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## Duality in the Spaces of Solutions of Elliptic Systems

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### Introduction

The aim of this paper is to give representations of the strong dual of the space of solutions of a linear elliptic system  $Pu = 0$  of partial differential equations on an open subset of  $\mathbb{R}^n$ . We consider both determined and overdetermined elliptic systems.

Let  $U$  be an open subset of the domain  $X \subset \mathbb{R}^n$  where the operator  $P$  is defined. Denote by  $\text{sol}(U, P)$  the vector space of all smooth solutions to the equation  $Pu = 0$  on  $U$ , with the usual Fréchet-Schwartz topology. We will write it simply  $\text{sol}(U)$  when no confusion can arise.

Denote by  $\text{sol}(U)'$  the dual space of  $\text{sol}(U)$ , i.e., the space of all continuous linear functionals on  $\text{sol}(U)$ . We tacitly assume that this dual space  $\text{sol}(U)'$  is endowed with the *strong topology*, i.e., the topology of uniform convergence on every bounded subset of  $\text{sol}(U)$ .

Any successful characterization of the dual space  $\text{sol}(U)'$  results in the analysis of solutions to  $Pu = 0$  (*Golubev series*, etc., see Havin [4], Tarkhanov [15]).

There are a few classical examples of representation of this dual space, such as *Grothendieck duality* and *Poincaré duality* (see for instance Tarkhanov [16, Ch. 5]). The Grothendieck duality is of analytical nature; it has been of particular interest in complex analysis. On the other hand, the Poincaré duality can be stated in an abstract framework.

For determined elliptic operators of the type  $P^*P$  we obtain in Section 3 an analogue of the duality result of Grothendieck [3] (cf. Mantovani and Spagnolo [7]). Note that the system  $P^*Pu = 0$  is a straightforward generalization of the *Laplace equation*. In this way we obtain what we shall call generalized harmonic functions, or simply *harmonic functions* when no confusion can arise.

Our main result for general elliptic systems is concerned with the case where the coefficients of  $P$  are real analytic and  $U$  is a relatively compact subdomain of  $X$  with real analytic boundary. In this case we prove the following theorem.

**THEOREM A.** *Let the coefficients of the operator  $P$  be real analytic on  $X$  and  $\mathcal{D} \Subset X$  be a domain with real analytic boundary. Suppose that, given any neighborhood  $U$  of  $\overline{\mathcal{D}}$ , there is a neighborhood  $U' \subset U$  of  $\overline{\mathcal{D}}$  such that  $\text{sol}(U')$  is dense in  $\text{sol}(\mathcal{D})$ . Then*

$$\text{sol}(\mathcal{D})' \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}).$$

In fact, in Sections 6, 7 below, we will formulate and prove a stronger statement with weaker assumptions on analyticity. Moreover, in these sections we provide also an explicit formula for the pairing.

In fact, there is a transparent heuristic explanation of this duality. Given any solution  $v \in \text{sol}(\overline{\mathcal{D}})$ , the *Petrovskii theorem* shows that  $v$  is real analytic in a neighborhood of  $\overline{\mathcal{D}}$ . On the other hand, each  $u \in \text{sol}(\mathcal{D})$  is real analytic in  $\mathcal{D}$ , and so  $u$  is a hyperfunction there. As the sheaf of hyperfunctions is *flabby*,  $u$  can be extended to a hyperfunction in  $X$  with support in the closure of  $\mathcal{D}$ . Thus,  $v$  can be paired with every  $u \in \text{sol}(\mathcal{D})$ .

By *Runge theorem*, the approximation assumption of Theorem A holds for every determined elliptic operator with real analytic coefficients or in the case where  $P$  is an elliptic operator with constant coefficients and  $\mathcal{D}$  is convex.

The approximation condition on the couple  $P$  and  $\mathcal{D}$  in this theorem is to some extent an analogue of the so-called *approximation property* introduced by Grothendieck [3]. In several complex variables a close concept is known as *Runge property* (cf. Hörmander [5]).

For the space of holomorphic functions in simply connected domains in  $\mathbb{C}$  and in  $(p, q)$ -circular domains in  $\mathbb{C}^2$  a similar result was obtained by Aizenberg and Gindikin [1]. For the spaces of harmonic and holomorphic functions a similar result was recently obtained by Stout [13]. However they constructed isomorphisms different from ours. The advantage of our approach is the fact that it highlights the close connection between the duality of Theorem A and the Grothendieck duality (see Section 3).

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## 1. – Preliminaries

Assume that  $X$  is an open set in  $\mathbb{R}^n$ , and  $E = X \times \mathbb{C}^k$ ,  $F = X \times \mathbb{C}^l$  are (trivial) vector bundles over  $X$ . Sections of  $E$  and  $F$  of a class  $\mathfrak{C}$  on an open set  $U \subset X$  can be interpreted as columns of complex valued functions from  $\mathfrak{C}(U)$ , that is,  $\mathfrak{C}(E|_U) \cong [\mathfrak{C}(U)]^k$ , and similarly for  $F$ .

Throughout the paper we will usually write the letters  $u, v$  for sections of  $E$ , and  $f, g$  for sections of  $F$ .

A differential operator  $P$  of order  $p \geq 1$  and type  $E \rightarrow F$  can be written in the form  $P(x, D) = \sum_{|\alpha| \leq p} P_\alpha D^\alpha$ , with suitable  $(l \times k)$ -matrices  $P_\alpha(x)$  of smooth functions on  $X$ .

The *principal symbol*  $\sigma(P)$  of  $P$  is a function on the cotangent bundle of  $X$  with values in the space of bundle morphisms  $E \rightarrow F$ . Given any  $(x, \xi) \in X \times \mathbb{R}^n$ , we have  $\sigma(P)(x, \xi) = \sum_{|\alpha|=p} P_\alpha(x) \xi^\alpha$ .

We say that  $P$  is *elliptic* if the mapping  $\sigma(P)(x, \xi) : \mathbb{C}^k \rightarrow \mathbb{C}^l$  is injective for every  $x \in X$  and  $\xi \in \mathbb{R}^n \setminus \{0\}$ . Hence it follows that  $l \geq k$ ; we say that  $P$  is *determined elliptic* if  $l = k$ , and *overdetermined elliptic* if  $l > k$ .

Every elliptic operator is hypoelliptic, i.e. all distribution sections satisfying  $Pu = 0$  on an open set  $U$  of  $X$  are infinitely differentiable there. If  $U$  is an open subset of  $X$ , then we denote by  $\text{sol}(U, P)$  the vector space of all  $C^\infty$  solutions of the equation  $Pf = 0$  on  $U$ . We will write it simply  $\text{sol}(U)$  when no confusion can arise.

We endow the space  $\text{sol}(U)$  with the topology of uniform convergence on compact subsets of  $U$ . This topology is generated by the family of seminorms

$$\|u\|_{C(E|_K)} = \sup_{x \in K} |u(x)|,$$

where  $K$  runs over all compact subsets of  $U$ .

LEMMA 1.1. *If  $U \subset X$  is open, then the topology in  $\text{sol}(U)$  coincides with that induced by  $C^\infty_{\text{loc}}(E|_U)$ . In particular,  $\text{sol}(U)$  is a Fréchet-Schwartz space.*

PROOF. By a priori estimates for solutions of elliptic equations, if  $K'$  and  $K''$  are compact subsets of  $U$  and  $K'$  is a subset of the interior of  $K''$ , then

$$(1.1) \quad \sup_{|\alpha| \leq j} \|D^\alpha u\|_{C(E|_{K'})} \leq c \|u\|_{C(E|_{K''})} \text{ for all } u \in \text{sol}(U),$$

with a constant  $c$  depending only on  $K'$ ,  $K''$  and  $j$ . Hence it follows that the original topology on  $\text{sol}(U)$  coincides with that induced by  $C^\infty_{\text{loc}}(E|_U)$ . To finish the proof we use the fact that  $C^\infty_{\text{loc}}(E|_U)$  is a Fréchet-Schwartz space.  $\square$

Throughout this paper we assume that the operator  $P$  possesses the following Unique Continuation Property:

$$(U)_s \quad \text{given any domain } \mathcal{D} \subset X, \text{ if } u \in \text{sol}(\mathcal{D}) \text{ vanishes on a non-empty open subset of } \mathcal{D}, \text{ then } u \equiv 0 \text{ on } \mathcal{D}.$$

Here and in the sequel, by a domain is meant any open connected subset of  $\mathbb{R}^n$ . This property holds, for instance, if the coefficients of the operator  $P$  are real analytic.

It is natural to consider solutions of the system  $Pu = 0$  on open sets. However, some problems require to consider solutions on sets  $\sigma \subset X$  which are not open. Here we are interested not simply in restrictions of solutions to the given set, but also in the *local solutions* of the system  $Pu = 0$  on  $\sigma$ , that is, solutions of the system in some (open) neighborhoods of  $\sigma$ .

If  $\sigma$  is a closed subset of  $X$ , then  $\text{sol}(\sigma)$  stands for the space of (equivalence classes of) local solutions of  $Pu = 0$  on  $\sigma$ . Two such solutions are *equivalent* if there is a neighborhood of  $\sigma$  where they are equal. In  $\text{sol}(\sigma)$ , a sequence  $\{u_\nu\}$  is said to converge if there exists a neighborhood  $\mathcal{N}$  of  $\sigma$  such that all the solutions are defined at least in  $\mathcal{N}$  and converge uniformly on compact subsets of  $\mathcal{N}$ .

Alternatively  $\text{sol}(\sigma)$  can be described as the inductive limit of the spaces  $\text{sol}(U_\nu)$ , where  $\{U_\nu\}$  is any decreasing sequence of open sets containing  $\sigma$  such that each neighborhood of  $\sigma$  contains some  $U_\nu$  and such that each connected component of each  $U_\nu$  intersects  $\sigma$ . (This latter condition guarantees that the maps  $\text{sol}(U_\nu) \rightarrow \text{sol}(\sigma)$  are injective. Then the space  $\text{sol}(\sigma)$  is necessarily a Hausdorff space.)

LEMMA 1.2. *Let the operator  $P$  possess the Unique Continuation Property  $(U)_s$ . Then the space  $\text{sol}(\sigma)$  is separated, a subset is bounded if and only if it is contained and bounded in some  $\text{sol}(U_\nu)$ , and each closed bounded set is compact.*

PROOF. This follows by the same method as in Köthe [6, p. 379].  $\square$

## 2. – Green’s function

Denote by  $E^* = X \times (\mathbb{C}^k)'$  the conjugate bundle of  $E$ , and similarly for  $F$ . For the operator  $P$ , we define the transpose  $P'$  as usual, so that  $P'$  is a differential operator of type  $F^* \rightarrow E^*$  and order  $p$  on  $X$ .

Fix the standard Hermitian structure in the fibers  $E_x = \mathbb{C}^k$  ( $x \in X$ ) of  $E$ :  $(u, v)_x = \sum_{j=1}^k u_j \bar{v}_j$  for  $u, v \in \mathbb{C}^k$ . This gives the conjugate linear bundle isomorphism  $\star_E : E \rightarrow E^*$  by  $\langle \star_E v, u \rangle_x = (u, v)_x$  for  $u, v \in E_x$ .

Using matrix operation conventions, we have  $\langle \star_E v, u \rangle_x = v^* u$  for  $u \in \mathbb{C}^k$  where  $v^*$  is the conjugate matrix: we have  $\star_E v = v^*$  under this identification.

The operator  $\star_E$  also acts on sections of  $E$  via  $(\star_E u)(x) = \star_E(u(x))$  for all  $x \in X$ . Thus, for a class  $\mathcal{C}$  of sections of  $E$  we have  $\star_E : \mathcal{C}(E) \rightarrow \mathcal{C}(E^*)$ .

The operator  $\star_E$  is similar to Hodge’s *star operator* on differential forms. We write simply  $\star$  when no confusion can arise.

We are now in a position to endow the spaces  $C_{\text{comp}}^\infty(E)$  and  $C_{\text{comp}}^\infty(F)$ , consisting of infinitely differentiable sections with compact supports of  $E$  and  $F$  respectively, with  $(L^2-)$  pre-Hilbert structures by  $(u, v)_X = \int_X \langle \star v, u \rangle_x dx$ .

Under these structures, the operator  $P$  has a formal adjoint operator which is denoted by  $P^*$ . This is the differential operator of type  $F \rightarrow E$  and order  $p$  on  $X$  given by  $P^*g(x) = \sum_{|\alpha| \leq p} D^\alpha (P_\alpha(x)^* g(x))$  for  $g \in C_{\text{comp}}^\infty(F)$ .

The relation between the transposed operator and its (formal) adjoint becomes clear by using the bundle isomorphism  $\star$ . Namely,  $P^* = \star_E^{-1} P' \star_F$  (see Tarkhanov [15, 4.1.4] for more details).

The operator  $\Delta = P^*P$  is usually referred to as the generalized *Laplacian* associated to  $P$ . It is easy to see that  $\Delta$  is an elliptic differential operator of type  $E \rightarrow E$  and order  $2p$  on  $X$ .

Throughout the paper we shall assume that the operator  $\Delta$  possesses the Unique Continuation Property  $(U)_s$ . Obviously, this implies that  $P$  does so.

If  $P$  is the gradient operator in  $\mathbb{R}^n$ , then  $\Delta = P^*P$  is the usual Laplace operator up to a  $-1$  factor. On the other hand, if  $P$  is the Cauchy-Riemann operator in  $\mathbb{C}^n$ , then  $\Delta = P^*P$  coincides with the usual Laplace operator on  $\mathbb{R}^{2n} \cong \mathbb{C}^n$  up to a  $-1/4$  factor.

In the general case, the solutions of the system  $\Delta u = 0$  are said to be generalized *harmonic functions*.

Let  $\mathcal{O} \Subset X$  be a domain with  $C^\infty$  boundary. Denote by  $n(x)$  the unit outward normal vector to the boundary  $\partial\mathcal{O}$  at a point  $x$ . The system of boundary operators  $\{(\partial/\partial n)^j\}_{j=0,1,\dots,p-1}$  is known to be a Dirichlet system of order  $p-1$  on  $\partial\mathcal{O}$ .

We formulate the Dirichlet problem for the generalized Laplacian  $\Delta$  in the following way.

**PROBLEM 2.1.** Given a section  $f$  of  $E$  over  $\mathcal{O}$ , find a section  $u$  of  $E$  over  $\mathcal{O}$  such that  $\Delta u = f$  in  $\mathcal{O}$  and  $(\partial/\partial n)^j u = 0$  on  $\partial\mathcal{O}$  for  $j = 0, 1, \dots, p-1$ .

As in the classical case, Problem 2.1 is verified to be an elliptic boundary value problem. Moreover, it is formally selfadjoint and possesses at most one solution in reasonable function spaces for  $u$ . So, this problem may be treated by standard tools in the scale  $\{H^s(E|_{\mathcal{O}})\}_{s \in \mathbb{R}}$  of Sobolev spaces on  $\mathcal{O}$  (see Roitberg [11]).

From this treatment, we briefly sketch the relevant material on Green's function. For more details we refer the reader to Roitberg [11] and Tarkhanov [15, 9.3.8].

It turns out that the inverse of the operator corresponding to Problem 2.1 is integral. Namely, there exists a unique kernel  $\mathcal{G}(x, y)$  on  $\mathcal{O} \times \mathcal{O}$  such that, for each data  $f \in H^{s-2p}(E|_{\mathcal{O}})$ , the function

$$(2.1) \quad u(x) = \int_{\mathcal{O}} \mathcal{G}(x, y) f(y) dy \quad (x \in \mathcal{O})$$

belongs to  $H^s(E|_{\mathcal{O}})$  and satisfies  $\Delta u = f$  in  $\mathcal{O}$  and  $(\partial/\partial n)^j u = 0$  on  $\partial\mathcal{O}$  for  $j = 0, 1, \dots, p-1$ . Such a kernel  $\mathcal{G}(x, y)$  is said to be the *Green's function* for Problem 2.1.

We will later give a precise meaning to the integrals in (2.1), specifying to which spaces the Green's function belongs.

The Green's function  $\mathcal{G}(\cdot, y)$  is alternatively defined as the solution to the Dirichlet problem with the data  $f = \delta_y$ , the Dirac *delta-function* supported at  $y \in \mathcal{O}$ . This data is easily verified to belong to all Sobolev spaces  $H^s(\mathcal{O})$  with  $s < -\frac{n}{2}$ .

**THEOREM 2.2.** *The kernel  $\mathcal{G}$  is a  $C^\infty$  section of the bundle  $E \otimes E^*|_{\overline{\mathcal{O}} \times \overline{\mathcal{O}}}$  away from the diagonal of  $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$ .*

PROOF. See Roitberg [11, 7.4]. □

A discussion of the singularity of  $\mathcal{G}(x, y)$  at the diagonal  $\{(x, x) : x \in \overline{\mathcal{O}}\}$  can be found in Roitberg [11, Th. 7.4.3]. For our purpose, it suffices to know that the mapping (2.1), when restricted to  $f \in C_{\text{comp}}^{\infty}(E|_{\mathcal{O}})$ , is a pseudodifferential operator of type  $E|_{\mathcal{O}} \rightarrow E|_{\mathcal{O}}$  and order  $-2p$ . Thus, if  $f$  is sufficiently smooth, the integral in (2.1) is actually a usual Lebesgue integral.

Green's formula enables us to prove that the Green's function is a solution of the adjoint boundary value problem in the  $y$  variable. To explain this more accurately, denote by  $I_k$  the identity  $(k \times k)$ -matrix.

THEOREM 2.3. *Given any  $x \in \mathcal{O}$ , we have:*

$$(2.2) \quad \begin{cases} \Delta'(y, D)\mathcal{G}(x, y) = \delta_x(y)I_k & \text{for } y \in \mathcal{O}, \\ (\partial/\partial n(y))^j \mathcal{G}(x, y) = 0 & \text{for } y \in \partial\mathcal{O} \quad (j = 0, 1, \dots, p-1). \end{cases}$$

PROOF. See Tarkhanov [15, Th. 9.3.24]. □

We are now in a position to state the symmetry of Green's function in the variables  $x$  and  $y$ . This symmetry could be expected from the fact that the Dirichlet problem is (formally) selfadjoint.

COROLLARY 2.4. *The matrix  $\mathcal{G}(x, y)$  is Hermitian, i.e.,  $\mathcal{G}(x, y)^* = \mathcal{G}(y, x)$  for all  $x, y \in \overline{\mathcal{O}}$ .*

PROOF. Indeed, since the solution of Problem 2.1 is unique, it follows from Theorem 2.3 that

$$\begin{aligned} \mathcal{G}(y, x) &= \star_x \mathcal{G}(x, y) \star_y^{-1} \\ &= \mathcal{G}(x, y)^*, \end{aligned}$$

as desired. □

### 3. – Grothendieck duality for harmonic functions

In the sequel, we shall denote by  $\mathcal{O}$  a fixed relatively compact domain in  $X$  with  $C^{\infty}$  boundary  $\partial\mathcal{O}$ , as in Section 2.

Inspired by the work of Grothendieck [3] who used solutions to  $\Delta v = 0$  at infinity, we shall consider the manifold with boundary  $\widehat{\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O}$  as the compactification of  $\mathcal{O}$ .

We use  $\widehat{\mathcal{O}}$  instead of  $\overline{\mathcal{O}}$  to conceptually distinguish this manifold with boundary from the closed subset  $\overline{\mathcal{O}}$  of  $X$ .

The topology of  $\widehat{\mathcal{O}}$  is given by the following neighborhoods bases:

- If  $x \in \mathcal{O}$ , then we take the usual basis of neighborhoods of  $x$  (for example, the family  $\{B \cap \mathcal{O}\}$ , where  $B$  runs over all balls in  $X$  centered at  $x$ ).

- If  $x \in \partial\mathcal{O}$ , then the basis of neighborhoods of  $x$  is defined to be the family  $\{B \cap (\mathcal{O} \cup \partial\mathcal{O})\}$ , where  $B$  runs over all balls in  $X$  centered at  $x$ .

We shall say that an open set  $U$  in  $\widehat{\mathcal{O}}$  is a neighborhood of infinity if  $U$  contains the part at infinity  $\partial\mathcal{O}$  of  $\widehat{\mathcal{O}}$ .

We shall also need the concept of a solution to  $\Delta u = 0$  in a neighborhood  $B \cap (\mathcal{O} \cup \partial\mathcal{O})$  of a point  $x \in \partial\mathcal{O}$ .

By this, we mean any solution to  $\Delta u = 0$  on  $B \cap \mathcal{O}$  (finite part) which is  $C^\infty$  up to  $B \cap \partial\mathcal{O}$  (infinite part) and satisfies  $(\partial/\partial n)^j u = 0$  on  $B \cap \partial\mathcal{O}$  for  $j = 0, 1, \dots, p - 1$ .

Given an open set  $U \subset \widehat{\mathcal{O}}$ , denote by  $\text{sol}(U, \Delta)$  the set of all solutions to  $\Delta u = 0$  on  $U$ .

LEMMA 3.1. *Let  $U$  be a neighborhood of infinity in  $\widehat{\mathcal{O}}$ . Then  $\text{sol}(U, \Delta)$  is a closed subspace of  $\text{sol}(U \cap \mathcal{O}, \Delta)$ .*

PROOF. Pick a sequence  $\{u_\nu\}$  in  $\text{sol}(U, \Delta)$  converging to a solution  $u_\infty$  in  $\text{sol}(U \cap \mathcal{O}, \Delta)$ . We shall establish the lemma by proving that  $u_\infty$  is  $C^\infty$  up to the boundary of  $\mathcal{O}$  and  $(\partial/\partial n)^j u_\infty = 0$  on  $\partial\mathcal{O}$  for  $j = 0, 1, \dots, p - 1$ .

To this end, let  $U'$  be a sufficiently thin open band close to the boundary in  $\mathcal{O}$ , so that  $\partial\mathcal{O} \subset \partial U'$  and  $U' \Subset U$ . We can certainly assume that the boundary of  $U'$  is of class  $C^\infty$ .

By the above, the Dirichlet problem for the Laplacian in  $U'$  is coercive. Hence for any integer  $s \geq p$  there is a constant  $c$  such that

$$(3.1) \quad \|u\|_{H^s(E|_{U'})} \leq c \left( \sum_{j=0}^{p-1} \|(\partial/\partial n)^j u\|_{H^{s-j-1/2}(E|_{\partial U'})}^2 \right)^{\frac{1}{2}}$$

whenever  $u \in H^s(E|_{U'}) \cap \text{sol}(U', \Delta)$ .

Let us apply this estimate to a solution  $u \in \text{sol}(U, \Delta)$ . Since the normal derivatives of  $u$  up to order  $p - 1$  vanish on the part  $\partial\mathcal{O}$  of the boundary of  $U'$ , we can assert that the norm of  $u$  in  $H^s(E|_{U'})$  is dominated by Sobolev norms of the normal derivatives of  $u$  up to order  $p - 1$  on the remaining part of the boundary of  $U'$ . What is especially important here is that this remaining part  $\partial U' \setminus \partial\mathcal{O}$  is a subset in  $U \cap \mathcal{O}$ . Hence combining the *Sobolev embedding theorem* with the interior a priori estimates (1.1) yields

$$(3.2) \quad \sup_{|\alpha| \leq j} \|D^\alpha u\|_{C(E|\_{\overline{U'}})} \leq c \|u\|_{C(E|_K)} \quad \text{for all } u \in \text{sol}(U, \Delta),$$

with  $K$  a compact subset of  $U \cap \mathcal{O}$ , whose interior contains  $\partial U' \setminus \partial\mathcal{O}$ , and  $c$  a constant depending only on  $U', k$  and  $j$ .

We can now return to the sequence  $\{u_\nu\}$ . It follows from (3.2) that, given any multi-index  $\alpha$ , the sequence of derivatives  $\{D^\alpha u_\nu\}$  is a Cauchy sequence in  $C(E|\_{\overline{U'}})$ . Therefore  $\{u_\nu\}$  converges to a section  $u \in C^\infty(E|\_{\overline{U'}})$  uniformly on  $\overline{U'}$  and together with all derivatives.

Obviously,  $u_\infty = u$  in  $U'$ . This shows at once that  $u_\infty$  is  $C^\infty$  up to the boundary of  $\mathcal{O}$  and  $(\partial/\partial n)^j u_\infty = 0$  on  $\partial\mathcal{O}$  for  $j = 0, 1, \dots, p - 1$ , as desired.  $\square$

In the case where  $U$  is an open subset of  $\widehat{\mathcal{O}}$  containing  $\partial\mathcal{O}$  we endow  $\text{sol}(U, \Delta)$  with the topology induced by  $\text{sol}(U \cap \mathcal{O}, \Delta)$ . Then Lemmas 1.1 and 3.1 show that  $\text{sol}(U, \Delta)$  is a Fréchet-Schwartz space. (For the moment we shall say nothing about the topology of  $\text{sol}(U, \Delta)$  in the general case.)

We now invoke the construction of the *inductive limit* of a sequence of Fréchet spaces in order to define the space  $\text{sol}(\sigma, \Delta)$  also for those closed sets  $\sigma$  in  $\widehat{\mathcal{O}}$  which are “approximable” by open subsets of  $\widehat{\mathcal{O}}$  containing  $\partial\mathcal{O}$ . These are nothing but the close subsets of  $\widehat{\mathcal{O}}$  containing the “infinitely far” hypersurface  $\partial\mathcal{O}$ .

Next we fix a *Green operator*  $G_P$  for the differential operator  $P$ . By definition,  $G_P$  is a bidifferential operator of type  $(F^*, E) \rightarrow \Lambda^{n-1}T^*(X)$  (where  $\Lambda^{n-1}T^*(X)$  is the bundle of exterior forms of degree  $n - 1$  on  $X$ ) and order  $p - 1$ , such that  $dG_P(\star g, u) = ((Pu, g)_x - (u, P^*g)_x)dx$  pointwise on  $X$ , for all smooth sections  $g$  of  $F$  and  $u$  of  $E$ .

We immediately obtain:

LEMMA 3.2. *A Green operator for the Laplacian  $\Delta$  is given by*

$$(3.3) \quad G_\Delta(\star v, u) = G_P(\star Pv, u) - \overline{G_P(\star Pu, v)}.$$

Having disposed of these preliminary steps, we fix now an open subset  $U$  of  $\mathcal{O}$  and turn to describing the dual space for  $\text{sol}(U, \Delta)$ .

Given any solution  $v \in \text{sol}(\mathcal{O} \setminus U, \Delta)$ , we define a linear functional  $\mathcal{F}_v$  on  $\text{sol}(U, \Delta)$  as follows.

There is an open set  $\mathcal{N}_v \Subset U$  with piecewise smooth boundary such that  $v$  is still defined and satisfies  $\Delta v = 0$  in a neighborhood of  $\mathcal{O} \setminus \mathcal{N}_v$ . Put

$$(3.4) \quad \langle \mathcal{F}_v, u \rangle = \int_{\partial\mathcal{N}_v} G_\Delta(\star v, u) \quad u \in \text{sol}(U, \Delta).$$

It follows from Stokes’ formula that the value  $\langle \mathcal{F}_v, u \rangle$  is independent of the particular choice of  $\mathcal{N}_v$  with the properties previously mentioned.

LEMMA 3.3. *The functional  $\mathcal{F}_v$  defined by (3.4) is a continuous linear functional on the space  $\text{sol}(U, \Delta)$ .*

PROOF. Use estimate (1.1) with  $K' = \partial\mathcal{N}_v$  and  $j = 2p - 1$ . □

The following result is related to the work of Grothendieck [3] where the concept of solution to  $\Delta v = 0$  regular at the point of infinity of the one-point compactification of  $\mathcal{O}$  was used.

THEOREM 3.4. *Let the operator  $P^*P$  possess the Unique Continuation Property  $(U)_s$  on  $X$ . Then for each open set  $U \subset \mathcal{O}$ , the correspondence  $v \mapsto \mathcal{F}_v$  induces a topological isomorphism*

$$\text{sol}(U, \Delta)' \stackrel{\text{top.}}{\cong} \text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta).$$

PROOF. Pick a continuous linear functional  $\mathcal{F}$  on  $\text{sol}(U, \Delta)$ . Since  $\text{sol}(U, \Delta)$  is a subspace of  $C_{\text{loc}}(E|_U)$ , the space of continuous sections of  $E$  over  $U$ , this functional can be extended, by the Hahn-Banach theorem, to an  $E^*$ -valued measure  $m$  with compact support in  $U$ . Set  $K = \text{supp } m$ .

Let  $\mathcal{N} \Subset U$  be any open set with piecewise smooth boundary such that  $K \subset \mathcal{N}$ . For each solution  $u \in \text{sol}(U, \Delta)$ , we have, by Green's formula,

$$u(x) = - \int_{\partial \mathcal{N}} G_{\Delta}(\mathcal{G}(x, y), u(y)) \quad (x \in \mathcal{N}).$$

(Here  $\mathcal{G}(x, y)$  is the Green's function of the Dirichlet problem for the Laplacian in  $\mathcal{O}$ , as in Section 2.) Therefore

$$\langle \mathcal{F}, u \rangle = \int_U \langle dm, u \rangle_x = \int_{\partial \mathcal{N}} G_{\Delta}(\star v, u),$$

where  $v(y) = - \star_y^{-1} \int_U \langle dm, \mathcal{G}(\cdot, y) \rangle_x$ .

Now we look more closely at the properties of this function  $v$  called the "Fantappiè indicatrix" of  $\mathcal{F}$ . Since  $\Delta'(y, D)\mathcal{G}(x, y) = \delta_x(y)I_k$ , we deduce that  $\Delta v = 0$  away from  $K$ .

Moreover, Theorems 2.2 and 2.3 show that  $v$  is  $C^\infty$  up to the boundary of  $\mathcal{O}$  and satisfies  $(\partial/\partial n)^j v = 0$  on  $\partial \mathcal{O}$  for  $j = 0, 1, \dots, p - 1$ .

From what has already been proved, it follows that  $v \in \text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$  and  $\mathcal{F} = \mathcal{F}_v$ . Our next claim is that such  $v$  is unique.

To this end, we let  $v \in \text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$  satisfy

$$(3.5) \quad \int_{\partial \mathcal{N}_v} G_{\Delta}(\star v, u) = 0 \quad \text{for all } u \in \text{sol}(U, \Delta),$$

where  $\mathcal{N}_v \Subset U$  is an open set with piecewise smooth boundary, such that  $v$  is still defined and satisfies  $\Delta v = 0$  in a neighborhood of  $\mathcal{O} \setminus \mathcal{N}_v$ .

We represent  $v$  in the complement of  $\mathcal{N}_v$  by Green's formula. This is possible because of  $(\partial/\partial n)^j v = 0$  on  $\partial \mathcal{O}$  for  $j = 0, 1, \dots, p - 1$ . We get

$$v(y) = - \star_y^{-1} \int_{\partial \mathcal{N}_v} G_{\Delta}(\star v(x), \mathcal{G}(x, y)) \quad \text{for } y \in \mathcal{O} \setminus \overline{\mathcal{N}_v}.$$

For any fixed  $y \in \mathcal{O} \setminus U$ , we have  $\mathcal{G}(\cdot, y) \in \text{sol}(U, \Delta)$ , and so  $v(y) = 0$  by condition (3.5). Since the operator  $P^*P$  possesses the Unique Continuation Property  $(U)_s$ ,  $v \equiv 0$  if  $\overline{U} \subset \mathcal{O}$ . To complete the proof in the case where  $\overline{U}$  is not contained in  $\mathcal{O}$ , we use the *Runge theorem* for solutions of the equation  $\Delta u = 0$  (cf. Tarkhanov [15, 5.1.6]).

There exists an open set  $\mathcal{N} \Subset U$  with the following properties:

- $\mathcal{N}_v \Subset \mathcal{N}$ , and
- the complement of  $\mathcal{N}$  has no compact connected components in  $U$ .

(The second property can always be achieved by adding all compact connected components of  $U \setminus \mathcal{N}$  to  $\mathcal{N}$ .)

Fix  $y \in \mathcal{O} \setminus \mathcal{N}$ . Then each column of the matrix  $\mathcal{G}(\cdot, y)$  is in  $\text{sol}(\mathcal{N}, \Delta)$ . According to the *Runge theorem*, it can be approximated uniformly on compact subsets of  $\mathcal{O}$  by solutions in  $\text{sol}(U, \Delta)$ . Let  $\{u_\nu\}$  be a resulting sequence for  $\mathcal{G}(\cdot, y)$ , so that the columns of  $u_\nu$  belong to  $\text{sol}(\mathcal{N}, \Delta)$  and  $u_\nu \rightarrow \mathcal{G}(\cdot, y)$  uniformly on compact subsets of  $\mathcal{O}$ .

Applying (1.1) we can assert that the derivatives up to order  $p - 1$  of  $u_\nu$  also converge to the corresponding derivatives of  $\mathcal{G}(\cdot, y)$  uniformly on compact subsets of  $\mathcal{N}$ . Therefore,

$$v(y) = - \lim_{\nu \rightarrow \infty} \int_{\partial \mathcal{N}_\nu} G_\Delta(\star v, u_\nu) = - \lim_{\nu \rightarrow \infty} 0 = 0.$$

Thus,  $v = 0$  in  $\mathcal{O} \setminus \mathcal{N}$ , i.e.,  $v$  is the zero element of  $\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$ .

We have proved that the correspondence  $v \mapsto \mathcal{F}_v$  induces the isomorphism of vector spaces

$$\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta) \xrightarrow{\cong} \text{sol}(U, \Delta)'.$$

We are now going to invoke an operator-theoretic argument to conclude that this algebraic isomorphism is in fact a topological one.

To this end, we note that the spaces  $\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$  and  $\text{sol}(U, \Delta)'$  are both spaces of type *DFS*. (For  $\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$ , see the proof of Theorem 1.5.5 in Morimoto [8, p. 13]. For  $\text{sol}(U, \Delta)'$ , see Lemma 1,1 above.) As the *Closed Graph theorem* is correct for linear maps between spaces of type *DFS* (see Corollary A.6.4. in Morimoto [8, p. 254]), to see that  $v \mapsto \mathcal{F}_v$  is a topological isomorphism, it suffices to show that it is continuous. This latter conclusion, however, is obvious from the way the inductive limit topology is defined, and the construction of  $\mathcal{F}_v$ . This completes the proof.  $\square$

One may conjecture that Theorem 3.4 is still true for *arbitrary* open sets  $U$  in  $\widehat{\mathcal{O}}$ . But we have not been able to prove this.

#### 4. – A corollary

In this section we derive the following consequence of Theorem 3.4.

**COROLLARY 4.1.** *Let  $\mathcal{D} \in \mathcal{O}$  be a domain with real analytic boundary. Assume that the operator  $\Delta$  satisfies the Unique Continuation Property  $(U)_s$  on  $X$  and its coefficients are real analytic in a neighborhood of the boundary of  $\mathcal{D}$ . Then it follows that*

$$(4.1) \quad \text{sol}(\mathcal{D}, \Delta)' \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, \Delta).$$

Before proving this corollary, we briefly discuss a result of Morrey and Nirenberg [9] to be used in the proof.

**THEOREM 4.2.** *Let  $\Delta$  be a determined strongly elliptic differential operator of order  $2p$  with real analytic coefficients on  $X$ . Assume that  $u$  is a solution to  $\Delta u = 0$  in a domain  $\mathcal{D} \subset X$ . If  $u$  vanishes up to order  $p - 1$  on an open real analytic portion  $S$  of the boundary of  $\mathcal{D}$ , then for each point  $x_0 \in S$  there is a neighborhood  $\mathcal{N}(x_0)$  on  $X$  depending only on the operator  $\Delta$  and the domain near  $x_0$ , such that  $u$  may be extended as a solution of  $\Delta u = 0$  from  $\mathcal{N}(x_0) \cap \mathcal{D}$  to the whole neighborhood  $\mathcal{N}(x_0)$ .*

**PROOF.** See Morrey and Nirenberg [9]. □

The important point to note is that the neighborhood  $\mathcal{N}(x_0)$  in Theorem 4.2 is independent of the particular solution  $u$ .

In fact, Morrey and Nirenberg [9] proved the existence of  $\mathcal{N}(x_0)$  by showing that there is a real  $r > 0$  such that, for any  $u \in \text{sol}(\mathcal{D}, \Delta)$  vanishing up to order  $p - 1$  on  $S$ , the Taylor series of  $u$  at  $x_0$  converges in the ball  $B(x_0, r)$ . Thus, the solution  $u$  holomorphically extends to a neighborhood  $\tilde{\mathcal{N}}_{x_0}$  of  $x_0$  in  $\mathbb{C}^n$ .

We are going to apply this corollary in the case where  $\Delta = P^*P$  is the generalized Laplacian. To this end, we have to verify that the Laplacian is strongly elliptic (this notion becomes clear below).

**LEMMA 4.3.** *If  $P$  is an elliptic differential operator of order  $p$ , then the operator  $\Delta = P^*P$  is strongly elliptic of order  $2p$ .*

**PROOF.** What has to be proved is that, given any non-zero vector  $v \in \mathbb{C}^k$ , we have

$$\text{Re } v^* \sigma(\Delta)(x, \xi) v \neq 0 \quad \text{for all } (x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\}).$$

Suppose the lemma is false. Then there is a non-zero vector  $v \in \mathbb{C}^k$  such that  $\text{Re } v^* \sigma(\Delta)(x, \xi) v = 0$  for some  $(x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\})$ . However,

$$\begin{aligned} \text{Re } v^* \sigma(\Delta)(x, \xi) v &= \text{Re}(\sigma(P)(x, \xi)v)^*(\sigma(P)(x, \xi)v) \\ &= |\sigma(P)(x, \xi)v|^2, \end{aligned}$$

and so  $v = 0$  because  $\sigma(P)(x, \xi) : \mathbb{C}^k \rightarrow \mathbb{C}^l$  is injective. This contradicts our assumption. □

We also need a slightly modified version of Theorem 4.2, a version which relates to *inhomogeneous* elliptic boundary value problems.

**LEMMA 4.4.** *We keep the assumptions of Theorem 4.2. Let  $\{B_j\}_{j=0,1,\dots,p-1}$  be a Dirichlet system of order  $p - 1$  with real analytic coefficients on  $S$ . If the Dirichlet data  $u_j = B_j u|_S$  ( $j = 0, 1, \dots, p - 1$ ) of a solution  $u$  to  $\Delta u = 0$  in  $\mathcal{D}$  are real analytic on  $S$ , then for each point  $x_0 \in S$  there exists a neighborhood  $\mathcal{N}(x_0)$  on  $X$  depending only on  $\Delta$ , the domain  $\mathcal{D}$  near  $x_0$  and  $\{u_j\}$ , such that  $u$  may be extended to a solution of  $\Delta u = 0$  on  $\mathcal{N}(x_0)$ .*

PROOF. For  $j = p, p+1, \dots, 2p-1$ , set  $B_j = (\partial/\partial n)^j$ , the  $j$ -th derivative along the unit outward normal vector to  $\mathcal{S}$ . This completes  $\{B_j\}_{j=0,1,\dots,p-1}$  to a Dirichlet system of order  $2p-1$  with real coefficients on  $\mathcal{S}$ .

By the *Cauchy-Kovalevskaya theorem*, there is a unique solution  $u'$  to the *Cauchy problem*

$$(4.2) \quad \begin{cases} \Delta u' = 0 & \text{in } \mathcal{N}, \\ B_j u' = u_j & \text{on } \mathcal{S} \quad (j = 0, 1, \dots, p-1), \\ B_j u' = 0 & \text{on } \mathcal{S} \quad (j = p, p+1, \dots, 2p-1), \end{cases}$$

defined on some neighborhood  $\mathcal{N}$  of  $\mathcal{S}$  in  $X$ . (We observe at once that  $u'$  is real analytic in  $\mathcal{N}$ .)

Let  $\mathcal{N}_{x_0}$  be the neighborhood of  $x_0$  which is guaranteed by Theorem 4.2. We can certainly assume that  $u'$  is defined in  $\mathcal{N}_{x_0}$ , for if not, we replace  $\mathcal{N}_{x_0}$  by  $\mathcal{N}_{x_0} \cap \mathcal{N}$ .

By (4.2), the difference  $u'' = u - u'$  satisfies the equation  $\Delta u'' = 0$  in  $\mathcal{D} \cap \mathcal{N}$  and vanishes up to order  $p-1$  on  $\mathcal{S}$ .

Repeated application of Theorem 4.2 enables us to assert that there is a neighborhood of  $x_0$  on  $X$  depending only on  $\Delta$  and the domain  $\mathcal{D} \cap \mathcal{N}$  near  $x_0$ , such that  $u''$  may be extended to a solution of  $\Delta u'' = 0$  in this neighborhood. To shorten notation, we continue to write  $\mathcal{N}_{x_0}$  for this new neighborhood. Obviously,  $u = u' + u''$  extends to  $\mathcal{N}_{x_0}$ , and the lemma follows.  $\square$

We are now able to prove Corollary 4.1.

PROOF. By Theorem 3.4, we establish the corollary if we prove that

$$(4.3) \quad \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta) \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, \Delta).$$

To this end, define a mapping  $\mathcal{E} : \text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$  in the following way (cf. Tarkhanov [15, 10.2.3]).

Given any  $u \in \text{sol}(\overline{\mathcal{D}}, \Delta)$ , there exists a unique solution  $v$  to the Dirichlet problem

$$(4.4) \quad \begin{cases} \Delta v = 0 & \text{in } \mathcal{O} \setminus \overline{\mathcal{D}}, \\ (\partial/\partial n)^j v = (\partial/\partial n)^j u & \text{on } \partial\mathcal{D} \quad (j = 0, 1, \dots, p-1), \\ (\partial/\partial n)^j v = 0 & \text{on } \partial\mathcal{O} \quad (j = 0, 1, \dots, p-1). \end{cases}$$

By the *regularity* of solutions to the Dirichlet problem,  $v$  is  $C^\infty$  up to the boundary of  $\mathcal{O} \setminus \overline{\mathcal{D}}$  and so  $v \in \text{sol}(\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}})$ .

Let us denote by  $\Omega$  the neighborhood of  $\partial\mathcal{D}$  where the coefficients of  $P^*P$  are real analytic. By the *Petrovskii theorem* there is a neighborhood  $\Omega'$  of  $\partial\mathcal{D}$  where  $u$  is real analytic. Since the Dirichlet data  $\{(\partial/\partial n)^j u\}_{j=0,1,\dots,p-1}$  are real analytic on the real analytic open portion  $\partial\mathcal{D}$  of the boundary of  $\Omega' \setminus \overline{\mathcal{D}}$  and  $\partial\mathcal{D}$  is compact, Lemma 4.4 shows that there is a neighborhood  $\mathcal{N}_v$  of  $\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}$

such that  $v$  extends as a solution of  $\Delta v = 0$  to  $\mathcal{N}_v$ . Moreover,  $\mathcal{N}_v$  depends only on  $\Delta$ , the domain  $\Omega' \setminus \overline{\mathcal{D}}$  near  $\partial\mathcal{D}$  and  $u$ .

For our case, we can derive a little bit more of information on  $\mathcal{N}_v$  than that given by Lemma 4.4. Namely,  $\mathcal{N}_v$  depends on the domain  $\Omega' \cup \mathcal{D} \supset \mathcal{N}_u \supset \overline{\mathcal{D}}$  of  $u$  rather than on  $u$ . Indeed, the difference  $v - u$  satisfies  $\Delta(v - u) = 0$  in the open set  $\mathcal{N}_u \setminus \overline{\mathcal{D}}$  and vanishes up to order  $p - 1$  on the real analytic portion  $\partial\mathcal{D}$  of its boundary. By Theorem 4.2, there is a neighborhood  $\mathcal{N}$  of  $\mathcal{N}_u \setminus \mathcal{D}$  depending only on  $\Delta$  and  $\mathcal{N}_v \setminus \overline{\mathcal{D}}$  near  $\partial\mathcal{D}$ , such that  $v - u$  extends to a solution on  $\mathcal{N}$ . Then  $v = u + (v - u)$  also extends to  $\mathcal{N}$ , and so we can add  $\mathcal{N}$  to  $\mathcal{N}_v$ .

It follows that  $v \in \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ . We set  $\mathcal{E}(u) = v$ , thus obtaining the mapping  $\mathcal{E}: \text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ .

Since the solution of the Dirichlet problem in  $\mathcal{D}$  is unique, the mapping  $\mathcal{E}$  is injective. On the other hand, since this problem is solvable for all Dirichlet data, the mapping  $\mathcal{E}$  is surjective. In other words,  $\mathcal{E}$  is an isomorphism of the vector spaces  $\text{sol}(\overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ .

We now argue as at the end of the proof of Theorem 3.4 to conclude that this algebraic isomorphism is in fact a topological one. Since  $\text{sol}(\overline{\mathcal{D}}, \Delta)$  and  $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$  are both spaces of type *DFS*, we are reduced to proving that  $\mathcal{E}$  is continuous.

To do this, pick a sequence  $\{u_\nu\}$  in  $\text{sol}(\overline{\mathcal{D}}, \Delta)$  converging to zero. By the definition of inductive limit topology, there is a neighborhood  $\mathcal{N}_{\{u_\nu\}}$  of  $\overline{\mathcal{D}}$  such that each  $u_\nu$  is defined in  $\mathcal{N}_{\{u_\nu\}}$  and  $u_\nu \rightarrow 0$  uniformly on compact subsets of  $\mathcal{N}_{\{u_\nu\}}$ .

Set  $v_\nu = \mathcal{E}(u_\nu)$ . From what has already been proved it follows that there is a neighborhood  $\mathcal{N}_{\{v_\nu\}}$  of  $\widehat{\mathcal{O}} \setminus \mathcal{D}$  such that all the  $v_\nu$  are defined in  $\mathcal{N}_{\{v_\nu\}}$ .

As the Dirichlet problem in  $\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}$  is well-posed, we can assert that  $v_\nu \rightarrow 0$  uniformly on  $\widehat{\mathcal{O}} \setminus \mathcal{D}$ . The same holds also for the derivatives of  $\{v_\nu\}$ . We have however to show that  $v_\nu \rightarrow 0$  uniformly on some neighborhood of  $\widehat{\mathcal{O}} \setminus \mathcal{D}$ .

For this purpose, we find an  $r > 0$  and a finite number of points  $x_1, \dots, x_J$  on  $\partial\mathcal{D}$  such that

- the balls  $\{B(x_j, r)\}_{j=1, \dots, J}$  cover  $\partial\mathcal{D}$ ; and
- for any  $\nu$  and  $j$ , the Taylor series of  $v_\nu$  at  $x_j$  converges in the ball  $(B(x_j, r))$ .

(That such  $r$  and  $\{x_j\}$  exists, follows from the comment to Theorem 4.2.)  $\square$

Let  $\mathcal{N} = (\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}) \cup (\bigcup_{j=1}^J B(x_j, r/2))$ . This is a neighborhood of  $\widehat{\mathcal{O}} \setminus \mathcal{D}$ , and we have

$$(4.5) \quad \sup_{x \in \mathcal{N}} |v_\nu(x)| \leq \sup_{x \in \widehat{\mathcal{O}} \setminus \mathcal{D}} |v_\nu(x)| + \sum_{j=1}^J \sup_{x \in B(x_j, r/2)} |v_\nu(x)|.$$

As mentioned,  $\sup_{x \in \widehat{\mathcal{O}} \setminus \mathcal{D}} |v_\nu(x)| \rightarrow 0$  when  $\nu \rightarrow \infty$ . It remains to estimate each term  $\sup_{x \in B(x_j, r/2)} |v_\nu(x)|$ .

Since the Taylor series of  $v_\nu$  at  $x_j$  converges in the ball of radius  $r$ , we obtain by the *Cauchy-Hadamard formula*

$$\left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \leq \text{const}(\nu) \left( \frac{1}{r} \right)^{|\alpha|} \quad \text{for all } \alpha \in \mathbb{Z}_+.$$

Therefore

$$\begin{aligned} \sup_{x \in B(x_j, r/2)} |v_\nu(x)| &= \sup_{x \in B(x_j, r/2)} \left| \sum_{\alpha} \frac{D^\alpha v_\nu(x_j)}{\alpha!} (x - x_j)^\alpha \right| \\ &\leq \sum_{\alpha} \left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \left( \frac{r}{2} \right)^{|\alpha|} \\ &\leq \text{const}(\nu) \sum_{\alpha} \left( \frac{1}{2} \right)^{|\alpha|}. \end{aligned}$$

We may now invoke the *Theorem on Dominated Convergence* to conclude that

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \sup_{x \in B(x_j, r/2)} |v_\nu(x)| &\leq \sum_{\alpha} \left( \lim_{\nu \rightarrow \infty} \left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \right) \left( \frac{r}{2} \right)^{|\alpha|} \\ &= 0, \end{aligned}$$

the last equality being a consequence of the fact that the derivatives of  $\{v_\nu\}$  converge to zero uniformly on  $\partial\mathcal{D}$ .

Thus, (4.5) shows that the sequence  $\{v_\nu\}$  converges to zero uniformly on  $\mathcal{N}$ . It follows that  $\{v_\nu\}$  converges to zero in the topology of  $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ , and so  $\mathcal{E}$  is continuous. This completes the proof.  $\square$

An advantage in describing duality by (4.1) is the fact that it also provides an explicit formula for the pairing.

**COROLLARY 4.5.** *Under the hypothesis of Corollary 4.1, let  $\mathcal{F}_\nu$  be defined by (3.4). Then the correspondence  $\nu \mapsto \mathcal{F}_{\mathcal{E}(\nu)}$  induces the topological isomorphism (4.1).*

**PROOF.** This follows from Theorem 3.4 and the proof of Corollary 4.1.  $\square$

## 5. – Miscellaneous

It turns out that the analyticity of the boundary of  $\mathcal{D}$  is essential for the validity of Corollary 4.5 (cf. Stout [13]).

**EXAMPLE 5.1.** If  $P$  is the Cauchy operator in  $X = \mathbb{R}^2$ , then  $P^*P$  is the usual Laplace operator  $\Delta$  in  $\mathbb{R}^2$  up to the factor  $-\frac{1}{4}$ . Assume that  $\mathcal{D}$  is a

bounded domain in  $\mathbb{R}^2$  with connected boundary  $\partial\mathcal{D}$  of class  $C^2$ . According to the *Riemann theorem*,  $\mathcal{D}$  is holomorphically equivalent to the unit ball  $B(0, 1)$  in  $\mathbb{R}^2$ , i.e., there exists a conformal mapping  $m : \mathcal{D} \rightarrow B(0, 1)$ . Moreover, it is known that  $m$  is of class  $C^1$  up to the boundary of  $\mathcal{D}$  and  $m' \neq 0$  on  $\overline{\mathcal{D}}$ . We denote by  $x^0$  the point of  $\mathcal{D}$  such that  $m(x^0) = 0$ . Let  $\mathcal{O} = B(x^0, R)$ , where  $R$  is a positive number, and  $\mathcal{D} \subseteq B(x^0, R)$ . For  $u(x) = \log \left| \frac{x-x_0}{Rm(x)} \right|$ , an easy verification shows that  $\mathcal{E}(u)(x) = \log \frac{|x-x^0|}{R}$  belongs to  $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ . Clearly,  $u$  is real analytic near the closure of  $\mathcal{D}$  if and only if  $m(x)$  is. Thus, if the boundary of  $\mathcal{D}$  is not real analytic, then  $u$  can fail to be real analytic near the closure of  $\mathcal{D}$ .  $\square$

However, Theorem A is still true for *certain* domains  $\mathcal{D}$  with non-analytic boundary.

EXAMPLE 5.2. Under the hypothesis of Example 5.1, the mapping  $m : \mathcal{D} \rightarrow B(0, 1)$  induces a topological isomorphism of  $\text{sol}(\mathcal{D}, \Delta) \xrightarrow{\cong} \text{sol}(B(0, 1), \Delta)$ . Arguing in a similar way, we see that the complement of  $\overline{\mathcal{D}}$  is holomorphically equivalent to the complement of the closed unit ball in  $\mathbb{R}^2$ . And the corresponding conformal mapping induces a topological isomorphism of  $\text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} \text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{B(0, 1)}, \Delta)$ . Using the Grothendieck duality and the reflexivity of the spaces  $\text{sol}(B(0, 1), \Delta)$  and  $\text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{B(0, 1)}, \Delta)$ , we conclude that  $\text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{B(0, 1)}, \Delta) \stackrel{\text{top.}}{\cong} \text{sol}(B(0, 1), \Delta)$ . Hence  $\text{sol}(\mathcal{D}, \Delta) \stackrel{\text{top.}}{\cong} \text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta)$ . Finally, because of the Grothendieck duality, we have

$$\begin{aligned} \text{sol}(\mathcal{D}, \Delta)' &\stackrel{\text{top.}}{\cong} \text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta)' \\ &\stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, \Delta). \end{aligned}$$

What is still lacking is an explicit description of this duality (cf. Aizenberg and Gindikin [1]).  $\square$

### 6. – Duality for solutions of $Pu = 0$

For a domain  $\mathcal{D} \in \mathcal{O}$  with real analytic boundary, the pairing corresponding to the duality (5.1) is explicitly defined as follows.

Let  $v \in \text{sol}(\overline{\mathcal{D}}, \Delta)$ . Denote by  $\mathcal{E}(v)$  the unique solution to the Dirichlet problem for the Laplacian in  $\mathcal{O} \setminus \overline{\mathcal{D}}$ , with Dirichlet data  $\{(\partial/\partial n)^j v\}_{j=0,1,\dots,p-1}$  on  $\partial\mathcal{D}$  and zero Dirichlet data on  $\partial\mathcal{O}$  (cf. (4.4)). There exists an open set  $\mathcal{N}_{\mathcal{E}(v)} \in \mathcal{D}$  with piecewise smooth boundary, such that  $\mathcal{E}(v)$  still satisfies  $\Delta\mathcal{E}(v) = 0$  in a neighborhood of  $\mathcal{O} \setminus \mathcal{N}_{\mathcal{E}(v)}$ . Set

$$(6.1) \quad \langle \mathcal{F}_{\mathcal{E}(v)}, u \rangle = \int_{\partial\mathcal{N}_{\mathcal{E}(v)}} G_{\Delta}(\star\mathcal{E}(v), u) \quad (u \in \text{sol}(\mathcal{D}, \Delta)).$$

Then the correspondence  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$  induces, by Corollary 4.5, the *topological* isomorphism  $\text{sol}(\overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} \text{sol}(\mathcal{D}, \Delta)'$

Since  $\Delta = P^*P$ , we have

$$\begin{aligned} \text{sol}(\mathcal{D}, P) &\hookrightarrow \text{sol}(\mathcal{D}, \Delta), \\ \text{sol}(\overline{\mathcal{D}}, P) &\hookrightarrow \text{sol}(\overline{\mathcal{D}}, \Delta) \end{aligned}$$

(and both subspaces are closed).

Moreover, equality (3.3) shows that the restriction of the functional (6.1) to the subspace  $\text{sol}(\mathcal{D}, P)$  is given by

$$(6.2) \quad \langle \mathcal{F}_{\mathcal{E}(v)}, u \rangle = \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P \mathcal{E}(v), u) \quad (u \in \text{sol}(\mathcal{D}, P)).$$

Again it follows from *Stokes' formula* that the value  $\langle \mathcal{F}_{\mathcal{E}(v)}, u \rangle$  is independent of the particular choice of  $\mathcal{N}_v$  with the properties previously mentioned. By the above,  $\mathcal{F}_{\mathcal{E}(v)}$  is a continuous linear functional on the space  $\text{sol}(\mathcal{D}, P)$ .

Of course, it is no longer true that to different solutions  $v_1$  and  $v_2$  in  $\text{sol}(\overline{\mathcal{D}}, \Delta)$  there correspond different functionals  $\mathcal{F}_{\mathcal{E}(v_1)}$  and  $\mathcal{F}_{\mathcal{E}(v_2)}$  on  $\text{sol}(\mathcal{D}, P)$  by (6.2). However, this still holds if we vary  $v$  within  $\text{sol}(\overline{\mathcal{D}}, P)$  only.

LEMMA 6.1. *If  $v \in \text{sol}(\overline{\mathcal{D}}, P)$  satisfies*

$$(6.3) \quad \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P \mathcal{E}(v), u) = 0 \quad \text{for all } u \in \text{sol}(\mathcal{D}, P),$$

*then  $v = 0$ .*

PROOF. Take  $u = v$  in (6.3). By Stokes' formula,

$$\begin{aligned} 0 &= \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P \mathcal{E}(v), v) \\ &= \int_{\partial \mathcal{D}} G_P(\star P \mathcal{E}(v), v) \\ &= \int_{\partial \mathcal{D}} G_P(\star P \mathcal{E}(v), \mathcal{E}(v)), \end{aligned}$$

the last equality being a consequence of the fact that  $v = \mathcal{E}(v)$  up to order  $p - 1$  on  $\partial \mathcal{D}$ . As  $\mathcal{E}(v)$  vanishes up to order  $p - 1$  on  $\partial \mathcal{O}$  and  $\Delta \mathcal{E}(v) = 0$  in  $\mathcal{O} \setminus \overline{\mathcal{D}}$ , we obtain by the definition of Green operators

$$\begin{aligned} 0 &= - \int_{\partial(\mathcal{O} \setminus \mathcal{D})} G_P(\star P \mathcal{E}(v), \mathcal{E}(v)) \\ &= - \int_{\mathcal{O} \setminus \mathcal{D}} |P \mathcal{E}(v)|^2 dx. \end{aligned}$$

Hence it follows that  $P\mathcal{E}(v) = 0$  in  $\mathcal{O} \setminus \overline{\mathcal{D}}$ .

Consider the section

$$\tilde{v} = \begin{cases} v & \text{in } \mathcal{D}, \\ \mathcal{E}(v) & \text{in } \mathcal{O} \setminus \overline{\mathcal{D}}. \end{cases}$$

It is of class  $C_{\text{loc}}^{p-1}(E|_{\mathcal{O}})$  and satisfies  $P\tilde{v} = 0$  away from the hypersurface  $\partial\mathcal{O}$ . A familiar argument on removable singularities (see for instance Tarkhanov [14, Theorem 3.2]) shows that  $\tilde{v}$  is actually a solution to  $P\tilde{v} = 0$  on the whole domain  $\mathcal{O}$ .

Since  $\tilde{v}$  vanishes up to order  $p - 1$  on  $\partial\mathcal{O}$ , it follows that  $\tilde{v} = 0$  in  $\mathcal{O}$ . Hence  $v = 0$  in  $\mathcal{D}$ , as desired.  $\square$

Thus, the correspondence  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ , provides us with an injective mapping  $\text{sol}(\overline{\mathcal{D}}, P) \rightarrow \text{sol}(\mathcal{D}, P)'$ . One may ask whether this mapping is surjective. We prove that this is the case if and only if the domain  $\mathcal{D}$  possesses a convexity property with respect to the operator  $P$ .

**THEOREM 6.2.** *Let  $\mathcal{D} \Subset \mathcal{O}$  be a domain with real analytic boundary. Assume that the operator  $P^*P$  possesses the Unique Continuation Property  $(U)_s$  and has real analytic coefficients in a neighborhood of  $\partial\mathcal{D}$ . If, given any neighborhood  $U$  of  $\overline{\mathcal{D}}$ , there is a neighborhood  $U' \subset U$  of  $\overline{\mathcal{D}}$  such that  $\text{sol}(U')$  is dense in  $\text{sol}(\mathcal{D})$  then the correspondence  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ , when restricted to  $v \in \text{sol}(\overline{\mathcal{D}}, P)$ , induces the topological isomorphism*

$$\text{sol}(\mathcal{D}, P)' \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, P).$$

This result sharpens Theorem A in the Introduction.

### 7. – Proof of the main theorem

The main step in the proof consists of verifying the surjectivity of the mapping  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ .

Let  $\mathcal{F}$  be a continuous linear functional on  $\text{sol}(\mathcal{D}, P)$ . Since  $\text{sol}(\mathcal{D}, P)$  is a subspace of  $C_{\text{loc}}(E|_{\mathcal{D}})$ , this functional can be extended, by the Hahn-Banach theorem, to an  $E^*$ -valued measure  $m$  with compact support in  $\mathcal{D}$ . We set  $K = \text{supp } m$ .

As in the previous section, we denote by  $\Omega$  the neighborhood of  $\partial\mathcal{D}$  where the coefficients of  $P^*P$  are real analytic. Fix an open set  $\mathcal{N} \Subset \mathcal{D}$  with piecewise smooth boundary, such that  $K \subset \mathcal{N}$  and  $\partial\mathcal{N} \subset \Omega$ . We first argue formally.

**SKETCH OF THE PROOF OF SURJECTIVITY.** For any  $u \in \text{sol}(\mathcal{D}, P)$ , we have by Green's formula

$$\begin{aligned} u(x) &= - \int_{\partial\mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u) \\ &= - \int_{\partial\mathcal{N}} G_P(\star P \mathcal{E}(\mathcal{E}^{-1}(\star^{-1} \mathcal{G}(x, \cdot))), u) \end{aligned}$$

whenever  $x \in \mathcal{N}$ .

Suppose that outside of a larger open set  $\mathcal{N}' \Subset \mathcal{D}$  with piecewise smooth boundary  $K(x, \cdot) = \mathcal{E}^{-1} \star^{-1} \mathcal{G}(x, \cdot)$  can be decomposed into the sum  $K(x, \cdot) = K_1(x, \cdot) + K_2(x, \cdot)$ , where  $K_1(x, \cdot) \in \text{sol}(\overline{\mathcal{D}}, P)$  is sufficiently smooth in  $x \in \mathcal{N}$ , and  $K_2(x, \cdot)$  is orthogonal to  $u$  under the pairing  $\int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_2(x, \cdot)), u)$ .

Then

$$u(x) = - \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_1(x, \cdot)), u), \quad x \in \mathcal{N},$$

and so

$$\langle \mathcal{F}, u \rangle = \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(v), u)$$

with  $v(y) = -\langle dm, K_1(\cdot, y) \rangle_{\mathcal{N}}$ .

Hence it follows that  $v \in \text{sol}(\overline{\mathcal{D}}, P)$  and  $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$ , as desired. □

We now proceed to give a rigorous proof. By Theorem 2.3, the columns of the Green's function  $\star_y^{-1} \mathcal{G}(x, y)$  belongs to  $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{N}, \Delta)$  in the variable  $y$ , for each fixed  $x \in \mathcal{N}$ . In the following we will apply different operators and notations to matrices, understanding that they hold for each of their columns.

Given any fixed  $x \in \mathcal{N}$ , let  $K(x, \cdot) = \mathcal{E}^{-1}(\star^{-1} \mathcal{G}(x, \cdot))$ , i.e.,  $K(x, y)$  be the unique solution to the following Dirichlet problem:

$$\begin{cases} \Delta(y, D)K(x, y) = 0 & \text{for } y \in \overline{\mathcal{D}}, \\ (\partial/\partial n(y))^j K(x, y) = (\partial/\partial n(y))^j (\star_y^{-1} \mathcal{G}(x, y)) & \text{for } y \in \partial \mathcal{D} \ (j=0, 1, \dots, p-1). \end{cases}$$

Since  $\overline{\mathcal{N}} \subset \mathcal{D}$ , it follows from Lemma 4.4 that there is a neighborhood  $U \Subset \Omega \cup D$  of  $\overline{\mathcal{D}}$  independent of  $x \in \mathcal{N}$ , such that  $K(x, \cdot)$  belongs to  $\text{sol}(U, \Delta)$ . (We use here the fact that the Green's function is real analytic away from the diagonal in  $\Omega \times \Omega$ .)

Moreover,  $K(x, \cdot)$  is real analytic in  $x \in \mathcal{N} \cap \Omega$  because of the *Poisson formula* for solutions of the Dirichlet problem (cf. Tarkhanov [15, (9.3.12)]).

As mentioned,  $\text{sol}(U, P)$  is a closed subspace of  $\text{sol}(U, \Delta)$ . Our next goal is to extract a summand from  $K(x, \cdot)$  which corresponds to this subspace, so that the rest is orthogonal to  $\text{sol}(\mathcal{D}, P)$  in a suitable sense. To this end, we invoke Hilbert space techniques.

We first recall a result of Nacinovich and Shlapunov [10].

LEMMA 7.1. *The Hermitian form*

$$(7.1) \quad h(u, v) = \int_{\mathcal{D}} (Pu, Pv)_x dx + \int_{\mathcal{O} \setminus \mathcal{D}} (P\mathcal{E}(u), P\mathcal{E}(v))_x dx \quad (u, v \in H^p(E|_{\mathcal{D}})),$$

defines a scalar product on  $H^p(E|_{\mathcal{D}})$ , and the topologies induced in  $H^p(E|_{\mathcal{D}})$  by  $h(\cdot, \cdot)$  and by the standard scalar product are equivalent.

PROOF. See *ibid* as well as in Tarkhanov [15, 10.2.3]. □

An easy calculation shows that if moreover  $v$  is sufficiently smooth up to the boundary of  $\mathcal{D}$  (it suffices  $v \in H^{2p}(E|_{\mathcal{D}})$ ) and  $u$  satisfies  $Pu = 0$  in  $\mathcal{D}$ , then

$$(7.2) \quad h(u, v) = - \int_{\partial\mathcal{D}} G_P(\star P\mathcal{E}(v), \mathcal{E}(u)).$$

By assumption, there is a neighborhood  $U' \Subset U$  of  $\overline{\mathcal{D}}$  such that  $\text{sol}(U', P)$  is dense in  $\text{sol}(\mathcal{D}, P)$ . We can certainly assume that  $H^p(E|_{U'}) \cap \text{sol}(U', P)$  is dense in  $\text{sol}(\mathcal{D}, P)$ , for if not, we replace  $U'$  by a smaller neighborhood.

Denote by  $H_2$  the closure of  $H^p(E|_{U'}) \cap \text{sol}(U', P)$  in  $H^p(E|_{\mathcal{D}})$ ; we endow  $H_2$  with scalar product (7.1).

The following result is a particular case of a general theorem of Shlapunov and Tarkhanov [12] (see also [15, 12.1.2]).

LEMMA 7.2. *There exists an orthonormal basis  $\{e_\nu\}$  in  $H^p(E|_{U'}) \cap \text{sol}(U', P)$  such that the restriction of  $\{e_\nu\}$  to  $\mathcal{D}$  is an orthogonal basis in  $H_2$ .*

PROOF. Consider the mapping  $R : H^p(E|_{U'}) \cap \text{sol}(U', P) \rightarrow H_2$  given by restricting sections over  $U'$  to  $\mathcal{D}$ . (It will cause no confusion if we use the same symbol for a section  $u \in H^p(E|_{U'}) \cap \text{sol}(U', P)$  and its restriction  $Ru$  to  $\mathcal{D}$ .)

By the Unique Continuation Property  $(U)_s$ ,  $R$  is injective. Moreover, by *Stiltjes-Vitali theorem*  $R$  is compact. It follows that  $R^*R$  is a compact selfadjoint operator of zero null-space in the Hilbert space  $H^p(E|_{U'}) \cap \text{sol}(U', P)$ . (Here  $R^*$  stands for the adjoint of  $R$  in the sense of Hilbert spaces.)

Let  $\{e_\nu\}$  be a complete orthonormal system of eigenfunctions of the operator  $R^*R$  in  $H^p(E|_{U'}) \cap \text{sol}(U', P)$  corresponding to eigenvalues  $\{\lambda_\nu\}$ . Since  $H^p(E|_{U'}) \cap \text{sol}(U', P)$  is dense in  $H_2$ , we can assert that

- $\{e_\nu\}$  is orthonormal basis in  $H^p(E|_{U'}) \cap \text{sol}(U', P)$ ; and
- the system  $\{Re_\nu\}$  is a basis in  $H_2$  orthogonal with respect to the scalar product  $h(\cdot, \cdot)$ .

Thus, the system  $\{e_\nu\}$  possesses the desired properties, and the lemma follows. □

Note that the Fourier coefficients of a section  $u \in H^p(E|_{U'}) \cap \text{sol}(U', P)$  with respect to the system  $\{e_\nu\}$  are given by

$$(7.3) \quad \begin{aligned} (u, e_\nu)_{H^p(E|_{U'})} &= \frac{1}{\lambda_\nu} (u, R^*Re_\nu)_{H^p(E|_{U'})} \\ &= \frac{1}{\lambda_\nu} h(Ru, Re_\nu) \\ &= \frac{1}{\lambda_\nu} h(u, e_\nu), \end{aligned}$$

where  $\lambda_\nu = h(e_\nu, e_\nu)$ .

Our next objective is to treat the “projection” of the kernel  $K(x, \cdot)$  on the space  $H^p(E|_{U'}) \cap \text{sol}(U', P)$ . To do this, we need the following technical lemma.

LEMMA 7.3. *Let  $\{e_\nu\}$  be an orthonormal system in a separable Hilbert space  $H$ , and  $K(x)$  be a continuous function on a topological space  $T$  with values in  $H$ . Then the Fourier series  $\sum_\nu (K(x), e_\nu)_H e_\nu$  converges in the norm of  $H$  uniformly in  $x$  on compact subset of  $T$ .*

PROOF. Denote by  $H_1$  the closure of the linear span of  $\{e_\nu\}$  in  $H$ . Pick a complete orthonormal system  $\{b_\mu\}$  in the orthogonal complement of  $H_1$  in  $H$ . Then  $\{e_\nu\} \cup \{b_\mu\}$  is an orthonormal basis in  $H$ .

Given any  $x \in T$ , decompose  $K(x)$  into the Fourier series with respect to this basis. Namely,

$$K(x) = \sum_{\nu=1}^{\infty} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{\infty} (K(x), b_\mu)_H b_\mu.$$

Hence it follows that

$$\begin{aligned} & \left\| K(x) - \left( \sum_{\nu=1}^N (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^N (K(x), b_\mu)_H b_\mu \right) \right\|_H^2 \\ &= \sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 + \sum_{\mu=N+1}^{\infty} |(K(x), b_\mu)_H|^2, \end{aligned}$$

and so

$$(7.4) \quad \sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 \leq \left\| K(x) - \left( \sum_{\nu=1}^N (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^N (K(x), b_\mu)_H b_\mu \right) \right\|_H^2$$

for all  $\nu = 1, 2, \dots$

Since the Fourier series converges in the norm  $H$ , for every  $x^0 \in T$  and  $\varepsilon > 0$  there is a number  $N^0$  depending on  $x^0$  and  $\varepsilon$ , such that

$$\left\| K(x^0) - \left( \sum_{\nu=1}^{N^0} (K(x^0), e_\nu)_H e_\nu + \sum_{\mu=1}^{N^0} (K(x^0), b_\mu)_H b_\mu \right) \right\|_H < \varepsilon.$$

Moreover, from the continuity of  $K(x)$  at  $x^0$  we deduce that the set

$$\mathcal{N}(x^0) = \left\{ x \in T : \left\| K(x) - \left( \sum_{\nu=1}^{N^0} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{N^0} (K(x), b_\mu)_H b_\mu \right) \right\|_H < \varepsilon \right\}$$

is an open neighborhood of  $x^0$ .

Applying (7.4) yields

$$\sum_{v=N+1}^{\infty} |(K(x), e_v)_H|^2 \leq \sum_{v=N^0+1}^{\infty} |(K(x), e_v)_H|^2 < \varepsilon^2,$$

for all  $N \geq N^0$  and  $x \in \mathcal{N}(x^0)$ . Therefore, the series  $\sum_v (K(x), e_v)_H e_v$  converges in the norm of  $H$  uniformly in  $x \in \mathcal{N}(x^0)$ .

As each compact subset of  $T$  can be covered by a finite number of such neighborhoods, the lemma follows.  $\square$

By the above,  $K(x, \cdot)$  is a continuous function of  $x \in \mathcal{N}$  with values in the Hilbert space  $H^p(E|_{U'}) \cap \text{sol}(U', \Delta)$ . Lemma 7.3 thus shows that the series

$$(7.5) \quad K_1(x, \cdot) = \sum_{v=1}^{\infty} (k(x, \cdot), e_v)_{H^p(E|_{U'})} e_v$$

converges in  $H^p(E|_{U'}) \cap \text{sol}(U', \Delta)$  uniformly in  $x$  on compact subsets of  $\mathcal{N}$ . As the same holds for the derivatives of  $K(x, \cdot)$  with respect to the  $x$  variables, we conclude that  $K_1(x, \cdot)$  is of class  $C^\infty$  in  $x \in \mathcal{N}$ .

We now apply the operator  $\mathcal{E}$  to both sides of equality (7.5). Since  $\mathcal{E}$  determines a topological isomorphism of  $\text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ , there is an open set  $\mathcal{N}' \Subset \mathcal{D}$  with piecewise smooth boundary, such that every  $\mathcal{E}(e_v)$  extends to a solution of  $\Delta v = 0$  in a neighborhood of  $\mathcal{O} \setminus \mathcal{N}'$ , and the series

$$\mathcal{E}(K_1(x, \cdot)) = \sum_{v=1}^{\infty} (K(x, \cdot), e_v)_{H^p(E|_{U'})} \mathcal{E}(e_v)$$

converges in  $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{N}', \Delta)$ . (By construction,  $\mathcal{N}'$  is larger than  $\mathcal{N}$ , since otherwise we obtain a gain in analyticity.)

LEMMA 7.4. For each  $u \in \text{sol}(\mathcal{D}, P)$ , it follows that

$$(7.6) \quad u(x) = - \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_1(x, \cdot)), u), \quad x \in \mathcal{N}.$$

PROOF. Pick a system  $\{b_\mu\}$  in  $H^p(E|_{\mathcal{D}}) \cap \text{sol}(\mathcal{D}, \Delta)$  such that  $\{e_v\} \cup \{b_\mu\}$  is a basis in this space orthogonal with respect to scalar product (7.1).

For a fixed  $x \in \mathcal{N}$ , we decompose  $K(x, \cdot)$  into the Fourier series with respect to this basis, i.e.,

$$(7.7) \quad \begin{aligned} K(x, \cdot) &= \sum_{v=1}^{\infty} \frac{h(K(x, \cdot), e_v)}{h(e_v, e_v)} e_v + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu \\ &= \sum_{v=1}^{\infty} (K(x, \cdot), e_v)_{H^p(E|_{U'})} e_v + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu \\ &= K_1(x, \cdot) + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu, \end{aligned}$$

the second equality being a consequence of (7.3). (Note that the series on the right-hand here converges in the norm of  $H^p(E|_{\mathcal{D}})$ .)

As  $\mathcal{E}$  is a topological isomorphism of

$$H^p(E|_{\mathcal{D}}) \cap \text{sol}(\mathcal{D}, \Delta) \rightarrow H^p(E|_{\mathcal{O} \setminus \overline{\mathcal{D}}}) \cap \text{sol}(\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}, \Delta),$$

we may apply  $\mathcal{E}$  to (7.7) termwise, thus obtaining

$$\star^{-1}\mathcal{G}(x, \cdot) = \mathcal{E}(K_1(x, \cdot)) + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_{\mu})}{h(b_{\mu}, e_{\mu})} \mathcal{E}(b_{\mu}), \quad x \in \mathcal{N},$$

the series converging in the norm of  $H^p(E|_{\mathcal{O} \setminus \overline{\mathcal{D}}})$ .

Having disposed of this preliminary step, we can now return to representation (7.6). Let  $u \in \text{sol}(\mathcal{D}, P)$ . By assumption, there exists a sequence  $\{u_j\}$  in  $H^p(E|_{U'}) \cap \text{sol}(U', P)$  converging to  $u$  together with all derivatives uniformly on compact subsets of  $\mathcal{D}$ . Given any  $x \in \mathcal{N}$ , we have therefore by Green's formula

$$\begin{aligned} u(x) &= - \int_{\partial \mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u) \\ &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u_j) \\ &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{D}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u_j), \end{aligned}$$

the second equality being a consequence of Lemma 1.1, and the third equality being a consequence of Stokes' formula.

On the boundary of  $\mathcal{D}$ , we have  $u_j = \mathcal{E}(u_j)$  up to order  $p - 1$ . Therefore

$$\begin{aligned} u(x) &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{D}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), \mathcal{E}(u_j)) \\ &= \lim_{j \rightarrow \infty} h(u_j, K(x, \cdot)), \end{aligned}$$

which is due to (7.2).

On the other hand, since every  $u_j$  is in  $H^p(E|_{U'}) \cap \text{sol}(U', P)$ , we may write

$$u_j = \sum_{\nu=1}^{\infty} (u_j, e_{\nu})_{H^p(E|_{U'})} e_{\nu},$$

where the series converges in the norm of  $H^p(E|_{U'})$ . Combining this with (7.7) yields

$$h(u_j, K(x, \cdot)) = h(u_j, K_1(x, \cdot)),$$

for the systems of sections  $\{e_{\nu}\}$  and  $\{b_{\mu}\}$  are pairwise orthogonal with respect to  $h(\cdot, \cdot)$ .

Thus

$$\begin{aligned} u(x) &= \lim_{j \rightarrow \infty} h(u_j, K(x, \cdot)) \\ &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{D}} G_P(\star P \mathcal{E}(K_1(x, \cdot)), \mathcal{E}(u_j)) \\ &= - \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_1(x, \cdot)), u), \end{aligned}$$

for  $x \in \mathcal{N}$ . This is precisely the assertion of the lemma. □

We are now in a position to finish the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. From Lemma 7.4 it follows that

$$\langle \mathcal{F}, u \rangle = \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(v), u) \quad \text{for all } u \in \text{sol}(\mathcal{D}, P),$$

where  $v(y) = -\langle dm, K_1(\cdot, y) \rangle_{\mathcal{N}}$ .

One easily verifies that  $Pv = 0$  in  $U'$ . Hence  $v \in \text{sol}(\overline{\mathcal{D}}, P)$  and  $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$ , which proves the surjectivity of the mapping  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ .

When combined with Lemma 6.1, this shows that the correspondence  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$  induces the isomorphism of vector spaces

$$\text{sol}(\overline{\mathcal{D}}, P) \xrightarrow{\cong} \text{sol}(\mathcal{D}, P)'.$$

We now argue as at end of the proof of Theorem 3.4 to conclude that this algebraic isomorphism is in fact a topological one.

For this purpose, we note that the spaces  $\text{sol}(\overline{\mathcal{D}}, P)$  and  $\text{sol}(\mathcal{D}, P)'$  are both spaces of type *DFS*. (For  $\text{sol}(\overline{\mathcal{D}}, P)$ , see the proof of Theorem 1.5.5 in Morimoto [8, p. 13]. For  $\text{sol}(\mathcal{D}, P)'$ , see Lemma 1.1 above.) As the *Closed Graph theorem* is correct for linear maps between spaces of type *DFS* (see Corollary A.6.4 in Morimoto [8, p. 24]), to see that  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$  is a topological isomorphism, it suffices to show that it is continuous.

The latter conclusion is however a consequence of the following two facts already proved:

- the mapping  $v \mapsto \mathcal{F}_v$  of  $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta) \rightarrow \text{sol}(\mathcal{D}, \Delta)$  is continuous (cf. Theorem 3.4); and
- the mapping  $v \mapsto \mathcal{E}(v)$  of  $\text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$  is continuous (cf. Corollary 4.1).

This completes the proof. □

Let us mention an important consequence of Theorem 6.2.

COROLLARY 7.5. *Under the hypotheses of Theorem 6.2, it follows that*

$$\text{sol}(\overline{\mathcal{D}}, P)' \stackrel{\text{top.}}{\cong} \text{sol}(\mathcal{D}, P).$$

PROOF. By Lemma 1.1,  $\text{sol}(\mathcal{D}, P)$  is a Fréchet-Schwartz space. Therefore, it is a *Montel space*. That  $\text{sol}(\mathcal{D}, P)$  is a Montel space implies that it is reflexive, i.e., under the natural pairing, we have

$$(\text{sol}(\mathcal{D}, P)')' \stackrel{\text{top.}}{\cong} \text{sol}(\mathcal{D}, P),$$

where both  $\text{sol}(\mathcal{D}, P)'$  and  $(\text{sol}(\mathcal{D}, P)')'$  are provided with the strong topology. Thus, the desired statement follows immediately from Theorem 6.2.  $\square$

## 8. – The converse theorem

Assume that  $\mathcal{D}$  is a relatively compact subdomain of  $\mathcal{O}$  with real analytic boundary.

We have proved that if, for any neighborhood  $U$  of  $\overline{\mathcal{D}}$ , there exists a neighborhood  $U' \subset U$  of  $\overline{\mathcal{D}}$  such that  $\text{sol}(U', P)$  is dense in  $\text{sol}(\mathcal{D}, P)$ , then the correspondence  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$  induces the topological isomorphism of  $\text{sol}(\overline{\mathcal{D}}, P)$  onto the dual space to  $\text{sol}(\mathcal{D}, P)$ .

We show that this condition is almost necessary.

**THEOREM 8.1.** *If the map  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$  of  $\text{sol}(\overline{\mathcal{D}}, P) \rightarrow \text{sol}(\mathcal{D}, P)'$  is surjective, then  $\text{sol}(\overline{\mathcal{D}}, P)$  is dense in  $\text{sol}(\mathcal{D}, P)$ .*

PROOF. Let  $\mathcal{F}$  be a continuous linear functional on  $\text{sol}(\mathcal{D}, P)$  vanishing on  $\text{sol}(\overline{\mathcal{D}}, P)$ . By the Hahn-Banach theorem, our statement will be proved once we show that  $\mathcal{F} \equiv 0$ .

By assumption, there is a  $v \in \text{sol}(\overline{\mathcal{D}}, P)$  such that  $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$ . It follows that

$$\begin{aligned} \langle \mathcal{F}_{\mathcal{E}(v)}, v \rangle &= \langle \mathcal{F}, v \rangle \\ &= 0, \end{aligned}$$

and so an argument similar to that in the proof of Lemma 6.1 shows that  $v = 0$  in  $\mathcal{D}$ . Hence  $\mathcal{F} \equiv 0$ , as desired.  $\square$

## 9. – Duality in complex analysis

Aizenberg and Gindikin [1] obtained Theorem A, formulated in the Introduction, in the case where  $P$  is the *Cauchy-Riemann operator* in  $\mathbb{C}^n$ , and  $n = 1, 2$  (for simply connected domains with real analytic boundary in  $\mathbb{C}$ , and for the so-called  $(p, q)$ -circular domains in  $\mathbb{C}^2$ ).

Stout [13] proved Theorem A for the Cauchy-Riemann operator in  $\mathbb{C}^n$  ( $n \geq 1$ ) and for domains  $\mathcal{D}$  possessing the following property:

- the Szegő Kernel  $\mathcal{K}(\cdot, \zeta)$  of  $\mathcal{D}$  has real analytic boundary values for each  $\zeta \in \mathcal{D}$ .

This condition is known to hold on some explicitly given domains. One supposes it to hold on *strictly pseudoconvex* domains with real analytic boundary. But, as far as Stout [13] has been able to determine, this result has not been written out anywhere.

However, the *approximation condition* in Theorem A holds true for strictly pseudoconvex domains in  $\mathbb{C}^n$  (cf. Hörmander [5]). Thus, our viewpoint sheds some new light on the result of Stout [13].

**THEOREM 9.1.** *Let  $\mathcal{D} \Subset \mathbb{C}^n$  ( $n \geq 2$ ) be a strictly pseudoconvex domain with real analytic boundary. Then the correspondence  $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ , when restricted to  $v \in \text{hol}(\overline{\mathcal{D}})$ , induces the topological isomorphism*

$$\text{hol}(\mathcal{D})' \stackrel{\text{top.}}{\cong} \text{hol}(\overline{\mathcal{D}}).$$

Here we use  $\text{hol}$  for the spaces of holomorphic functions.

**PROOF.** This follows immediately by combining Theorem 6.2 with the *Runge theorem* as stated in Hörmander [5].  $\square$

We note that, because the Cauchy-Riemann operator in  $\mathbb{C}$  is determined elliptic, Theorem 9.1 holds true for spaces of holomorphic functions in every bounded domain in  $\mathbb{C}$  with real analytic boundary.

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