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1. – Introduction

The aim of this paper is to present descriptions of the envelopes of holomorphy of certain classes of subsets of $\mathbb{C}^2$, namely:

a) the open subsets which are complements of noncompact closed domains bounded by strictly Levi-convex real hypersurfaces of class $C^2$;

b) the compact subsets which lie on the boundaries of closed domains – either compact or noncompact – bounded by strictly Levi-convex real hypersurfaces of class $C^2$.

More generally we shall consider an arbitrary two-dimensional Stein manifold $M^2$ as the ambient space, rather than $\mathbb{C}^2$.

Let us recall that the envelope of holomorphy $E(S)$ of an arbitrary subset $S$ of a Stein manifold $M$ can be defined as the union of the components of $\tilde{S} = \text{spec}(\mathcal{O}(S))$ which meet $S$. For a non-open subset $S \subset M$, $\tilde{S}$ need not be embedded in a complex manifold in any natural way. On the other hand, if there exists a holomorphically convex set $S' \subset M$ containing $S$, with the property that the restriction map $\mathcal{O}(S') \to \mathcal{O}(S)$ is bijective, then $E(S)$ may be identified with $S'$. In this connection we also recall that if a subset of a complex manifold admits a fundamental system of Stein neighborhoods, then it is holomorphically convex. (We refer to [12] for all these facts.)

The mentioned descriptions require us to take into considerations certain holomorphic hulls of some subsets of $M^2$ which are not compact sets. If $S$ is an arbitrary subset of $M^2$ and $K \subset S$ is a compact set, let us use the notation

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that \( h_{\mathcal{O}(S)}(K) \) denotes the \( \mathcal{O}(S) \)-hull of \( K \), i.e.,

\[
h_{\mathcal{O}(S)}(K) = \bigcap_{f \in \mathcal{O}(S)} \{ z \in S : |f(z)| \leq \|f\|_K \}.
\]

Now, let \( T \) be an arbitrary subset of \( S \). Then we define the \( \mathcal{O}(S) \)-hull of \( T \), \( h_{\mathcal{O}(S)}(T) \), to be the union of the \( \mathcal{O}(S) \)-hulls of all compact subsets of \( T \), i.e.,

\[(1.1) \quad h_{\mathcal{O}(S)}(T) = \bigcup_{K \subset T} h_{\mathcal{O}(S)}(K), \]

where \( K \) ranges through the family of compact subsets of \( T \). We have already used this notion in our previous paper [19], where one can find results related to the subject which is being discussed here.

Moreover we find it convenient to introduce, for a closed set \( F \subset M^2 \), a notion of “hull at infinity”, in the following way. If \( S \subset M \) is an arbitrary set containing \( F \), we define the \( \mathcal{O}(S) \)-hull at infinity of \( F \) \( h_{\mathcal{O}(S)}(F) \), to be the intersection of the \( \mathcal{O}(S) \)-hulls of the subsets of \( F \) which are complements of compact sets, that is

\[(1.2) \quad h_{\mathcal{O}(S)}(F) = \bigcap_{G \subset F} h_{\mathcal{O}(S)}(F \setminus G), \]

where \( G \) ranges through the family of compact subsets of \( F \). Plainly, if \( F \) is compact, \( h_{\mathcal{O}(S)}(F) = \emptyset \), but if \( F \) is noncompact, \( h_{\mathcal{O}(S)}(F) \) may be nonempty; for example, if there is a one-dimensional complex-analytic subvariety \( V \) of \( M^2 \) with \( V \subset F \), then \( V \subset h_{\mathcal{O}(S)}(F) \). We have been led to consider the preceding notion of hulls at infinity by some analogy with the notion of cohomology of the ideal boundary of a noncompact space \( X \), which is known to be the inductive limit of the cohomology of \( X \setminus G \) as \( G \) ranges through the compact subsets of \( X \) (see [6]), and is sometimes also called the cohomology at infinity of \( X \) and denoted by \( H^\infty(X) \).

That being stated, we can formulate our main results.

**Theorem 1.** Let \( D \subset M^2 \) be an open domain of holomorphy, whose boundary \( bD \) is a real hypersurface of class \( C^2 \), strictly Levi-convex with respect to \( D \). Put \( \Omega = M^2 \setminus \overline{D} \). Then the envelope of holomorphy of \( \Omega \), \( E(\Omega) \), is given by

\[
E(\Omega) = M^2 \setminus h_{\mathcal{O}(D)}(\overline{D}) = \Omega \cup [h_{\mathcal{O}(D)}(bD) \setminus h_{\mathcal{O}(D)}(bD)] = h_{\mathcal{O}(M^2)}(\Omega) \setminus h_{\mathcal{O}(D)}(bD).
\]

In particular \( E(\Omega) \) is single-sheeted over \( \Omega \).
THEOREM 2. Let $D$ be as in Theorem 1. Let $K$ be a compact subset of $bD$. Then the envelope of holomorphy of $K$, $E(K)$, is given by

$$E(K) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K) = h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).$$

Indeed the sets $h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ and $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ are a same Stein compactum containing $K$, $\tilde{K}$, say, such that the restriction map $\mathcal{O}(\tilde{K}) \rightarrow \mathcal{O}(K)$ is bijective. In particular $E(K)$ is single-sheeted over $K$.

Moreover, if $K$ is holomorphically convex, then $E(K) = K$, i.e., $K$ is a Stein compactum.

THEOREM 3. Let $D$ be as in the preceding theorems. Let $K$ be a compact subset of $bD$. Assume that $K$ has a neighborhood basis $\mathcal{N}$, in $bD$, such that each $N \in \mathcal{N}$ is a relatively compact open subset of $bD$ (possibly disconnected), whose boundary $bN$ is the union of finitely many pairwise disjoint topological 2-spheres of class $C^2$. Then it follows that $E(K) = h_{\mathcal{O}(\overline{D})}(K)$.  

We emphasize that in the preceding three theorems $\overline{D}$ may be either compact or noncompact and in the latter case $bD$ is allowed to be disconnected. However in the compact case, as $h_{\mathcal{O}(\overline{D})}(\overline{D}) = 0$, Theorem 1 yields only a result equivalent to Hartogs’s extension theorem.

Moreover let us recall that $K$ is said to be holomorphically convex if the evaluation map $K \rightarrow \text{spec}(\mathcal{O}(K))$ is bijective, or, equivalently, if $H^1(K, \mathcal{F}) = H^2(K, \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on $K$ (see [12]).

Some further comments are in order.

Since every $f \in \mathcal{O}(bD)$ can be written as $f = f_1 - f_2$, with $f_1 \in \mathcal{O}(\overline{\Omega})$ and $f_2 \in \mathcal{O}(\overline{D})$ and the restriction map $\mathcal{O}(\overline{\Omega}) \rightarrow \mathcal{O}(\Omega)$ is surjective, Theorem 1 is equivalent to the following result:

COROLLARY 1. Let $D$ be as in Theorem 1. Then the envelope of holomorphy of $bD$, $E(bD)$, is given by

$$E(bD) = \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD) = [h_{\mathcal{O}(\mathcal{M}^2)(\Omega) \cap \overline{D}} \setminus h_{\mathcal{O}(\overline{D})}(bD)].$$

In particular $E(bD)$ is single-sheeted over $bD$.

Moreover, since

$$h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD) = \bigcup_{K \subset bD} [h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)].$$

(1) We wish to point out that a sufficient condition which implies the mentioned property of $bN$ is, besides $bN$ being of class $C^2$, that $H_1(\overline{N}, \mathbb{Z}) = 0$. This will be shown at the end of Section 5.
with \( K \) ranging through the family of compact subsets of \( bD \), it is not difficult to see that the equality \( E(bD) = h^{\infty}_{\mathcal{O}(\overline{D})}(bD) \setminus h^{\infty}_{\mathcal{O}(\overline{D})}(bD) \) is also a consequence of the first statement of Theorem 2.

The second statement of Theorem 2 seems to deserve some interest in connection with the question raised by Harvey and Wells [12, p. 515] whether every holomorphically convex compact set in a Stein manifold should be a Stein compactum. This question was answered in the negative by Björk [5], who exhibited examples of compact holomorphically convex sets in \( \mathbb{C}^n, n \geq 2 \), which are not Stein compacta. On the other hand Theorem 2 gives a positive answer to the question at least for the holomorphically convex compact sets which lie on \( bD \), when \( n = 2 \). In this connection we also recall that, if \( D \) is compact, a compact subset of \( bD \) is holomorphically convex if and only if it is “weakly removable” (see [18, Corollary 2]).

Combining Theorem 3 with the second statement of Theorem 2 gives in particular the following result:

**Corollary 2.** Let \( D \) be as in the preceding theorems. Let \( K \subset bD \) be a holomorphically convex compact set endowed with a neighborhood basis, in \( bD \), of topological 3-cells. Then \( K \) is \( O(\overline{D}) \)-convex, i.e., \( h_{\mathcal{O}(\overline{D})}(K) = K \).

Here the requirement that the boundaries of the topological 3-cells should be of class \( C^2 \) is not necessary, since known results on 3-manifolds and smoothing of homeomorphisms ([20, Theorem 4] and [21, Theorem 6.3]) imply the existence also of a neighborhood basis of \( K \), in \( bD \), of topological 3-cells with boundaries of class \( C^2 \), the essential point being the fact that two homeomorphic 3-manifolds of class \( C^2 \) are \( C^2 \)-diffeomorphic. Corollary 2 is close to a theorem of Forstnerič and Stout [9], which yields the same conclusion, in the case that \( D \) is relatively compact, under the additional assumption that the set \( K \) should have a Stein open neighborhood \( X \) in which it is \( O(X) \)-convex. The first result in this direction is due to Jörnitsche [14], who obtained the equivalent result that \( K \) is “removable” (see [7], [18], [23]) in the case that \( K \) is a compact totally real disk of class \( C^2 \). Forstnerič and Stout resorted, for the proof of their theorem, to the work of Bedford and Klingenberg [4] on the envelopes of holomorphy of 2-spheres, and also our proof of Theorem 3 depends on that work, in that we need a result from [4] to prove the vanishing of the two-dimensional holomorphic de Rham cohomology of a topological 2-sphere of class \( C^2 \) embedded in the boundary of a strongly pseudoconvex domain (Section 5, Proposition 8).

Theorem 3 is also useful to obtain more information in the direction of Theorem 1 and Corollary 1, under some reasonably general additional conditions on \( bD \).

**Corollary 3.** Let \( D \) be as in the preceding theorems. Assume that \( bD \) can be exhausted by an increasing sequence \( \{N_n\} \) of relatively compact \( C^2 \)-bounded open subsets (possibly disconnected), such that each boundary \( bN_n \) is the union of finitely many pairwise disjoint topological 2-spheres of class \( C^2 \) (which is true in particular in case \( bD \) is homeomorphic to \( \mathbb{R}^3 \)). Then

\[
E(\Omega) = \Omega \cup h_{\mathcal{O}(\overline{D})}(bD) = h_{\mathcal{O}(M^2)}(\Omega) \quad E(bD) = h_{\mathcal{O}(\overline{D})}(bD) = h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D}.
\]
In other words, $h^{\infty}_{\mathcal{O}(\overline{D})}(bD)$ is empty.

Finally, a reason of interest in respect of the above results is, in our opinion, the circumstance that they do not extend to higher dimensions, in the sense that, if one replaces $M^2$ by a Stein manifold of dimension $\geq 3$ as the ambient space, the corresponding statements become false. We shall discuss this point at the end of the article, in Section 6; in particular we will exhibit an example, inspired by one of Chirka and Stout [7], which shows that for all dimensions $\geq 3$ $E(\Omega)$ may be multi-sheeted\(^{(2)}\). On the other hand, at the beginning of Section 6 we will also mention the weaker results which can be obtained in the positive, for dimensions $\geq 3$, in the direction contemplated here (Theorem 4 and Theorem 5).

2. – Preliminaries

Consider a domain $D$ as in the statements of Theorem 1 and Theorem 2. Let us fix once for all a $C^\infty$ strongly plurisubharmonic exhaustion function $\Phi : M^2 \rightarrow \mathbb{R}$ and an increasing divergent sequence $\{c_n\}_{n \in \mathbb{N}}$ of positive real numbers all of which are regular values for both of the functions $\Phi$ and $\Phi|_{bD}$; moreover let us put, for every $n \in \mathbb{N}$,

$$B_n = \{z \in M^2 : \Phi(z) < c_n\}, \quad D_n = B_n \cap D , \quad \Gamma_n = B_n \cap bD , \quad \Delta_n = bB_n \cap \overline{D} .$$

Then $D_n$ is a relatively compact Stein open set in $M^2$, such that $bD_n = \Gamma_n \cup \Delta_n$.

It is known that, since $bD$ is strictly Levi-convex, the closed domain $\overline{D}$ admits a neighborhoods basis of Stein open sets (for the noncompact case see [24, Lemme 2]). Then, since $\overline{D}_n$ is an $O(M^2)$-convex Stein compactum, it is readily seen that $\overline{D}_n$ is $O(\overline{D})$-convex, i.e. the restriction map $O(\overline{D}) \rightarrow O(\overline{D}_n)$ has dense image, and consequently the following property, which will be used repeatedly throughout the continuation of this paper, holds:

\[(2.1) \quad h_{O(\overline{D})}(G) = h_{O(\overline{D}_n)}(G) \text{ for every compact set } G \subset \overline{D}_n .\]

We shall also apply several times a pseudoconvexity result which refines slightly a result of Slodkowski (see [16] and the references cited there), namely:

\[(2.2) \text{ Let } C \subset M^2 \text{ be a compact set, } X \subset M^2 \text{ a Stein open set containing } C \text{ and } S \subset M^2 \text{ a Stein open set such that } C \cap S \text{ is empty. Then the open set } S \setminus h_{O(X)}(C) \text{ is Stein.}\]

Moreover we need to recall a result on holomorphic extension of CR-functions (see [23], [17] and references cited there):

\[^{(2)}\]The original example of [7] is suitable for the same conclusion only as regards the even dimensions $\geq 4$, thus excluding in particular dimension 3.
Let \( D \subset \subset M^2 \) be an open domain and \( K \subset bD \) a compact set. Assume that \( bD \setminus K \) is a \( C^1 \)-smooth real hypersurface of \( M^2 \setminus K \) and that \( \overline{D} \) admits a Stein open neighborhood \( X \) in which it is an \( \mathcal{O}(X) \)-convex Stein compactum. Then every continuous CR-function on \( bD \setminus K \) has a unique extension to a continuous function on \( \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(K) \) holomorphic on \( D \setminus h_{\mathcal{O}(\overline{D})}(K) \).

That being stated, we collect in a lemma three further properties that will come directly in the proofs of our theorems.

**Lemma.** For each \( n \in \mathbb{N} \) the following properties are valid:

(2.4) Every continuous CR-function on \( \Gamma_n \) extends uniquely to a continuous function on \( \overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \) holomorphic on \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \).

(2.5) \( h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) = \overline{D}_n \).

(2.6) \( h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m) \), for every \( m \in \mathbb{N} \) with \( m \leq n \).

**Proof.** By (2.1), in proving the lemma we may replace the \( \mathcal{O}(D) \)-hulls by the corresponding \( \mathcal{O}(\overline{D}_n) \)-hulls.

Now, to prove (2.4), let \( D^i_n, i \in \mathcal{I} \) be the connected components of \( D_n \) and put, for each \( i \in \mathcal{I} \), \( \Delta_n^i = bD^i_n \cap \Delta_n \) and \( \Gamma_n^i = bD^i_n \setminus \Delta_n^i = bD^i_n \cap bD \). Then each \( D^i_n \) is a Stein domain in \( M^2 \), such that \( \overline{D}_n^i \) is a Stein compactum, and \( \Gamma_n^i \) is a real hypersurface of class \( C^2 \) in \( M^2 \setminus \Delta_n^i \). In this situation we may apply (2.3): every continuous CR-function on \( \Gamma_n^i \) has a unique extension to a continuous function on \( \overline{D}_n^i \setminus h_{\mathcal{O}(\overline{D}_n^i)}(\Delta_n^i) \) which is holomorphic on \( D_n^i \setminus h_{\mathcal{O}(\overline{D}_n^i)}(\Delta_n) \). Then, since \( \Gamma_n \) is the disjoint union of the \( \Gamma_n^i \)'s, \( i \in \mathcal{I} \), and \( \bigcup_{i \in \mathcal{I}} h_{\mathcal{O}(\overline{D}_n^i)}(\Delta_n^i) = h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \), it is also true that every continuous CR-function on \( \Gamma_n \) extends uniquely to a continuous function on \( \overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \) which is holomorphic on \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \). Hence we see that (2.4) holds.

Next we prove (2.5). It suffices to prove that the inclusion

\[
\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\Delta_n) \subset h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n)
\]

is valid. Since \( \Gamma_n \) is strictly Levi-convex at each point with respect to \( D_n \), we can construct a relatively compact Stein open set \( D'_n \subset M^2 \) such that \( \overline{D}_n \setminus \Delta_n \subset D'_n \), \( \Delta_n \subset bD'_n \) and \( \overline{D}_n \setminus \Delta_n \) is \( \mathcal{O}(D'_n) \)-convex. Indeed \( D'_n \) can be obtained by pushing \( \overline{\Gamma}_n \) away from \( D_n \) by a small \( C^2 \)-perturbation that leaves \( b\overline{\Gamma}_n \) fixed pointwise. Then consider the open set \( D'_n \setminus \overline{D}_n \) and make its \( \mathcal{O}(D'_n) \)-hull \( h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n) \). The latter is a Stein and Runge open subset of \( D'_n \), such that

\[
h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n) = h_{\mathcal{O}(\overline{D}_n \setminus \Delta_n)}(\Gamma_n) = (\overline{D}_n \setminus \Delta_n) \cap h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n)
\]
(see [19]). Since \( h_{\mathcal{O}(\mathbb{D}^n)}(D_n^* \setminus \overline{D}_n) \) is a Stein set containing \( \Gamma_n \), one can find CR-functions on \( \Gamma_n \) (of class \( C^2 \)) which cannot be holomorphically extended through any boundary point, in \( D_n \), of \( h_{\mathcal{O}(\mathbb{D}^n)}(D_n^* \setminus \overline{D}_n) \), namely the restrictions to \( \Gamma_n \) of the functions holomorphic on \( h_{\mathcal{O}(\mathbb{D}^n)}(D_n^* \setminus \overline{D}_n) \) which do not admit holomorphic continuations to any larger open set; hence, granted the validity of (**) if (*) were not true, this would lead to a contradiction to (2.3).

Finally let us prove (2.6). Since we can choose a Stein open neighborhood \( X \) of \( \overline{D}_n \), such that \( \overline{D}_n \) is \( \mathcal{O}(X) \)-convex, and consequently \( h_{\mathcal{O}(\overline{D}_n)}(C) = h_{\mathcal{O}(X)}(C) \) for every compact set \( C \subset \overline{D}_n \), (2.2) implies that the three open sets \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \), \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \), and \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n \setminus \Delta_n) \) are Stein. Moreover, by (2.5), the union \( [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \cup \Delta_n] \) is disjoint and hence it is a Stein open set as well. On account of the latter fact, by a reasoning analogous to that used above to prove (†), one can show that:

(†) There exist CR-functions on \( bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_n) \) (of class \( C^2 \)) which cannot be holomorphically extended through any boundary point, in \( D_n \), of \( [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \cup \Delta_n] \).

On the other hand, (2.3) can also be applied to derive the property, parallel to (2.4), in which one considers \( bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_n) \) in place of \( \Gamma_n \), and \( \overline{\Gamma}_n \setminus \Gamma_n \) in place of \( \Delta_n \), respectively. Hence the following is true too:

(††) Every continuous CR-function on \( bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_n) \) admits a continuous extension to \( \overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \) holomorphic on \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \).

Combining (†) and (††), we see that \( \overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \subset [\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [\overline{D}_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \cup \Delta_n] \). This amounts to having \( h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n) \subset h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_n \setminus \Delta_n) \), which yields the desired conclusion.

For the proof of Theorem 3 we shall need two results from [19] (see Corollary 5 and Corollary 7 therein). For the convenience of the reader we restate the results here.

(2.7) Let \( D \subset M^2 \) be a \( C^2 \)-bounded strongly pseudoconvex domain and \( K \subset bD \) a compact set, and put \( \Gamma = bD \setminus K \). Then for a continuous CR-function \( f \) on \( \Gamma \) the following two conditions are equivalent:

- \( f \) extends uniquely to a function in \( C^0(h_{\mathcal{O}(\overline{D})(\Gamma)}(\overline{\Gamma})) \cup \mathcal{O}(h_{\mathcal{O}(\overline{D})(\Gamma)}(\Gamma) \setminus \Gamma) \).

(2.8) Let \( D, K \) and \( \Gamma \) be as in (2.7). Then the following three conditions are equivalent:

- \( E(K) = h_{\mathcal{O}(\overline{D})(K)} \).
- \( E(\Gamma) = h_{\mathcal{O}(\overline{D})(\Gamma)} \).
- \( h_{\mathcal{O}(\overline{D})(\Gamma)} = \overline{D} \setminus h_{\mathcal{O}(\overline{D})(K)} \).
REMARK. We point out that all the properties discussed in this section, except (2.2) and (2.8), remain valid if $M^2$ is replaced by a Stein manifold $M'$ of dimension $r \geq 2$ as the ambient space. As regards (2.2) and (2.8), on the contrary, for $r \geq 3$ it is not true in general that $S \setminus h(O(X))(C)$ is Stein, nor that the three properties of (2.8) are equivalent, whereas it is still true that $H^{r-1}(S \setminus h(O(X))(C), \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on $M'$ (see [16] and [17]). Since we have applied (2.2) in the proof of (2.6), the given proof of (2.6) does not work for $r \geq 3$. However it is possible to prove (2.6) for general $r \geq 3$, in a different way, by generalizing a result of Basener [3] relative to the polynomial hulls of compact subsets of $bB_r$. Basener’s proof of his result appears to be tied up the ball case only in that it invokes an earlier result of H. Alexander [1] on the connectivity properties of the polynomial hulls of compact subsets of $bB_r$. Since it is now known that Alexander’s result generalizes from the ball case so as to cover classes of domains of a Stein manifold $M'$ which include the connected components of the above $D_n$’s (see [2], [15]), it turns out that Basener’s result generalizes as well, so as to imply the validity of (2.6) for $r \geq 2$.

3. – Proof of Theorem 1

We divide the proof of Theorem 1 into the proofs of four propositions.

**PROPOSITION 1.** The hull at infinity $h^\infty_{O(O(D))}(\overline{D})$ is a closed set in $M^2$ such that

$$\overline{D} \setminus h_{O(O(D))}(bD) \subset h^\infty_{O(O(D))}(\overline{D}) \subset D.$$ 

**PROOF.** It follows immediately from the definition, (1.2), of a hull at infinity that

$$h^\infty_{O(O(D))}(\overline{D}) = \bigcap_{n \in \mathbb{N}} h_{O(O(D))}(\overline{D} \setminus D_n),$$

hence to show that $h^\infty_{O(O(D))}(\overline{D})$ is closed in $M^2$, it suffices to prove that so is

$$h_{O(O(D))}(\overline{D} \setminus D_n)$$

for each $n \in \mathbb{N}$. Since the restriction map $O(\overline{D} \setminus B_n) \to O(\overline{D} \setminus D_n)$ is surjective, it follows (arguing as in [19, Lemma 4]) that $\Delta_n \subset h_{O(O(D))}(\overline{D} \setminus D_n)$, and hence

$$h_{O(O(D))}(\overline{D} \setminus D_n) = h_{O(O(D))}(\overline{D} \setminus B_n).$$

Let us show that

$$h_{O(O(D))}(\Delta_n) = \overline{D}_n \cap h_{O(O(D))}(\overline{D} \setminus D_n).$$

In view of (2.1), the inclusion of the left hand side set in the right hand side set follows at once from the above. As regards the reverse inclusion, consider a compact set $G \subset \overline{D} \setminus B_n$. The local maximum modulus principle implies that

$$\overline{B}_n \cap h_{O(O(D))}(G) \subset h_{O(O(D))}(bD_n \cap h_{O(O(D))}(G)).$$

Then, since

$$\bigcup_{G \subset \overline{D} \setminus B_n} h_{O(O(D))}(bB_n \cap h_{O(O(D))}(G)) = h_{O(O(D))}(\Delta_n),$$

we have

$$h_{O(O(D))}(\overline{D} \setminus \Delta_n) = h_{O(O(D))}(\overline{D}) \setminus h_{O(O(D))}(\Delta_n).$$
the reverse inclusion holds as well. It follows that
\[
(3.2) \quad h_{\partial(D)}(\overline{D} \setminus \overline{D}_n) = (\overline{D} \setminus \overline{D}_n) \cup h_{\partial(D)}(\Delta_n) = (\overline{D} \setminus B_n) \cup h_{\partial(D)}(\Delta_n),
\]
which shows $h_{\partial(D)}(\overline{D} \setminus \overline{D}_n)$ to be closed in $M^2$.

Next, since $bD$ is strictly Levi-convex with respect to $D$, every compact set $G \subset \overline{D}$ verifies $bD \cap h_{\partial(D)}(G) = bD \cap G$. Therefore, if $z$ is a point of $bD$ and $n$ is a positive integer large enough that $z \in D_n$, it follows that
\[
z \notin h_{\partial(D)}(\overline{D} \setminus \overline{D}_n) = \bigcup_{G \subset \overline{D} \setminus \overline{D}_n} h_{\partial(D)}(G).
\]
Consequently, $z \notin h_{\partial(D)}^\infty(\overline{D})$. This proves that $bD \cap h_{\partial(D)}^\infty(\overline{D}) = \emptyset$, and hence that $h_{\partial(D)}^\infty(\overline{D}) \subset D$.

Finally, let $z \in \overline{D} \setminus h_{\partial(D)}(bD)$ and choose a positive integer $m$ large enough so that $z \in D_n$ for $n \geq m$. In view of (3.2) it is plain that
\[
h_{\partial(D)}^\infty(\overline{D}) = \bigcap_{n=m}^{\infty} [(\overline{D} \setminus \overline{D}_n) \cup h_{\partial(D)}(\Delta_n)].
\]
On the other hand, by (2.5),
\[
D_n \subset h_{\partial(D)}(\Gamma_n) \cup h_{\partial(D)}(\Delta_n),
\]
for every $n \in \mathbb{N}$. Then, as $z \notin h_{\partial(D)}(\Gamma_n)$ for every $n \in \mathbb{N}$ and $z \in D_n$ for $n \geq m$, it follows that $z \in h_{\partial(D)}(\Delta_n)$ for $n \geq m$, and hence $z \in h_{\partial(D)}^\infty(\overline{D})$. This proves that $\overline{D} \setminus h_{\partial(D)}(bD) \subset h_{\partial(D)}^\infty(\overline{D})$.

**PROPOSITION 2.** The hull at infinity $h_{\partial(D)}^\infty(bD)$ verifies
\[
h_{\partial(D)}^\infty(bD) = h_{\partial(D)}(bD) \cap h_{\partial(D)}^\infty(\overline{D}).
\]

**PROOF.** Clearly, only the inclusion $h_{\partial(D)}(bD) \cap h_{\partial(D)}^\infty(\overline{D}) \subset h_{\partial(D)}^\infty(bD)$ has to be proved. On account of (2.6) for $m = n$, we have, for each $n \in \mathbb{N}$,
\[
h_{\partial(D)}(\Gamma_n) \cap h_{\partial(D)}(\Delta_n) \subset h_{\partial(D)}(bD \setminus \Gamma_n);
\]
hence, in view of (3.1) and (3.2), we see that
\[
h_{\partial(D)}(\Gamma_n) \cap h_{\partial(D)}(\overline{D} \setminus \overline{D}_n) \subset h_{\partial(D)}(bD \setminus \Gamma_n),
\]
from which, since $h_{\partial(D)}^\infty(\overline{D}) \subset h_{\partial(D)}(\overline{D} \setminus \overline{D}_n)$, we infer that, for each $n \in \mathbb{N}$,
\[
h_{\partial(D)}(\Gamma_n) \cap h_{\partial(D)}^\infty(\overline{D}) \subset h_{\partial(D)}(bD \setminus \Gamma_n).
\]
On the other hand, since for any choice of \( m \in \mathbb{N} \),
\[
h_{\mathcal{O}(\overline{D})}(bD) = \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n),
\]
we also have, for each \( m \in \mathbb{N} \),
\[
h_{\mathcal{O}(\overline{D})}(bD) \cap h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) = \bigcup_{n=m}^{\infty} \left[ h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \right];
\]
and therefore, in view of (*), it follows that
\[
h_{\mathcal{O}(\overline{D})}(bD) \cap h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \subset \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n) = h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m).
\]
Since this is true for each \( m \in \mathbb{N} \), we may conclude that
\[
h_{\mathcal{O}(\overline{D})}(bD) \cap h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \subset \bigcap_{m=\infty}^{\infty} h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m) = h_{\mathcal{O}(\overline{D})}^\infty(bD).
\]

**PROPOSITION 3.** The following two properties hold:

(i) \( M^2 \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) = \Omega \cup [h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D})] = h_{\mathcal{O}(M^2)}(\Omega) \setminus h_{\mathcal{O}(\overline{D})}^\infty(bD); \)

(ii) Every \( f \in \mathcal{O}(\Omega) \) extends uniquely to an \( F \in \mathcal{O}(M^2 \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D})). \)

**PROOF.** Since, by Proposition 1, \( D = h_{\mathcal{O}(\overline{D})}(bD) \cup h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \), it follows that
\[
\overline{D} \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus [h_{\mathcal{O}(\overline{D})}(bD) \cap h_{\mathcal{O}(\overline{D})}^\infty(\overline{D})].
\]
By Proposition 2, the last term is \( h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \), and hence the first equality of (i) follows at once. Moreover, since the restriction map \( \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega) \) is surjective, it follows that \( h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D} = h_{\mathcal{O}(\overline{D})}(bD) \) (see [19, Lemma 4]), hence
\[
h_{\mathcal{O}(M^2)}(\Omega) = \Omega \cup [h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D}] = \Omega \cup h_{\mathcal{O}(\overline{D})}(bD),
\]
which implies immediately the second equality of (i).

Next, to prove (ii), let \( \tilde{f} \) denote a holomorphic extension of \( f \) to an open neighborhood of \( \Omega \) and consider its restriction to \( bD \), which is a CR-function on \( bD \) of class \( \mathcal{C}^2 \). It suffices to prove that the latter has a unique continuous extension, \( g \), say, to \( \overline{D} \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \), which is holomorphic on \( D \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \). Then \( F \) will be given by \( F = f \) on \( \Omega \) and \( F = g \) on \( \overline{D} \setminus h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \). By (2.3), for each \( n \in \mathbb{N} \) there exists a unique extension of \( \tilde{f}|_{\Gamma_n} \) to a continuous function
on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$ holomorphic on $D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$, $g_n$, say. Moreover, for each $n \in \mathbb{N}$,

$$
\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n) \subset \overline{D}_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\Delta_{n+1});
$$

for, by the local maximum modulus principle,

$$
\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_{n+1}) \subset h_{\mathcal{O}(\overline{D})}(bB_n \cap h_{\mathcal{O}(\overline{D})}(\Delta_{n+1}));
$$

hence $g_{n+1} = g_n$ on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$, for each $n \in \mathbb{N}$, and this implies the existence of a unique continuous extension of $\hat{f}|_{bD}$ to $\bigcup_{n \in \mathbb{N}}[\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)]$ which is holomorphic on $\bigcup_{n \in \mathbb{N}}[D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)]$, namely the coherent union of the $g_n$'s. Finally, on account of (3.2), we have

$$
\overline{D} \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = \bigcup_{n \in \mathbb{N}}[\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n)]
$$

$$
= \bigcup_{n \in \mathbb{N}}[\overline{D} \setminus (\overline{D} \setminus D_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n)]
$$

$$
= \bigcup_{n \in \mathbb{N}}[\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)],
$$

and hence we conclude that the coherent union of the $g_n$'s defines the function $g$ as is required above. \qed

**Proposition 4.** The open set $M^2 \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D})$ is Stein.

**Proof.** For each $n \in \mathbb{N}$ we put

$$
G_n = bB_n \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}).
$$

Let us first prove that

$$
(*) \quad B_n \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus h_{\mathcal{O}(\overline{D})}(G_n).
$$

It is readily seen that

$$
B_n \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus \bigcap_{C \supset bB_n} \overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C),
$$

where $C$ ranges through the family of the compact subsets of $M^2$ which contain $bB_n$. By the local maximum modulus principle, for each such $C$ and for each $n \in \mathbb{N}$, we have

$$
\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C) = h_{\mathcal{O}(\overline{D})}(bB_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)).
$$

Therefore what we have to prove is that

$$
h_{\mathcal{O}(\overline{D})}(G_n) = \bigcap_{C \supset bB_n} h_{\mathcal{O}(\overline{D})}(\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)).
$$
The validity of the inclusion of the left hand side set in the right hand side set is evident. Conversely, let \( z \) be an arbitrary point in the right hand side set. Then, if \( f \in \mathcal{O}(\overline{D}_n) \), it follows that \( |f(z)| \leq |f(\zeta)| \), for every \( \zeta \in \overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C) \) whichever be the compact set \( C \) containing \( \overline{B}_n \). Hence \( |f(z)| \leq |f(\zeta)| \) for every \( \zeta \in \bigcap_{C \supseteq \overline{B}_n} \left[ \overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C) \right] = G_n \), i.e., \( z \in h_{\mathcal{O}(\overline{D}_n)}(G_n) \). This proves \((*)\).

Now, we can readily infer that the open set \( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) is Stein, by resorting to (2.2). Since we can choose a Stein open neighborhood \( X \) of \( \overline{D}_n \), such that the restriction map \( \mathcal{O}(X) \to \mathcal{O}(\overline{D}_n) \) has dense image, and consequently \( h_{\mathcal{O}(\overline{D}_n)}(G_n) = h_{\mathcal{O}(X)}(G_n) \), by (3.1) and \((*)\) we have \( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus h_{\mathcal{O}(X)}(G_n) \), and hence we see at once that \( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) is Stein.

Moreover, since

\[
B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) = (B_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D})) \cap B_n = \{ z \in B_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) : \Phi(z) < c_n \},
\]

\( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) is Runge in \( B_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) (see [13]).

Hence we may conclude that \( M^2 \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \), being the union of an increasing sequence of Stein open subsets, each of which is Runge in the subsequent, is itself a Stein open subset of \( M^2 \) (see [11, p. 215]).

**Remark.** The first three propositions of this section remain valid in the setting of a Stein manifold \( M' \) of dimension \( r \geq 2 \) as the ambient space, as a direct inspection of the corresponding proofs shows, in view of the remark at the end of Section 2 too. On the contrary Proposition 4 becomes false for \( r \geq 3 \), as will be seen in Section 6. On account of the result of [16], it is likely that for \( r \geq 3 \) it should be still true that \( H^{r-1}(M' \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}), \mathcal{F}) = 0 \), for every coherent analytic sheaf, \( \mathcal{F} \), on \( M' \); however this does not seem to deserve a relevant interest in connection with the subject of this paper.

### 4. Proof of Theorem 2

Let \( K \) be a compact subset of \( bD \) as in the statement of Theorem 2 and put

\[
\tilde{K} = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).
\]

We divide the proof of Theorem 2 into the proofs of three propositions.

**Proposition 5.** The set \( \tilde{K} \) is compact.

**Proof.** Let us first prove that, if \( m, n \in \mathbb{N} \) and \( m < n \), then

\[
h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \setminus \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m) \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m).
\]
Indeed, in view of (2.5) and (3.1), one has
\[ \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m) \subset (\overline{D} \setminus \overline{D}_m) \cup h_{\mathcal{O}(\overline{D})}(\Delta_m) = h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m), \]
and since \( h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) = h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \), it follows that
\[ h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)] \subset h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \cap \overline{D}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m). \]
Moreover, since \( bD_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m) = (\Gamma_n \setminus \Gamma_m) \cup \Delta_n \), by the local maximum modulus principle,
\[ \overline{D}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_m) \subset h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n). \]
On the other hand, by (2.6),
\[ h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m), \]
and then (4.1) follows at once. Now, since, for any fixed \( m \in \mathbb{N} \),
\[ \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) = h_{\mathcal{O}(\overline{D})}(bD) \quad \text{and} \quad \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \Gamma_m) = h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m), \]
(3.1) implies that, for each \( m \in \mathbb{N} \),
\[ h_{\mathcal{O}(\overline{D})}(bD) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)] \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m), \]
and consequently
\[ h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m) \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m). \]
Then, taking \( m \) large enough that the given compact set \( K \) is contained in \( \Gamma_m \), it follows that
\[ \tilde{K} \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m), \]
and consequently that
\[ \tilde{K} = h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K). \]
Since \( h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m) \) is a compact subset of \( \overline{D} \), in order to conclude the proof it suffices to show that the set \( h_{\mathcal{O}(\overline{D})}(bD \setminus K) \) is open in \( \overline{D} \). As a matter of fact, consider, for \( n \geq m \), a Stein open set \( D'_n \) such that \( D_n \setminus (K \cup \Delta_n) \subset D'_n \), \( bD_n \cap bD'_n = K \cup \Delta_n \) and \( D_n \setminus (K \cup \Delta_n) \) is \( \mathcal{O}(D'_n) \)-convex, as can be obtain by pushing \( \Gamma_n \) away from \( D_n \) by a small \( C^2 \)-perturbation that leaves \( K \) and \( b\Gamma_n \) fixed pointwise. Then \( h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n) \) is an open (Stein and Runge) subset of \( D'_n \), such that
\[ h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K) = h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n \setminus K) = \overline{D}_n \cap h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n) = \overline{D} \cap h_{\mathcal{O}(D'_n)}(D'_n \setminus \overline{D}_n), \]
(see [19]). Therefore \( h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K) \) is open in \( \overline{D} \). Since
\[ h_{\mathcal{O}(\overline{D})}(bD \setminus K) = \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K), \]
it follows that \( h_{\mathcal{O}(\overline{D})}(bD \setminus K) \) is open in \( \overline{D} \). \( \square \)
PROPOSITION 6. The restriction map \( \mathcal{O}(\tilde{K}) \to \mathcal{O}(K) \) is bijective. Consequently, \( \tilde{K} \) is also equal to the set \( h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K) \). Moreover, if \( K \) is holomorphically convex, then \( \tilde{K} = K \).

PROOF. Consider the two sets \( h_{\mathcal{O}(\overline{D})}(bD \setminus bD) \) and \( h_{\mathcal{O}(\overline{D})}(bD \setminus K) \setminus (bD \setminus K) \), and, for brevity, call them \( X \) and \( Y \), respectively. Both of these sets are open in \( M^2 \) and \( X \) is Stein. Indeed \( X = D \cap \overline{h_{\mathcal{O}(M^2)}(M^2 \setminus \overline{D})} \), and \( h_{\mathcal{O}(M^2)}(M^2 \setminus \overline{D}) \) is a Stein open set in \( M^2 \) (see [19]); moreover at the end of the proof of Proposition 4 we have shown that \( Y \cup (bD \setminus K) \) is open in \( D \). Furthermore, \( Y \) is a Stein and Runge open set in \( X \). As a matter of fact, given a compact set \( G \subset Y \), \( h_{\mathcal{O}(X)}(G) \) is contained in \( h_{\mathcal{O}(D)}(G) \), which is a compact subset of \( D \). On the other hand, by definition of \( h_{\mathcal{O}(\overline{D})}(bD \setminus K) \), there is a compact set \( E \subset bD \setminus K \) with \( G \subset h_{\mathcal{O}(\overline{D})}(E) \), and consequently \( h_{\mathcal{O}(X)}(G) \subset h_{\mathcal{O}(\overline{D})}(E) \). It follows that \( h_{\mathcal{O}(X)}(G) \) is contained in \( h_{\mathcal{O}(D)}(G) \cap h_{\mathcal{O}(\overline{D})}(E) \), which is a compact subset of \( Y \) and hence it is itself a compact subset of \( Y \). We claim that consequently

\[ (*) \quad H^0_c(X \setminus Y, \mathcal{O}) = 0 \text{ and } H^1_c(X \setminus Y, \mathcal{O}) = 0. \]

As a matter of fact, there is an exact cohomology sequence with compact supports

\[ 0 \to H^0_c(Y, \mathcal{O}) \to H^0_c(X, \mathcal{O}) \to H^0_c(X \setminus Y, \mathcal{O}) \to H^1_c(X, \mathcal{O}) \to H^1_c(X \setminus Y, \mathcal{O}) \to 0. \]

Plainly \( H^0_c(Y, \mathcal{O}) = 0 \) and \( H^0_c(X, \mathcal{O}) = 0 \), and it is known that, since \( X \) and \( Y \) are Stein, \( H^1_c(Y, \mathcal{O}) = 0 \) and \( H^1_c(X, \mathcal{O}) = 0 \). Moreover it is also known that, since \( Y \) is Runge in \( X \), the map \( H^2_c(Y, \mathcal{O}) \to H^2_c(X, \mathcal{O}) \) is injective. In view of these facts, the preceding exact sequence implies at once the validity of \( (*) \). Now, we have

\[ \text{(***)} \quad X \setminus Y = \tilde{K} \setminus K, \]

and since \( K \) and \( \tilde{K} \) are compact, there is also an exact cohomology sequence with compact supports

\[ 0 \to H^0_c(\tilde{K} \setminus K, \mathcal{O}) \to H^0(\tilde{K}, \mathcal{O}) \to H^0(K, \mathcal{O}) \to H^1_c(\tilde{K} \setminus K, \mathcal{O}) \to \cdots, \]

from which, on account of \( (*) \) and \( (***) \), we infer that the restriction map \( \mathcal{O}(\tilde{K}) \to \mathcal{O}(K) \) is bijective.

The first assertion of the proposition implies that \( \tilde{K} \subset h_{\mathcal{O}(\overline{D})}(K) \), for, if \( z \) is a point in \( \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(K) \), there exists \( f \in \mathcal{O}(\overline{D}) \) with \( f(z) = 1 \) and \( \max_K |f| < 1 \); then \( (1 - f)^{-1} \in \mathcal{O}(K) \) and hence \( (1 - f)^{-1} \) extends to be holomorphic on a neighborhood of \( \tilde{K} \), which means that \( z \in \overline{D} \setminus \tilde{K} \). It follows that \( \tilde{K} = h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K) \).
Next, suppose that $K$ is holomorphically convex. Then $H^1(K, \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on $M^2$; in particular $H^1(K, \Omega^2) = 0$, with $\Omega^2$ being the sheaf of germs of holomorphic 2-forms, and hence, by the exact cohomology sequence with compact supports

$$\cdots \rightarrow H^1(K, \Omega^2) \rightarrow H^2_c(\tilde{K} \setminus K, \Omega^2) \rightarrow H^2(\tilde{K}, \Omega^2) = 0 \rightarrow \cdots,$$

it follows that

$$H^2_c(\tilde{K} \setminus K, \Omega^2) = H^2_c(X \setminus Y, \Omega^2) = 0.$$

In this connection let us recall that, by a result of Greene and Wu [10], every noncompact (connected) complex-analytic manifold $M$ of dimension $r \geq 1$ is $(r - 1)$-complete, and hence $H^r(M, \mathcal{S}) = 0$, for every coherent analytic sheaf, $\mathcal{S}$, on $M$. Consequently, an inductive limit consideration gives that also $H^r(\mathcal{E}, \mathcal{S}) = 0$, for every subset $\mathcal{E} \subset M$, which is the reason why $H^r(\tilde{K}, \Omega^2) = 0$.

Now, the vanishing of $H^2_c(X \setminus Y, \Omega^2)$ is equivalent to having $h_{\mathcal{O}(X)}(Y) = X$ (see [19, Theorem 4]), and since $Y$ is Runge in $X$, so that $h_{\mathcal{O}(X)}(Y) = Y$, the latter property just amounts to saying that $Y = X$, i.e., $\tilde{K} = K$. \(\square\)

**Proposition 7.** The set $\tilde{K}$ is a Stein compactum.

**Proof.** Let $C$ be a compact neighborhood of $K$ in $bD$, and consider the set $h_{\mathcal{O}(\overline{D})}(bD \setminus C)$. This is a relatively open subset of $D$, as follows from the final part of the proof of Proposition 5, taking in it $C$ in place of $K$. Hence the set $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus C)$ is compact. Since $\tilde{K}$ can be obtained as the intersection of a decreasing sequence of sets like this, it suffices to prove that $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus C)$ is a Stein compactum. Indeed, since the set $h_{\mathcal{O}(\overline{D})}(K)$ is a Stein compactum, it admits a neighborhood basis $\mathcal{V}$ of relatively compact Stein open sets, and since $bD \cap h_{\mathcal{O}(\overline{D})}(K) = K$, we can choose $\mathcal{V}$ such that $bD \cap V \subset C$, for each $V \in \mathcal{V}$. Moreover let us fix an exhausting family $\mathcal{G}$ of compact subsets of $bD \setminus C$. Given $G \in \mathcal{G}$, we can find a Stein open neighborhood $X$ of $G$, such that $h_{\mathcal{O}(\overline{D})}(G) = h_{\mathcal{O}(X)}(G)$. Then, by resorting again to (2.2), we infer that, for every $V \in \mathcal{V}$ and $G \in \mathcal{G}$, the open set $V \setminus h_{\mathcal{O}(\overline{D})}(G)$ is Stein. Since

$$h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus C) = \bigcap_{G \in \mathcal{G}} \bigcap_{V \in \mathcal{V}} [V \setminus h_{\mathcal{O}(\overline{D})}(G)],$$

we reach the desired conclusion. \(\square\)

**Remark.** Proposition 5 and Proposition 6 are also valid in the setting of a Stein manifold $M'$ of dimension $r \geq 2$ as the ambient space, rather than $M^2$, as a direct inspection of the corresponding proofs shows, in view of the remarks at the end of Section 2 and Section 3 too. Actually, as regards the $r$-dimensional extension of Proposition 6, the assumption that $H^{r-1}(K, \mathcal{F}) = 0$ is sufficient to imply that $\tilde{K} = K$. On the contrary Proposition 7 becomes false for $r \geq 3$. 
as will be seen in Section 6. On account of the result of [16], it is probably
ture that, also for \( r \geq 3 \), \( H^{r-1}(\mathcal{D}, \mathcal{F}) = 0 \); however, as the parallel property
of \( M' \setminus h^\infty(\mathcal{O}(D)) \), this does not seem to be a relevant information for our
purposes.

5. – Proof of Theorem 3

We may limit ourselves to deal with the case that the domain \( D \) is relatively
compact. Indeed, if \( D \) is not relatively compact, given any compact subset \( K \)
of \( bD \), one can, by the procedure of [24], construct a Stein open set \( D' \) with \( C^2 \)
boundary, which is the disjoint union of finitely many relatively compact strongly
pseudoconvex domains, such that \( bD' \) contains a neighborhood, in \( bD \), of \( K \),
and \( D' \) is \( \mathcal{O}(D) \)-convex. Then, clearly, it suffices to prove Theorem 3 for any
connected component of \( D' \).

The following proposition is the essential point of the proof.

**Proposition 8.** Let \( D \subset M^2 \) be a \( C^2 \)-bounded strongly pseudoconvex domain
and \( S \subset bD \) a topological 2-sphere of class \( C^2 \). Then, if \( \omega \) is a holomorphic 2-form
defined on a neighborhood of \( S \), it follows that

\[
\int_S \omega = 0.
\]

In other words, the holomorphic de Rham cohomology \( H^2_{\text{hol}}(S) = \Omega^2(\omega) = 0 \).

**Proof.** Let \( U \) be an open neighborhood of \( S \) such that \( \omega \in \Omega^2(U) \). By
applying to \( bD \) a standard smoothing result for manifolds of class \( C^r \) (\( 1 \leq r \))
imbedded in manifolds of class \( C^\infty \) (see [22, Theorem 4.8]), we can find a \( C^\infty \)-
bounded strongly pseudoconvex domain \( D_1 \), with \( bD_1 \) being \( C^2 \) diffeomorphic
and \( C^2 \) isotopically equivalent to \( bD \), and so close to \( bD \) that the diffeomorphic
image of \( S \), \( S_1 \), say, is contained in \( U \). Moreover we may assume that \( S_1 \)
is generically imbedded in \( M^2 \), so that it has only finitely many complex
tangencies, all of which are either elliptic or hyperbolic. Then, we can apply to
\( S_1 \) the result of Bedford and Klingenberg [4, Theorem 1](3), according to which
there is a small \( C^2 \) perturbation \( S'_1 \), of \( S_1 \) on \( bD_1 \), which has, in particular,
the following property: there is a smooth 3-manifold \( B' \) in \( D_1 \), such that
\( B' \cup S'_1 = B' = E(S'_1) \). Then, it follows that the form \( \omega \) extends to a holomorphic

(3) Note of the editor. The author considers an arbitrary two-dimensional Stein manifold \( M \). It is
to be observed that for the validity of Proposition 8, \( M \) should equal \( C^2 \). Proposition 8 is based
on the Bedford and Klingenberg theorem that is proved in fact only for \( C^2 \).
form $\tilde{\omega}$ on a neighborhood of $\overline{B}'$, and hence, by Stokes’s theorem,

$$\int_S \omega = \int_{S_1} \omega = \int_{S_1'} \omega = \int_{\overline{B}} d\tilde{\omega} = 0. \quad \square$$

Now we can prove:

**PROPOSITION 9.** Let $D$ be as in the preceding proposition and let $K$ be a compact subset of $bD$. Assume that $K$ has a neighborhood basis $N$, in $bD$, such that each $N \in N$ is a relatively compact open subset of $bD$ (possibly disconnected), whose boundary $bN$ is the union of finitely many pairwise disjoint topological 2-spheres of class $C^2$. Put $\Gamma = bD \setminus K$. Then, if $f$ is any continuous CR-function on $\Gamma$, it follows that $\int_{\overline{\Gamma}} f\alpha = 0$, for every $C^\infty$ $\overline{\partial}$-closed $(2, 1)$-form $\alpha$ on a neighborhood of $\overline{D}$ such that $\text{supp}(\alpha) \cap K = \emptyset$. Consequently, $E(K) = h_{\Omega(\overline{D})(K)}$.

**PROOF.** Since $D$ is a Stein compactum, there exists a $C^\infty$ $(2, 0)$-form $\beta$ on a neighborhood of $\overline{D}$ such that $\alpha = \overline{\partial}\beta = d\beta$, and since $\text{supp}(\alpha) \cap K = \emptyset$, there exists a neighborhood $U$ of $K$ in $M^2$ such $\overline{\partial}\beta = 0$ on $U$, i.e., $\beta$ is a holomorphic 2-form on $U$. By assumption there exists $N \in N$ such that $\overline{N} \subset U$, and hence, on account of Proposition 8, it is readily seen that $\int_{\overline{N}} f\beta = 0$. Then, by Stokes’s theorem, we have

$$\int_{\overline{bD}} f\alpha = \int_{\overline{bD}} f\overline{\partial}\beta = \int_{\overline{bD} \setminus N} f\overline{\partial}\beta = -\int_{\overline{bN}} f\beta = 0.$$

It follows, in view of (2.7), that $E(\Gamma) = h_{\Omega(\overline{D})(\Gamma)}$, and hence, in view of (2.8), we achieve the desired conclusion. $\square$

**REMARKS.** (i) In connection with the assumption of Theorem 3, we point out that, if $M$ is an orientable topological 3-manifold with boundary, such that $H_1(M, \mathbb{Z}) = 0$, then it follows that the boundary $bM$ of $M$ is a union of topological 2-spheres. Indeed, the vanishing of the homology group $H_1(M, \mathbb{Z})$ implies that the cohomology group $H^1(M, \mathbb{Z})$ is null (recall that $H^q(\cdot, \mathbb{Z})$ is isomorphic to $\text{Hom}\mathbb{Z}(H_q(\cdot, \mathbb{Z}), \mathbb{Z})$, provided $H_{q-1}(\cdot, \mathbb{Z})$ is a free $\mathbb{Z}$-module). Then, by the Poincaré duality for compact manifolds with boundary (see [8, Proposition 9.1]), also the relative homology group $H_2(M, bM; \mathbb{Z})$ is null. By the exact sequence of relative homology

$$\cdots \rightarrow H_2(M, bM; \mathbb{Z}) \rightarrow H_1(bM, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z}) \rightarrow \cdots,$$

it follows that $H_1(bM, \mathbb{Z}) = 0$. This implies first that the connected components of $bM$ are orientable (see [8, Proposition 2.12]) and then, being orientable compact surfaces of genus zero, that these connected components are topological 2-spheres.
(ii) It is simple to show that Theorem 3 does not extend to higher dimensions. Consider in $\mathbb{C}^r$ for $r \geq 3$, the open unit ball $B$ and in $S^{2r-1} = bB$ the two disjoint closed semi-spheres

$$
\Sigma_1^2 = \{ z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 = 1, \ Re z_2 \geq 0, \ Re z_2 = 0, z_3 = \cdots = z_r = 0 \},
$$

$$
\Sigma_2^2 = \{ z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \ Re z_{r-1} = 0, \ Re z_{r-1} \geq 0, (\Re z_{r-1})^2 + |z_r|^2 = 1 \},
$$

and put $K = \Sigma_1^2 \cup \Sigma_2^2$. It is evident that $K$ verifies the assumption of Theorem 3. On the other hand, since the intersection $h_{O(B)}(\Sigma_1^2) \cap h_{O(B)}(\Sigma_2^2)$ is nonempty, as it contains at least the origin, it is trivially not true that every $f \in O(K)$ may have a holomorphic extension to a neighborhood of $h_{O(B)}(K)$. In the preceding counterexample $K$ is disconnected, but this does not affect its validity, since also in Theorem 3 $K$ is allowed to be disconnected. On the other hand in Section 6 we shall be able to show a less trivial counterexample in which $K$ is connected.

6. – Non-extendability to higher dimensions

In the first place we state the weaker extension theorems that generalize Theorem 1 and Theorem 2 to the setting of a Stein manifold $M'$ of dimension $r \geq 2$, rather than $r = 2$. In view of the remarks at the ends of Section 2, Section 3 and Section 4, we have:

**THEOREM 4.** Let $D \subset M'$ be an open domain of holomorphy, whose boundary $bD$ is a real hypersurface of class $C^2$, strictly Levi-convex with respect to $D$. Put $\Omega = M' \setminus \overline{D}$. Then the three sets $M' \setminus h_{O(D)}^{\infty}(bD), \Omega \cup [h_{O(D)}^{\infty}(bD) \setminus h_{O(D)}^{\infty}(bD)]$ and $h_{O(M')}^{\infty}(\Omega) \setminus h_{O(D)}^{\infty}(bD)$ are a same open subset of $M'$, $\Omega$, say, such that the restriction map $O(\Omega) \to O(\Omega)$ is bijective.

**THEOREM 5.** Let $D$ be as in Theorem 4. Let $K$ be an arbitrary compact subset of $bD$, and put $\tilde{K} = h_{O(D)}^{\infty}(bD) \setminus h_{O(D)}^{\infty}(bD \setminus K)$. Then $\tilde{K}$ is a compact set containing $K$, such that the restriction map $O(\tilde{K}) \to O(K)$ is bijective. Consequently, $\tilde{K}$ is also equal to the set $h_{O(D)}^{\infty}(K) \setminus h_{O(D)}^{\infty}(bD \setminus K).

Furthermore, if $H^{r-1}(K, \mathcal{F}) = 0$, for every coherent analytic sheaf, $\mathcal{F}$, on $K$, then $\tilde{K} = K$.

Now we wish to show that for $r \geq 3$ the open set $\tilde{\Omega}$ of Theorem 4 may not be Stein, as well as the compact set $\tilde{K}$ of Theorem 5 may not be a Stein compactum.

Preliminarily, consider a $C^2$- bounded strongly pseudoconvex domain $D \subset \subset \mathbb{C}'$ and a compact set $\mathfrak{K} \subset bD$. Let us push $bD$ away from $D$ by a small $C^2$
perturbation which leaves \( \mathcal{R} \) fixed pointwise, so as to obtain a Stein domain, call it \( M' \), with \( \overline{\mathcal{D}} \setminus \mathcal{R} \subset M' \) and \( bM' \cap \overline{\mathcal{D}} = \mathcal{R} \). We may consider \( \mathcal{D} \) as an unbounded open domain of holomorphy in the Stein manifold \( M' \). Then we change the notations, so that \( D \) denotes the domain \( \mathcal{D} \) when it is regarded as a domain in \( M' \) rather than in \( \mathbb{C}^r \), whereas \( bD \) and \( \overline{D} \) denote the boundary and the closure of \( D \) in \( M' \). Then \( b\mathcal{D} = bD \cup \mathcal{R} \) and \( \overline{\mathcal{D}} = \overline{D} \cup \mathcal{R} \). We claim that

\[
(6.1) \quad h^\infty_{\mathcal{C}(\overline{D})}(\overline{D}) = h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}).
\]

As a matter of fact, consider the open sets \( D_n, n \in \mathbb{N} \) defined at the beginning of Section 2. It is evident that, for each \( n \in \mathbb{N} \), \( h^\infty_{\mathcal{C}(\overline{D})}(\overline{D} \setminus D_n) \subset h^\infty_{\mathcal{C}(\overline{D})}(\overline{D} \setminus D_n) \), whereas the local maximum modulus principle implies that \( h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}) \cap \overline{D}_n \subset h^\infty_{\mathcal{C}(\overline{D})}(bD_n \cap h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R})) \subset h^\infty_{\mathcal{C}(\overline{D})}(\overline{D} \setminus D_n) \). Hence, making the intersections for all \( n \in \mathbb{N} \) gives the two inclusions \( h^\infty_{\mathcal{C}(\overline{D})}(\overline{D}) \subset h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}) \) and \( h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}) \setminus \mathcal{R} \subset h^\infty_{\mathcal{C}(\overline{D})}(\overline{D}) \) and (6.1) follows at once.

That being stated, to produce an example, for \( r \geq 3 \), in which \( \bar{\Omega} = M' \setminus h^\infty_{\mathcal{C}(\overline{D})}(\overline{D}) \) is not Stein, it suffices to consider the preceding construction, taking as \( \mathcal{R} \) the intersection of \( b\mathcal{D} \) with any complex-analytic subvariety \( V \) of \( \mathbb{C}^r \), of codimension \( q \) in the range \( 2 \leq q \leq r - 1 \), passing through \( \mathcal{D} \) since in this case \( h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}) = \mathcal{R} \cup (V \cap \mathcal{D}) \), it follows, in view of (6.1), that \( \bar{\Omega} = M' \setminus V \), which is not a Stein manifold. Moreover, if we choose a suitably small open neighborhood \( U \) of the variety \( V \), also the interior of \( M' \setminus U \) is not Stein, and hence the compact set \( \overline{D} \setminus U \) is not a Stein compactum. We can take as \( U \) a Stein open neighborhood of \( V \) which is Runge in \( \mathbb{C}^r \), so as to have \( U \cap \overline{\mathcal{D}} = h^\infty_{\mathcal{C}(\overline{D})}(U \cap b\mathcal{D}) \). Then the compact set \( K = bD \setminus U \) verifies \( \bar{K} = \overline{D} \setminus U \), thus providing an example, for \( r \geq 3 \), of a compact set \( K \subset bD \) such that \( \bar{K} \) is not a Stein compactum.

Next we show that for \( r \geq 3 \) the envelope of holomorphy of \( \Omega \) (which, by Theorem 3, coincides with the envelope of holomorphy of \( \bar{\Omega} \)) may be multi-sheeted. Indeed Chirka and Stout [7, 4.5] exhibited a \( C^\infty \)-bounded strongly pseudoconvex domain \( \mathcal{D} \subset \subset \mathbb{C}^{2m} \), \( m \geq 2 \), a compact set \( \mathcal{R} \subset b\mathcal{D} \) (with \( b\mathcal{D} \setminus \mathcal{R} \) being connected) and a function \( f \in \mathcal{O}(\overline{\mathcal{D}}) \setminus h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}) \) in such a way that \( f \) can be continued analytically in the sense of Weierstrass to the whole \( \overline{\mathcal{D}} \setminus \mathcal{R} \), so as to give rise to two different determinations at each point of \( h^\infty_{\mathcal{C}(\overline{D})}(\mathcal{R}) \setminus \mathcal{R} \). Therefore, by applying in this case the procedure described above, we can obtain at once, on account of (6.1), a counterexample to \( E(\Omega) \) being single-sheeted, valid for all even dimensions \( \geq 4 \). On the other hand, we can obtain also a counterexample valid for all dimensions \( \geq 3 \), rather than only for the even ones, by modifying in a suitable manner the construction of Chirka and Stout. For the convenience of the reader we give a complete description of the modified construction, parallel to the description of the original construction given in [7, 4.5]. Consider in \( \mathbb{C}^r \), \( r \geq 3 \), the open unit ball \( \mathbb{B} \), and in \( S^{2r-1} = b\mathbb{B} \) the two
disjoint closed 2-spheres

$S^1_1 = \{ z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 = 1, z_2 = 0, z_3 = \cdots = z_r = 0 \}$,
$S^1_2 = \{ z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \Re z_{r-1} = 0, (\Im z_{r-1})^2 + |z_r|^2 = 1 \}$.

Let $\Gamma_1$ and $\Gamma_2$ be connected open neighborhoods, in $S^{2r-1}$, of $S^1_1$ and $S^1_2$, respectively, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and put $\mathcal{R} = S^{2r-1} \setminus (\Gamma_1 \cup \Gamma_2)$. Then let $\gamma$ be a smooth arc in $(C' \setminus \mathcal{B}) \cup \{(1, 0, \ldots, 0), (0, \ldots, 0, 1)\}$, which connects the points $(1, 0, \ldots, 0)$ and $(0, \ldots, 0, 1)$, is orthogonal to $\partial \mathcal{B}$ at these points, and verifies the following two conditions: a) if $\phi$ is the function on $C'$ given by $\phi(z_1, \ldots, z_r) = z_1 - z_r$, then $\gamma_1 = \phi(\gamma)$ is a smooth arc in the upper half plane $\Pi \subset C$, which connects the points 1 and $-1$; b) the point $2i$ belongs to the relatively compact component of $\Pi \setminus \gamma_1$. Since $|z_1 - z_r| \leq 1$ on $S^1_1 \cup S^1_2$, we may assume that $\Gamma_1$ and $\Gamma_2$ have been chosen so small that $|z_1 - z_r| < 2$ on $\Gamma_1 \cup \Gamma_2$. Hence we can define a continuous argument of $z_1 - z - 2i$ on $\Gamma_1 \cup \Gamma_2 \cup \gamma$ which takes values in the interval $(-\pi, 0)$ on $\Gamma_1$ and takes values in the interval $(\pi, 2\pi)$ on $\Gamma_2$. Consequently, the function $f$ defined by

$$(6.2) \quad f(z_1, \ldots, z_r) = (z_1 - z_r - 2i)^{\frac{1}{2}} = \sqrt{|z_1 - z_r - 2i|} e^{i\arg(z_1 - z_r - 2i)},$$

with the above mentioned argument function, is holomorphic on a neighborhood of $\Gamma_1 \cup \Gamma_2 \cup \gamma$, and $\Im f < 0$ on $\Gamma_1$, $\Im f > 0$ on $\Gamma_2$. The envelope of holomorphy of $\Gamma_1$ contains the compact 3-ball $\mathcal{B}^3_1 = \{ z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 \leq 1, \Re z_2 = 0, z_3 = \cdots = z_r = 0 \}$, and the envelope of holomorphy of $\Gamma_2$ contains the compact 3-ball $\mathcal{B}^3_2 = \{ z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \Re z_{r-1} = 0, (\Im z_{r-1})^2 + |z_r|^2 \leq 1 \}$.

Therefore the function $f$ extends holomorphically into a neighborhood of $\mathcal{B}^3_1$ and into a neighborhood of $\mathcal{B}^3_2$ with different values: $\Im f < 0$ on $\mathcal{B}^3_1$ and $\Im f > 0$ on $\mathcal{B}^3_2$. It follows that the domain of holomorphy of $f$ has two different sheets at least on a neighborhood of $\mathcal{B}^3_1 \cap \mathcal{B}^3_2$. Finally, given a small neighborhood $V$ of $\gamma$ in $C'$, such that the above function $f$ is holomorphic on $V$ and $V \cap \mathcal{R} = \emptyset$, consider a $C^\infty$-bounded strongly pseudoconvex domain $\mathcal{D}$ with $\mathcal{B} \cup V \subset \mathcal{D} \subset \mathcal{B} \cup V$. Then $\mathcal{R}$ is a compact subset of $\partial \mathcal{D}$ such that $\mathcal{B} \setminus \mathcal{R}$ is connected and $f$ is a function holomorphic on a neighborhood of $\mathcal{B} \setminus \mathcal{R}$ whose domain of holomorphy is not single-sheeted. It follows that, by applying in this case the procedure described above, we can obtain, on account of (6.1), a counterexample to $\Omega$ being single-sheeted which is valid for all dimensions $\geq 3$.

We conclude the paper by providing a counterexample to the possibility of extending Theorem 3 to higher dimensions, in which, unlike in the final remark of Section 5, the compact set $K$ is connected. By perturbing slightly the arc $\gamma$ of the preceding construction, we can find a smooth arc $\gamma' \subset C'$, contained in a neighborhood of $\Gamma_1 \cup \Gamma_2 \cup \gamma$ where the function $f$ of (6.2) is single-valued, which connects two points $p_1, p_2 \in \partial \mathcal{B} \setminus (S^1_1 \cup S^1_2)$, close to $(1, 0, \ldots, 0)$, $(0, \ldots, 0, 1)$, respectively, which is orthogonal to $\partial \mathcal{B}$ at these points and is
contained in $(C' \setminus \overline{B}) \cup \{p_1, p_2\}$. Then, given a small neighborhood $V'$ of $\gamma'$ in $C'$, such that $f$ is holomorphic on $\overline{V'}$ and $\overline{V'} \cap (\Sigma^2_1 \cup \Sigma^2_2) = \emptyset$, consider a $C^\infty$-bounded strongly pseudoconvex domain $D$ with $\mathbb{B} \cup \gamma' \subset D \subset \mathbb{B} \cup \overline{V}'$, analogous to the preceding domain $\mathcal{D}$. Then $\Sigma^2_1 \cup \Sigma^2_2 \subset \partial D$, and we can find a smooth arc $\gamma'' \subset (\partial D \setminus (\Sigma^2_1 \cup \Sigma^2_2)) \cup \{(1, 0, \ldots, 0), (0, \ldots, 0, 1)\}$, joining the points $(1, 0, \ldots, 0)$ and $(0, \ldots, 0, 1)$, such that $f$ is single-valued on a neighborhood of $\Sigma^2_1 \cup \Sigma^2_2 \cup \gamma''$. Now consider again the two closed semi-2-spheres of the final remark of Section 5, and put $K = \Sigma^2_1 \cup \Sigma^2_2 \cup \gamma''$. It is evident that $K$ verifies the assumption of Theorem 3 and is connected; however the domain of holomorphy of the function $f$ is not single-sheeted over a neighborhood of the origin, and consequently $E(K) \neq h_{\mathcal{O}(\overline{D})}(K)$.

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