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A Series of Smooth Irregular Varieties in Projective Space

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Abstract. One of the simplest examples of a smooth, non degenerate surface in $\mathbb{P}^4$ is the quintic elliptic scroll. It can be constructed from an elliptic normal curve E by joining every point on E with the translation of this point by a non-zero 2-torsion point. The same construction can be applied when E is replaced by a (linearly normally embedded) abelian variety A. In this paper we ask the question when the resulting scroll Y is smooth. If A is an abelian surface embedded by a line bundle L of type (d_1, d_2) and $r = d_1 d_2$, then we prove that for general A the scroll Y is smooth if r is at least 7 with the one exception where $r = 8$ and the 2-torsion point is in the kernel $K(L)$ of L. In this case Y is singular. The case $r = 7$ is particularly interesting, since then Y is a smooth threefold in $\mathbb{P}^6$ with irregularity 2. The existence of this variety seems not to have been noticed before. One can also show that the case of the quintic elliptic scroll and the above case are the only possibilities where Y is smooth and the codimension of Y is at most half the dimension of the surrounding projective space.

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0. - Introduction

One of the simplest examples of a smooth, non degenerate surface in $\mathbb{P}^4$ is the quintic elliptic scroll Y. Its construction goes as follows. Let A be an elliptic normal curve of degree 5 in $\mathbb{P}^4$ and let ϵ be a non zero point of order two on A. Then the union of all the lines joining pairs of points of type x and $x + \epsilon$ on A is an elliptic quintic scroll.

Exactly the same construction can be repeated starting from any abelian variety A of dimension n, with A linearly normally embedded in a projective space $\mathbb{P}^N$ via a very ample line bundle $\mathcal{L}$, and from any non trivial point $\epsilon \in A$ of order two. We investigate this construction in the present paper. In this way we get a scroll Y of dimension $n + 1$ in $\mathbb{P}^N$ related to the above data

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$(A, \epsilon, \mathcal{L})$ and the first interesting question is: when is Y smooth? It is well known that this is the case if $n = 1$ and $N \geq 4$. So the next interesting case is that of surfaces, i.e. $n = 2$, embedded in $\mathbb{P}^{r-1}$ via a (d_1, d_2)-polarization, with $r = d_1 \cdot d_2$. If $r \leq 6$ there is no hope for Y to be smooth because of Lefschetz's hyperplane section theorem. So the question becomes relevant as soon as $r \geq 7$. In fact the main part of this paper is devoted to proving that if A is general in its moduli space (it is enough to assume that $\mathrm{End}(A) \simeq \mathbb{Z}$ or $NS(A) \simeq \mathbb{Z}$ depending on the case under consideration), and if $r \geq 7$ and $r \neq 8$, then Y is smooth. This is particularly remarkable in the case $r = 7$, since Y is then an irregular, codimension three manifold in $\mathbb{P}^6$, whose existence does not seem to have been previously noticed. As we remark at the end of Section 2, for no other dimension of A, but 1 and 2, and $N = 4$ and $N = 6$ respectively, Y can be smooth of codimension $c \leq \frac{N}{2}$ in $\mathbb{P}^N$. The case $d = 8$ is also interesting. If A is a general abelian surface with a polarization of type $(1, 8)$, then Y is smooth, unless the translation by the point ϵ of order two fixes the polarization, in which case Y is singular. If the polarization is of type $(2, 4)$, then the translation by ϵ automatically fixes the polarization and Y is again singular.

The paper is organised as follows. In Section 1 we present the construction of a suitable projective bundle X over $\bar{A} = A/\epsilon$ which maps to Y via its tautological line bundle. In Section 2 we prove that this map is finite and we compute the double point cycle of the composite map of $X \to Y$ with a general projection in a $\mathbb{P}^l$, with $n + 1 \leq l \leq N$. From Section 3 on we restrict our attention to the case of abelian surfaces. In particular in Section 3 we prove that Y is smooth as soon as $r \geq 10$. This comes as a consequence of the fact that, in this situation, if A is general enough, then it has no quadrisecant plane. A property which, in turn, follows as an application of Reider's method. Finally in Section 4 we prove that Y is smooth if $r = 7$, in Section 5 we analyse the case $r = 8$ and in Section 6 the case $r = 9$. The idea for the proof that Y is smooth and the tools we use in the cases $r = 7$, $r = 8$ and the polarization is of type $(1, 8)$ with ϵ not fixing it, and $r = 9$ and the polarization is the triple of a principal polarization (which is the only critical case for $r = 9$) are the same: we first bound dimension and degree of the possible singular locus of Y by using geometric arguments and the double point formula, then we use the action of the Heisenberg group to give a lower bound for the degree of the singular locus, finally contradicting the previous estimate.

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1. – Some projective bundles over abelian varieties

Let A be an abelian variety of dimension n with a polarization $\Theta \in NS(A)$ of type $(d_1, \dots, d_n)$ with $d_1 | \dots | d_n$ (our general reference for the theory of abelian varieties will be [LB]). Let us take a non trivial point $\epsilon \in A$ of order two.

Let $K(\Theta)$ be the kernel of the isogeny $\lambda_\Theta : A \to \hat{A} = \mathrm{Pic}^0(A)$ determined by the polarization. Recall that $K(\Theta) \simeq (\mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_n})^2$ and that, if d_1 is even, then Θ is divisible by two in $NS(A)$ and every point of order two of A is an element of $K(\Theta)$.

Let $\mathcal{L}$ be a line bundle on A representing Θ. Then we have:

$$t_\epsilon^* \mathcal{L} \simeq \mathcal{L} \otimes \mathcal{L}_0$$

where t_x is the translation by a point $x \in A$ and $\mathcal{L}_0 \in \mathrm{Pic}^0(A)$ is the point of order two given by $\lambda_\Theta(\epsilon)$. Hence $\mathcal{L}_0$ is trivial if and only if $\epsilon \in K(\Theta)$.

Let $\bar{A}$ be the quotient A/ϵ and let $\pi : A \to \bar{A}$ be the quotient map, which is an isogeny of degree 2. If $\epsilon \in K(\Theta)$, then there is a line bundle $\overline{\mathcal{L}}$ on $\bar{A}$, such that $\pi^*(\overline{\mathcal{L}}) = \mathcal{L}$. The line bundle $\overline{\mathcal{L}}$ represents a polarization $\overline{\Theta}$ on $\bar{A}$, of type $(\overline{d_1}, \dots, \overline{d_n})$, such that:

$$2 \cdot \overline{d_1} \cdot \dots \cdot \overline{d_n} = d_1 \cdot \dots \cdot d_n\,,$$

a relation which is obtained from $\Theta^n = \pi^*(\overline{\Theta})^n = 2\overline{\Theta}^n$. In particular, if $d_1 = \dots = d_{n-1} = 1$, $d_n = d$, then d is even and $\overline{\Theta}$ is of type $(1, \dots, 1, \frac{d}{2})$.

One has:

$$\pi_* \mathcal{O}_A = \mathcal{O}_{\bar{A}} \oplus \mathcal{M}_1$$

where $\mathcal{M}_1$ is a non trivial 2-torsion point in $\mathrm{Pic}^0(\bar{A})$. The induced map $\pi^* : \mathrm{Pic}^0(\bar{A}) \to \mathrm{Pic}^0(A)$ is also an isogeny of degree 2, whose kernel is generated by $\mathcal{M}_1$. Therefore we have two line bundles $\mathcal{M}_2, \mathcal{M}_3 \in \mathrm{Pic}^0(\bar{A})$ such that:

$$\pi^*(\mathcal{M}_2) = \pi^*(\mathcal{M}_3) = \mathcal{L}_0$$

and one has $\mathcal{M}_2 = \mathcal{M}_3 \otimes \mathcal{M}_1$. The elements $\mathcal{M}_i$, $i = 1, 2, 3$, and the trivial bundle form a subgroup G of $\mathrm{Pic}^0(\bar{A})$, which is the inverse image via π^* of the subgroup generated by $\mathcal{L}_0$.

We have the:

LEMMA 1.1. *If $\epsilon \in K(\Theta)$ then G is the group of order two generated by $\mathcal{M}_1$. Otherwise G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

PROOF. The first assertion is clear. To prove the second assertion let L be a lattice which defines $A = \mathbb{C}^g/L$. Then the point ϵ is represented by an element $e \in \frac{1}{2}L$. The fact that ϵ is not in $K(\Theta)$ is equivalent to the existence of some element $f \in L$ such that for the pairing defined by the polarization $(e, f) = \frac{1}{2} \bmod \mathbb{Z}$. The lattice $\bar{L}$ which defines the quotient $\bar{A}$ is the lattice generated by L and e. We denote by $L^\vee$ the dual lattice of L. This defines the dual variety $\mathrm{Pic}^0(A) = \mathbb{C}^g/L^\vee$ of A. The element e is not contained in $L^\vee$ and represents the line bundle $\mathcal{L}_0$ in $\mathrm{Pic}^0(A)$. Similarly $f \notin \overline{L}^\vee$, but $2f \in \overline{L}^\vee$. The element f corresponds to the line bundle $\mathcal{M}_1$ in $\mathrm{Pic}^0(\bar{A})$ which is 2-torsion. The element e also defines a line bundle in $\mathrm{Pic}^0(\bar{A})$, whose pullback to A is $\mathcal{L}_0$ and this corresponds to $\mathcal{M}_2$ or $\mathcal{M}_3$. The claim follows if we can show that $\mathcal{M}_2$ or $\mathcal{M}_3$ is 2-torsion. But this follows since $2e \in \overline{L}^\vee$. (A different proof will follow from Proposition 1.5 below (see Remark 1.6)). $\square$

Let us set:

$$\mathcal{E} = \mathcal{E}(A, \epsilon, \mathcal{L}) = \pi_*\mathcal{L}\,.$$

This is a rank 2 vector bundle on $\bar{A}$, and we can consider the associated projective bundle:

$$X = X(A, \epsilon, \mathcal{L}) = \mathbb{P}(\mathcal{E}) = \mathbf{Proj}(\oplus_{i=0}^{\infty} \mathrm{Sym}^i(\mathcal{E}))$$

with its tautological line bundle $\mathcal{O}_X(1)$ and its structure map $p : X \to \bar{A}$. We will denote by F a fibre of p and by H a divisor in $|\mathcal{O}_X(1)|$. We will use the same notation to denote their classes in the homology ring of X.

If $\epsilon \in K(\Theta)$ then, since $\pi^*(\overline{\mathcal{L}}) = \mathcal{L}$, the projection formula tells us that:

$$\mathcal{E} = \overline{\mathcal{L}} \oplus (\overline{\mathcal{L}} \otimes \mathcal{M}_1)\,.$$

By contrast, as we shall see in a moment, if $\epsilon \notin K(\Theta)$, then the bundle $\mathcal{E}$ in general does not split.

The natural map $\pi^*\mathcal{E} \to \mathcal{L}$ defines an inclusion i of A into X such that H restricts to $\mathcal{L}$ on $i(A)$. The image $i(A)$ is a 2-section over $\bar{A}$. If there is no danger of confusion we shall denote $i(A)$ also by A. Let

$$\widetilde{X} = \mathbb{P}(\pi^*\mathcal{E})\,.$$

We have a natural étale map $f : \widetilde{X} \to X$ of degree 2. The inverse image of the 2-section A in X under the map f consists of 2 sections of $\widetilde{X}$ corresponding to the 2 projections $\pi^*\mathcal{E} \to \mathcal{L}$ and $\pi^*\mathcal{E} \to t_\epsilon^*\mathcal{L}$ whose existence follows from the construction of $\mathcal{E}$. This shows that $\pi^*\mathcal{E}$ splits, more precisely

$$\pi^*\mathcal{E} \simeq \mathcal{L} \oplus t_\epsilon^*\mathcal{L} = \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}_0)\,. \tag{1}$$

Notice that, if $\epsilon \in K(\Theta)$, then $\widetilde{X} = A \times \mathbb{P}^1$ is trivial.

Lemma 1.2. *One has:*

(i) $\mathcal{O}_X(1)_{|A} \simeq \mathcal{L}$;
(ii) $\mathcal{O}_A(A) \simeq \mathcal{L}_0$;
(iii) $\mathcal{O}_X(-2A) \simeq \mathcal{O}_X(2K_X)$.

Moreover, if $\epsilon \notin K(\Theta)$ *then:*

(iv) *there is no section* A' *of* X *over* $\bar{A}$ *which is disjoint from* A.

Hence if $\epsilon \notin K(\Theta)$ *and* A *does not contain elliptic curves, then* $\mathcal{E}$ *does not split.*

Proof. (i) follows by the definition of the tautological bundle.

(ii) There are two sections $\tilde{A}_1$ and $\tilde{A}_2$ of $\widetilde{X}$ over A, which map both isomorphically to A via f. These sections correspond to the splitting of $\pi^*(\mathcal{E}) = \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}_0)$. Since the normal bundle of both these sections in $\widetilde{X}$ is given by $\mathcal{L}_0$, and since f is étale, we have the assertion.

(iii) Since $A \cdot F = -K_X \cdot F = 2$, there is a line bundle $\mathcal{M}$ on $\bar{A}$ such that $\mathcal{O}_X(-A) \simeq \mathcal{O}_X(K_X) \otimes \pi^*(\mathcal{M})$. On the other hand, by adjunction, one has $\mathcal{O}_A(-A) \simeq \mathcal{O}_A(K_X)$. This implies that either $\mathcal{M} \simeq \mathcal{O}_{\bar{A}}$ or $\mathcal{M} \simeq \mathcal{M}_1$. This immediately yields the assertion.

(iv) Suppose A' is disjoint from A. Then A' would pull back to a section $\tilde{A}'$ of $\widetilde{X}$, disjoint from $\tilde{A}_1$ and $\tilde{A}_2$, which would give another way of splitting $\pi^*(\mathcal{E} \otimes \mathcal{L}^*)$. This is impossible under the assumption $\epsilon \notin K(\Theta)$.

If $\mathcal{E}$ splits, we have two sections A' and A'' of X which do not meet. However they both meet A and they must cut out two divisors C' and C'' on A which do not meet each other. Hence C' and C'' are pull-backs from an elliptic curve and so A contains an elliptic curve. □

Notice that the map $p : X \to \bar{A}$ induces an isomorphism $p^* : \mathrm{Pic}^0(\bar{A}) \to \mathrm{Pic}^0(X)$. We will identify $\mathrm{Pic}^0(\bar{A})$ and $\mathrm{Pic}^0(X)$ using p^*. We have the:

Proposition 1.3. *Let η be an element of $\mathrm{Pic}^0(X)$. One has:*

- (i) *if $\epsilon \in K(\Theta)$ then $h^0(X, \mathcal{O}_X(A) \otimes \eta) = 0$ unless $\eta = \mathcal{O}_X$, in which case $h^0(X, \mathcal{O}_X(A)) = 2$, and $\eta = \mathcal{M}_1$, in which case $h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 1$;*
- (ii) *if $\epsilon \notin K(\Theta)$ then $h^0(X, \mathcal{O}_X(A) \otimes \eta) = 0$ unless $\eta = \mathcal{O}_X, \mathcal{M}_2, \mathcal{M}_3$, in which cases $h^0(X, \mathcal{O}_X(A) \otimes \eta) = 1$.*

Proof. We have:

$$p_*\mathcal{O}_X(1) \simeq \mathcal{E}, \quad p_*\mathcal{O}_X(2) \simeq \mathrm{Sym}^2 \mathcal{E}.$$

Therefore, by using (1) we have:

$$\pi^* p_* \mathcal{O}_X(2) \simeq \mathrm{Sym}^2 \pi^* \mathcal{E} \simeq \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 2} \oplus (\mathcal{L}^{\otimes 2} \otimes \mathcal{L}_0). \tag{2}$$

Moreover, since $A \cdot F = 2$, there is a line bundle $\mathcal{N}$ on $\bar{A}$ such that:

$$\mathcal{O}_X(A) \simeq \mathcal{O}_X(2) \otimes p^*\mathcal{N}. \tag{3}$$

Hence, by the projection formula, one has:

$$\pi^* p_*(\mathcal{O}_X(A) \otimes \eta) \simeq \pi^*(p_*\mathcal{O}_X(2) \otimes (\mathcal{N} \otimes \eta)) \simeq \pi^* p_* \mathcal{O}_X(2) \otimes \pi^*(\mathcal{N} \otimes \eta). \tag{4}$$

From Lemma 1.2, (i), (ii) and restricting (3) to A, we obtain:

$$\pi^* \mathcal{N}^* \simeq \mathcal{L}^{\otimes 2} \otimes \mathcal{L}_0. \tag{5}$$

Now, by (2), (4) and (5), we get:

$$\pi^* p_*(\mathcal{O}_X(A) \otimes \eta) \simeq (\mathcal{L}_0 \oplus \mathcal{L}_0 \oplus \mathcal{O}_A) \otimes \pi^* \eta$$

and therefore if $\epsilon \in K(\Theta)$:

$$(6) \qquad h^0(A, \pi^* p_* \mathcal{O}_X(A \otimes \eta)) = \begin{cases} 0 & \text{if } \pi^*\eta \neq \mathcal{O}_A \\ 3 & \text{if } \pi^*\eta = \mathcal{O}_A, \text{ i.e. } \eta \simeq \mathcal{O}_{\bar{A}}, \mathcal{M}_1 \end{cases}$$

whereas:

$$(6') \qquad h^0(A, \pi^* p_* \mathcal{O}_X(A \otimes \eta)) = \begin{cases} 0 & \text{if } \pi^*\eta \neq \mathcal{O}_A, \mathcal{L}_0 \\ 1 & \text{if } \pi^*\eta = \mathcal{O}_A, \text{ i.e. } \eta \simeq \mathcal{O}_{\bar{A}}, \mathcal{M}_1 \\ 2 & \text{if } \pi^*\eta = \mathcal{L}_0, \text{ i.e. } \eta \simeq \mathcal{M}_2, \mathcal{M}_3 \end{cases}$$

otherwise. Notice that:

$$\begin{aligned} h^0(A, \pi^* p_* \mathcal{O}_X(A \otimes \eta)) &= h^0(\bar{A}, \pi_* \pi^* p_* \mathcal{O}_X(A \otimes \eta)) \\ &= h^0(\bar{A}, p_* \mathcal{O}_X(A \otimes \eta)) + h^0(\bar{A}, (p_* \mathcal{O}_X(A \otimes \eta)) \otimes \mathcal{M}_1) \\ &= h^0(X, \mathcal{O}_X(A \otimes \eta)) + h^0(X, \mathcal{O}_X(A \otimes \eta) \otimes \mathcal{M}_1) \,. \end{aligned}$$

Then, by (6), resp. (6′) if $\eta \neq \mathcal{O}_{\bar{A}}, \mathcal{M}_i$, $i = 1, 2, 3$, we have:

$$h^0(X, \mathcal{O}_X(A \otimes \eta)) + h^0(X, \mathcal{O}_X(A \otimes \eta) \otimes \mathcal{M}_1) = 0 \,,$$

in particular $h^0(X, \mathcal{O}_X(A \otimes \eta)) = 0$.

Let $\epsilon \in K(\Theta)$. If $\eta = \mathcal{O}_{\bar{A}}, \mathcal{M}_1$, we find:

$$h^0(X, \mathcal{O}_X(A)) + h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 3 \,.$$

We claim that $h^0(X, \mathcal{O}_X(A)) = 2$ and $h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 1$. Let $\tilde{A}$ be a trivial section of $\widetilde{X} = A \times \mathbb{P}^1$ over A. Of course $|\tilde{A}|$ is a base point free pencil on $\widetilde{X}$. The image of this pencil under the map $f : \widetilde{X} \to X$ is a system of divisors on X which is contained in a linear system. Since A is in this system, we see that $h^0(X, \mathcal{O}_X(A)) \geq 2$. on the other hand we cannot have $h^0(X, \mathcal{O}_X(A)) \geq 3$ because of Lemma 1.2, (ii). Hence the assertion follows, proving (i).

Let now $\epsilon \notin K(\Theta)$. If $\eta = \mathcal{O}_{\bar{A}}, \mathcal{M}_1$, we find:

$$h^0(X, \mathcal{O}_X(A)) + h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 1 \,,$$

and since $h^0(X, \mathcal{O}_X(A)) \geq 1$, we have $h^0(X, \mathcal{O}_X(A)) = 1$ and $h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 0$. Finally, if $\eta = \mathcal{M}_i$, $i = 2, 3$, we have:

$$h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_2) + h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_3) = 2 \,.$$

We claim that both summands are smaller than 2. Otherwise the linear system, say, $|\mathcal{O}_X(A) \otimes \mathcal{M}_2|$ would be a pencil, and therefore we would find an element of it meeting A. But by Lemma 1.2, (ii), the restriction of $\mathcal{O}_X(A) \otimes \mathcal{M}_2$ to A is trivial. This would yield that A itself is an element of the pencil, implying $h^0(X, \mathcal{M}_2) > 0$ and hence that $\mathcal{M}_2$ is trivial on X, a contradiction. In conclusion we have:

$$h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_2) = h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_3) = 1$$

which finishes our proof. □

REMARKS 1.4. (i) First we consider the case $\epsilon \in K(\Theta)$. We reconsider the relation between the pencils $|\tilde{A}|$ on $\tilde{X}$ and $|A|$ on X. The map f sends each element of $|\tilde{A}|$ to an element of $|A|$. As we saw in the proof of Lemma 1.2, (ii), we have $f^*(A) = \tilde{A}_1 + \tilde{A}_2$. Hence f is two-to-one between $|\tilde{A}|$ and $|A|$. This means that all, but two, elements of $|A|$ are smooth, irreducible, isomorphic to A, and that there are two elements of $|A|$ of type $2\bar{A}^+$, $2\bar{A}^-$ with $\bar{A}^\pm$ sections of X over $\bar{A}$. One moment of reflection shows that these two sections, which do not meet, correspond to the splitting of $\mathcal{E}$. Of course $\bar{A}^\pm$ are isomorphic to $\bar{A}$ and one has $\mathcal{O}_{\bar{A}^\pm}(\bar{A}^\pm) \simeq \mathcal{M}_1$. In addition $2\bar{A}^\pm \equiv A$ but of course $\bar{A}^+ \not\equiv \bar{A}^-$. Hence $\bar{A}^+ - \bar{A}^-$ gives a non trivial point of order two in $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(\bar{A})$. By restricting to $\bar{A}^\pm$, we see that this point of order two is $\mathcal{M}_1$. Hence $\mathcal{O}_X(\bar{A}^+) \simeq \mathcal{O}_X(\bar{A}^-) \otimes \mathcal{M}_1$ and therefore $\mathcal{O}_X(A) \simeq \mathcal{O}_X(2\bar{A}^+) \simeq \mathcal{O}_X(\bar{A}^+ + \bar{A}^-) \otimes \mathcal{M}_1$, whence $\mathcal{O}_X(A) \otimes \mathcal{M}_1 \simeq \mathcal{O}_X(\bar{A}^+ + \bar{A}^-)$, which fully explains the meaning of part (i) of Proposition 1.3.

Notice that all the smooth abelian varieties in $|A|$ play a symmetric role in the construction of X and of its tautological line bundle.

One more obvious remark. Let $\bar{A}^-$ correspond to the quotient $\mathcal{E} \to \overline{\mathcal{L}}$ and $\bar{A}^+$ to the quotient $\mathcal{E} \to \overline{\mathcal{L}} \otimes \mathcal{M}_1$. Then $\mathcal{O}_{\bar{A}^-}(1) \simeq \overline{\mathcal{L}}$ and $\mathcal{O}_{\bar{A}^+}(1) \simeq \overline{\mathcal{L}} \otimes \mathcal{M}_1$.

(ii) Now we take up the case $\epsilon \notin K(\Theta)$. Consider the varieties A_2, A_3 which are the unique divisors in the linear systems $|\mathcal{O}_X(A) \otimes \mathcal{M}_3|$, $|\mathcal{O}_X(A) \otimes \mathcal{M}_2|$, respectively. As we saw in the proof of Proposition 1.3, we have $A \cap A_2 = A \cap A_3 = \emptyset$. Then, by Lemma 1.2, (iv), A_2 and A_3 are irreducible. We shall see in Proposition 1.5 that these varieties are smooth abelian. We also set $A_1 = A$.

PROPOSITION 1.5. *One has:*

(i) *if* $\epsilon \in K(\Theta)$, *then* $h^0(X, \mathcal{O}_X(2A)) = 3$;

(ii) *if* $\epsilon \notin K(\Theta)$, *then* $h^0(X, \mathcal{O}_X(2A)) = 2$. *Moreover the pencil* $|2A|$ *has exactly* 3 *singular elements namely* $2A_i$ *for* $i = 1, 2, 3$. *All other elements* D *in* $|2A|$ *are smooth abelian. The reduced varieties* A_i *are also smooth abelian.*

PROOF. In case (i) the linear system is composed with the pencil $|A|$, hence the assertion.

Let us consider case (ii). Since $2A \equiv 2A_2 \equiv 2A_3$, it is clear that $h^0(X, \mathcal{O}_X(2A)) \geq 2$. Suppose $h^0(X, \mathcal{O}_X(2A)) = r + 1 \geq 3$. Then the linear system $|2A|$ would have dimension $r \geq 2$. Moreover $\mathcal{O}_A(2A)$ is trivial. Therefore the linear system $|A|$ would have dimension at least $r - 1 \geq 1$, contradicting Proposition 1.3. We now look at the pencil $|2A|$. We have already seen that the A_i are irreducible. The same is true for the elements D. In fact D does not meet A and by Lemma 1.2, (iv) D cannot have a component which is a section. In addition, there cannot be a non trivial component which is a 2-section either, because such a 2-section would be numerically equivalent to A, hence would be equal to A_2 or A_3, which is not possible since $A_2 + A_3$ is not equivalent to $2A$. It follows from adjunction that the square of the dualizing sheaf ω_{A_i} is trivial and that ω_D is trivial. Our assertion follows if we can show that the projection onto $\bar{A}$ defines an étale $2:1$ cover from A_i to $\bar{A}$, resp. an étale $4:1$ cover from D to $\bar{A}$. To see this we look at the pencil of degree 4 cut out by $|2A|$ on each ruling. This is base point free and has at least 3 singular elements consisting of 2 double points each, corresponding to the 2-sections A_i. By the Hurwitz formula there can be no worse singularities and this gives the claim. □

REMARKS 1.6. (i) Assume that $\epsilon \notin K(\Theta)$. Then we have just seen that A_2 and A_3 are smooth abelian varieties isogenous to $\bar{A}$ via the degree 2 maps π_2, π_3 induced by p. In addition we have $A_2 \cap A_3 = \emptyset$. Hence $\pi_2^* \mathcal{M}_2 \simeq \mathcal{O}_{A_2}$ and $\pi_3^* \mathcal{M}_3 \simeq \mathcal{O}_{A_3}$, which gives another proof of Lemma 1.1 in the present case.

(ii) In this situation the projection p induces on every smooth element $D \in |2A|$ an isogeny $\delta : D \to \bar{A}$ of degree 4. We have just seen that $\pi_2^* \mathcal{M}_2 \simeq \mathcal{O}_{A_2}$ and $\pi_3^* \mathcal{M}_3 \simeq \mathcal{O}_{A_3}$. It follows from this that $\delta^*(\mathcal{M}_i)$ is trivial for every $i = 1, 2, 3$. Hence D is constant in moduli and it is the unique degree 4 cover of $\bar{A}$ with this property. We also remark that, in view of the above description, the isogeny δ factors through degree 2 isogenies $\delta_i : D \to A_i$, $i = 1, 2, 3$.

(iii) We consider the line bundles $\mathcal{L}_i := \mathcal{O}_X(1)_{|A_i}$, and the corresponding polarizations Θ_i, $i = 1, 2, 3$. Of course $\mathcal{E} \simeq \pi_{i*}\mathcal{L}_i$, $i = 1, 2, 3$, and the abelian varieties A_i play a symmetric role in the construction of X and of its tautological line bundle $\mathcal{O}_X(1)$.

PROPOSITION 1.7. *One has:*

(i) *if* $\epsilon \in K(\Theta)$, *then* $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A)$;
(ii) *if* $\epsilon \notin K(\Theta)$, *then* $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A_i) \otimes \mathcal{M}_i$, *for* $i = 1, 2, 3$.

PROOF. (i) By the proof of Lemma 1.2, we know that either $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A) \otimes \mathcal{M}_1$ or $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A)$. The assertion follows by restricting to $\bar{A}^{\pm}$.

(ii) As above the proof of Lemma 1.2 tells us that either $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A_i) \otimes \mathcal{M}_i$ or $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A_i)$. Suppose that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A)$. Then the adjunction formula tells us that $\mathcal{O}_{A_2} \simeq \mathcal{O}_X(K_X + A_2) \otimes \mathcal{O}_{A_2} \simeq \mathcal{O}_X(A_2 - A_1) \otimes \mathcal{O}_{A_2} \simeq \mathcal{M}_3 \otimes \mathcal{O}_{A_2}$, which is a contradiction, since only $\mathcal{M}_2 \otimes \mathcal{O}_{A_2}$ is trivial on A_2. □

Let us now consider the action of $K(\Theta)$ on A. We will assume Θ is a primitive polarization, i.e. it is an indivisible element of $NS(A)$, of type $(d_1, d_2, \ldots, d_n)$. This is equivalent to $d_1 = 1$.

LEMMA 1.8. *Let us suppose that the Neron-Severi group of A is generated by Θ. Let γ be an effective divisor on A fixed by $K(\Theta)$. Then there is a positive integer a such that $\gamma = a \cdot d_n \cdot \Theta$ in the Neron-Severi group of A.*

PROOF. Let $\hat{A}$ be the polarized dual variety of A. The primitive dual polarization $\widehat{\Theta}$ is of type $(1, \frac{d_n}{d_{n-1}}, \ldots, \frac{d_n}{d_2}, d_n)$ and it generates the Neron-Severi group of $\hat{A}$. We have the map $\lambda_\Theta : A \to \hat{A}$. Then an easy computation using self-intersection numbers shows that:

$$\lambda_\Theta^*(\widehat{\Theta}) = d_n \Theta\,.$$

On the other hand we have $\gamma = \lambda_\Theta^*(\hat{\gamma})$, where $\hat{\gamma}$ is an effective divisor on $\hat{A}$. Therefore we have $\hat{\gamma} = a\widehat{\Theta}$ for some positive integer a. By pulling this back to A via λ_Θ, we get the assertion. □

LEMMA 1.9. *Assume that* $\epsilon \notin K(\Theta)$ *and that the Neron-Severi group of* A *is generated by* Θ. *Let* D *be any irreducible element of the pencil* $|-2K_X|$ *and let* H_D *be the element of the Neron-Severi group of* D *given by the restriction of* H *to* D. *Then* $H_D = \delta_i^*(\Theta_i)$, $i = 1, 2, 3$ *and* H_D *is at most divisible by* 2 *in the Neron-Severi group of* D.

PROOF. The assumption that $NS(A) \simeq \mathbb{Z}$ implies that also $NS(D) \simeq \mathbb{Z}$. The assertion $H_D = \delta_i^*(\Theta_i)$ is then purely numerical and follows from the fact that $D = 2A$ in $NS(X)$. Since the maps δ_i are $2:1$ covers the maps ${\delta_i}^* : NS(A_i) \simeq \mathbb{Z} \to NS(D) \simeq \mathbb{Z}$ have a cokernel which is torsion of order at most 2. □

In the case of abelian surfaces we can extend Lemma 1.8 above in the following way:

LEMMA 1.10. *Let* A *be an abelian surface with a polarization* Θ *of type* $(1, 2n)$, *resp.* $(2, n)$ *and assume that* $NS(A) \simeq \mathbb{Z}$. *If* C *is a curve invariant under a group* $G \simeq \mathbb{Z}_n \times \mathbb{Z}_n$ *which acts on* A *by translation, then* $C = a \cdot \Theta$ *where* a *is a multiple of* n, *resp.* $n/2$.

PROOF. By the assumption $NS(A) \simeq \mathbb{Z}$ the curve C is a multiple of Θ, resp. $\Theta/2$. Since C is invariant under the group G the associated line bundle descends to A/G. But this implies that G is a totally isotropic subgroup with respect to the Weil pairing of $K(a\Theta)$ (cf. [LB, Corollary 6.3.5]). This is only possible if a is divisible by n, resp. $n/2$ (cf. the description of the Weil pairing given in [LB, Example 7.7.4]). □

2. – Some scrolls of secant lines to abelian varieties

Let us consider a linearly normal abelian variety $A \subset \mathbb{P}^{r-1}$ of dimension n, embedded via a very ample line bundle $\mathcal{L}$ belonging to a polarization Θ of type $(d_1, d_2, \dots, d_n)$. Then $r = d_1 \cdot \ldots \cdot d_n$ and the degree of A equals $n! \cdot d_1 \cdot \ldots \cdot d_n$. Let ϵ be a non trivial 2-torsion point on A. We are interested in the $n+1$-dimensional scroll:

$$Y = Y(A, \epsilon, \mathcal{L}) = \cup_{x \in A} L(x, x+\epsilon)$$

where $L(a, b)$ is the line joining two distinct points a, b in projective space. We notice that, unless $n = 1$ and $r = d = 3$, Y is a proper subvariety in $\mathbb{P}^{r-1}$.

As we saw in Section 1, from which we keep the notation, we can associate to this situation a $\mathbb{P}^1$-bundle X on $\bar{A} = A/\epsilon$. The relation between X and Y is described in the following lemma:

LEMMA 2.1. *One has the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[rr] \arrow[d, "\phi_{\mathcal{L}}"'] & & X \arrow[d, "\phi := \phi_{\mathcal{O}_X(1)}"] \\
\mathbb{P}^{r-1} = \mathbb{P}(H^0(A, \mathcal{L})^*) & \simeq & \mathbb{P}(H^0(X, \mathcal{O}_X(1))^*)
\end{tikzcd}$$

Moreover the map ϕ is a morphism and its image is Y.

PROOF. Since $p_* \mathcal{O}_X(1) \simeq \mathcal{E} \simeq \pi_* \mathcal{L}$ we have a canonical isomorphism of vector spaces $H^0(A, \mathcal{L}) \simeq H^0(X, \mathcal{O}_X(1))$. Moreover, by Lemma 1.2, (ii) we have $\mathcal{O}_X(1)_{|A} \simeq \mathcal{L}$. This shows the existence of the commutative diagram above, which in turn, implies $\phi(X) = Y$.

Let F be any ruling of X and let $x, x + \epsilon$ be the two points where F intersects A. Since $\mathcal{L}$ is very ample on A, the linear system $|\mathcal{O}_X(1)|$ separates these two points and hence $|\mathcal{O}_X(1)|$ has no base points on X, i.e. ϕ is a morphism. □

We can now prove the following propositions:

PROPOSITION 2.2. *The map ϕ is finite, i.e. $\mathcal{O}_X(1)$ is ample.*

PROOF. Assume that there is an irreducible curve C which is contracted under ϕ. The curve C can only meet a ruling F once, i.e. the projection of C to $\bar{A}$ is birational onto its image. After possibly replacing C by its normalization we obtain a smooth curve $\tilde{C}$ and a morphism $f : \tilde{C} \to \bar{A}$ which is birational onto its image such that

$$f^* \mathcal{E} = \mathcal{O}_{\tilde{C}} \oplus \xi$$

for some suitable line bundle ξ on $\tilde{C}$. The $\mathbb{P}^1$-bundle $V = \mathbb{P}(f^* \mathcal{E})$ over $\tilde{C}$ is mapped to a cone W in Y. Let C_0 be the section of V which is mapped to the vertex of W and let C_1 be the 2-section which is the pullback of the 2-section A of X. We denote by f the class of a fibre in V. Then $C_1 = 2C_0 + af$ in the Neron-Severi group of V for some integer a. Since C_0 is contracted it follows that $e = C_0^2 < 0$. On the other hand it follows from Lemma 1.2 (ii) that $C_1^2 = 0$ and hence $a = -e$. But then $C_0.C_1 = e < 0$, i.e. C_0 and C_1 have a common component. This contradicts the fact that the abelian variety A is embedded by the map ϕ. □

PROPOSITION 2.3. *One has:*

$$\deg(\phi) \cdot \deg(Y) = \frac{1}{2}(n+1)! \cdot r\,.$$

If $n = 1$ and $r \geq 4$, then ϕ is birational onto its image and it is an embedding as soon as $r \geq 5$.

PROOF. The result is well known for $n = 1$ (see e.g. [CH, Proposition 1.1 and Proposition 1.2]). So we assume $n \geq 2$. We have:

$$\deg(\phi) \cdot \deg(Y) = \mathcal{O}_X(1)^{n+1} = \frac{1}{2}(f^*\mathcal{O}_X(1))^{n+1} = \frac{1}{2}(\mathcal{O}_{\pi^*(\mathcal{E})}(1))^{n+1} .$$

We can then deduce from formula (1) of Section 1 that:

$$\mathcal{O}_X(1)^{n+1} = \frac{1}{2}(n+1)\Theta^n = \frac{1}{2}(n+1)! \cdot r$$

and hence the assertion. □

In what follows we will need a formula for the double point cycle $\mathbb{D}(X, l)$ (see [F, p. 166]) of a map $f : X \to \mathbb{P}^l$, where $l \leq r - 1$ and f is the composition of ϕ with a general projection $\mathbb{P}^{r-1} \to \mathbb{P}^l$. We assume $l \geq n + 1$. In this situation the map f is finite.

THEOREM 2.4. *One has:*

$$\mathbb{D}(X, l) = \left(\frac{1}{2}(n+1)! \cdot r - \binom{l+1}{n+2}\right) H^{l-n-1} + \binom{l+1}{n+3} H^{l-n-2} \cdot A$$

in the homology ring of X.

PROOF. By applying Theorem 9.3 from [F, p. 166], one has:

$$\mathbb{D}(X, l) = f^*f_*[X] - c_{l-n-1}(N_f) \cap [X]$$

where N_f is the normal sheaf to the map f which is defined by the exact sequence:

$$0 \to T_X \to f^*T_{\mathbb{P}^l} \to N_f \to 0 .$$

By Proposition 2.3 and finiteness of f, we have:

$$f^*f_*[X] = \left(\frac{1}{2}(n+1)! \cdot r\right) H^{l-n-1} .$$

Moreover from the Euler sequence, we see that $c(f^*T_{\mathbb{P}^l}) = (1 + H)^{l+1}$. Also by the exact sequence:

$$0 \to p^*T_{\bar{A}} \to T_X \to T_{X|\bar{A}} \to 0$$

we deduce that $c(T_X) = 1 + A$. Since $A^2 = 0$ in homology, the assertion follows by an easy computation. □