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# Counterexamples to the Gleason Problem

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**Abstract.** For a domain $\Omega$ in $\mathbb{C}^n$ we consider the Gleason Problem for the Banach algebras $H^\infty(\Omega)$ and $A(\Omega)$. We prove that pseudoconvex counterexamples can be constructed and that such domains may have the additional property of being $H^\infty$-domains of holomorphy. A sufficient condition for a domain in $\mathbb{C}^n$ to have the Gleason property is also presented.

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## 1. – Introduction

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. By $H^\infty(\Omega)$ we denote the ring of bounded holomorphic functions on $\Omega$ and by $A(\Omega)$ the ring consisting of functions holomorphic on $\Omega$ and continuous on the closure of $\Omega$. Throughout this paper $\mathfrak{R}(\Omega)$ will denote either $H^\infty(\Omega)$ or $A(\Omega)$.

We study a problem which in the literature is known as the Gleason Problem. The problem is to decide whether the maximal ideal in $\mathfrak{R}(\Omega)$ consisting of functions vanishing at a point $z^0 \in \Omega$ is algebraically finitely generated by the coordinate functions $(z_1 - z_1^0), \dots, (z_n - z_n^0)$. We say that a domain $\Omega$ has the Gleason $\mathfrak{R}$-property if the problem has an affirmative solution for all points in $\Omega$.

The problem arose in connection with the search for multidimensional analytic structure in spectra of commutative complex Banach algebras. One of the first results was found by A. M. Gleason [Gle] who proved that near the complex homomorphism corresponding to an algebraically finitely generated maximal ideal, the spectrum can be endowed with an analytic structure, in terms of which the Gelfand transforms of the elements in the Banach algebra are holomorphic functions. However, in most situations it turns out to be difficult to determine whether a maximal ideal is finitely generated. Even in the case when $\Omega$ is the unit ball in $\mathbb{C}^n$, it is not at all obvious how to decide whether the maximal ideal in $\mathfrak{R}(\Omega)$ consisting of functions vanishing at the origin is

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generated by the coordinate functions, although analytic structure near such an ideal is trivially present. The difficulty of answering this question was mentioned by A. M. Gleason in [Gle] and subsequently solved by Z. L. Leibenzon [Hen]. The method of proof that Leibenzon used yields in fact that all convex domains in $\mathbb{C}^n$ with $C^2$-boundary have the Gleason $\mathfrak{R}$-property.

A number of authors, using different methods, have contributed to settle the strictly pseudoconvex case. The result that has been obtained is that every strictly pseudoconvex domain has the Gleason $\mathfrak{R}$-property. Sheaf-theoretic methods were used by N. Kerzman and A. Nagel [KN] to prove that strictly pseudoconvex domains in $\mathbb{C}^2$ with $C^4$-boundary have the Gleason $A$-property. I. Lieb [Lie] and G. M. Henkin [Hen] independently proved the same result in $\mathbb{C}^n$, for $C^5$ and $C^3$-boundary respectively, using integral representation formulas and precise estimates for $\overline{\partial}$. The general case with $C^2$-boundary was treated by N. Øvrelid [Øvr].

F. Beatrous Jr [Bea] showed that weakly pseudoconvex domains in $\mathbb{C}^2$ with $C^\infty$-boundary and with the additional property that there is a complex line through the base point $z^0$ which intersects the boundary only in strictly pseudoconvex points, have the Gleason $A$-property at $z^0$. Studying the structure of the weakly pseudoconvex boundary points and using F. Beatrous Jr's result, J. E. Fornaess and N. Øvrelid [FØ] proved that pseudoconvex domains in $\mathbb{C}^2$ with real analytic boundary have the Gleason $A$-property. A. Noell [Noe] used information on the weakly pseudoconvex boundary points obtained by D. Catlin [Cat] and methods from [FØ] to show that pseudoconvex domains in $\mathbb{C}^2$ with boundary of finite type have the Gleason $A$-property.

The authors proved in [BF] that bounded pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$ with $C^2$-boundary have the Gleason $A$-property.

In this paper we prove that pseudoconvex counterexamples can be constructed and that such domains may have the additional property of being $H^\infty$-domains of holomorphy. A sufficient condition for a domain in $\mathbb{C}^n$ to have the Gleason $\mathfrak{R}$-property is also presented.

## 2. – Counterexamples

In this section we give counterexamples to the Gleason Problem.

DEFINITION 2.1. A bounded domain $\Omega$ in $\mathbb{C}^n$ is said to have the Gleason $\mathfrak{R}$-property at $z^0 \in \Omega$ if the maximal ideal in $\mathfrak{R}(\Omega)$, consisting of functions vanishing at $z^0$ is algebraically generated by the coordinate functions $z_1 - z_1^0, \dots, z_n - z_n^0$. If $\Omega$ has the Gleason $\mathfrak{R}$-property at every point $z \in \Omega$, then $\Omega$ is said to have the Gleason $\mathfrak{R}$-property.

Every domain in the complex plane $\mathbb{C}$ has the Gleason $\mathfrak{R}$-property. In higher dimension it is easy to construct domains without this property. In fact, in every domain with nonschlicht $\mathfrak{R}$-envelope of holomorphy one can find points where the domain fail to have the Gleason $\mathfrak{R}$-property. As an example, consider the following domain $\Omega$:

Let $D_1$ and $D_2$ be the following Reinhardt domains in $\mathbb{C}^2$:

$$D_1 = \left\{z \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 1\right\} \cup \left\{z \in \mathbb{C}^2 : 2 < |z_1| < 3, |z_2| < 3\right\}$$
$$D_2 = \left\{z \in \mathbb{C}^2 : |z_1| < 1, 2 < |z_2| < 3\right\}$$

To get the domain $\Omega$, connect $D_1$ and $D_2$ by a small open connected neighbourhood $U$ of a curve outside $D_1 \cup D_2$ joining a point in the boundary of $D_1$ to a point in the boundary of $D_2$, so that $\Omega = D_1 \cup U \cup D_2$. It is easy to see that the $\mathfrak{R}$-envelope of holomorphy of $\Omega$ is nonschlicht over $D_2$. It follows from Proposition 2.1 below that $\Omega$ cannot have the Gleason $\mathfrak{R}$-property. The domain $\Omega$ is not a domain of holomorphy.

Before the statements and proofs of the theorems, we need some definitions and a proposition. Recall that the set of nonzero multiplicative complex homomorphisms on $\mathfrak{R}(\Omega)$ is called the spectrum of the Banach algebra $\mathfrak{R}(\Omega)$, when it is equipped with the weak*-topology. We denote by $\mathcal{M}^{\mathfrak{R}(\Omega)}$ the spectrum of $\mathfrak{R}(\Omega)$ and by $\pi$ the projection from $\mathcal{M}^{\mathfrak{R}(\Omega)}$ to $\mathbb{C}^n$ defined by

$$\pi(m) = (m(z_1), ..., m(z_n))$$

for every $m \in \mathcal{M}^{\mathfrak{R}(\Omega)}$. If $z_0 \in \Omega$, then the set

$$\pi^{-1}(z^0) = \left\{m \in \mathcal{M}^{\mathfrak{R}(\Omega)} : \pi(m) = z^0\right\}$$

is called the fibre over $z^0$. Observe that if $z \in \Omega$, then the point evaluation $m_z$, defined by $m_z(f) = f(z)$ for $f \in \mathfrak{R}(\Omega)$, is an element in the fibre over $z$.

DEFINITION 2.2 A domain $\Omega$ in $\mathbb{C}^n$ is said to be $\mathfrak{R}$-spectrumschlicht at $z^0 \in \Omega$ if the fibre over $z^0$ contains exactly one element, i.e. if

$$\left\{m \in \mathcal{M}^{\mathfrak{R}(\Omega)} : \pi(m) = z^0\right\} = \{m_{z^0}\}$$

If $\Omega$ is $\mathfrak{R}$-spectrumschlicht at every point $z \in \Omega$, then $\Omega$ is said to be $\mathfrak{R}$-spectrumschlicht.

PROPOSITION 2.1. *If a domain $\Omega$ in $\mathbb{C}^n$ has the Gleason $\mathfrak{R}$-property at $z^0 \in \Omega$, then $\Omega$ is $\mathfrak{R}$-spectrumschlicht at $z^0$.*

PROOF. Suppose that $z_0 \in \Omega$ and that $\Omega$ is not $\mathfrak{R}$-spectrumschlicht at $z^0$. This means that apart from the point evaluation $m_{z^0}$ there is at least one element, $m' \neq m_{z^0}$, in the spectrum $\mathcal{M}^{\mathfrak{R}(\Omega)}$ such that

$$\pi(m') = \left(m'(z_1), \dots, m'(z_n)\right) = z^0$$

Let $f$ be a function in the kernel of $m_{z^0}$ such that $m'(f) \neq 0$. Since $\Omega$ has the Gleason $\mathfrak{R}$-property at $z^0$ there exist $f_j \in \mathfrak{R}(\Omega)$, $1 \leq j \leq n$, such that

$$f(z) = \sum_{j=1}^{n} (z_j - z_j^0) f_j(z), \qquad z \in \Omega$$

We get

$$\begin{aligned} 0 &\neq m'(f) - m_{z^0}(f) \\ &= m' \left( \sum_{j=1}^{n} (z_j - z_j^0) f_j \right) - m_{z^0} \left( \sum_{j=1}^{n} (z_j - z_j^0) f_j \right) \\ &= \sum_{j=1}^{n} (m'(z_j) - z_j^0) \cdot m'(f_j) - \sum_{j=1}^{n} (m_{z^0}(z_j) - z_j^0) \cdot m_{z^0}(f_j) = 0 \end{aligned}$$

since $\pi(m_{z^0}) = \pi(m')$ implies that $m_{z^0}(z_j) = m'(z_j)$, $1 \leq j \leq n$.
This shows that $\pi^{-1}(z^0)$ cannot contain more than one element. □

We now describe how one can construct a domain of holomorphy with nonschlicht $\mathfrak{R}$-envelope of holomorphy:

THEOREM 2.2. *There exists a domain of holomorphy in $\mathbb{C}^2$ with nonschlicht $\mathfrak{R}$-envelope of holomorphy.*

PROOF. Let $\mathcal{S}$ be the domain constructed by N. Sibony in [Si1]. It is a pseudoconvex subdomain of the unit polydisk $\Delta^2$ in $\mathbb{C}^2$ such that $\Delta^2 \setminus \overline{\mathcal{S}} \neq \emptyset$ and such that all bounded holomorphic functions on $\mathcal{S}$ extend to $\Delta^2$. The domain is constructed in the following way: Choose a discrete sequence $\{a_\nu\}_{\nu=1}^{\infty}$ in the unit disk $\Delta \subset \mathbb{C}$ such that every boundary point of the disk is a non-tangential limit of a subsequence and define $\lambda : \Delta \to \mathbb{R} \cup \{-\infty\}$ by

$$\lambda(z) = \sum_{\nu=1}^{\infty} \varepsilon_\nu \log \left| \frac{z - a_\nu}{2} \right|$$

where $\varepsilon_\nu \searrow 0$ rapidly so that $\lambda \not\equiv -\infty$ and is subharmonic on $\Delta$. $\mathcal{S}$ is defined as

$$\mathcal{S} = \left\{ (z, w) \in \Delta^2 : \rho(z, w) = |w| e^{e^{\lambda(z)}} - 1 < 0 \right\}$$

and is pseudoconvex since $\rho$ is plurisubharmonic on $\Delta \times \mathbb{C}$. Hartogs series and Fatou's lemma yield that every bounded holomorphic function on $\mathcal{S}$ extends to $\Delta^2$ and by computing the Levi form for $\rho$ one gets that there are lots of strictly pseudoconvex points in the boundary of $\mathcal{S}$.

Let $p_0$ be a strictly pseudoconvex point in the boundary of $\mathcal{S}$ and let $\mathcal{B}_0$ be an open ball centred at $p_0$ such that $\overline{\mathcal{S}} \cap \overline{\mathcal{B}}_0$ is connected and $\rho$ is strictly plurisubharmonic in $\mathcal{B}_0$. Furthermore, let $\mathcal{B}_1 \subset\subset \mathcal{B}_0$ be an open ball also centred at $p_0$ and $\mathcal{O}$ an open connected neighbourhood of $p_0$ such that $\mathcal{O} \subset \mathcal{B}_0$, $(\mathcal{O} \setminus \mathcal{B}_1) \setminus \mathcal{S} \neq \emptyset$ and $\partial\mathcal{S} \cap \mathcal{O} \subset\subset \partial\mathcal{S} \cap \mathcal{B}_1$.

Choose a function $\varphi \in C_0^\infty(\mathcal{O})$ and a number $\varepsilon > 0$ so that $0 \le \varphi \le 1$, $\varphi \equiv 1$ near $p_0$, $\rho_\varepsilon = \rho - \varepsilon\varphi$ is strictly plurisubharmonic in $\mathcal{O}$, $\mathcal{V} = \{z : \rho_\varepsilon(z) < 0\}$ is a pseudoconvex domain, $(\mathcal{V} \setminus \mathcal{S}) \setminus \mathcal{B}_1 \neq \emptyset$ and so that $(\mathcal{V} \setminus \mathcal{S}) \setminus \mathcal{B}_1$ consists of one connected component. Moreover, choose a point $p = (p_1, p_2) \in \partial\mathcal{B}_1 \setminus \overline{\mathcal{S}}$ such that $\rho_\varepsilon(p) < 0$, a point $q = (q_1, q_2)$ in $\Delta^2 \setminus \overline{(\mathcal{S} \cup \mathcal{B}_0)}$ and a ball $\mathcal{B}$ in $\mathbb{C}^2$ such that $q \in \partial\mathcal{B}$ and $\overline{\mathcal{B}} \cap \overline{(\mathcal{S} \cup \mathcal{B}_0)} = \emptyset$.

Let $s \in \mathbb{C}$ with $|s| > 1$ and let $L : [0, 1] \to \mathbb{C}$ be the curve defined by $L(t) = tp_1 + (1-t)q_1$. Choose a $C^\infty$ simple curve $\gamma$ in $\mathbb{C}^2$ going from $p$ to $q$ so that $\gamma$ is transversal to $\partial\mathcal{B}_1$ and to $\partial\mathcal{B}$, $\gamma$ does not intersect $\overline{(\mathcal{S} \cup \mathcal{B}_1 \cup \mathcal{B})} \setminus \{p, q\}$, the image of the closed curve $\Gamma = \gamma_{\mathrm{Pr}_{z_1}} + L$ does not contain $s$ and so that the winding number $n(\Gamma, s) \neq 0$. ($\gamma_{\mathrm{Pr}_{z_1}}$ is the projection curve of $\gamma$ from $p_1$ to $q_1$ in the $z_1$-plane.)

Furthermore, let $\mathcal{U}$ be an open neighbourhood of $\overline{\mathcal{B}} \cup \gamma$ such that $\mathcal{B}_1 \cap \mathcal{U} \subset \mathcal{V}$. In order to get the domain $\mathcal{S}_0$ we now choose $\mathcal{W}$ to be a sufficiently small pseudoconvex domain containing $\mathcal{B} \cup \gamma \cup \mathcal{B}_1$ so that

$$\mathcal{S}_0 = \mathcal{S} \cup (\mathcal{V} \cap \mathcal{W}) \cup (\mathcal{W} \cap \mathcal{U})$$

is pseudoconvex and so that the closure of the projection of $\mathcal{W}$ on the $z_1$-plane does not intersect $\{s\}$. This can be done since $\mathcal{B} \cup \gamma \cup \mathcal{B}_1$ has a Stein neighbourhood basis ([FZ]; Theorem 4), $\mathcal{V}$ is pseudoconvex and since pseudoconvexity is a local concept.

We complete the proof by showing that the $\mathfrak{R}$-envelope of holomorphy of $\mathcal{S}_0$ is nonschlicht.

For every function $f$ in $\mathfrak{R}(\mathcal{S}_0)$ the restriction of $f$ to $S$ extends to a function in $\mathfrak{R}(\Delta^2)$ and the extended values may differ from the given values of $f$ on $(\mathcal{S}_0 \cap \Delta^2) \setminus S$. For example, there is a branch such that the function $f_0$ defined by $f_0 = \log(z_1 - s)$ belongs to $\mathfrak{R}(\mathcal{S}_0)$. The values of $f_0$ differ on parts of $(\mathcal{S}_0 \cap \Delta^2) \setminus S$ from the extended values from $S$. Hence the $\mathfrak{R}$-envelope of holomorphy of $\mathcal{S}_0$ is nonschlicht. □

Since a domain with nonschlicht $\mathfrak{R}$-envelope of holomorphy is not $\mathfrak{R}$-spectrumschlicht, we get the following corollary:

COROLLARY 2.3. *There exists a domain of holomorphy which does not have the Gleason $\mathfrak{R}$-property.*

PROOF. It follows from Proposition 2.1 that the domain $\mathcal{S}_0$ constructed in the proof of Theorem 2.2 does not have the Gleason $\mathfrak{R}$-property. □

We now prove that there exists an $H^\infty$-domain of holomorphy which does not have the Gleason $\mathfrak{R}$-property. If $\Omega$ is a domain in $\mathbb{C}^n$ and $V$ a plurisubharmonic function on $\Omega$, then we define $V^1$ to be the plurisubharmonic function

$$V^1(z) = (\sup\{c\log|f| : c > 0, f \in H(\Omega) \text{ such that } c\log|f| \le V \text{ on } \Omega\})^*$$

where the asterisk as usual denotes upper semicontinuous regularization.
We define $\mathcal{P}(\Omega)$ to be the convex cone of plurisubharmonic functions $V$ on $\Omega$ satisfying $V^1 = V$, i.e.

$$\mathcal{P}(\Omega) = \left\{ V \in \mathrm{PSH}(\Omega) : V = V^1 \text{ on } \Omega \right\}$$

The domain obtained in the proof of Theorem 2.2 will now be used to construct an $H^\infty$-domain of holomorphy which is not $\mathfrak{R}$-spectrumschlicht.

Theorem 2.4. *There exists an $H^\infty$-domain of holomorphy in $\mathbb{C}^3$ which is not $\mathfrak{R}$-spectrumschlicht.*

Proof. Let $\mathcal{S}$ and $\mathcal{S}_0$ be the domains described in the proof of Theorem 2.2 and let

$$\delta_{\mathcal{S}_0} = d(z, \mathcal{S}_0^c)$$

where $\mathcal{S}_0^c$ denotes the complement of $\mathcal{S}_0$ and $d$ is the Euclidean distance. Since $\mathcal{S}_0$ is pseudoconvex the function $-\log \delta_{\mathcal{S}_0}$ is plurisubharmonic. It follows from [Si3] that $-\log \delta_{\mathcal{S}_0}$ belongs to the class $\mathcal{P}(\mathcal{S}_0)$.

We now define the Hartogs domain $M(\mathcal{S}_0, \delta_{\mathcal{S}_0})$ to be

$$M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) = \left\{ (z, w) \in \mathbb{C}^2 \times \mathbb{C} : z \in \mathcal{S}_0, |w| < \delta_{\mathcal{S}_0}(z) \right\}.$$

Since $-\log \delta_{\mathcal{S}_0}(z)$ belongs to $\mathcal{P}(\mathcal{S}_0)$, it follows from a result due to N. Sibony [Si2] that $M(\mathcal{S}_0, \delta_{\mathcal{S}_0})$ is an $H^\infty$-domain of holomorphy. For every function $f$ in $\mathfrak{R}(M(\mathcal{S}_0, \delta_{\mathcal{S}_0}))$ the restriction of $f$ to $\left\{(z, w) \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) : z \in \mathcal{S}, w = 0\right\}$ extends to a function in $\mathfrak{R}\left(\{(z, w) \in \mathbb{C}^2 \times \mathbb{C} : z \in \Delta^2, w = 0\}\right)$ and the extended values may differ from the given values of $f$ on $\{(z, w) \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) : z \in (\mathcal{S}_0 \cap \Delta^2) \setminus \mathcal{S}, w = 0\}$. For example, there is a branch such that the function $f_0$ defined by $f_0 = \log(z_1 - s)$ belongs to $\mathfrak{R}\left(M(\mathcal{S}_0, \delta_{\mathcal{S}_0})\right)$. The values of the restriction of $f_0$ to $\left\{(z, w) \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) : z \in \mathcal{S}_0, w = 0\right\}$ differ on parts of $\{(z, w) \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) : z \in (\mathcal{S}_0 \cap \Delta^2) \setminus \mathcal{S}, w = 0\}$ from the extended values from $\{(z, w) \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) : z \in \mathcal{S}, w = 0\}$.

This means that there is a point $p \in \Big\{(z, w) \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0}) : z \in (\mathcal{S}_0 \cap \Delta^2) \setminus \mathcal{S}, w = 0\Big\}$ and an element $m \in \mathcal{M}^{\mathfrak{R}(M(\mathcal{S}_0, \delta_{\mathcal{S}_0}))}$ such that $\pi(m) = p \in M(\mathcal{S}_0, \delta_{\mathcal{S}_0})$ and $m(f_0) \neq m_p(f_0) = f_0(p)$. □

The following corollary is now obvious:

Corollary 2.5. *There exists an $H^\infty$-domain of holomorphy which does not have the Gleason $\mathfrak{R}$-property.*

Proof. It follows from Proposition 2.1 that the domain constructed in the proof of Theorem 2.4 does not have the Gleason $\mathfrak{R}$-property. □

It is not known whether there exist $H^\infty$-domains of holomorphy in $\mathbb{C}^2$ which does not have the Gleason $\mathfrak{R}$-property.

## 3. – A sufficient condition

In this section we prove a sufficient condition for a domain $\Omega$ in $\mathbb{C}^n$ to have the Gleason $\mathfrak{R}$-property. We also give an example of a domain which has the Gleason $\mathfrak{R}$-property but for which there are no other known general tools than the condition given in this paper to decide that.

THEOREM 3.1. *Suppose that $\Omega$ is a bounded domain in $\mathbb{C}^n$, $z^0 \in \Omega$ and that the following holds:*

i) *There is a domain $\omega \subset \mathbb{C}^n$ and a biholomorphic mapping*

$$\Phi : \Omega \to \omega, \Phi(z) = (\Phi_1(z), ..., \Phi_n(z)),$$

*such that there are functions $g_j^k \in \mathfrak{R}(\Omega)$, $1 \le k \le n$, $1 \le j \le n$, with*

$$\Phi_k(z) - \Phi_k(z^0) = \sum_{j=1}^{n} (z_j - z_j^0) g_j^k(z), \qquad z \in \Omega$$

ii) *The domain $\omega$ has the Gleason $\mathfrak{R}$-property at $\zeta^0 = \Phi(z^0)$.*

*Then $\Omega$ has the Gleason $\mathfrak{R}$-property at $z^0$.*

PROOF. Let $f$ be an arbitrary function in $\mathfrak{R}(\Omega)$ such that $f(z^0) = 0$ and let $\zeta$ denote $\Phi(z)$ for $z \in \Omega$. Since $\Phi$ is biholomorphic, the function $f \circ \Phi^{-1}$ belongs to $\mathfrak{R}(\omega)$. Moreover, the fact that $\omega$ has the Gleason $\mathfrak{R}$-property at $\zeta^0$ implies that there are functions $h_k \in \mathfrak{R}(\omega)$, $1 \le k \le n$, such that

$$\begin{aligned} f \circ \Phi^{-1}(\zeta) &= \sum_{k=1}^{n} h_k(\zeta)(\zeta_k - \zeta_k^0) \\ &= \sum_{k=1}^{n} h_k \circ \Phi(z) \left(\Phi_k(z) - \Phi_k(z^0)\right) \\ &= \sum_{k=1}^{n} h_k \circ \Phi(z) \left(\sum_{j=1}^{n} (z_j - z_j^0) g_j^k(z)\right) \end{aligned}$$

Since $h_k \circ \Phi \in \mathfrak{R}(\Omega)$ and $g_j^k \in \mathfrak{R}(\Omega)$ it follows that there are functions $f_k \in \mathfrak{R}(\Omega)$, $1 \le k \le n$, such that

$$f(z) = \sum_{k=1}^{n} f_k(z) \left(z_k - z_k^0\right)$$

This shows that $\Omega$ has the Gleason $\mathfrak{R}$-property at $z^0$. □

We give an example in order to illustrate how the theorem can be applied.

EXAMPLE 1. Consider the bounded domain $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ where

$$\rho(z) = \begin{cases} |z_1| - 1, & |z_1| > \left|z_2 - z_1^2\right| \\ \left|z_2 - z_1^2\right| - 1, & |z_1| \leq \left|z_2 - z_1^2\right| \end{cases}.$$

Then $\Omega$ is a weakly pseudoconvex domain and the boundary of $\Omega$ is not of class $C^1$. Define the mapping $\Phi$ by

$$\Phi(z) = \left(z_1, z_2 - z_1^2\right)$$

This is a biholomorphic mapping from $\Omega$ to the unit bidisk in $\mathbb{C}^2$. Since the unit bidisk has the Gleason $\mathfrak{R}$-property it follows from Theorem 3.1 that $\Omega$ also has the Gleason $\mathfrak{R}$-property.

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