=r_n$ for some positive constant $\varepsilon_0>0$. Let

$$v_n(y)=u_n(r_ny)-u_n(x_n).$$

Then v_n is bounded in $L^\infty_{\rm loc}(\mathbb{R}^2\backslash\{0\})$ by Harnack inequality and satisfies

$$\Delta v_n + K_n(y)e^{2v_n} = 0 \quad \text{in} \quad \mathbb{R}^2\,,$$

where $K_n(y) = e^{2u_n(x_n)}K(r_n y)r_n^2$. By the assumption on K and (3.20) for any $r_0 > 0$, we have for $|y| \geq r_0$,

$$K_n(y) \leq 2e^{2u_n(x_n)}r_n^{2-b}|y|^{-b}$$

for large n. If $\lim_{n\longrightarrow+\infty} e^{u_n(x_n)}r_n^{2-b} = 0$, then using (3.21) and the same argument of Step 4 of Theorem 1.2, $v_n(y)$ converges to $v_0(y) = \alpha_0\log|y| + C_0$ in $L^\infty_{\rm loc}(\mathbb{R}^2\backslash\{0\})$ where α_0 and C_0 are constant. Since $v_0(y)$ is radially symmetric, it yields a contradiction.

If $\lim_{n\longrightarrow+\infty} e^{2u_n(x_n)}r_n^{2-b} = s > 0$, then $K_n(y) \longrightarrow s|y|^{-b}$ uniformly in any compact set of $\mathbb{R}^2\backslash\{0\}$. Then $v_0(y)$ satisfies

$$\begin{cases} \Delta v_0(y) + s|y|^{-b}e^{2v_0(y)} = \beta\delta(0) & \text{in} \quad \mathbb{R}^2\,, \\ v_0(y) = \dfrac{\beta}{2\pi}\log|y| + \mathrm{O}(1) & \text{as} \quad y \longrightarrow 0\,, \end{cases}$$

where $\beta \in \mathbb{R}$ and $\delta(0)$ is the Dirac measure. For any $r_0 > 0$,

$$\int_{|y|=r_0} \frac{\partial v_0(y)}{\partial \nu}d\sigma = \lim_{n\longrightarrow+\infty}\int_{|y|=r_0}\frac{\partial v_n}{\partial\nu}d\sigma = -\lim_{n\longrightarrow+\infty}\int_{|y|\leq r_0} K_n(y)e^{2v_n}dy \leq 0\,.$$

Thus either $v_0(y)$ is regular at 0 or $v_0(y) \longrightarrow +\infty$ as $|y| \longrightarrow +\infty$. Since $|y|^{-b}e^{2v_0(y)} \in L^1(\mathbb{R}^2)$ and $v_0(y) = \alpha\log|y| + \mathrm{O}(1)$ as $|y| \longrightarrow +\infty$, for some $\alpha \in \mathbb{R}$, we have $2\alpha - b < -2$, i.e. $|y|^{-b}e^{2v_0(y)} = \mathrm{o}(1)|y|^{-2}$ as $|y| \longrightarrow +\infty$. Hence, we can apply the method of moving planes as in [CL] and [CLn] to prove $v_0(y)$ is radially symmetric with respect to the origin, which obviously yields a contradiction. Hence the uniformly asymptotic symmetry of u_n is proved.

To finish the proof of Theorem 1.3, we set

$$F_n(r) = \int_{B_r} K(x)e^{2u_n(x)}dx\,,$$

and,

$$\bar{K}_n(r) = (2\pi r)^{-1}e^{-2\bar{u}_n(r)}\int_{|x|=r} K(x)e^{2u_n(x)}dx\,.$$

As in (3.19), we have

$$\left(\frac{r^{1-b}F_n'(r)}{\bar{K}_n(r)}\right)' = \frac{-F_n'(r)}{r^b\bar{K}_n(r)}\left[\frac{F_n(r)}{\pi} - (2-b)\right]\,. \tag{3.23}$$

For any $\varepsilon > 0$, let $R = R(\varepsilon)$ be large such that

$$e^{-3\varepsilon} r^{-b} \leq \bar{K}_n(r) \leq e^{3\varepsilon} r^{-b}$$

holds for $r \geq R$. This immediately follows from the uniformly asymptotic symmetry of u_n and the assumption on K. Let r_n satisfy $F_n(r_n) = \pi(2-b)$. Suppose $\lim_{n \longrightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx < (2-b)$ first. Then we can follow the same proof as Step 5 in Theorem 1.2 to obtain

$$(2-b)F(\infty) - (2\pi)^{-1} F^2(\infty) \leq 0\,,$$

where $F(\infty) = \lim_{n \longrightarrow +\infty} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx$. Obviously, the above yields $F(\infty) \geq 2\pi(2-b)$, a contradiction.

Suppose $\lim_{n \longrightarrow +\infty} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx > 2\pi(2-b)$. Then by (3.23), we have the reverse inequality

$$\left(\frac{r^{1-b} F_n'(r)}{\bar{K}_n(r)} \right)' \leq \begin{cases} e^{3\varepsilon}[(2-b) - F_n(r)/\pi] F_n'(r) & \text{for} \quad R \leq r \leq r_n\,, \\ e^{-3\varepsilon}[(2-b) - F_n(r)/\pi] F_n'(r) & \text{for} \quad r \geq r_n\,. \end{cases}$$

Integrating the above and letting $n \longrightarrow +\infty$ first and then $\varepsilon \longrightarrow 0$, we have

$$(2-b)F(\infty) - \frac{F^2(\infty)}{2\pi} \geq 0\,,$$

which implies

$$F(\infty) \leq 2\pi(2-b)\,.$$

Obviously, it yields a contradition. Hence the boundedness of u_n in $L^\infty_{\text{loc}}(\mathbb{R}^2)$ is proved.

To prove (1.9), we may assume $u_n \longrightarrow u_0$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. Obviously, u_0 satisfies (1.1) and has a finite total curvature. In particular,

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_0(x)} dx > \frac{2-b}{2}\,.$$

Hence, there exists $R_0 > 0$, $\varepsilon_0 > 0$ and n_0 such that

$$\frac{1}{2\pi} \int_{B_r} K(x) e^{2u_n(x)} dx > \left(\frac{2-b}{2} + \varepsilon_0 \right)$$

for all $r \geq R_0$ and $n \geq n_0$. Integratiang (1.1), we have

$$\frac{d}{dr} \bar{u}_n(r) < -\left(\frac{2-b}{2} + \varepsilon_0 \right) r^{-1}$$

for all $r \geq R_0$ and $n \geq n_0$, where $\bar{u}_n(r) = \frac{1}{2\pi r}\int_{|x|=r} u_n(x)d\sigma$. Thus,

$$\bar{u}_n(r) \leq \bar{u}_n(R_0) - \left(\frac{2-b}{2} + \varepsilon_0\right) \log r/R_0 .$$

Applying the Harnack inequality, we have

$$u_n(x) \leq \bar{u}_n(|x|) + C_1 \leq C_2 - \left(\frac{2-b}{2} + \varepsilon_0\right) \log r$$

for $r \geq R_0$ and $n \geq n_0$ where C_1 and C_2 are constants independent of n and r. In particular,

$$\int_{|y|\geq r} K(x)e^{2\bar{u}_n(x)}dx \leq C_3 \int_{|y|\geq r} |x|^{-(2+2\varepsilon_0)}dx$$

could be arbitraily small provided that r is large. Thus, (1.9) follows immediately. And the proof of Theorem 1.3 is finished. □

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