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Liouville-Gelfand Type Problems for the $N$-Laplacian on Bounded Domains of $\mathbb{R}^N$

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Abstract. In this article it is used minimax methods to establish some results of existence and multiplicity of solutions for the $N$-Laplacian equation on bounded domains of $\mathbb{R}^N$, with Dirichlet boundary conditions, when the nonlinearity has exponential growth. The subcritical and critical cases are considered.

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1. – Introduction

In this article, we study the existence and multiplicity of solutions for the following quasilinear elliptic problem

\[
(P)_\lambda \begin{cases}
-\Delta u = -\text{div}(\nabla |\nabla u|^{N-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega, \\
u \geq 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$ with boundary $\partial \Omega$, $\lambda > 0$ is a real parameter, and the nonlinearity $f(x,u)$ satisfies

\( (f_1) f : \tilde{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a continuous function and $f(x,0) > 0$, for every $x \in \Omega$, and the growth condition

\( (f_{\alpha_0}) \) There exists $\alpha_0 \geq 0$ such that

\[
\lim_{s \to \infty} \frac{|f(x,s)|}{\exp(\alpha s^{N/N-1})} = \begin{cases}
0, & \forall \alpha > \alpha_0, \text{ unif. on } \tilde{\Omega}, \\
+\infty, & \forall \alpha < \alpha_0, \text{ unif. on } \tilde{\Omega}.
\end{cases}
\]

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In the literature [1], [10], [11], \( f(x, s) \) is said to have subcritical or critical growth when \( \alpha_0 = 0 \) or \( \alpha_0 > 0 \), respectively. We note that such notion is motivated by Trudinger-Moser estimates [17], [23] which provide

\[
\exp(\alpha|u|^{\frac{N}{N-1}}) \in L^1(\Omega), \quad \forall \, u \in W^{1,N}_0(\Omega), \quad \forall \, \alpha > 0,
\]

and

\[
\sup_{\|u\|_{W^{1,N}_0(\Omega)} \leq 1} \int_\Omega \exp(\alpha|u|^{\frac{N}{N-1}}) \, dx \leq C(N) \in \mathbb{R}, \quad \forall \, \alpha \leq \alpha_N = Nw_{N-1}^{\frac{1}{N-1}},
\]

where \( w_k \) is the volume of \( S^k \). We also observe that a typical and relevant case to be considered for problem \((P)_\lambda\) is given by \( f(x, s) = \exp(\alpha_0 s^{\frac{N}{N-1}}) \).

In our first result, we establish the existence of a solution for \((P)_x\) when \( h > 0 \) is sufficiently small,

**Theorem 1.1.** Suppose \( f(x, s) \) satisfies \((f_1)\) and \((f)_{\alpha_0}\). Then, there exists \( \lambda > 0 \) such that problem \((P)_\lambda\) possesses at least one solution for every \( \lambda \in (0, \lambda) \).

To obtain the existence of a second solution for problem \((P)_\lambda\) in the subcritical case, we assume that \( f(x, s) \) satisfies

\((f_2)\) There are constants \( \theta > N \) and \( R > 0 \) such that

\[
0 < \theta F(x, s) \leq sf(x, s), \quad \forall \, x \in \Omega, \quad s \geq R.
\]

**Theorem 1.2 (Second solution: Subcritical case).** Suppose \( f(x, s) \) satisfies \((f_1)\), \((f_2)\) and \((f)_{\alpha_0}\), with \( \alpha_0 = 0 \). Then, there exists \( \bar{\lambda} > 0 \) such that problem \((P)_{\bar{\lambda}}\) possesses at least two solutions for every \( \lambda \in (0, \bar{\lambda}) \).

Note that \((f_2)\) is the version of the famous Ambrosetti-Rabinowitz condition [3] for the \( N \)-Laplacian. It implies, in particular, that \( f(x, s)/s^{N} \to \infty \), as \( s \to \infty \), uniformly on \( \Omega \).

In our next result, we provide the existence of two solutions for \((P)_\lambda\) when \( f(x, s) \) has critical growth. In that case, we shall need to suppose a stronger version of condition \((f_2)\),

\((f_2')\) For every \( \theta > N \), there exists \( R(\theta) > 0 \) such that

\[
0 < \theta F(x, s) \leq sf(x, s), \quad \forall \, x \in \Omega, \quad s \geq R(\theta).
\]

Assuming the following further restriction on the growth of \( f(x, s) \),

\((f_3)\) There exists a non-empty open set \( \tilde{\Omega} \subset \Omega \) such that

\[
\liminf_{s \to \infty, x \in \tilde{\Omega}} \frac{f(x, s)}{\exp(\alpha_0 s^{\frac{N}{N-1}})} = \infty,
\]

we obtain
THEOREM 1.3 (Second solution: Critical case). Suppose \( f(x, s) \) satisfies (f1), (f2), (f3) and \((f)\alpha_0\), with \( \alpha_0 > 0 \). Then, there exists \( \bar{\lambda} > 0 \) such that problem \((P)_\lambda\) possesses at least two solutions for every \( \lambda \in (0, \bar{\lambda}) \).

Exploiting the convexity of the primitive \( F(x, s) \), in our final result we are able to consider a weaker version of (f3), obtaining the same conclusion of Theorem 1.3. More specifically, we suppose

1. There exist a non-empty open set \( \Omega \subset \Omega \)

\[
\lim_{s \to \infty} \inf_{x \in \Omega} \frac{f(x, s)s^\beta}{\exp(\alpha_0 s^{N/N-1})} = \infty,
\]

where \( \beta = \frac{1}{2(N-1)} \) if \( N = 3 \), and \( \beta = \frac{1}{N-1} \) otherwise.

2. \( f(x, .) \) is convex on \([0, \infty)\) for every \( x \in \Omega \subset \Omega \), \( \Omega \) given by (f3),

THEOREM 1.4 (Second solution: Convex Critical case). Suppose \( f(x, s) \) satisfies (f1), (f2), (f3), (f4) and \((f)\alpha_0\), with \( \alpha_0 > 0 \). Then, there exists \( \bar{\lambda} > 0 \) such that problem \((P)_\lambda\) possesses at least two solutions for every \( \lambda \in (0, \bar{\lambda}) \).

We observe that Theorem 1.4 establishes the existence of two solutions of \((P)_\lambda\) for \( \lambda > 0 \) sufficiently small when \( f(x, s) = \exp(\alpha_0 s^{N/N-1}) \).

As it is well known, the classical Liouville-Gelfand problem is given by

\[
\begin{aligned}
-Lu &= \lambda e^u, & & \text{in } \Omega, \\
u &> 0, & & \text{in } \Omega, \\
u &= 0, & & \text{on } \partial\Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \) with boundary \( \partial\Omega \), and \( \lambda > 0 \) is a real parameter. First considered by Liouville [16], for the case \( N = 1 \), and afterwards by Bratu [4], for \( N = 2 \), and Gelfand [13], for \( N \geq 1 \), this problem has been extensively studied during the last three decades (See [7], [8], [12] and references therein). As observed in [12], problem \((LG)_\lambda\) is of great relevance since it appears in mathematical models associated with astrophysical phenomena and to problems in combustion reactions.

In [8], Crandall and Rabinowitz used bifurcation theory to establish the existence of one solution for problem \((LG)_\lambda\), for \( \lambda > 0 \) sufficiently small, and a nonlinearity \( f(x, s) \) replacing \( e^s \). In [8] no growth restriction on \( f(x, s) \) is assumed. To obtain such result, those authors assume \( f(x, s) \in C^3(\Omega \times \mathbb{R}, \mathbb{R}) \), \( f_s(x, 0) > 0 \) and \( f_{ss}(x, s) > 0 \), for every \( x \in \Omega \) and \( s > 0 \) Supposing that \( f(x, s) \) has a subcritical growth, they show that this solution is a local minimum for the associated functional. Then, using critical point theory, they are able to prove the existence of a second solution. We note that Theorems 1.3 and 1.4 improve the last mentioned result of [8] when \( N = 2 \) since they allow \( f(x, s) \) to have critical growth. In particular, we may consider \( f(x, s) = e^{s^2} \).
In [12], Garcia Azorero and Peral Alonso proved the existence of solutions for \((LG)_{\lambda}\), with \(\lambda > 0\) sufficiently small, when the Laplacian is replaced by a \(p\)-Laplacian operator. The nonexistence of solutions for \((LG)_{\lambda}\) for this more general class of operators, when \(\lambda > 0\) is sufficiently large, was also established in [12]. We should also mention the article by Clément, Figueiredo and Mitidieri [7], where the exact number of solutions for an operator more general that the \(p\)-Laplacian is established when \(\Omega\) is an open ball of \(\mathbb{R}^N\). In [7], it is not assumed any growth restriction on \(f(x, s)\).

We note that the solutions mentioned in Theorems 1.1-1.4 are weak solutions of \((P)_{\lambda}\) (See [19]). We also observe that in this article, we use minimax methods to derive such solutions.

To prove Theorem 1.1, we first provide an abstract result that establishes the existence of a critical point for a functional of class \(C^1\) defined on a real Banach space assuming a version of the famous Palais-Smale condition for the weak topology (See Definitions and Proposition 2.2 in Section 2). Motivated by the argument used in [18], we prove that the associated functional satisfies such condition under hypotheses \((f_1)\) and \((f)_{\alpha_0}\). Taking \(\lambda > 0\) sufficiently small, we are able to apply the mentioned abstract result. In our proof of Theorem 1.2, we use condition \((f_2)\) to verify that the associated functional satisfies the Palais-Smale condition. As in [8], this provides the existence of a second solution for \((P)_{\lambda}\) via the Mountain Pass Theorem [3].

In the proofs of Theorems 1.3 and 1.4, we argue by contradiction, assuming that Theorem 1.1 provides the only possible solution of \((P)_{\lambda}\). This assumption and condition \((\hat{f}_2)\) allow us to use the argument of Brezis and Nirenberg [6] and a result of Lions [15] to verify that the associated functional satisfies the Palais-Smale condition on a given interval of the real line. We use conditions \((f_3)\) and \((\hat{f}_3)\), respectively, to establish that the level associated with the Mountain Pass Theorem belongs to this interval. As in the proof of Theorem 1.2, that implies the existence of a second solution.

Finally, we should mention that the existence of a nonzero solution for \((P)_{\lambda}\) when \(f(x, 0) \equiv 0\) has been intensively studied in recent years (See [1], [2], [10], [11] and references therein). As it is shown in [1] (See also [10]), when \(f(x, s) \geq 0\), for \(s \geq 0\), a weaker version of \((f_3)\) may be considered. We also observe that our method may be used to improve such results since in those articles a stronger version of \((\hat{f}_2)\) is assumed. Condition \((f_3)\) can also be used in that setting to study the case where \(f(x, s)\) may assume negative values.

The article is organized in the following way: In Section 2, we introduce the notion of Palais-Smale condition for the weak topology and establish two abstract results which are used to prove our results. There, we also recall the variational framework associated with \((P)_{\lambda}\) and state a version of Trudinger-Moser inequality (1.2) for \(W^{1,N}(\Omega)\) when \(\Omega\) is an open ball in \(\mathbb{R}^N\). In section 2, we also state a result by Lions [15] that will be used, via contradiction, to verify \((PS)_{\epsilon}\), for \(c\) below a given level, when condition \((f)_{\alpha_0}\) holds with \(\alpha_0 > 0\). In Section 3, we prove the weak version of Palais-Smale condition for
the associated functional. In Section 4, we prove Theorems 1.1 and 1.2. In Section 5 we establish the estimates that are used to prove Theorem 1.3. In Section 6, we prove Theorem 1.3. In Section 7, we establish the estimates for the associated functional when conditions \((f_3)\) and \((f_4)\) are assumed. There, we also present the proof of Theorem 1.4. In Appendix A, we prove the Trudinger-Moser inequality mentioned in Section 2. Finally, in Appendix B, we prove an inequality for vector fields on \(\mathbb{R}^N\), used in Section 7 to establish the necessary estimates.

2. – Preliminaries

Given \(E\) a real Banach space and \(\Phi\) a functional of class \(C^1\) on \(E\), we recall that \(\Phi\) satisfies Palais-Smale condition at level \(c \in \mathbb{R}\) [Denoted \((PS)_c\)] on an open set \(\mathcal{O} \subset E\) if every sequence \((u_n) \subset \mathcal{O}\) for which (i) \(\Phi(u_n) \to c\) and (ii) \(\Phi'(u_n) \to 0\), as \(n \to \infty\), possesses a converging subsequence. We also observe that \(\Phi\) satisfies \((PS)_c\) if it satisfies \((PS)_c\) on \(E\), and we say that \(\Phi\) satisfies \((PS)\) when it satisfies \((PS)_c\) for every \(c \in \mathbb{R}\). Finally, we note that every sequence \((u_n) \subset E\) satisfying (i) and (ii) is called a Palais-Smale \((PS)\) sequence.

To establish the existence of a critical point when the functional is bounded from below on a closed convex subsets of \(E\), we introduce a version of the Palais-Smale condition for the weak topology.

**Definition 2.1.** Given \(c \in \mathbb{R}\), we say that \(\Phi \in C^1(E, \mathbb{R})\) satisfies the \((wPS)_c\) on \(A \subset E\) if every sequence \((u_n) \subset A\) for which \(\Phi(u_n) \to c\) and \(\Phi'(u_n) \to 0\), as \(n \to \infty\), possesses a subsequence converging weakly to a critical point of \(\Phi\). We say that \(\Phi\) satisfies \((wPS)\) on \(A\) if \(\Phi\) satisfies \((wPS)_c\) on \(A\), for every \(c \in \mathbb{R}\). When \(\Phi\) satisfies \((wPS)\) on \(E\), we simply say that \(\Phi\) satisfies \((wPS)\).

Assuming

\((\Phi_1)\) There exist a closed bounded set \(A \subset E\), constants \(\gamma \leq b \in \mathbb{R}\), and \(u_0 \in A\) such that

- (i) \(\Phi(u) \geq \gamma\), \(\forall\ u \in A\),
- (ii) \(\Phi(u) \geq b \geq \Phi(u_0)\), \(\forall\ u \in \partial A\),

we define

\[
(2.1) \quad c_1 = \inf_{u \in A} \Phi(u).
\]

The following abstract result provides a critical point for \(\Phi\) under conditions \((\Phi_1)\) and \((wPS)\).
PROPOSITION 2.2. Let $E$ be a real Banach space. Suppose $\Phi \in C^1(E, \mathbb{R})$ satisfies $(\Phi_1)$, with $A$ a closed bounded convex subset of $E$. Then, $\Phi$ possesses a critical point $u \in A$ provided it satisfies $(wPS)_{c_1}$ on $A$.

PROOF. Arguing by contradiction, we suppose that $\Phi$ does not have a critical point $u \in A$. Under this assumption, we claim that $\Phi$ satisfies $(PS)_{c_1}$ on $\overset{\circ}{A}$. Effectively, given a sequence $(u_n) \subset \overset{\circ}{A}$ such that $\Phi(u_n) \to c_1$ and $\Phi'(u_n) \to 0$, as $n \to \infty$, by $(wPS)_{c_1}$, $(u_n)$ possesses a subsequence converging weakly to a critical point $u$. Furthermore, $u \in A$ since $A$ is a closed convex subset of $E$. This contradicts our assumption and proves the claim.

We note that $\gamma \leq c_1 \leq \Phi(u_0)$. If $c_1 = \Phi(u_0)$, the conclusion is immediate. Thus, we may assume $c_1 < \Phi(u_0) \leq b$. In this case, we take $0 < \varepsilon < \Phi(u_0) - c_1$. Then, we argue as in Proposition 2.7 of [19], using a local version of the Deformation Lemma [21], to obtain a contradiction with the definition of $c_1$. Proposition 2.2 is proved. \(\square\)

REMARK 2.3. When $\Phi$ satisfies $(PS)_{c_1}$ on $\overset{\circ}{A}$, the second part of the proof of Proposition 2.2 shows that actually $\Phi$ possesses a local minimum $u \in \overset{\circ}{A}$ such that $\Phi(u) = c_1$.

Taking $b \in \mathbb{R}$ and $A$, given by $(\Phi_1)$, we consider

$(\Phi_2)$ There exists $e \in E \setminus A$ such that

$$\Phi(e) \leq b \leq \Phi(u), \quad \forall u \in \partial A,$$

and we define

$$c_2 = \inf_{g \in \Gamma} \max_{u \in g} \Phi(u) \geq b,$$

where

$$\Gamma = \{ g \in C([0, 1], E); \quad g(0) = u_0, \quad g(1) = e \}.$$

As a consequence of Proposition 2.2, Remark 2.3 and the argument employed in [21], we obtain the following version of the Mountain Pass Theorem [3].

PROPOSITION 2.4. Let $E$ be a real Banach space. Suppose $\Phi \in C^1(E, \mathbb{R})$ satisfies $(\Phi_1)$, with $A$ closed and convex subset of $E$, and $(\Phi_2)$. Then, $\Phi$ possesses at least two critical points provided it satisfies $(PS)_c$, for every $c \leq c_2$.

PROOF. By Proposition 2.2 and Remark 2.3, $\Phi$ possesses a local minimum $u_1 \in \overset{\circ}{A}$ such that $\Phi(u_1) = c_1$. Furthermore, if $\Phi$ does not have any critical point on $\partial A$, we may invoke the local version of the Deformation Lemma [21] one more time to obtain a neighbourhood $V$ of $u_0$ and $\varepsilon > 0$ such that $u_0 \in V$, $e \notin V$ and

$$c_1 \leq \max\{\Phi(u_0), \Phi(e)\} < \inf_{u \in \partial A} \Phi(u) + \varepsilon \leq \inf_{u \in \partial V} \Phi(u) \leq c_2.$$
Consequently, by the Mountain Pass Theorem [3], $c_2$ is a critical value of $\Phi$. The proposition is proved.

Observe that when $c_1 = c_2$, by the above proof, $\Phi$ must have a critical point $u \in \partial A$ such that $\Phi(u) = c_1$.

Now, we recall the variational framework associated with problem $(P)_\lambda$. Considering the Sobolev space $W^{1,N}_0(\Omega)$ endowed with the norm

$$\|u\| = \left( \int_\Omega |\nabla u|^N \, dx \right)^{1/N}, \quad \forall \, u \in W^{1,N}_0(\Omega),$$

the functional associated with $(P)_\lambda$, $I_\lambda : W^{1,N}_0(\Omega) \to \mathbb{R}$ is given by

$$I_\lambda(u) = \frac{1}{N} \int_\Omega |\nabla u|^N \, dx - \lambda \int_\Omega F(x, u) \, dx, \quad \forall \, u \in W^{1,N}_0(\Omega),$$

where we assume $f(x, s) = f(x, 0)$, for every $x \in \bar{\Omega}$, $s < 0$, and we take $F(x, s) = \int_0^s f(x, t) \, dt$, for $x \in \bar{\Omega}$, $s \in \mathbb{R}$. Under the hypothesis $(f)_{\alpha_0}$, the functional $I_\lambda$ is well defined and belongs to $C^1(W^{1,N}_0(\Omega), \mathbb{R})$ (See [1], [10]). Furthermore,

$$I'_\lambda(u)v = \int_\Omega |\nabla u|^{N-2} \nabla u \nabla v \, dx - \lambda \int_\Omega f(x, u)v \, dx, \quad \forall \, u, v \in W^{1,N}_0(\Omega).$$

Thus, every critical point of $I_\lambda$ is a weak solution of $(P)_\lambda$.

We also remark that if $f(x, s)$ satisfies conditions $(f_1)$ and $(f)_{\alpha_0}$, then, for every $\beta > \alpha_0$, there exists $C = C(\beta) > 0$ such that

$$\max\{|f(x, s)|, |F(x, s)|\} \leq C \exp(\beta |s|^N), \quad \forall \, x \in \bar{\Omega}, \quad s \geq 0.$$

As a direct consequence of (1.1) and (2.3), we obtain that $F(x, u(x)) \in L^1(\Omega)$ and $f(x, u(x)) \in L^q(\Omega)$, for every $q \geq 1$, whenever $u \in W^{1,N}_0(\Omega)$.

The following lemma establishes a version of Trudinger-Moser inequality (1.2) for $W^{1,N}(\Omega)$ when $\Omega$ is an open ball in $\mathbb{R}^N$.

**Lemma 2.5.** Let $B(x_0, R)$ be an open ball in $\mathbb{R}^N$ with radius $R > 0$ and center $x_0 \in \mathbb{R}^N$. Then, there exist constants $\tilde{\alpha} = \tilde{\alpha}(N) > 0$ and $C(N, R) > 0$ such that

$$\int_{B(x_0, R)} \exp(\tilde{\alpha}|u|^{N/(N-1)}) \, dx \leq C(N, R),$$

for every $u \in W^{1,N}(B(x_0, R))$ such that $\|u\|_{W^{1,N}(B(x_0, R))} \leq 1$.

**Proof.** For the sake of completeness, we present the proof of Lemma 2.5 in Appendix A. □

Finally, we state a theorem due to Lions [15] which will be essential to verify, via contradiction, that the functional $I_\lambda$ satisfies $(PS)_c$, for $c$ below a given level, when $f(x, s)$ satisfies the critical growth condition.
THEOREM 2.6. Let \( \{ u_n \} \in W^{1,N}_0(\Omega) \) be a sequence in \( W^{1,N}_0(\Omega) \) converging weakly to a nonzero function \( u \). Then, for every \( 0 < p < (1 - \|u\|^N)^{-1} \), we have
\[
\sup_{n \in \mathbb{N}} \int_\Omega \exp\left( p \alpha_N |u_n|^{\frac{N}{N-1}} \right) \, dx < \infty.
\]

3. - (wPS) condition

In this section, we shall prove a technical result that will be used to establish (wPS) condition for the functional \( I_\lambda(u) \), defined by (2.2), when the nonlinearity \( f(x,s) \) satisfies the critical growth condition, 

\( (f_5) \) There exist \( \alpha, C > 0 \) such that
\[
|f(x,s)| \leq C \exp\left( \alpha |s|^{\frac{N}{N-1}} \right), \quad \forall \, x \in \tilde{\Omega}, \ s \in \mathbb{R}.
\]

Our objective is to verify that any bounded sequence \( \{ u_n \} \subset W^{1,N}_0(\Omega) \) such that \( I'_\lambda(u_n) \rightarrow 0 \), as \( n \rightarrow \infty \), possesses a subsequence converging weakly to a solution of (P). Such result provides (wPS) condition for the functional \( I_\lambda \).

Considering that next result is independent of the parameter \( \lambda > 0 \), we denote by (P) and \( I \) the problem \( (P)_\lambda \) and the functional \( I_\lambda \), respectively.

The proof of the following proposition is based on the argument used in [18] for the Neumann problem (See also [10]).

PROPOSITION 3.1. Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \). Suppose \( f(x,s) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R}) \) satisfies \( (f_5) \). Then, any bounded sequence \( \{ u_n \} \subset W^{1,N}_0(\Omega) \) such that \( I'(u_n) \rightarrow 0 \), as \( n \rightarrow \infty \), possesses a subsequence converging weakly to a solution of (P).

REMARK 3.2 (i) Note that Proposition 3.1 generalizes to the N-Laplacian a well known fact for the Laplacian operator on \( \Omega \subset \mathbb{R}^N, \ N > 2 \), when the nonlinearity \( f(x,s) \) satisfies the polynomial critical growth condition. (ii) We also observe that in Proposition 3.1 it is not assumed that \( \{ u_n \} \) is a Palais-Smale sequence since \( I(u_n) \) may be unbounded. (iii) Finally, we note that in [22], we prove a similar result for the \( p \)-Laplacian on \( \Omega = \mathbb{R}^N \).

The proof of Proposition 3.1 will be carried out in a series of steps. First, by the Sobolev Embedding Theorem, Banach-Alaoglu Theorem and the characterization of \( C(\Omega)^* \), given by the Riesz Representation Theorem [20], we may suppose that \( \{ u_n \} \) has a subsequence, still denoted by \( \{ u_n \} \), and there exist
\( u \in W^{1,N}_0(\Omega) \) and \( \mu \in \mathcal{M}(\bar{\Omega}) \), the space of regular Borel measure on \( \bar{\Omega} \), such that

\[
\begin{aligned}
&u_n \rightharpoonup u, \quad \text{weakly in } W^{1,N}_0(\Omega), \\
&|\nabla u_n|^N \rightharpoonup \mu, \quad \text{weakly}^{\star} \text{ in } \mathcal{M}(\bar{\Omega}),
\end{aligned}
\]

(3.1)

Now, we fix \( 0 < \sigma < \infty \) such that \( a_\sigma N^{-1} < \hat{\alpha} \), with \( \hat{\alpha} \) given by Lemma 2.5. Setting \( \Omega_\sigma = \{ x \in \Omega \mid \mu(x) \geq \sigma \} \), we have that \( \Omega_\sigma \) is a finite set since \( \mu \) is a bounded nonnegative measure on \( \bar{\Omega} \). Furthermore,

**Lemma 3.3.** Let \( K \subset (\Omega \setminus \Omega_\sigma) \) be a compact set. Then, there exist \( q > 1 \) and \( M = M(K) > 0 \) such that

\[
\int_K |f(x, u_n(x))|^q \, dx \leq M, \quad \forall \, n \in \mathbb{N}.
\]

**Proof.** To prove such result, we take \( q > 1 \) such that \( a_\sigma N^{-1} < \hat{\alpha} \) and consider \( r_1 = \text{dist}(K, \partial \Omega \cup \Omega_\sigma) > 0 \), the distance between \( K \) and \( \partial \Omega \cup \Omega_\sigma \). For every \( x \in K \), there exists \( 0 < r_x < r_1 \) such that

\[
\mu(B(x, 2r_x)) + \|u\|_{L^N(B(x,2r_x))}^N < \sigma^N.
\]

Using the compactness of \( K \), we find \( j \in \mathbb{N} \) so that

\[
K \subset \bigcup_{i=1}^j B(x_i, r_{x_i}) = \bigcup_{i=1}^j B_i.
\]

Applying (3.1) and (3.2), we find \( n_0 \in \mathbb{N} \) such that

\[
\|u_n\|_{W^{1,N}(B_i)}^N \leq \sigma^N, \quad \forall \, n \geq n_0, \quad 1 \leq i \leq j.
\]

Consequently, from Lemma 2.5, (f5), (3.3) and our choice of \( q \), there exists \( M > 0 \) such that

\[
\int_K |f(x, u_n(x))|^q \, dx \leq \sum_{i=1}^N \int_{B_i} \exp \left( \hat{\alpha} \left( \frac{|u_n(x)|}{\|u_n\|_{W^{1,N}(B_i)}} \right)^\frac{N}{N-1} \right) \, dx \leq M,
\]

for every \( n \geq n_0 \). This proves the lemma. \( \square \)

**Lemma 3.4.** Let \( K \subset (\Omega \setminus \Omega_\sigma) \) be a compact set. Then, \( \nabla u_n \to \nabla u \), strongly in \( (L^N(K))^N \), as \( n \to \infty. \)
PROOF. Taking $\psi \in C_0^\infty(\Omega \setminus \Omega_\sigma)$ such that $\psi \equiv 1$, on $K$, and $0 \leq \psi \leq 1$, and considering that
\begin{equation}
(|a|^{N-2}a - |b|^{N-2}b) \cdot (a - b) \geq 2^{2-N}|a - b|^N, \quad \forall \ a, b \in \mathbb{R}^N,
\end{equation}
we obtain
\begin{equation}
2^{2-N}\|\nabla u_n - \nabla u\|_{L^N(K)}^N
\leq \int_{\Omega} \left[ (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u) \cdot (\nabla u_n - \nabla u) \right] \psi \, dx
\end{equation}
\begin{equation}
= \int_{\Omega} \left[ |\nabla u_n|^N\psi - |\nabla u_n|^{N-2}(\nabla u_n, \nabla u)\psi
- |\nabla u|^{N-2}(\nabla u, \nabla (u_n - u))\psi \right] \, dx.
\end{equation}
As $I'(u_n) \to 0$, as $n \to \infty$, we have
\begin{equation}
\int_{\Omega} \left[ |\nabla u_n|^{N-2} ((\nabla u_n, \nabla u)\psi + (\nabla u_n, \nabla \psi)u) - \psi f(x, u_n)u \right] \, dx = o(1),
\end{equation}
as $n \to \infty$. Moreover, since $(\psi u_n)$ is a bounded sequence in $W_{0}^{1,N}(\Omega)$, we also have
\begin{equation}
\int_{\Omega} \left[ |\nabla u_n|^N\psi + |\nabla u_n|^{N-2}(\nabla u_n, \nabla \psi)u_n - \psi f(x, u_n)u_n \right] \, dx = o(1),
\end{equation}
as $n \to \infty$. Combining (3.5)-(3.7), we obtain
\begin{equation}
2^{2-N}\|\nabla u_n - \nabla u\|_{L^N(K)}^N
\leq \int_{\Omega} \psi f(x, u_n)(u_n - u) \, dx + \int_{\Omega} |\nabla u_n|^{N-2}(u - u_n)(\nabla u_n, \nabla \psi) \, dx
+ \int_{\Omega} |\nabla u|^{N-2}(\nabla u, \nabla (u - u_n))\psi \, dx + o(1), \quad \text{as } n \to \infty.
\end{equation}
Applying Lemma 3.3, for the compact set $\text{supp} \psi \subset (\Omega \setminus \Omega_\sigma)$, and using Hölder’s inequality, we get
\begin{equation}
2^{2-N}\|\nabla u_n - \nabla u\|_{L^N(K)}^N
\leq \|
abla \psi\|_{L^\infty(\Omega)} \left\{ \frac{1}{q} \|u_n - u\|_{L^{q^{-1}}(\Omega)}^{\frac{1}{q-1}} + \|
abla \psi\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^N(\Omega)}^{N-1} \|u - u_n\|_{L^N(\Omega)} \right\}
+ \int_{\Omega} |\nabla u|^{N-2}\psi(\nabla u, \nabla (u - u_n)) \, dx + o(1), \quad \text{as } n \to \infty.
\end{equation}
The hypothesis that $(u_n) \subset W_{0}^{1,N}(\Omega)$ is bounded and (3.1) show that $\nabla u_n \to \nabla u$, strongly in $(L^N(K))^N$, as desired. The lemma is proved.

As a direct consequence of Lemma 3.4, we have
Corollary 3.5. The sequence \((u_n) \subset W^{1,N}_0(\Omega)\) possesses a subsequence \((u_{n_i})\) satisfying \(\nabla u_{n_i}(x) \rightarrow \nabla u(x)\), for almost every \(x \in \Omega\).

The following Lemma shows that \(I'(u)\) restricted to \(W^{1,N}_0(\Omega \setminus \Omega_\sigma)\) is the null operator.

Lemma 3.6.

\((I'(u), \phi) = \int_\Omega |\nabla u|^{N-2}(\nabla u \cdot \nabla \phi) \, dx - \int_\Omega f(x,u)\phi \, dx = 0,\)

for every \(\phi \in C^\infty_0(\Omega \setminus \Omega_\sigma)\).

Proof. Given \(\phi \in C^\infty_0(\Omega \setminus \Omega_\sigma)\), by Hölder’s inequality and the fact that \((u_n) \subset W^{1,N}_0(\Omega)\) is a bounded sequence, we have that \((|\nabla u_{n_i}|^{N-2}\nabla u_{n_i} \cdot \nabla \phi)\) is a family of uniformly integrable functions in \(L^1(\Omega)\). Thus, by Vitali’s Theorem [20] and Corollary 3.5, we get

\((3.9) \int_\Omega |\nabla u_{n_i}|^{N-2}(\nabla u_{n_i} \cdot \nabla \phi) \, dx \rightarrow \int_\Omega |\nabla u|^{N-2}(\nabla u \cdot \nabla \phi) \, dx, \text{ as } i \rightarrow \infty.\)

We also assert that

\((3.10) \int_\Omega f(x,u_{n_i})\phi \, dx \rightarrow \int_\Omega f(x,u)\phi \, dx, \text{ as } i \rightarrow \infty,\)

for every \(\phi \in C^\infty_0(\Omega \setminus \Omega_\sigma)\). Effectively, by Lemma 3.3, there exist \(q > 1\) and \(M_1 > 0\) so that

\((3.10) \int_{K_2} |f(x,u_n)|^q \, dx \leq M_1,\)

where \(K_2 = \text{supp} \phi\). Given \(\epsilon > 0\), from (3.1) and Egoroff’s Theorem, there exists \(E \subset \Omega\) such that \(|E| < \epsilon\) and \(u_n(x) \rightarrow u(x)\), uniformly on \((\Omega \setminus E)\).

Using Hölder’s inequality, (3.11), and \((f_5)\), we get \(M_2 > 0\) such that

\[\int_{\Omega \setminus E} |f(x,u_n) - f(x,u)|\phi dx \leq \int_{\Omega \setminus E} |f(x,u_n) - f(x,u)|\phi dx + M_2 \epsilon^{\frac{q-1}{q}}.\]

As \(\epsilon > 0\) can be chosen arbitrarily small and \(f(x,u_n(x)) \rightarrow f(x,u(x))\), uniformly on \(\Omega \setminus E\), we derive (3.10). Now, we use (3.9), (3.10) and the fact that \(I'(u_n) \rightarrow 0\), as \(n \rightarrow \infty\), to verify that (3.8) holds.

In the following, we conclude the proof of Proposition 3.1. In view of (1.1), \((f_5)\) and the density of \(C^\infty_0(\Omega)\) in \(W^{1,N}_0(\Omega)\), it suffices to show that relation (3.8) holds for every \(\phi \in C^\infty_0(\Omega)\).

Given \(\phi \in C^\infty_0(\Omega)\) such that \(\text{supp} \phi \cap \Omega_\sigma \neq \emptyset\), we take \(K = \text{supp} \phi, \hat{\Omega}_\sigma = \Omega_\sigma \cap K = \{y_1, \ldots, y_l\}, 1 \leq l \leq j,\) and \(r_1 > 0\) such that \(2r_1 < |y_i - y_m|, i \neq m,\)
and $2r_1 < \text{dist}(K, \partial \Omega)$. We consider, $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$, on $[0, 1]$, and $\psi \equiv 0$, on $[2, \infty)$, and we define

$$
\psi_{i,r}(x) = \psi \left( \frac{|x - y_i|}{r} \right), \quad \forall \ x \in \Omega, \ 1 \leq i \leq l, \ 0 < r < r_1.
$$

We also set $\psi_{l+1,r}(x) = 1 - \sum_{i=1}^{l} \psi_{i,r}(x)$, for every $x \in \Omega$. Hence, $\phi(x) \equiv \sum_{i=1}^{l+1} \phi_{i,r}(x)$ and $\phi \psi_{l+1,r} \in C_0^\infty(\Omega \setminus \Omega_0)$. From Lemma 3.6, we have

$$(I'(u), \phi) = \sum_{i=1}^{l} \int_{\Omega} |\nabla u|^{N-2}(\nabla u, \nabla (\phi \psi_{i,r})) \, dx \tag{3.12}$$

$$- \sum_{i=1}^{l} \int_{\Omega} f(x, u) \phi \psi_{i,r} \, dx, \ \forall \ 0 < r < r_1.$$

Applying Hölder’s inequality, for every $1 \leq i \leq l$, we get

$$
\left| \int_{\Omega} |\nabla u|^{N-2}(\nabla u, \nabla (\phi \psi_{i,r})) \, dx \right|
\leq \left[ \int_{B_i} |\nabla u|^{N} \, dx \right]^\frac{N-1}{N} \left[ \|\nabla \phi\|_{L^\infty(\Omega)} \|\psi_{i,r}\|_{L^N(B_i)} \right] + \|\phi\|_{L^\infty(\Omega)} \|\nabla \psi_{i,r}\|_{L^N(B_i)}, \tag{3.13}
$$

where $B_i = B(y_i, 2r_1), 0 < r < r_1$. On the other hand, from the first Trudinger-Moser inequality (1.1) and $(f_3)$, we find $M_3 > 0$ such that, for every $1 \leq i \leq l$,

$$
\int_{\Omega} f(x, u) \phi \psi_{i,r} \, dx \leq M_3\|\phi\|_{L^\infty(\Omega)} |B_i|^\frac{N-1}{2N} \|\psi_{i,r}\|_{L^N(\Omega)}. \tag{3.14}
$$

We use our definition of $\psi_{i,r}$ to get $M_4 > 0$ so that

$$
\|\psi_{i,r}\|_{W^{1,N}(B_i)} \leq M_4, \ \forall \ 0 < r < r_1, \ 1 \leq i \leq l.
$$

Consequently, given $\epsilon > 0$, by Lebesgue’s Dominated Convergence Theorem, (3.13) and (3.14), we find $0 < r_2 < r_1$ so that

$$
\int_{\Omega} |\nabla u|^{N-2}(\nabla u, \nabla (\phi \psi_{i,r})) \, dx < \epsilon, \tag{3.15}
$$

for every $0 < r < r_2, 1 \leq i \leq l$. From (3.12), (3.15) and the fact that $\epsilon > 0$ can be chosen arbitrarily small, we obtain that (3.8) holds for every $\phi \in C_0^\infty(\Omega)$. This concludes the proof of Proposition 3.1. □

As a direct consequence of Proposition 3.1, we have the following results:
**Corollary 3.7.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$. Suppose that $f(x, s) \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies $(f_5)$. Then, $I$ satisfies (wPS) on $A$, for every bounded set $A \subset W^{1, N}_0(\Omega)$.

**Corollary 3.8.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$. Suppose that $f(x, s) \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies $(f_5)$. Then, $I$ satisfies (wPS) provided every (PS) sequence associated with $I$ possesses a bounded subsequence.

### 4. – Theorems 1.1 and 1.2

In this section, we apply the abstract results described in Section 2 to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** The weak solution of problem $(P_\lambda)$ will be established with the aid of Proposition 2.2. For this, it suffices to verify that $I_\lambda$, for $\lambda > 0$ sufficiently small, satisfies $(\Phi_1)$ and (wPS)$_{c_1}$ on the closure of $B(0, \rho)$, denoted by $B[0, \rho]$, for some appropriate value of $\rho > 0$.

Given $\beta > \alpha_0$, we take $\rho \in (0, (\frac{\alpha_N}{\beta})^{\frac{N}{N-1}})$ and use (2.3) to obtain $C_1 > 0$ such that

$$I_\lambda(u) \geq \frac{1}{N} ||u||^N - \lambda C_1 \int_\Omega \exp(\beta ||u||^{\frac{N}{N-1}}) \, dx$$

$$= \frac{1}{N} ||u||^N - \lambda C_1 \int_\Omega \exp\left(\beta ||u||^{\frac{N}{N-1}} \left(\frac{|u|}{||u||}\right)^{\frac{N}{N-1}}\right) \, dx,$$

for every $u \in W^{1, N}_0(\Omega)$ such that $||u|| \leq \rho$. Hence, by Trudinger-Moser inequality (1.2), we find $C_2(N) > 0$ such that

$$I_\lambda(u) \geq \frac{1}{N} ||u||^N - \lambda C_2(N),$$

for every $u \in B[0, \rho]$.

Taking $\bar{\lambda} = N^{-1} C_2(N)^{-1} \rho^N$, $u_0 = 0$, $\gamma = -\bar{\lambda} C_2(N)$, $b = 0$, and considering $c_\lambda = c_1$, $c_1$ given by (2.1), we have that $I_\lambda$ satisfies condition $(\Phi_1)$, for every $0 < \lambda < \bar{\lambda}$.

Finally, we observe that conditions $(f_1)$, $(f)_{a_0}$ and Corollary 3.7 imply that $I_\lambda$ satisfies (wPS) condition on $B[0, \rho]$. Theorem 1.1 is proved. \(\square\)

Before proving Theorem 1.2, we note that, from $(f_1)$ and $(f_2)$, there exists a constant $C > 0$ such that

$$(4.1) \quad F(x, s) \geq C|s|^\beta - C, \quad \forall \, x \in \bar{\Omega}, \ s \geq 0.$$

**Proof of Theorem 1.2.** Considering $\bar{\lambda} > 0$, given in the proof of Theorem 1.1, we have that the functional $I_\bar{\lambda}$ satisfies $(\Phi_1)$, for every $\lambda \in (0, \bar{\lambda})$. Thus,
by Proposition 2.4, it suffices to verify that \( I_\lambda \) satisfies (\( \Phi_2 \)) and (PS) for such values of \( \lambda \).

Choosing \( u \in W_0^{1,N}(\Omega) \setminus \{0\} \) such that \( u(x) > 0 \), for every \( x \in \Omega \), from (4.1), we obtain

\[
I_\lambda(tu) \leq \frac{t^N}{N} ||u||^N - \lambda C t^\theta \int_\Omega u^\theta \, dx + C|\Omega|.
\]

Therefore, \( I_\lambda(tu) \to -\infty \), as \( t \to +\infty \), since \( C > 0 \) and \( \theta > N \). Consequently, \( I_\lambda \) satisfies \( \Phi_2 \).

Now, we shall verify that \( I_\lambda \) satisfies (PS). Let \( (u_n) \subset W_0^{1,N}(\Omega) \) be a sequence such that \( (I_\lambda(u_n)) \subset \mathbb{R} \) is bounded, and \( I_\lambda'(u_n) \to 0 \), as \( n \to \infty \), i.e,

\[
\left| \frac{1}{N} \int_\Omega |\nabla u_n|^N \, dx - \lambda \int_\Omega F(x, u_n) \, dx \right| \leq C < \infty , \quad \forall \ n \in \mathbb{N},
\]

and

\[
\int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla v \, dx - \lambda \int_\Omega f(x, u_n) v \, dx \leq \varepsilon_n ||v||,
\]

for every \( v \in W_0^{1,N}(\Omega) \), where \( \varepsilon_n \to 0 \), as \( n \to \infty \). Taking \( \theta > N \), given by (f2), we use (4.2) and (4.3) to get

\[
\int_\Omega |\nabla u_n|^N \, dx - \lambda \int_\Omega (\theta F(x, u_n) - f(x, u_n) u_n) \, dx \leq C + \varepsilon_n ||u_n||.
\]

From this inequality, (f2), and our definition of \( f(x,s) \) for \( s \leq 0 \), we conclude that \( (u_n) \) is a bounded sequence in \( W_0^{1,N}(\Omega) \). Consequently, we may assume that

\[ u_n \rightharpoonup u \text{ weakly in } W_0^{1,N}(\Omega), \quad u_n \to u \text{ strongly in } L^q(\Omega), \quad \forall \ q > 1. \]

From (4.3), with \( v = u_n - u \), we have

\[
\lim_{n \to \infty} \left\{ \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u) \, dx - \lambda \int_\Omega f(x, u_n)(u_n - u) \, dx \right\} = 0.
\]

Using Hölder’s inequality, we may estimate the second integral in the above equation,

\[
\left| \int_\Omega f(x, u_n)(u_n - u) \, dx \right| \leq \left( \int_\Omega |f(x, u_n)|^p \, dx \right)^{1/p} ||u_n - u||_{L^q},
\]
where \( p, q > 1 \) are fixed with \( \frac{1}{q} + \frac{1}{p} = 1 \). Noting that \((u_n)\) is a bounded sequence, we may find \( \beta > \alpha_0 = 0 \) such that \( \beta p\|u_n\|^{N/N-1} < \alpha_N \), for every \( n \in \mathbb{N} \). Hence, by (2.3), we have

\[
\int_{\Omega} |f(x, u_n)(u_n - u)| \, dx \leq C \left\{ \int_{\Omega} \exp \left( p\beta \|u_n\|^{N/N-1} \left( \frac{|u_n|}{\|u_n\|} \right)^{N/N-1} \right) \right\}^{\frac{1}{p}} \|u_n - u\|_{L^q}.
\]

Thus, by Trudinger-Moser inequality (1.2), we obtain \( C_2 > 0 \) such that

\[
\left| \int_{\Omega} f(x, u_n)(u_n - u) \, dx \right| \leq C_2 \|u_n - u\|_{L^q}.
\]

Since \( u_n \to u \) strongly in \( L^q(\Omega) \), from (4.4) and the above inequality, we have

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u) \, dx = 0.
\]

On the other hand,

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u|^{N-2} \nabla u \nabla (u_n - u) \, dx = 0,
\]

because \( u_n \rightharpoonup u \) weakly in \( W^{1,N}_0(\Omega) \). Consequently,

\[
\lim_{n \to \infty} \int_{\Omega} (|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u)(\nabla u_n - \nabla u) \, dx = 0.
\]

Thus, by inequality (3.4), we have

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n - \nabla u|^N \, dx = 0.
\]

This implies that \( I_\lambda \) satisfies (PS) condition. Theorem 1.2 is proved. \(\square\)

**Remark 4.1.** As it is shown in [9] (See also [14]), any solution of \((P)_\lambda\) is in \( C^{1,\alpha}(\Omega) \), for \( N \geq 3 \), and in \( C^{2,\alpha}(\Omega) \), for \( N = 2 \).
5. - Estimates

We start this section with the definition of Moser functions (See [17]). Let $x_0 \in \Omega$ and $R > 0$ be such that the ball $B(x_0, R)$ of radius $R$ centered at $x_0$ is contained in $\Omega$. The Moser functions are defined for $0 < r < R$ by

$$M_r(x) = \begin{cases} \left( \log \frac{R}{r} \right) \frac{N-1}{N}, & \text{if } 0 \leq |x - x_0| \leq r, \\ \log \left( \frac{R}{|x-x_0|} \right), & \text{if } r \leq |x - x_0| \leq R, \\ 0, & \text{if } |x - x_0| \geq R. \end{cases}$$

Then, $M_r \in W_0^{1,N}(\Omega)$, $||M_r|| = 1$ and supp $(M_r)$ is contained in $B(x_0, R)$. Considering $\Omega$ given by $(f_3)$, we take $x_0 \in \Omega$ and consider the Moser sequence $M_n(x) = M_{R_n}(x)$ where $R_n = (\log n)^{1/N}$, for every $n \in \mathbb{N}$. Without loss of generality, we may suppose that supp $(M_n) \subset \Omega$, for every $n \in \mathbb{N}$.

Taking $\lambda > 0$ and $u_{\lambda}$, for $\lambda \in (0, \lambda)$, given in the proof of Theorem 1.1, we have

**Proposition 5.1.** Suppose $f(x, s)$ satisfies $(f_1)$, $(f_{a_0})$, with $a_0 > 0$, and $(f_3)$. Then, for every $\lambda \in (0, \bar{\lambda})$, there exists $n \in \mathbb{N}$ such that

$$\max \{ I_{\lambda}(u_{\lambda} + tM_n) \mid t \geq 0 \} \leq I_{\lambda}(u_{\lambda}) + \frac{1}{N} \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1}.$$ 

The proof of Proposition 5.1 will be carried out through the verification of several steps. First, we suppose by contradiction that, for every $n$, we have

$$\max \{ I_{\lambda}(u_{\lambda} + tM_n) \mid t \geq 0 \} \geq I_{\lambda}(u_{\lambda}) + \frac{1}{N} \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1}. \quad (5.1)$$

Now, we apply the argument employed in the proof of Theorem 1.2 to conclude that $I_{\lambda}(u_{\lambda} + tM_n) \to -\infty$, as $t \to \infty$, for every $n \in \mathbb{N}$. Thus, there exists $t_n > 0$ such that

$$I_{\lambda}(u_{\lambda} + t_nM_n) = \max \{ I_{\lambda}(u_{\lambda} + tM_n) \mid t \geq 0 \}. \quad (5.2)$$

The following lemmas provide estimates for the value of $t_n$.

**Lemma 5.2.** The sequence $(t_n) \subset \mathbb{R}$ is bounded.
PROOF. Since \( \frac{d}{dt}[I_\lambda(u_\lambda + t M_n)] = 0 \) for \( t = t_n \), it follows that

\[
\int_\Omega |\nabla(u_\lambda + t_n M_n)|^{N-2} \nabla(u_\lambda + t_n M_n) \cdot \nabla M_n \, dx = \lambda \int_\Omega f(x, u_\lambda + t_n M_n) M_n \, dx.
\]

Invoking Holder's inequality, we obtain

\[
\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda \int_\Omega f(x, u_\lambda + t_n M_n) M_n \, dx.
\]

(5.3)

We observe that given \( M > 0 \), from (f3), there exists a positive constant \( C \) such that

\[
f(x, s) \geq M \exp \left( a_0|s|^{N/N-1} \right) - C, \quad \forall \, s \geq 0, x \in \hat{\Omega}.
\]

(5.4)

Thus, from (5.3)-(5.4), the definition of the function \( M_n \) and the nonnegativity of \( u_\lambda \), we have

\[
\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda M \int_{B(x_0, R_n)} \exp \left( a_0|t_n M_n|^{N/N-1} \right) M_n \, dx - \lambda C \int_{B(x_0, R_n)} M_n \, dx.
\]

Using the definition of the function \( M_n \) one more time, we find \( \hat{C} > 0 \) such that

\[
\|u_\lambda + M_n\|^{N-1} \geq \lambda M \int_{B(x_0, R_n)} \exp \left( a_0|t_n M_n|^{N/N-1} \right) M_n \, dx - \lambda \hat{C} R_n^N
\]

\[
= \frac{\lambda M w_{N-1}^{N-1}}{N} \exp \left[ \left( \frac{\alpha_0}{\alpha_N} t_n^{N-1} - 1 \right) N \log n \right] R_n^N \left( \log n \right)^{N-1} - \lambda \hat{C} R_n^N.
\]

Hence, from the definition of \( R_n \), we get

\[
\|u_\lambda + M_n\|^{N-1} \geq \frac{\lambda M w_{N-1}^{N-1}}{N} \exp \left[ \left( \frac{\alpha_0}{\alpha_N} t_n^{N-1} - 1 \right) N \log n \right] - \lambda \hat{C} R_n^N.
\]

(5.5)

Since \( R_n \to 0 \), as \( n \to \infty \), from (5.5), we conclude that \( (t_n) \subset \mathbb{R} \) is a bounded sequence. Lemma 5.2 is proved. \( \Box \)
LEMMA 5.3. There exist a positive constant $C = C(\lambda, \alpha_0, N)$ and no $n_0 \in \mathbb{N}$ such that
\[ t_n^{N/N-1} \geq \frac{\alpha_n}{\alpha_0} - \frac{CR_n^N}{(\log n)^{1/N}}, \forall n \geq n_0. \]

PROOF. From equation (5.1),
\[ I_\lambda(u_\lambda) + \frac{1}{N} \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1} \leq \frac{1}{N} \int_\Omega |\nabla (u_\lambda + t_n M_n)|^N dx - \lambda \int_\Omega F(x, u_\lambda + t_n M_n) dx. \]
Hence,
\[ \frac{1}{N} \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1} \leq \frac{t_n^N}{N} - \lambda \int_\Omega (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda)) dx + \frac{1}{N} \sum_{k=1}^{N-1} \binom{N}{k} t_n^k \int_\Omega |\nabla u_\lambda|^{N-k} |\nabla M_n|^k dx. \]
Furthermore,
\[ F(x, u_\lambda + t_n M_n) - F(x, u_\lambda) = \int_0^{t_n M_n} f(x, s + u_\lambda) ds \geq -m t_n M_n, \]
where $m \geq 0$ is given by $(f_1)$ and $(f)_{\alpha_0}$. Consequently,
\[ \frac{1}{N} \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1} \leq \frac{t_n^N}{N} + \lambda m t_n \int_\Omega M_n dx \]
(5.6)
\[ + \frac{1}{N} \sum_{k=1}^{N-1} \binom{N}{k} t_n^k \int_\Omega |\nabla u_\lambda|^{N-k} |\nabla M_n|^k dx. \]
On the other hand, from the definition of the sequence $(M_n)$, we have
\[ \int_\Omega M_n dx = \frac{R_n^N w_{n-1}^{N-1}}{N} \left\{ \frac{2(\log n)^{N-1}}{n^N} + \frac{1}{N} \left( 1 - \frac{1}{n^N} \right) \frac{1}{(\log n)^{1/N}} \right\} \]
(5.7)
\[ \int_\Omega |\nabla u_\lambda|^{n-k} |\nabla M_n|^k dx \leq C(N, n, \lambda, k) \frac{1}{(\log n)^{k/N}}, \]
(5.8)
where $C(N, n, \lambda, k) = \frac{R_n^{N-k} w_{n-1}^{(\lambda-k)}}{(N-k)} \left( 1 - \frac{1}{n^{N-k}} \right) ||\nabla u_\lambda||_{L^{\infty}}^{N-k}$.
Using (5.6)-(5.8) and Lemma 5.2, we find a constant $C > 0$ such that
\[ t_n^{N/N-1} \geq \left( \frac{\alpha_n}{\alpha_0} \right) \left( 1 - \frac{C \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1} R_n^N}{(\log N)^{1/n}} \right)^{1/(N-1)}. \]
A direct application of Mean Value Theorem to the function \( h(s) = (1-s)^{1/(N-1)} \) on the above relation provides the conclusion of Lemma 5.3.

Now, we shall use Lemmas 5.2 and 5.3 to derive the desired contradiction. From (5.5), Lemma 5.3 and the definition of \( R_n \), we obtain

\[
\|u_\lambda + t_n M_n\|^{N-1} \geq \frac{\lambda M w_{N-1}^{N-1}}{N} \exp \left( -\frac{\alpha_0 CN}{\alpha_N} \right) - \lambda \hat{C} R_n^N.
\]

Thus,

\[
\frac{\lambda M w_{N-1}^{N-1}}{N} \exp \left( -\frac{\alpha_0 CN}{\alpha_N} \right) \leq (\|u_\lambda\| + t_n)^{N-1} + \lambda \hat{C} R_n^N.
\]

But, this contradicts Lemma 5.2, since \( M \) can be arbitrarily chosen and \( R_n \to 0 \), as \( n \to \infty \). Proposition 5.1 is proved.

6. – Theorem 1.3

In this section, after the verification of some preliminary results, we prove Theorem 1.3.

**Lemma 6.1.** Suppose \( f(x, s) \) satisfies (\( f_1 \)), (\( f_2 \)) and (\( f \)\(a_0 \)). Then, any (PS) sequence \( (u_n) \subset W_0^{1,N}(\Omega) \) associated with \( I_\lambda \) possesses a subsequence \( (u_{n_i}) \) converging weakly in \( W_0^{1,N}(\Omega) \) to a solution \( u \) of (\( P_\lambda \)). Furthermore,

\[
\int_\Omega F(x, u_{n_i}(x)) \, dx \to \int_\Omega F(x, u(x)) \, dx, \text{ as } n \to \infty.
\]

**Remark 6.2.** We note that Lemma 6.1 also holds when \( f(x, s) \) satisfies (\( f_2 \)) and (\( f \)\(a_0 \)) for \( s \leq -R(\theta) \), and \( s \leq 0 \), respectively.

**Proof.** Consider a sequence \( (u_n) \subset W_0^{1,N}(\Omega) \) such that

\[
\begin{cases}
I_\lambda(u_n) \to c, \\
I_\lambda'(u_n) \to 0, \text{ as } n \to \infty.
\end{cases}
\]

Arguing as in Section 4, we obtain that \( (u_n) \) is a bounded sequence. Therefore, by Proposition 3.1, there exists a subsequence, that we continue to denote by \( (u_n) \), converging weakly in \( W_0^{1,N}(\Omega) \) to a solution \( u \) of (\( P_\lambda \)). Moreover, we may assume that \( u_n(x) \to u(x) \), for almost every \( x \in \Omega \). From (6.1) and (\( f_1 \)), we get

\[
\|u_n\|^N \leq \|u_n\|^N + \lambda \int_\Omega f(x, 0) u_n^-(x) \, dx \leq \|I_\lambda'(u_n)\| \|u_n^-\| \to 0.
\]
as \( n \to \infty \). Hence, \( u_n(x) \to u(x) \geq 0 \), as \( n \to \infty \), for almost every \( x \in \Omega \).

Now, we fix \( \theta_1 > N \), and we consider \( R_1 = R(\theta_1) > 0 \) given by \((\tilde{f}_2)\). From (6.1) and \((\tilde{f}_2)\), we find \( M_1 > 0 \) such that

\[
\int_{\{u_n(x) \geq R_1\}} \left[ \frac{1}{\theta_1} f(x, u_n(x))u_n(x) - F(x, u_n(x)) \right] \, dx \leq M_1.
\]

Observing that \(|\{x \in \Omega \mid u_n(x) \leq -R_1\}| \to 0\), as \( n \to \infty \), from (2.3), (6.2), (6.3) and Hölder's inequality, we have

\[
\int_{\{u_n^+(x) \geq R_1\}} \left[ \frac{1}{\theta_1} f(x, u_n^+(x))u_n^+(x) - F(x, u_n^+(x)) \right] \, dx \leq M_1.
\]

Given \( \epsilon > 0 \), we take \( \theta_2 > \theta_1 \) such that \( \frac{\theta_1 M_1}{\theta_2 - \theta_1} \leq \epsilon \) and \( R_2 > \max\{R_1, R(\theta_2)\} \), \( R(\theta_2) \) given by \((\tilde{f}_2)\). Applying (6.4) and \((\tilde{f}_2)\), we obtain

\[
\int_{\{u_n^+(x) \geq R_2\}} |F(x, u_n^+(x))| \, dx \leq \epsilon.
\]

Applying Egoroff’s Theorem, we find \( E \subset \Omega \) such that \(|E| < \epsilon \) and \( u_n(x) \to u(x) \), as \( n \to \infty \), uniformly on \((\Omega \setminus E)\). Hence, from (2.3) and (6.2), we have

\[
\left| \int_{\Omega} [F(x, u_n(x)) - F(x, u(x))] \, dx \right| \\
\leq \int_{E} |F(x, u_n^+(x))| \, dx + \int_{E \setminus E}[F(x, u(x))] \, dx + o(1), \ \text{as } n \to \infty.
\]

Fixed \( q > 1 \), we use (2.3) and Hölder’s inequality to get \( M_2 > 0 \) such that

\[
\int_{E} |F(x, u(x))| \, dx \leq M_2 \epsilon^{\frac{1}{q}}.
\]

From (6.5), (6.7) and Lesbegue’s Dominated Convergence Theorem, we have

\[
\int_{E} |F(x, u_n^+(x))| \, dx \leq \epsilon + \int_{E \setminus \{0 \leq u_n(x) \leq R_2\}} |F(x, u_n^+(x))| \, dx \\
\leq \epsilon + \int_{E \setminus \{0 \leq u_n(x) \leq R_2\}} |F(x, u(x))| \, dx + o(1) \\
\leq \epsilon + M_2 \epsilon^{\frac{1}{q}} + o(1), \ \text{as } n \to \infty.
\]

The above inequality, (6.6), (6.7) and the fact that \( \epsilon > 0 \) can be chosen arbitrarily provide the conclusion of the proof of Lemma 6.1.

Considering \( c_\lambda = I_\lambda(u_\lambda) + \frac{1}{N} \left( \frac{a_{N, \lambda}}{a_0} \right)^{N-1} \), with \( u_\lambda \) given by the proof of Theorem 1.1, we shall verify that \( I_\lambda \) satisfies (PS) condition below the level \( c_\lambda \), whenever we suppose that \( u = u_\lambda \) is the only possible solution of \((P)_\lambda\).
LEMMA 6.3. Suppose \( f(x, s) \) satisfies (f1), (f2), with \( \alpha_0 > 0 \), and (f3). Assume that \( u_\lambda \) is the only possible solution of \((P)_\lambda\) for \( 0 < \lambda < \tilde{\lambda} \). Then, \( I_\lambda \) satisfies (PS)c, for every \( c < c_\lambda = I_\lambda(u_\lambda) + \frac{\alpha_N}{\alpha_0} \left( \frac{\alpha_N}{\beta} \right)^{N-1} \).

PROOF. Let \( (u_n) \subset W_0^{1,N}(\Omega) \) be a sequence such that

\[
\begin{cases}
I_\lambda(u_n) \to c < c_\lambda, \\
I_\lambda'(u_n) \to 0, \quad \text{as} \quad n \to \infty.
\end{cases}
\]

Since \( u_\lambda \) is the only solution of \((P)_\lambda\), by Lemma 6.1, we may assume that \( u_n \rightharpoonup u_\lambda \), as \( n \to \infty \), weakly in \( W_0^{1,N}(\Omega) \) and

\[
\int_{\Omega} F(x, u_n(x)) \, dx \to \int_{\Omega} F(x, u_\lambda(x)) \, dx, \quad \text{as} \quad n \to \infty.
\]

From (6.8) and (6.9), we have

\[
\|u_n\|^N \to N \left( c + \lambda \int_{\Omega} F(x, u_\lambda(x)) \, dx \right), \quad \text{as} \quad n \to \infty.
\]

Taking \( v_n = \frac{u_n}{\|u_n\|^N} \), we get that

\[
v_n \rightharpoonup v = \frac{u_\lambda}{[N(c + d)]^{-1}}.
\]

where \( d = \lambda \int_{\Omega} F(x, u_\lambda(x)) \, dx \). Considering \( \beta > \alpha_0 \) such that

\[
c < I_\lambda(u_\lambda) + \frac{1}{N} \left( \frac{\alpha_N}{\beta} \right)^{N-1},
\]

by (2.3), we find \( q > 1 \) and \( C > 0 \) so that

\[
|f(x, s)|^q \leq C \exp \left( \beta |s|^\frac{N}{N-1} \right), \quad \forall \ x \in \Omega, \ s \in \mathbb{R}.
\]

Thus,

\[
\int_{\Omega} |f(x, u_n(x))|^q \, dx \leq C \int_{\Omega} \exp \left( \beta \|u_n\|^\frac{N}{N-1} |v_n(x)|^\frac{N}{N-1} \right) \, dx,
\]

for every \( n \in \mathbb{N} \). On the other hand, by (6.11),

\[
1 - \|v\|^N < \left( \frac{\alpha_N}{\beta} \right)^{N-1} \frac{1}{N(c + d)}.
\]
Consequently, from (6.10), there exists \( p > 0 \) such that
\[
\frac{\beta}{\alpha_N} \|u_n\|^N < p < \left(1 - \|v\|^N\right)^{\frac{1}{N-1}}.
\]

Hence, by Theorem 2.6 and (6.12), there exists \( M > 0 \) such that
\[
\int_{\Omega} |f(x, u_n(x))|^q \, dx \leq M, \quad \forall \ n \in \mathbb{N}.
\]

Applying Egoroff’s Theorem, the above inequality and the argument employed in the proof of Proposition 3.1, we obtain
\[
\int_{\Omega} f(x, u_n(x))u_n(x) \, dx \to \int_{\Omega} f(x, u_\lambda(x))u_\lambda(x) \, dx, \quad \text{as} \ n \to \infty.
\]

Therefore, by (6.8),
\[
\|u_n\|^N \to \lambda \int_{\Omega} f(x, u_\lambda(x))u_\lambda(x) \, dx = \|\lambda\|^N.
\]

The Lemma 6.3 is proved. \( \square \)

Now, we may conclude the proof of Theorem 1.3. Arguing by contradiction, we suppose that \( u_\lambda, \) for \( 0 < \lambda < \hat{\lambda} \), is the only possible solution of \((P)\). By Lemma 6.3, \( I_{\lambda} \) satisfies \((PS)\) for every \( c < c_\lambda \). Furthermore, by the argument employed in the proof of Theorem 1.1, \( I_{\lambda} \) satisfies \((\Phi_1)\) on \( B(0, \rho) \), for \( \rho > 0 \) sufficiently small. Hence, Proposition 2.2 and Remark 2.3 imply \( u_\lambda \in B(0, \rho) \). Invoking Propositions 5.1 and 2.4 and Lemma 6.3, we conclude that \( I_{\lambda} \) possesses at least two critical points. However, this contradicts the fact that \( u_\lambda \) is the only critical point of \( I_{\lambda} \). Theorem 1.3 is proved. \( \square \)

7. - Theorem 1.4

In this section we establish a proof of Theorem 1.4. The key ingredient is the verification of Proposition 5.1 under conditions \((\hat{f}_3)\) and \((\hat{f}_4)\). To obtain such result we exploit the convexity of the function \( F(x, s) \) and the fact that \( u_{\lambda}, \) for \( \lambda \in (0, \hat{\lambda}) \), is a solution of \((P)\).

First, we state a basic result that will be used in our estimates.

**Lemma 7.1.** Let \( a, b \in \mathbb{R}^N, N \geq 2, \) and \( (.,.) \) the standard scalar product in \( \mathbb{R}^N \). Then, there exists a nonnegative polynomial \( p_N(x, y) \) \( (p_2 \equiv 0) \) such that
\[
|a + b|^N \leq |a|^N + N|a|^{N-2}(a, b) + |b|^N + p_N(|a|, |b|).
\]

Furthermore, the smallest exponent of the variable \( y \) of \( p_N(x, y) \) is \( 3/2 \) for \( N = 3 \) and \( 2 \) for \( N \geq 4 \), and the greatest exponent of \( y \) is strictly smaller than \( N \).
PROOF. We present a proof of Lemma 7.1 in Appendix B. □

Now, we are ready to establish the version of Proposition 5.1. Consider $\hat{\Omega}$ given by $(\hat{f}_3)$. Let $x_0 \in \hat{\Omega}$ and the Moser sequence associated $M_n = M R_n$, where $R_n = (\log n)^{\frac{(N-1)(1-\beta)}{N^2}}$ if $N \geq 3$, and $R_n = R$ if $N = 2$, where $R > 0$ is chosen so that $B(x_0, R) \subset \hat{\Omega}$.

**PROPOSITION 7.2.** Suppose $f(x, s)$ satisfies $(f_1), (f\alpha_0)$, with $\alpha_0 > 0$, $(\hat{f}_3)$, and $(f_4)$. Then, for every $\lambda \in (0, \hat{\lambda})$, there exists $n \in \mathbb{N}$ such that

$$\max\{I_\lambda(u_\lambda + t M_n) | t \geq 0\} < I_\lambda(u_\lambda) + \frac{1}{N} \left(\frac{\alpha N}{\alpha_0}\right)^{N-1}.$$ 

Arguing as in the proof of Proposition 5.1, we suppose by contradiction that for every $n \in \mathbb{N}$, (5.1) holds. As before, there exists $t_n \in \mathbb{R}$ satisfying equation (5.2). The following two results are versions of Lemmas 5.2 and 5.3 for this new situation.

**LEMMA 7.3.** The sequence $(t_n) \subset \mathbb{R}$ is bounded.

**PROOF.** Arguing as in the proof of Lemma 5.2, we have that equation (5.3) must hold. By $(f_1)$ and $(f_4)$, for every $x \in \hat{\Omega}$, the function $f(x, \cdot)$ is positive on $[0, \infty)$ and nondecreasing. Thus, from (5.3),

$$(7.2) \quad \|u_\lambda + t_n M_n\|^{N-1} \geq \lambda \int_{B(x_0, R_n)} f(x, t_n M_n) M_n \, dx.$$ 

Now, by $(\hat{f}_3)$, given $M > 0$ there exists $R_M > 0$ such that

$$(7.3) \quad s^\beta f(x, s) \geq M \exp\left(\alpha_0 s^{\frac{N}{N-1}}\right), \quad \forall \ s \geq R_M, \ x \in \hat{\Omega}.$$ 

Consequently, by the definition of $M_n$, for $n$ sufficiently large, we get

$$t_n \|u_\lambda + t_n M_n\|^{N-1} \geq \lambda M w^{N-1}_{N-1} \exp\left[\left(\frac{\alpha_0}{\alpha N t_n^{\frac{N}{N-1}}} - 1 + \frac{\log R_n^N}{N \log n}\right)^{N-1} \log n\right].$$

Now, from definition of $R_n$, we have

$$\frac{\log R_n^N}{N \log n} = \frac{(\beta - 1)(N - 1) \log(\log n)}{N \log n} \to 0, \quad \text{as} \quad n \to \infty.$$ 

Thus, we conclude that $(t_n) \subset \mathbb{R}$ is a bounded sequence. The lemma is proved. □
LEMMA 7.4. There exist $n_0 > 0$, and a positive constant $C(\lambda, \alpha_0, N) \geq 0$ ($C(\lambda, \alpha_0, 2) = 0$) such that

$$t_n^{N/N-1} \geq \frac{\alpha_N}{\alpha_0} \frac{CR_n^N}{(\log n)^{\gamma/N}}, \quad \forall \ n \geq n_0,$$

where $\gamma = 3/2$ if $N = 3$, and $\gamma = 2$ if $N \geq 4$.

PROOF. From equations (5.1)-(5.2), we have

$$I_{\lambda}(u_\lambda) + \frac{1}{N} \left( \frac{\alpha_0}{\alpha_N} \right)^{N-1} \leq I_{\lambda}(u_\lambda + t_n M_n).$$

Consequently,

$$\frac{1}{N} \left( \frac{\alpha_0}{\alpha_N} \right)^{N-1} \leq \frac{1}{N} \int_{\Omega} (|\nabla u_\lambda + \nabla(t_n M_n)|^N - |\nabla u_\lambda|^N) \, dx$$

$$- \lambda \int_{\Omega} (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda)) \, dx.$$

Using Lemma 7.1 with $a = \nabla u_\lambda(x)$, and $b = \nabla(t_n M_n(x))$, we have

$$\int_{\Omega} (|\nabla u_\lambda + \nabla(t_n M_n)|^N - |\nabla u_\lambda|^N) \, dx$$

$$\leq \int_{\Omega} \left( N|\nabla u_\lambda|^{N-2} \nabla u_\lambda \nabla(t_n M_n) + |\nabla(t_n M_n)|^N + p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|) \right) \, dx.$$

From (7.4), (7.5), $\|M_n\| = 1$, and the fact that $u_\lambda$ is a solution of $(P)_\lambda$, we obtain

$$\frac{1}{N} \left( \frac{\alpha_0}{\alpha_N} \right)^{N-1} \leq \frac{t_n^N}{N} \lambda \int_{\Omega} \left[ (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda) - f(x, u_\lambda) t_n M_n) \right.$$  

$$\left. + p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|) \right] \, dx.$$

Hence, from (7.6) and $(f_\lambda)$, we get

$$\frac{1}{N} \left( \frac{\alpha_0}{\alpha_N} \right)^{N-1} \leq \frac{t_n^N}{N} + \int_{\Omega} p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|) \, dx.$$

In the particular case $N = 2$, from Lemma 7.1, we have that $p_2 = 0$. From (7.7), we obtain

$$t_n^{N-1} \geq \frac{\alpha_0}{\alpha_N}.$$
Thus, it suffices to consider $N \geq 3$. Using the definition of the function $M_n$, we obtain the following estimates

\[(7.8) \quad \int_{\Omega} |\nabla u_\lambda| |\nabla M_n|^k \, dx \leq \|\nabla u_\lambda\|_{L^\infty(\Omega)}^i \frac{R_n^N w_{N-1}^{N-k}}{(N-k)(\log n)^{k/N}}, \quad \forall i \geq 0, \quad 1 \leq k < N.\]

Now, from (7.8), Lemma 7.3, and the definition of the polynomial $p_n(x, y)$ (See Lemma 7.1.), there exists a positive constant $C$ such that

\[(7.9) \quad \int_{\Omega} p_n(|\nabla u_\lambda|, |\nabla (t_n M_n)|) \, dx \leq \frac{CR_n^N}{(\log n)^{\gamma/N}},\]

where $\gamma = 3/2$, for $N = 3$, and $\gamma = 2$, for $N \geq 4$. Hence, from (7.7), (7.9), we have

\[t_n^\beta \|u_\lambda + t_n M_n\| \geq \frac{1}{N} \left( \frac{a_0}{a_N} \right)^{N-1} - \frac{CR_n^N}{(\log n)^{\gamma/N}}.\]

Arguing as in the proof of Lemma 5.2, we get the conclusion of Lemma 7.4. \(\square\)

To prove Proposition 7.2, we use Lemmas 7.3 and 7.4 to derive the desired contradiction. From (7.7) and (7.9), for $n$ sufficiently large, we have

\[t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M \int_{B(x_0, R_n^{N/N})} \exp(\alpha_0 |t_n M_n|^{N-1}) M_n^{1-\beta} \, dx.\]

Using the definition of $M_n$ and Lemma 7.4, we get

\[(7.10) \quad t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M w_{N-1}^{N-1-\beta} \exp \left( \frac{-N C R_n^N \log n}{(\log n)^{\gamma/N}} \right) R_n^N \left( \frac{(N-1)(1-\beta)}{N} \right),\]

for $N \geq 3$, and

\[t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M w_{N-1}^{N-1} R_n^{N} \exp(\left( \frac{\alpha_0}{a_N} t_n^{N-1} - 1 \right) N \log n)\]

\[\geq \lambda M w_{N-1}^{N-1} R_n^N,\]

for $N = 2$.

From the definition of $R_n$, we obtain

\[(7.12) \quad R_n^N \left( \frac{(N-1)(1-\beta)}{N} \right) = 1, \quad \text{and} \quad \frac{R_n^N \log n}{(\log n)^{\gamma/N}} = 1.\]

From (7.10) or (7.11) and (7.12), we have a contradiction because the left hand sides of (7.10) and (7.11) are bounded and $M$ can be chosen arbitrarily large. This proves Proposition 7.2. \(\square\)

Finally, we observe that the proof of Theorem 1.4 follows the same argument employed in the proof of Theorem 1.3, with Proposition 7.2 replacing Proposition 5.1.
8. – Appendix A

In this Appendix, we prove Lemma 2.5. First, we note that, without loss of generality, we may suppose \( B(x_0, R) = B(0, R) \equiv B_R \).

Setting \( u_M = \frac{1}{B_R} \int_{B_R} u(x) \, dx \), we may apply Lemma 7.16 in [14] to find \( C = C(N) > 0 \) such that

\[
|u(x) - u_M| \leq C(N) \int_{B_R} \frac{|\nabla u(y)|}{|x - y|^{N-1}} \, dy, \text{ a.e. in } B_R.
\]

Taking \( v(x) = u(x) - u_M, h \in L^p(B_R), p > 1, q = \frac{p}{p-1} \), and we use Hölder’s inequality, as in [23], to obtain

\[
\int_{B_R} |h(x)||v(x)| \, dx \\
\leq C(N) \left[ \int_{B_R \times B_R} \frac{|h(x)|}{|x - y|^{N-\frac{1}{q}}} \, dxdy \right]^{\frac{N-1}{N}} \left[ \int_{B_R \times B_R} \frac{|\nabla u(x)||h(x)|}{|x - y|^{\frac{N-1}{q}}} \, dxdy \right]^{\frac{1}{N}}.
\]

Observing that the diameter of \( B_R \) is equal to 2R, we get a constant \( C_1(N) > 0 \) such that

\[
\int_{B_R \times B_R} \frac{|h(x)|}{|x - y|^{N-\frac{1}{q}}} \, dxdy \leq C_1(N)q \|h\|_{L^p(B_R)} R^{\frac{N+1}{q}}.
\]

Applying Hölder’s inequality one more time, we find \( C_2(N) > 0 \) such that

\[
\int_{B_R} \frac{|h(x)|}{|x - y|^{\frac{N-1}{q}}} \, dx \leq C_2(N)\|h\|_{L^p(B_R)} R^{\frac{1}{q}}.
\]

Combining the above inequalities, we find \( C_3(N) > 0 \) such that

\[
\int_{B_R} |h(x)||v(x)| \, dx \leq C_3(N)q \frac{N-1}{N} R^{\frac{N}{q}} \|h\|_{L^p(B_R)} \|\nabla u\|_{L^N(B_R)},
\]

for every \( h \in L^p(B_R) \). Therefore,

\[
\|v\|_{L^q(B_R)} \leq C_3(N)q \frac{N-1}{N} R^{\frac{N}{q}} \|\nabla u\|_{L^N(B_R)},
\]

for every \( q > 1 \). Consequently, there exists \( C_4(N) > 0 \) so that

\[
\int_{B_R} |u - u_M|^q \, dx \leq C_4(N)q R^N,
\]
whenever $u \in W^{1,N}(B_R)$, \( \|u\|_{W^{1,N}(B_R)} \leq 1 \). Now, we use the power series expansion of $\psi(t) = e^t$ and the above inequality to derive

$$\int_{B_R} \exp(\alpha |u-u_M|^{N-1}) \, dx \leq |B_R| + \alpha \int_{B_R} |u-u_M|^{N-1} \, dx + \beta N \sum_{q=2}^{\infty} \frac{\alpha^q C_4(N) q^q}{q!},$$

if $\|u\|_{W^{1,N}(B_R)} \leq 1$. Hence, there exist $\alpha = \alpha(N) > 0$, and $C_5(N) > 0$ such that

$$\int_{B_R} \exp \left( \alpha 2^{\frac{1}{N}} |u-u_M|^{N-1} \right) \, dx \leq C_5(N) R^N + \alpha 2^{\frac{1}{N}} \|u-u_M\|_{L^{N-1}(B_R)}^{N-1}.$$ 

Since

$$|u_M| \leq C_6(N) R^{-1} \|u\|_{L^N(B_R)}, \quad \forall \; u \in W^{1,N}(B_R),$$

for some $C_6(N) > 0$, we may use the convexity of the function $\psi(t) = t^{N-1}$ to obtain $C(N, R) > 0$ such that

$$\int_{B_R} \exp \left( \alpha |u|^{N-1} \right) \, dx \leq C(N, R),$$

for every $u \in W^{1,N}(B_R)$ satisfying $\|u\|_{W^{1,N}(B_R)} \leq 1$. Lemma 2.5 is proved. \( \square \)

9. – Appendix B

In this Appendix we prove Lemma 7.1. First, we establish an inequality that will be necessary in the sequel.

**Lemma 9.1.** Let $x, y$ be real numbers with $x > 0$ and $x + y \geq 0$. Consider $k = \frac{N}{2}$, where $N \in \mathbb{N}$, and $N \geq 3$. Then, there exist nonnegative constants $C_1, C_2$ such that

$$(x+y)^k \leq x^k + kx^{k-1}y + |y|^k + C_1 |x| |y|^{k-1} + C_2 x^{k-2} y^2.$$ 

Furthermore, $C_1 = C_2 = 0$ if $N = 3$ and 4, and $C_1 = 0$ when $N = 5$.

**Proof.** Since $x > 0$ and $(x+y)^k = x^k(1+xy^{-1})^k$, it suffices to consider $(1+z)^k$ for every $z \geq -1$.

(i) Case $N = 3$. Let $g(z) = 1 + \frac{3}{2} z + |z|^{\frac{3}{2}} - (1 + z)^{\frac{3}{2}}$, for every $z \geq -1$. We must show that the function $g$ is nonnegative. Direct calculation shows that $g'(z) \geq 0$ for every $z \geq 0$, and $g(0) = 0$. When $z \in [-1, 0]$, we consider $r = |z|$. Thus, $g(z) = h(r) = 1 - \frac{3}{2} r + r^{\frac{3}{2}} - (1 - r)^{\frac{3}{2}}$, and $h'(r) \geq 0$, and $h(0) = 0$. Hence, $g(z) \geq 0$ for every $z \geq -1$. 


(ii) Case $N = 4$. The proof is immediate.

(iii) Case $N = 5$. Consider the polynomial function:

$$P(z) = (1 + z)^{\frac{5}{2}} - (1 + \frac{5}{2}z + |z|^{\frac{5}{2}}).$$

By L’Hospital’s Theorem, we have

$$\lim_{|z| \to 0} \frac{P(z)}{z^2} = \frac{15}{8}.$$ 

Moreover, by Mean Value Theorem, we get

$$\lim_{|z| \to \infty} \frac{P(z)}{z^2} = 0.$$ 

Consequently, there exists a nonnegative constant $C$ such that

$$(1 + z)^{k} \leq 1 + \frac{5}{2}z + |z|^{\frac{5}{2}} + Cz^2, \quad \forall \ z \in \mathbb{R}.$$ 

(iv) Case $N \geq 6$. Consider the polynomial function:

$$P_k(z) = (1 + z)^k - (1 + kz + |z|^k), \quad \text{where} \quad k = \frac{N}{2}, \quad \text{and} \quad z \geq -1.$$ 

Arguing as above, we have

$$\lim_{|z| \to 0} \frac{P_k(z)}{z^2} = \frac{k(k - 1)}{2},$$

and

$$\lim_{|z| \to \infty} \frac{P_k(z)}{|z|^{k-1}} = k.$$ 

Consequently, there exist nonnegative constants $C_1, C_2$ such that

$$(1 + z)^k \leq 1 + kz + |z|^k + C_1|z|^{k-1} + C_2z^2, \quad \forall \ z \in \mathbb{R}.$$ 

Lemma 7.3 is proved. \qed

Proof of Lemma 7.1. The proof is immediate when $N = 2$. Thus, it suffices to verify the lemma for $N \geq 3$. Writing

$$|a + b|^N = (|a + b|^2)^{N/2} = (|a|^2 + 2(a, b) + |b|^2)^{N/2},$$
and using Lemma 9.1 with $x = |a|^2$, and $y = 2\langle a, b \rangle + |b|^2$, we have

$$|a + b|^N \leq |a|^N + N|a|^{N-2}\langle a, b \rangle + \frac{N}{2}|a|^{N-2}|b|^2 + (2|a||b| + |b|^2)^{N/2}$$

$$+ 2^{(N-2)/2}C_1|a|^2(2|a||b|)^{(N-2)/2} + 2^{(N-2)/2}C_1|a|^2|b|^{N-2}$$

$$+ 4C_2|a|^{N-2}|b|^2 + 4C_2|a|^{N-3}|b|^3 + C_2|a|^{N-4}|b|^4.$$ 

Applying Lemma 9.1 one more time, we obtain

$$|a + b|^N \leq |a|^N + N|a|^{N-2}\langle a, b \rangle + |b|^N + p_N(|a|, |b|),$$

where

$$p_N(|a|, |b|) = \left( \frac{N}{2} + 4C_2 \right)|a|^{N-2}|b|^2 + N2^{(N-4)/2}|a|^{(N-2)/2}|b|^{(N+2)/2}$$

$$+ 2^{N-2}C_1|a|^{(N+2)/2}|b|^{(N-2)/2} + 2^{(N-2)/2}C_1|a|^2|b|^{N-2}$$

$$+ 2^{N/2}|a|^{N/2}|b|^{N/2} + 2^{(N-4)/2}C_2|a|^{(N-4)/2}|b|^{(N+4)/2}$$

$$+ 2C_1|a||b|^{N-1} + 4C_2|a|^{N-3}|b|^3 + C_2|a|^{N-4}|b|^4.$$ 

Finally, since $C_1 = C_2 = 0$ if $N = 3$ and 5, and $C_1 = 0$ when $N = 5$, from the definition of $p_N$ we conclude that the smallest exponent of $|b|$ is $3/2$, for $N = 3$, and 2, for $N \geq 4$, and the greatest exponent of $|b|$ is strictly smaller than $N$. Lemma 7.1 is proved. \qed

REFERENCES


[16] J. LIOUVILLE, Sur l’equation aux différences partielles $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{u^2} = 0$, J. Math. Pures Appl. 18 (1853), 71-72.