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1. – Introduction

The aim of this paper is to present descriptions of the envelopes of holomorphy of certain classes of subsets of $\mathbb{C}^2$, namely:

a) the open subsets which are complements of noncompact closed domains bounded by strictly Levi-convex real hypersurfaces of class $C^2$;

b) the compact subsets which lie on the boundaries of closed domain – either compact or noncompact – bounded by strictly Levi-convex real hypersurfaces of class $C^2$.

More generally we shall consider an arbitrary two-dimensional Stein manifold $M^2$ as the ambient space, rather than $\mathbb{C}^2$.

Let us recall that the envelope of holomorphy $E(S)$ of an arbitrary subset $S$ of a Stein manifold $M$ can be defined as the union of the components of $\hat{S} = \text{spec}(\mathcal{O}(S))$ which meet $S$. For a non-open subset $S \subset M$, $\hat{S}$ need not be embedded in a complex manifold in any natural way. On the other hand, if there exists a holomorphically convex set $S' \subset M$ containing $S$, with the property that the restriction map $\mathcal{O}(S') \rightarrow \mathcal{O}(S)$ is bijective, then $E(S)$ may be identified with $S'$. In this connection we also recall that if a subset of a complex manifold admits a fundamental system of Stein neighborhoods, then it is holomorphically convex. (We refer to [12] for all these facts.)

The mentioned descriptions require us to take into considerations certain holomorphic hulls of some subsets of $M^2$ which are not compact sets. If $S$ is an arbitrary subset of $M^2$ and $K \subset S$ is a compact set, let us use the notation

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that $h_{\mathcal{O}(S)}(K)$ denotes the $\mathcal{O}(S)$-hull of $K$, i.e.,

$$h_{\mathcal{O}(S)}(K) = \bigcap_{f \in \mathcal{O}(S)} \{ z \in S : |f(z)| \leq \|f\|_K \}. $$

Now, let $T$ be an arbitrary subset of $S$. Then we define the $\mathcal{O}(S)$-hull of $T$, $h_{\mathcal{O}(S)}(T)$, to be the union of the $\mathcal{O}(S)$-hulls of all compact subsets of $T$, i.e.,

$$(1.1) \quad h_{\mathcal{O}(S)}(T) = \bigcup_{K \subseteq T} h_{\mathcal{O}(S)}(K),$$

where $K$ ranges through the family of compact subsets of $T$. We have already used this notion in our previous paper [19], where one can find results related to the subject which is being discussed here.

Moreover we find it convenient to introduce, for a closed set $F \subset M^2$, a notion of “hull at infinity”, in the following way. If $S \subset M$ is an arbitrary set containing $F$, we define the $\mathcal{O}(S)$-hull at infinity of $F$, $h_{\mathcal{O}(S)}^\infty (F)$, to be the intersection of the $\mathcal{O}(S)$-hulls of the subsets of $F$ which are complements of compact sets, that is

$$(1.2) \quad h_{\mathcal{O}(S)}^\infty (F) = \bigcap_{G \subseteq F} h_{\mathcal{O}(S)}(F \setminus G),$$

where $G$ ranges through the family of compact subsets of $F$. Plainly, if $F$ is compact, $h_{\mathcal{O}(S)}^\infty (F) = \emptyset$, but if $F$ is noncompact, $h_{\mathcal{O}(S)}^\infty (F)$ may be nonempty; for example, if there is a one-dimensional complex-analytic subvariety $V$ of $M^2$ with $V \subset F$, then $V \subset h_{\mathcal{O}(S)}^\infty (F)$. We have been led to consider the preceding notion of hulls at infinity by some analogy with the notion of cohomology of the ideal boundary of a noncompact space $X$, which is known to be the inductive limit of the cohomology of $X \setminus G$ as $G$ ranges through the compact subsets of $X$ (see [6]), and is sometimes also called the cohomology at infinity of $X$ and denoted by $H^\infty_\omega (X)$.

That being stated, we can formulate our main results.

**Theorem 1.** Let $D \subset M^2$ be an open domain of holomorphy, whose boundary $\partial D$ is a real hypersurface of class $C^2$, strictly Levi-convex with respect to $D$. Put $\Omega = M^2 \setminus \overline{D}$. Then the envelope of holomorphy of $\Omega$, $E(\Omega)$, is given by

$$E(\Omega) = M^2 \setminus h_{\mathcal{O}(\overline{D})}^\infty (\overline{D})$$

$$= \Omega \cup [h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}^\infty (bD)]$$

$$= h_{\mathcal{O}(M^2)}(\Omega) \setminus h_{\mathcal{O}(\overline{D})}^\infty (bD).$$

In particular $E(\Omega)$ is single-sheeted over $\Omega$. 
THEOREM 2. Let $D$ be as in Theorem 1. Let $K$ be a compact subset of $bD$. Then the envelope of holomorphy of $K$, $E(K)$, is given by

$$E(K) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K) = h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).$$

Indeed the sets $h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ and $h_{\mathcal{O}(\overline{D})}(K) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)$ are a same Stein compactum containing $K$, $\tilde{K}$, say, such that the restriction map $\mathcal{O}(\tilde{K}) \rightarrow \mathcal{O}(K)$ is bijective. In particular $E(K)$ is single-sheeted over $K$.

Moreover, if $K$ is holomorphically convex, then $E(K) = K$, i.e., $K$ is a Stein compactum.

THEOREM 3. Let $D$ be as in the preceding theorems. Let $K$ be a compact subset of $bD$. Assume that $K$ has a neighborhood basis $\mathcal{N}$, in $bD$, such that each $N \in \mathcal{N}$ is a relatively compact open subset of $bD$ (possibly disconnected), whose boundary $bN$ is the union of finitely many pairwise disjoint topological 2-spheres of class $C^2$. Then it follows that $E(K) = \mathcal{h}(D)(K)$.

We emphasize that in the preceding three theorems $\overline{D}$ may be either compact or noncompact and in the latter case $bD$ is allowed to be disconnected. However in the compact case, as $h_{\mathcal{O}(\overline{D})}(bD) = \emptyset$, Theorem 1 yields only a result equivalent to Hartogs’s extension theorem. Moreover let us recall that $K$ is said to be holomorphically convex if the evaluation map $K \rightarrow \text{spec} (\mathcal{O}(K))$ is bijective, or, equivalently, if $H^1(K, \mathcal{F}) = H^2(K, \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on $K$ (see [12]).

Some further comments are in order. Since every $f \in \mathcal{O}(bD)$ can be written as $f = f_1 - f_2$, with $f_1 \in \mathcal{O}(\overline{\Omega})$ and $f_2 \in \mathcal{O}(\overline{D})$ and the restriction map $\mathcal{O}(\overline{\Omega}) \rightarrow \mathcal{O}(\Omega)$ is surjective, Theorem 1 is equivalent to the following result:

COROLLARY 1. Let $D$ be as in Theorem 1. Then the envelope of holomorphy of $bD$, $E(bD)$, is given by

$$E(bD) = \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D})$$

$$= h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD)$$

$$= [h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{D}] \setminus h_{\mathcal{O}(\overline{D})}(bD).$$

In particular $E(bD)$ is single-sheeted over $bD$.

Moreover, since

$$h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD) = \bigcup_{K \subset bD} [h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K)],$$

\(^{(1)}\) We wish to point out that a sufficient condition which implies the mentioned property of $bN$ is, besides $bN$ being of class $C^2$, that $H_1(\overline{N}, \mathbb{Z}) = 0$. This will be shown at the end of Section 5.
with $K$ ranging through the family of compact subsets of $bD$, it is not difficult to see that the equality $E(bD) = h_{O(D)}(bD) \setminus h_{O(D)}^\infty(bD)$ is also a consequence of the first statement of Theorem 2.

The second statement of Theorem 2 seems to deserve some interest in connection with the question raised by Harvey and Wells [12, p. 515] whether every holomorphically convex compact set in a Stein manifold should be a Stein compactum. This question was answered in the negative by Björk [5], who exhibited examples of compact holomorphically convex sets in $\mathbb{C}^n$, $n \geq 2$, which are not Stein compacta. On the other hand Theorem 2 gives a positive answer to the question at least for the holomorphically convex compact sets which lie on $bD$, when $n = 2$. In this connection we also recall that, if $\overline{D}$ is compact, a compact subset of $bD$ is holomorphically convex if and only if it is “weakly removable” (see [18, Corollary 2]).

Combining Theorem 3 with the second statement of Theorem 2 gives in particular the following result:

**Corollary 2.** Let $D$ be as in the preceding theorems. Let $K \subset bD$ be a holomorphically convex compact set endowed with a neighborhood basis, in $bD$, of topological $3$-cells. Then $K$ is $O(\overline{D})$-convex, i.e., $h_{O(\overline{D})}(K) = K$.

Here the requirement that the boundaries of the topological $3$-cells should be of class $\mathcal{C}^2$ is not necessary, since known results on $3$-manifolds and smoothing of homeomorphisms ([20, Theorem 4] and [21, Theorem 6.3]) imply the existence also of a neighborhood basis of $K$, in $bD$, of topological $3$-cells with boundaries of class $\mathcal{C}^2$, the essential point being the fact that two homeomorphic $3$-manifolds of class $\mathcal{C}^2$ are $\mathcal{C}^2$-diffeomorphic. Corollary 2 is close to a theorem of Forstnerič and Stout [9], which yields the same conclusion, in the case that $D$ is relatively compact, under the additional assumption that the set $K$ should have a Stein open neighborhood $X$ in which it is $O(X)$-convex. The first result in this direction is due to Jörnicke [14], who obtained the equivalent result that $K$ is “removable” (see [7], [18], [23]) in the case that $K$ is a compact totally real disk of class $\mathcal{C}^2$. Forstnerič and Stout resorted, for the proof of their theorem, to the work of Bedford and Klingenberg [4] on the envelopes of holomorphy of $2$-spheres, and also our proof of Theorem 3 depends on that work, in that we need a result from [4] to prove the vanishing of the two-dimensional holomorphic de Rham cohomology of a topological $2$-sphere of class $\mathcal{C}^2$ embedded in the boundary of a strongly pseudoconvex domain (Section 5, Proposition 8).

Theorem 3 is also useful to obtain more information in the direction of Theorem 1 and Corollary 1, under some reasonably general additional conditions on $bD$.

**Corollary 3.** Let $D$ be as in the preceding theorems. Assume that $bD$ can be exhausted by an increasing sequence $\{N_n\}$ of relatively compact $\mathcal{C}^2$-bounded open subsets (possibly disconnected), such that each boundary $bN_n$ is the union of finitely many pairwise disjoint topological $2$-spheres of class $\mathcal{C}^2$ (which is true in particular in case $bD$ is homeomorphic to $\mathbb{R}^3$). Then

$$E(\Omega) = \Omega \cup h_{O(\overline{D})}(bD) = h_{O(M^2)}(\Omega). \quad E(bD) = h_{O(\overline{D})}(bD) = h_{O(M^2)}(\Omega) \cap \overline{D}.$$
In other words, \( h^{\infty}_{\mathcal{O}(\overline{D})}(bD) \) is empty.

Finally, a reason of interest in respect of the above results is, in our opinion, the circumstance that they do not extend to higher dimensions, in the sense that, if one replaces \( M^2 \) by a Stein manifold of dimension \( \geq 3 \) as the ambient space, the corresponding statements become false. We shall discuss this point at the end of the article, in Section 6; in particular we will exhibit an example, inspired by one of Chirka and Stout [7], which shows that for all dimensions \( \geq 3 \) \( E(\Omega) \) may be multi-sheeted\(^{(2)}\). On the other hand, at the beginning of Section 6 we will also mention the weaker results which can be obtained in the positive, for dimensions \( \geq 3 \), in the direction contemplated here (Theorem 4 and Theorem 5).

2. – Preliminaries

Consider a domain \( D \) as in the statements of Theorem 1 and Theorem 2. Let us fix once for all a \( C^\infty \) strongly plurisubharmonic exhaustion function \( \Phi : M^2 \to \mathbb{R} \) and an increasing divergent sequence \( \{c_n\}_{n \in \mathbb{N}} \) of positive real numbers all of which are regular values for both of the functions \( \Phi \) and \( \Phi|_{bD} \); moreover let us put, for every \( n \in \mathbb{N} \),

\[
B_n = \{z \in M^2 : \Phi(z) < c_n\}, \quad D_n = B_n \cap D, \quad \Gamma_n = B_n \cap bD, \quad \Delta_n = bB_n \cap \overline{D}.
\]

Then \( D_n \) is a relatively compact Stein open set in \( M^2 \), such that \( bD_n = \Gamma_n \cup \Delta_n \).

It is known that, since \( bD \) is strictly Levi-convex, the closed domain \( \overline{D} \) admits a neighborhoods basis of Stein open sets (for the noncompact case see [24, Lemme 2]). Then, since \( \overline{B}_n \) is an \( \mathcal{O}(M^2) \)-convex Stein compactum, it is readily seen that \( \overline{D}_n \) is \( \mathcal{O}(\overline{D}) \)-convex, i.e. the restriction map \( \mathcal{O}(\overline{D}) \to \mathcal{O}(\overline{D}_n) \) has dense image, and consequently the following property, which will be used repeatedly throughout the continuation of this paper, holds:

\[
(2.1) \quad h_{\mathcal{O}(\overline{D})}(G) = h_{\mathcal{O}(\overline{D}_n)}(G) \text{ for every compact set } G \subset \overline{D}_n.
\]

We shall also apply several times a pseudoconvexity result which refines slightly a result of Slodkowski (see [16] and the references cited there), namely:

\[
(2.2) \quad \text{Let } C \subset M^2 \text{ be a compact set, } X \subset M^2 \text{ a Stein open set containing } C \text{ and } S \subset M^2 \text{ a Stein open set such that } C \cap S \text{ is empty. Then the open set } S \setminus h_{\mathcal{O}(X)}(C) \text{ is Stein.}
\]

Moreover we need to recall a result on holomorphic extension of CR-functions (see [23], [17] and references cited there):

\(^{(2)}\)The original example of [7] is suitable for the same conclusion only as regards the even dimensions \( \geq 4 \), thus excluding in particular dimension 3.
Let $D \subset M^2$ be an open domain and $K \subset \partial D$ a compact set. Assume that $\partial D \setminus K$ is a $C^1$-smooth real hypersurface of $M^2 \setminus K$ and that $\partial D$ admits a Stein open neighborhood $X$ in which it is an $O(X)$-convex Stein compactum. Then every continuous CR-function on $\partial D \setminus K$ has a unique extension to a continuous function on $\partial D \setminus h_{O(\partial D)}(K)$ holomorphic on $D \setminus h_{O(\partial D)}(K)$.

That being stated, we collect in a lemma three further properties that will come directly in the proofs of our theorems.

**Lemma.** For each $n \in \mathbb{N}$ the following properties are valid:

(2.4) Every continuous CR-function on $\Gamma_n$ extends uniquely to a continuous function on $\overline{D}_n \setminus h_{O(\partial D_n)}(\Delta_n)$ holomorphic on $D_n \setminus h_{O(\partial D_n)}(\Delta_n)$.

(2.5) $h_{O(\partial D)}(\overline{\Gamma}_n) \cup h_{O(\partial D)}(\Delta_n) = \overline{D}_n$.

(2.6) $h_{O(\partial D)}(\overline{\Gamma}_n) \cap h_{O(\partial D)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{O(\partial D)}(\overline{\Gamma}_n \setminus \Gamma_m)$, for every $m \in \mathbb{N}$ with $m \leq n$.

**Proof.** By (2.1), in proving the lemma we may replace the $O(D)$-hulls by the corresponding $O(D_n)$-hulls.

Now, to prove (2.4), let $D_n^i$, $i \in I$ be the connected components of $D_n$ and put, for each $i \in I$, $\Delta_n^i = bD_n^i \cap \Delta_n$ and $\Gamma_n^i = bD_n^i \setminus \Delta_n^i = bD_n^i \cap bD$. Then each $D_n^i$ is a Stein domain in $M^2$, such that $\overline{D}_n^i$ is a Stein compactum, and $\Gamma_n^i$ is a real hypersurface of class $C^2$ in $M^2 \setminus \Delta_n^i$. In this situation we may apply (2.3): every continuous CR-function on $\Gamma_n^i$ has a unique extension to a continuous function on $\overline{D}_n^i \setminus h_{O(\partial D_n^i)}(\Delta_n^i)$ which is holomorphic on $D_n^i \setminus h_{O(\partial D_n^i)}(\Delta_n^i)$. Then, since $\Gamma_n$ is the disjoint union of the $\Gamma_n^i$’s, $i \in I$, and $\bigcup_{i \in I} h_{O(\partial D_n^i)}(\Delta_n^i) = h_{O(\partial D_n)}(\Delta_n)$, it is also true that every continuous CR-function on $\Gamma_n$ extends uniquely to a continuous function on $\overline{D}_n \setminus h_{O(\partial D_n)}(\Delta_n)$ which is holomorphic on $D_n \setminus h_{O(\partial D_n)}(\Delta_n)$. Hence we see that (2.4) holds.

Next we prove (2.5). It suffices to prove that the inclusion

\[(\ast) \quad \overline{D}_n \setminus h_{O(\partial D_n)}(\Delta_n) \subset h_{O(\partial D_n)}(\Gamma_n)\]

is valid. Since $\Gamma_n$ is strictly Levi-convex at each point with respect to $D_n$, we can construct a relatively compact Stein open set $D'_n \subset M^2$ such that $\overline{D}_n \setminus \Delta_n \subset D'_n$, $\Delta_n \subset bD'_n$, and $\overline{D}_n \setminus \Delta_n$ is $O(D'_n)$-convex. Indeed $D'_n$ can be obtained by pushing $\overline{\Gamma}_n$ away from $D_n$ by a small $C^2$-perturbation that leaves $b\Gamma_n$ fixed pointwise. Then consider the open set $D'_n \setminus \overline{D}_n$ and make its $O(D'_n)$-hull $h_{O(D'_n)}(D'_n \setminus \overline{D}_n)$. The latter is a Stein and Runge open subset of $D'_n$, such that

\[(\ast\ast) \quad h_{O(D'_n)}(\Gamma_n) = h_{O(D'_n \setminus \Delta_n)}(\Gamma_n) = (\overline{D}_n \setminus \Delta_n) \cap h_{O(D'_n)}(D'_n \setminus \overline{D}_n)\]
Since \( h_{\mathcal{O}(D_n')} \) is a Stein set containing \( \Gamma_n \), one can find CR-functions on \( \Gamma_n \) (of class \( C^2 \)) which cannot be holomorphically extended through any boundary point, in \( D_n \), of \( h_{\mathcal{O}(D_n')}(D_n \setminus \overline{D}_n) \), namely the restrictions to \( \Gamma_n \) of the functions holomorphic on \( h_{\mathcal{O}(D_n)}(D_n \setminus \overline{D}_n) \) which do not admit holomorphic continuations to any larger open set; hence, granted the validity of (**), if (*) were not true, this would lead to a contradiction to (2.3).

Finally let us prove (2.6). Since we can choose a Stein open neighborhood \( X \) of \( \overline{D}_n \), such that \( D_n \) is \( \mathcal{O}(X) \)-convex, and consequently \( h_{\mathcal{O}(\overline{D}_n)}(C) = h_{\mathcal{O}(X)}(C) \) for every compact set \( C \subset \overline{D}_n \), (2.2) implies that the three open sets \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \), \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) \) and \( D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n \setminus \Gamma_m) \) are Stein. Moreover, by (2.5), the union \([D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)]\) is disjoint and hence it is a Stein open set as well. On account of the latter fact, by a reasoning analogous to that used above to prove (**), one can show that:

\[ (\dagger) \text{ There exist CR-functions on } bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_m) \text{ (of class } C^2 \text{) which cannot be holomorphically extended through any boundary point, in } D_n, \text{ of } [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n)] \cup [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)]. \]

On the other hand, (2.3) can also be applied to derive the property, parallel to (2.4), in which one considers \( bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_m) \) in place of \( \Gamma_n \), and \( \overline{\Gamma}_n \setminus \Gamma_m \) in place of \( \Delta_n \), respectively. Hence the following is true too:

\[ (\dagger\dagger) \text{ Every continuous CR-function on } bD_n \setminus (\overline{\Gamma}_n \setminus \Gamma_m) \text{ admits a continuous extension to } D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n). \]

Combining (\dagger) and (\dagger\dagger), we see that \( (\overline{\Gamma}_n \setminus \Gamma_m) \subset [D_n \setminus h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n)]. \) This amounts to having \( h_{\mathcal{O}(\overline{D}_n)}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) \subset h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) \), which yields the desired conclusion.

For the proof of Theorem 3 we shall need two results from [19] (see Corollary 5 and Corollary 7 therein). For the convenience of the reader we restate the results here.

\[ (2.7) \text{ Let } D \subset \subset M^2 \text{ be a } C^2 \text{-bounded strongly pseudoconvex domain and } K \subset bD \text{ a compact set, and put } \Gamma = bD \setminus K. \text{ Then for a continuous CR-function } f \text{ on } \Gamma \text{ the following two conditions are equivalent:} \]
\[ - \int_{\Gamma} f \alpha = 0 \text{ for every } C^\infty \text{-closed } (2,1)-\text{form } \alpha \text{ on a neighborhood of } \overline{D} \text{ such that } \text{supp}(\alpha) \cap K = \emptyset. \]
\[ - f \text{ extends uniquely to a function in } C^0(h_{\mathcal{O}(\overline{D})}(\Gamma)) \cup C^0(h_{\mathcal{O}(\overline{D})}(\Gamma) \setminus \Gamma). \]

\[ (2.8) \text{ Let } D, K \text{ and } \Gamma' \text{ be as in } (2.7). \text{ Then the following three conditions are equivalent:} \]
\[ - E(K) = h_{\mathcal{O}(\overline{D})}(K). \]
\[ - E(\Gamma') = h_{\mathcal{O}(\overline{D})}(\Gamma'). \]
\[ - h_{\mathcal{O}(\overline{D})}(\Gamma') = \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(K). \]
REMARK. We point out that all the properties discussed in this section, except (2.2) and (2.8), remain valid if $M^2$ is replaced by a Stein manifold $M'$ of dimension $r \geq 2$ as the ambient space. As regards (2.2) and (2.8), on the contrary, for $r \geq 3$ it is not true in general that $S \setminus h_{\mathcal{O}(X)}(C)$ is Stein, nor that the three properties of (2.8) are equivalent, whereas it is still true that $H^{-1}(S \setminus h_{\mathcal{O}(X)}(C), \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on $M'$ (see [16] and [17]). Since we have applied (2.2) in the proof of (2.6), the given proof of (2.6) does not work for $r \geq 3$. However it is possible to prove (2.6) for general $r \geq 3$, in a different way, by generalizing a result of Basener [3] relative to the polynomial hulls of compact subsets of $b\mathbb{B}_r$. Basener’s proof of his result appears to be tied up the ball case only in that it invokes an earlier result of H. Alexander [1] on the connectivity properties of the polynomial hulls of compact subsets of $b\mathbb{B}_r$. Since it is now known that Alexander’s result generalizes from the ball case so as to cover classes of domains of a Stein manifold $M'$ which include the connected components of the above $D_n$’s (see [2], [15]), it turns out that Basener’s result generalizes as well, so as to imply the validity of (2.6) for $r \geq 2$.

3. – Proof of Theorem 1

We divide the proof of Theorem 1 into the proofs of four propositions.

PROPOSITION 1. The hull at infinity $h_{\mathcal{O}(\overline{D})}^\infty(\overline{D})$ is a closed set in $M^2$ such that

$$\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(bD) \subset h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) \subset D.$$ 

PROOF. It follows immediately from the definition, (1.2), of a hull at infinity that

$$h_{\mathcal{O}(\overline{D})}^\infty(\overline{D}) = \bigcap_{n \in \mathbb{N}} h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n),$$

hence to show that $h_{\mathcal{O}(\overline{D})}^\infty(\overline{D})$ is closed in $M^2$, it suffices to prove that so is $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$ for each $n \in \mathbb{N}$. Since the restriction map $\mathcal{O}(\overline{D} \setminus B_n) \to \mathcal{O}(\overline{D} \setminus \overline{D}_n)$ is surjective, it follows (arguing as in [19, Lemma 4]) that $\Delta_n \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$, and hence $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n) = h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus B_n)$. Let us show that

$$h_{\mathcal{O}(\overline{D})}(\Delta_n) = \overline{D}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n).$$

In view of (2.1), the inclusion of the left hand side set in the right hand side set follows at once from the above. As regards the reverse inclusion, consider a compact set $G \subset \overline{D} \setminus B_n$. The local maximum modulus principle implies that $\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(G) \subset h_{\mathcal{O}(\overline{D})}(bD_n \cap h_{\mathcal{O}(\overline{D})}(G))$. Then, since

$$\bigcup_{G \subset \overline{D} \setminus B_n} h_{\mathcal{O}(\overline{D})}(bB_n \cap h_{\mathcal{O}(\overline{D})}(G)) = h_{\mathcal{O}(\overline{D})}(\Delta_n),$$
the reverse inclusion holds as well. It follows that

\[(3.2) \quad h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) = (\overline{D} \setminus D_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) = (\overline{D} \setminus B_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) , \]

which shows \( h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) \) to be closed in \( M^2 \).

Next, since \( bD \) is strictly Levi-convex with respect to \( D \), every compact set \( G \subset \overline{D} \) verifies \( bD \cap h_{\mathcal{O}(\overline{D})}(G) = bD \cap G \). Therefore, if \( z \) is a point of \( bD \) and \( n \) is a positive integer large enough that \( z \in D_n \), it follows that

\[ z \notin h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) = \bigcup_{G \subset \overline{D} \setminus D_n} h_{\mathcal{O}(\overline{D})}(G) . \]

Consequently, \( z \notin h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \). This proves that \( bD \cap h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) = \emptyset \), and hence that \( h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \subset D \).

Finally, let \( z \in \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(bD) \) and choose a positive integer \( m \) large enough so that \( z \in D_n \) for \( n \geq m \). In view of (3.2) it is plain that

\[ h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) = \bigcap_{n=m}^{\infty} \{ (\overline{D} \setminus D_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) \} . \]

On the other hand, by (2.5),

\[ D_n \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) , \]

for every \( n \in \mathbb{N} \). Then, as \( z \notin h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \) for every \( n \in \mathbb{N} \) and \( z \in D_n \) for \( n \geq m \), it follows that \( z \in h_{\mathcal{O}(\overline{D})}(\Delta_n) \) for \( n \geq m \), and hence \( z \in h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \).

This proves that \( \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(bD) \subset h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \). \( \square \)

**Proposition 2.** The hull at infinity \( h_{\mathcal{O}(\overline{D})}^{\infty}(bD) \) verifies

\[ h_{\mathcal{O}(\overline{D})}^{\infty}(bD) = h_{\mathcal{O}(\overline{D})}(bD) \cap h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) . \]

**Proof.** Clearly, only the inclusion \( h_{\mathcal{O}(\overline{D})}(bD) \cap h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \subset h_{\mathcal{O}(\overline{D})}^{\infty}(bD) \) has to be proved. On account of (2.6) for \( m = n \), we have, for each \( n \in \mathbb{N} \),

\[ h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}(\Delta_n) \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n) ; \]

hence, in view of (3.1) and (3.2), we see that

\[ h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n) , \]

from which, since \( h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) \), we infer that, for each \( n \in \mathbb{N} \),

\[ h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h_{\mathcal{O}(\overline{D})}^{\infty}(\overline{D}) \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n) . \]
On the other hand, since for any choice of \( m \in \mathbb{N} \),
\[
h_{\mathcal{O}(\overline{D})}(bD) = \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n),
\]
we also have, for each \( m \in \mathbb{N} \),
\[
h_{\mathcal{O}(\overline{D})}(bD) \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = \bigcup_{n=m}^{\infty} [h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D})];
\]
and therefore, in view of (*), it follows that
\[
h_{\mathcal{O}(\overline{D})}(bD) \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) \subset \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_n) = h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m).
\]
Since this is true for each \( m \in \mathbb{N} \), we may conclude that
\[
h_{\mathcal{O}(\overline{D})}(bD) \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) \subset \bigcap_{m \in \mathbb{N}} h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m) = h^\infty_{\mathcal{O}(\overline{D})}(bD).
\]

**Proposition 3.** The following two properties hold:

(i) \( M^2 \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = \Omega \cup [h_{\mathcal{O}(\overline{D})}(bD) \setminus h^\infty_{\mathcal{O}(\overline{D})}(bD)] = h_{\mathcal{O}(M^2)}(\Omega) \setminus h^\infty_{\mathcal{O}(\overline{D})}(bD); \)

(ii) Every \( f \in \mathcal{O}(\Omega) \) extends uniquely to an \( F \in \mathcal{O}(M^2 \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D})). \)

**Proof.** Since, by Proposition 1, \( \overline{D} = h_{\mathcal{O}(\overline{D})}(bD) \cup h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) \), it follows that
\[
\overline{\Omega} \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{D})}(bD) \setminus [h_{\mathcal{O}(\overline{D})}(bD) \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D})].
\]
By Proposition 2, the last term is \( h_{\mathcal{O}(\overline{D})}(bD) \setminus h^\infty_{\mathcal{O}(\overline{D})}(bD) \), and hence the first equality of (i) follows at once. Moreover, since the restriction map \( \mathcal{O}(\overline{\Omega}) \to \mathcal{O}(\Omega) \) is surjective, it follows that \( h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{\Omega} = h_{\mathcal{O}(\overline{D})}(bD) \) (see [19, Lemma 4]), hence
\[
h_{\mathcal{O}(M^2)}(\Omega) = \Omega \cup [h_{\mathcal{O}(M^2)}(\Omega) \cap \overline{\Omega}] = \Omega \cup h_{\mathcal{O}(\overline{D})}(bD),
\]
which implies immediately the second equality of (i).

Next, to prove (ii), let \( \tilde{f} \) denote a holomorphic extension of \( f \) to an open neighborhood of \( \overline{\Omega} \) and consider its restriction to \( bD \), which is a CR-function on \( bD \) of class \( \mathcal{C}^2 \). It suffices to prove that the latter has a unique continuous extension, \( g \), say, to \( \overline{D} \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) \) which is holomorphic on \( D \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) \). Then \( F \) will be given by \( F = \tilde{f} \) on \( \Omega \) and \( F = g \) on \( \overline{D} \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) \). By (2.3), for each \( n \in \mathbb{N} \) there exists a unique extension of \( \tilde{f} |_{\Gamma_n} \) to a continuous function
on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$ holomorphic on $D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$, $g_n$, say. Moreover, for each $n \in \mathbb{N}$,

$$\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n) \subset \overline{D}_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\Delta_{n+1});$$

for, by the local maximum modulus principle,

$$\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_{n+1}) \subset h_{\mathcal{O}(\overline{D})}(bB_n \cap h_{\mathcal{O}(\overline{D})}(\Delta_{n+1}));$$

hence $g_{n+1} = g_n$ on $\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)$, for each $n \in \mathbb{N}$, and this implies the existence of a unique continuous extension of $f|_{bD}$ to $\bigcup_{n \in \mathbb{N}}(\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n))$ which is holomorphic on $\bigcup_{n \in \mathbb{N}}(D_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n))$, namely the coherent union of the $g_n$'s. Finally, on account of (3.2), we have

$$\overline{D} \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = \bigcup_{n \in \mathbb{N}}(\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n))$$

$$= \bigcup_{n \in \mathbb{N}}(\overline{D} \setminus ((\overline{D} \setminus \overline{D}_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n)))$$

$$= \bigcup_{n \in \mathbb{N}}(\overline{D}_n \setminus h_{\mathcal{O}(\overline{D})}(\Delta_n)),$$

and hence we conclude that the coherent union of the $g_n$'s defines the function $g$ as is required above. \hfill $\Box$

**Proposition 4.** The open set $M^2 \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D})$ is Stein.

**Proof.** For each $n \in \mathbb{N}$ we put

$$G_n = bB_n \cap h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}).$$

Let us first prove that

$$B_n \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus h_{\mathcal{O}(\overline{D}_n)}(G_n).$$

It is readily seen that

$$B_n \setminus h^\infty_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus \bigcap_{C \supset B_n} h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C),$$

where $C$ ranges through the family of the compact subsets of $M^2$ which contain $\overline{B}_n$. By the local maximum modulus principle, for each such $C$ and for each $n \in \mathbb{N}$, we have

$$\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C) = h_{\mathcal{O}(\overline{D}_n)}(bB_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)).$$

Therefore what we have to prove is that

$$h_{\mathcal{O}(\overline{D}_n)}(G_n) = \bigcap_{C \supset \overline{B}_n} h_{\mathcal{O}(\overline{D})}(\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)).$$
The validity of the inclusion of the left hand side set in the right hand side set is evident. Conversely, let \( z \) be an arbitrary point in the right hand side set. Then, if \( f \in \mathcal{O}(\Delta_n) \), it follows that \(|f(z)| \leq |f(\zeta)|\), for every \( \zeta \in \overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C) \) whichever be the compact set \( C \) containing \( \overline{B}_n \). Hence \(|f(z)| \leq |f(\zeta)| \) for every \( \zeta \in \cap_{C \supset \overline{B}_n} [\overline{B}_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus C)] = G_n \), i.e., \( z \in h_{\mathcal{O}(\overline{D})}(G_n) \). This proves (*).

Now, we can readily infer that the open set \( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) is Stein, by resorting to (2,2). Since we can choose a Stein open neighborhood \( X \) of \( \Delta_n \), such that the restriction map \( \mathcal{O}(X) \rightarrow \mathcal{O}(\Delta_n) \) has dense image, and consequently \( h_{\mathcal{O}(\overline{D})}(G_n) = h_{\mathcal{O}(X)}(G_n) \), by (3,1) and (*) we have \( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) = B_n \setminus h_{\mathcal{O}(\overline{X})}(G_n) \), and hence we see at once that \( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) is Stein.

Moreover, since

\[
B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) = [B_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D})] \cap B_n = \{ z \in B_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) : \Phi(z) < c_n \},
\]

\( B_n \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) is Runge in \( B_{n+1} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \) (see [13]).

Hence we may conclude that \( M^2 \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}) \), being the union of an increasing sequence of Stein open subsets, each of which is Runge in the subsequent, is itself a Stein open subset of \( M^2 \) (see [11, p. 215]). □

**Remark.** The first three propositions of this section remain valid in the setting of a Stein manifold \( M' \) of dimension \( r \geq 2 \) as the ambient space, as a direct inspection of the corresponding proofs shows, in view of the remark at the end of Section 2 too. On the contrary Proposition 4 becomes false for \( r \geq 3 \), as will be seen in Section 6. On account of the result of [16], it is likely that for \( r \geq 3 \) it should be still true that \( H^{r-1}(M' \setminus h_{\mathcal{O}(\overline{D})}(\overline{D}), \mathcal{F}) = 0 \), for every coherent analytic sheaf, \( \mathcal{F} \), on \( M' \); however this does not seem to deserve a relevant interest in connection with the subject of this paper.

4. – Proof of Theorem 2

Let \( K \) be a compact subset of \( \partial D \) as in the statement of Theorem 2 and put

\[
\tilde{K} = h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K).
\]

We divide the proof of Theorem 2 into the proofs of three propositions.

**Proposition 5.** The set \( \tilde{K} \) is compact.

**Proof.** Let us first prove that, if \( m, n \in \mathbb{N} \) and \( m < n \), then

\[
(4.1) \quad h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_m)] \subset h_{\mathcal{O}(\overline{D})}(\overline{\Gamma}_n \setminus \overline{\Gamma}_m).
\]
Indeed, in view of (2.5) and (3.1), one has
\[ \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(T_n) \subset (\overline{D} \setminus D_n) \cup h_{\mathcal{O}(\overline{D})}(\Delta_n) = h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n), \]
and since \( h_{\mathcal{O}(\overline{D})}(T_n) = h_{\mathcal{O}(\overline{D})}(T_n) \), it follows that
\[ h_{\mathcal{O}(\overline{D})}(T_n) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(T_n)] \subset h_{\mathcal{O}(\overline{D})}(T_n) \cap \overline{D} \setminus h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n). \]
Moreover, since \( bD_n \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) = (\Gamma_n \setminus \Gamma_m) \cup \Delta_n \), by the local maximum modulus principle,
\[ \overline{D} \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n)((\Gamma_n \setminus \Gamma_m) \cup \Delta_n) = h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n). \]
On the other hand, by (2.6),
\[ h_{\mathcal{O}(\overline{D})}(T_n) \cap h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) = h_{\mathcal{O}(\overline{D})}(T_n), \]
and then (4.1) follows at once. Now, since, for any fixed \( m \in \mathbb{N} \),
\[ \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(T_n) = h_{\mathcal{O}(\overline{D})}(bD) \] and \( \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(T_n \setminus \Gamma_m) = h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m), \)
(3.1) implies that, for each \( m \in \mathbb{N} \),
\[ h_{\mathcal{O}(\overline{D})}(bD) \cap [\overline{D} \setminus h_{\mathcal{O}(\overline{D})}(T_n)] \subset h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m), \]
and consequently
\[ h_{\mathcal{O}(\overline{D})}(bD) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus \Gamma_m) \subset h_{\mathcal{O}(\overline{D})}(T_n). \]
Then, taking \( m \) large enough that the given compact set \( K \) is contained in \( \Gamma_m \), it follows that
\[ \tilde{K} \subset h_{\mathcal{O}(\overline{D})}(T_n), \]
and consequently that
\[ \tilde{K} = h_{\mathcal{O}(\overline{D})}(T_n) \setminus h_{\mathcal{O}(\overline{D})}(bD \setminus K). \]
Since \( h_{\mathcal{O}(\overline{D})}(T_n) \) is a compact subset of \( \overline{D} \), in order to conclude the proof it suffices to show that the set \( h_{\mathcal{O}(\overline{D})}(bD \setminus K) \) is open in \( \overline{D} \). As a matter of fact, consider, for \( n \geq m \), a Stein open set \( D'_n \) such that \( \overline{D}_n \setminus (K \cup \Delta_n) \subset D'_n \), \( bD_n \cap bD'_n = K \cup \Delta_n \) and \( D_n \setminus (K \cup \Delta_n) \) is \( \mathcal{O}(\overline{D}_n) \)-convex, as can be obtained by pushing \( \Gamma_n \) away from \( D_n \) by a small \( C^2 \)-perturbation that leaves \( K \) and \( b\Gamma_n \) fixed pointwise. Then \( h_{\mathcal{O}(\overline{D}_n)}(D'_n \setminus D_n) \) is an open (Stein and Runge) subset of \( D'_n \), such that
\[ h_{\mathcal{O}(\overline{D}_n)}(\Gamma_n \setminus K) = h_{\mathcal{O}(\overline{D}_n)}((\Gamma_n \setminus K) \setminus h_{\mathcal{O}(\overline{D}_n)}(D'_n \setminus D_n) = \overline{D} \cap h_{\mathcal{O}(\overline{D}_n)}(D'_n \setminus D_n), \]
(see [19]). Therefore \( h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K) \) is open in \( \overline{D} \). Since
\[ h_{\mathcal{O}(\overline{D})}(bD \setminus K) = \bigcup_{n=m}^{\infty} h_{\mathcal{O}(\overline{D})}(\Gamma_n \setminus K), \]
it follows that \( h_{\mathcal{O}(\overline{D})}(bD \setminus K) \) is open in \( \overline{D} \). \( \square \)
PROPOSITION 6. The restriction map $\mathcal{O}(\overline{K}) \to \mathcal{O}(K)$ is bijective. Consequently, $\overline{K}$ is also equal to the set $h_{\mathcal{O}(D)}(K) \setminus h_{\mathcal{O}(D)}(bD \setminus K)$. Moreover, if $K$ is holomorphically convex, then $\overline{K} = K$.

PROOF. Consider the two sets $h_{\mathcal{O}(D)}(bD) \setminus bD$ and $h_{\mathcal{O}(D)}(bD \setminus K) \setminus (bD \setminus K)$, and, for brevity, call them $X$ and $Y$, respectively. Both of these sets are open in $M^2$ and $X$ is Stein. Indeed $X = D \cap h_{\mathcal{O}(M^2)}(M^2 \setminus D)$, and $h_{\mathcal{O}(M^2)}(M^2 \setminus D)$ is a Stein open set in $M^2$ (see [19]); moreover at the end of the proof of Proposition 4 we have shown that $Y \cup (bD \setminus K)$ is open in $D$. Furthermore, $Y$ is a Stein and Runge open set in $X$. As a matter of fact, given a compact set $G \subset Y$, $h_{\mathcal{O}(X)}(G)$ is contained in $h_{\mathcal{O}(D)}(G)$, which is a compact subset of $D$. On the other hand, by definition of $h_{\mathcal{O}(D)}(bD \setminus K)$, there is a compact set $E \subset bD \setminus K$ with $G \subset h_{\mathcal{O}(D)}(E)$, and consequently $h_{\mathcal{O}(X)}(G) \subset h_{\mathcal{O}(D)}(E)$. It follows that $h_{\mathcal{O}(X)}(G)$ is contained in $h_{\mathcal{O}(D)}(G) \cap h_{\mathcal{O}(D)}(E)$, which is a compact subset of $Y$ and hence it is itself a compact subset of $Y$. We claim that consequently

\[(*) \quad H^0_c(X \setminus Y, \mathcal{O}) = 0 \text{ and } H^1_c(X \setminus Y, \mathcal{O}) = 0.\]

As a matter of fact, there is an exact cohomology sequence with compact supports

\[
0 \to H^0_c(Y, \mathcal{O}) \to H^0_c(X, \mathcal{O}) \to H^0_c(X \setminus Y, \mathcal{O}) \to H^1_c(Y, \mathcal{O}) \to H^1_c(X, \mathcal{O}) \\
H^1_c(X \setminus Y, \mathcal{O}) \to H^2_c(X, \mathcal{O}) \to H^2_c(X \setminus Y, \mathcal{O}) \to 0.
\]

Plainly $H^0_c(Y, \mathcal{O}) = 0$ and $H^0_c(X, \mathcal{O}) = 0$, and it is known that, since $X$ and $Y$ are Stein, $H^1_c(Y, \mathcal{O}) = 0$ and $H^1_c(X, \mathcal{O}) = 0$. Moreover it is also known that, since $Y$ is Runge in $X$, the map $H^2_c(Y, \mathcal{O}) \to H^2_c(X, \mathcal{O})$ is injective. In view of these facts, the preceding exact sequence implies at once the validity of $(*)$. Now, we have

\[(**) \quad X \setminus Y = \overline{K} \setminus K,\]

and since $K$ and $\overline{K}$ are compact, there is also an exact cohomology sequence with compact supports

\[
0 \to H^0_c(\overline{K} \setminus K, \mathcal{O}) \to H^0(\overline{K}, \mathcal{O}) \to H^0(K, \mathcal{O}) \to H^1_c(\overline{K} \setminus K, \mathcal{O}) \to \cdots,
\]

from which, on account of $(*)$ and $(**)$, we infer that the restriction map $\mathcal{O}(\overline{K}) \to \mathcal{O}(K)$ is bijective.

The first assertion of the proposition implies that $\overline{K} \subset h_{\mathcal{O}(D)}(K)$, for, if $z$ is a point in $\overline{D} \setminus h_{\mathcal{O}(D)}(K)$, there exists $f \in \mathcal{O}(D)$ with $f(z) = 1$ and $\max_K |f| < 1$; then $(1 - f)^{-1} \in \mathcal{O}(K)$ and hence $(1 - f)^{-1}$ extends to be holomorphic on a neighborhood of $\overline{K}$, which means that $z \in \overline{D} \setminus \overline{K}$. It follows that $\overline{K} = h_{\mathcal{O}(D)}(K) \setminus h_{\mathcal{O}(D)}(bD \setminus K)$. 
Next, suppose that $K$ is holomorphically convex. Then $H^1(K, \mathcal{F}) = 0$ for every coherent analytic sheaf, $\mathcal{F}$, on $M^2$; in particular $H^1(K, \Omega^2) = 0$, with $\Omega^2$ being the sheaf of germs of holomorphic 2-forms, and hence, by the exact cohomology sequence with compact supports

$$\cdots \to H^1(K, \Omega^2) \to H^2_c(\tilde{K} \setminus K, \Omega^2) \to H^2_c(\tilde{K}, \Omega^2) = 0 \to \cdots,$$

it follows that

$$H^2_c(\tilde{K} \setminus K, \Omega^2) = H^2_c(\tilde{K} \setminus K, \Omega^2) = 0.$$

In this connection let us recall that, by a result of Greene and Wu [10], every noncompact (connected) complex-analytic manifold $\mathcal{M}$ of dimension $r \geq 1$ is $(r - 1)$-complete, and hence $H''(\mathcal{M}, \mathcal{G}) = 0$, for every coherent analytic sheaf, $\mathcal{G}$, on $\mathcal{M}$. Consequently, an inductive limit consideration gives that also $H''(\mathcal{E}, \mathcal{G}) = 0$, for every subset $\mathcal{E} \subset \mathcal{M}$, which is the reason why $H^2_c(\tilde{K}, \Omega^2) = 0$.

Now, the vanishing of $H^2_c(X \setminus Y, \Omega^2)$ is equivalent to having $h_{\mathcal{O}(X)}(Y) = X$ (see [19, Theorem 4]), and since $Y$ is Runge in $X$, so that $h_{\mathcal{O}(X)}(Y) = Y$, the latter property just amounts to saying that $Y = X$, i.e., $\tilde{K} = K$.

**Proposition 7.** The set $\tilde{K}$ is a Stein compactum.

**Proof.** Let $C$ be a compact neighborhood of $K$ in $bD$, and consider the set $h_{\mathcal{O}(\tilde{D})}(bD \setminus C)$. This is a relatively open subset of $\tilde{D}$, as follows from the final part of the proof of Proposition 5, taking in it $C$ in place of $K$. Hence the set $h_{\mathcal{O}(\tilde{D})}(\tilde{K}) \setminus h_{\mathcal{O}(\tilde{D})}(bD \setminus C)$ is compact. Since $\tilde{K}$ can be obtained as the intersection of a decreasing sequence of sets like this, it suffices to prove that $h_{\mathcal{O}(\tilde{D})}(\tilde{K}) \setminus h_{\mathcal{O}(\tilde{D})}(bD \setminus C)$ is a Stein compactum. Indeed, since the set $h_{\mathcal{O}(\tilde{D})}(\tilde{K})$ is a Stein compactum, it admits a neighborhood basis $\mathcal{V}$ of relatively compact Stein open sets, and since $bD \cap h_{\mathcal{O}(\tilde{D})}(\tilde{K}) = K$, we can choose $\mathcal{V}$ such that $bD \cap V \subset C$, for each $V \in \mathcal{V}$. Moreover let us fix an exhausting family $\mathcal{G}$ of compact subsets of $bD \setminus C$. Given $G \in \mathcal{G}$, we can find a Stein open neighborhood $X$ of $G$, such that $h_{\mathcal{O}(\tilde{D})}(G) = h_{\mathcal{O}(X)}(G)$. Then, by resorting again to (2.2), we infer that, for every $V \in \mathcal{V}$ and $G \in \mathcal{G}$, the open set $V \setminus h_{\mathcal{O}(\tilde{D})}(G)$ is Stein. Since

$$h_{\mathcal{O}(\tilde{D})}(K) \setminus h_{\mathcal{O}(\tilde{D})}(bD \setminus C) = \bigcap_{G \in \mathcal{G}} \bigcap_{V \in \mathcal{V}} [V \setminus h_{\mathcal{O}(\tilde{D})}(G)],$$

we reach the desired conclusion.

**Remark.** Proposition 5 and Proposition 6 are also valid in the setting of a Stein manifold $M'$ of dimension $r \geq 2$ as the ambient space, rather than $M^2$, as a direct inspection of the corresponding proofs shows, in view of the remarks at the end of Section 2 and Section 3 too. Actually, as regards the $r$-dimensional extension of Proposition 6, the assumption that $H^{r-1}(K, \mathcal{F}) = 0$ is sufficient to imply that $\tilde{K} = K$. On the contrary Proposition 7 becomes false for $r \geq 3,
as will be seen in Section 6. On account of the result of [16], it is probably true that, also for $r \geq 3$, $H^{r-1}(\tilde{K}, \mathcal{F}) = 0$; however, as the parallel property of $M^r \setminus h^\infty_{\mathcal{O}(\mathcal{D})}(\tilde{D})$, this does not seem to be a relevant information for our purposes.

5. – Proof of Theorem 3

We may limit ourselves to deal with the case that the domain $D$ is relatively compact. Indeed, if $D$ is not relatively compact, given any compact subset $\tilde{K}$ of $bD$, one can, by the procedure of [24], construct a Stein open set $D'$ with $C^2$ boundary, which is the disjoint union of finitely many relatively compact strongly pseudoconvex domains, such that $bD'$ contains a neighborhood, in $bD$, of $K$, and $D'$ is $O(D)$-convex. Then, clearly, it suffices to prove Theorem 3 for any connected component of $D'$.

The following proposition is the essential point of the proof.

PROPOSITION 8. Let $D \subset \subset M^2$ be a $C^2$-bounded strongly pseudoconvex domain and $S \subset bD$ a topological 2-sphere of class $C^2$. Then, if $\omega$ is a holomorphic 2-form defined on a neighborhood of $S$, it follows that

$$\int_S \omega = 0.$$

In other words, the holomorphic de Rham cohomology $H^2_{\text{hol}}(S) = \frac{\Omega^2(S)}{d\Omega^2(S)} = 0$.

PROOF. Let $U$ be an open neighborhood of $S$ such that $\omega \in \Omega^2(U)$. By applying to $bD$ a standard smoothing result for manifolds of class $C^r$ ($1 \leq r$) imbedded in manifolds of class $C^\infty$ (see [22, Theorem 4.8]), we can find a $C^\infty$-bounded strongly pseudoconvex domain $D_1$, with $bD_1$ being $C^2$ diffeomorphic and $C^2$ isotropically equivalent to $bD$, and so close to $bD$ that the diffeomorphic image of $S$, $S_1$, say, is contained in $U$. Moreover, we may assume that $S_1$ is generically imbedded in $M^2$, so that it has only finitely many complex tangencies, all of which are either elliptic or hyperbolic. Then, we can apply to $S_1$ the result of Bedford and Klingenberg [4, Theorem 1] (3), according to which there is a small $C^2$ perturbation $S'_1$, of $S_1$ on $bD_1$, which has, in particular, the following property: there is a smooth 3-manifold $B'$ in $D_1$, such that $B' \cup S'_1 = \overline{B} = E(S'_1)$. Then, it follows that the form $\omega$ extends to a holomorphic

(3) Note of the editor. The author considers an arbitrary two-dimensional Stein manifold $M$. It is to be observed that for the validity of Proposition 8, $M$ should equal $C^2$. Proposition 8 is based on the Bedford and Klingenberg theorem that is proved in fact only for $C^2$. 

GUIDO LUPOCCIOLOU
form $\tilde{\omega}$ on a neighborhood of $\overline{B'}$, and hence, by Stokes's theorem,

$$\int_S \omega = \int_{S_1} \omega = \int_{S'_1} \omega = \int_{\overline{B}} d\tilde{\omega} = 0. \quad \Box$$

Now we can prove:

**PROPOSITION 9.** Let $D$ be as in the preceding proposition and let $K$ be a compact subset of $\partial D$. Assume that $K$ has a neighborhood basis $\mathcal{N}$ in $\partial D$, such that each $N \in \mathcal{N}$ is a relatively compact open subset of $\partial D$ (possibly disconnected), whose boundary $\partial N$ is the union of finitely many pairwise disjoint topological 2-spheres of class $C^2$. Put $\Gamma = \partial D \setminus K$. Then, if $f$ is any continuous CR-function on $\Gamma$, it follows that $\int_\Gamma f \alpha = 0$, for every $C^\infty$ $\partial$-closed $(2, 1)$-form $\alpha$ on a neighborhood of $\overline{D}$ such that $\text{supp}(\alpha) \cap K = \emptyset$. Consequently, $E(K) = h_{\mathcal{O}(\overline{D})}(K)$.

**PROOF.** Since $D$ is a Stein compactum, there exists a $C^\infty$ $(2,0)$-form $\beta$ on a neighborhood of $\overline{D}$ such that $\alpha = \overline{\partial} \beta = d\beta$, and since $\text{supp}(\alpha) \cap K = \emptyset$, there exists a neighborhood $U$ of $K$ in $\mathcal{M}^2$ such $\overline{\partial} \beta = 0$ on $U$, i.e., $\beta$ is a holomorphic 2-form on $U$. By assumption there exists $N \in \mathcal{N}$ such that $\overline{N} \subset U$, and hence, on account of Proposition 8, it is readily seen that $\int_{bN} f \beta = 0$. Then, by Stokes's theorem, we have

$$\int_{bD} f \alpha = \int_{bD} f d\beta = \int_{bD \setminus N} f d\beta = -\int_{bN} f \beta = 0.$$

It follows, in view of (2.7), that $E(\Gamma) = h_{\mathcal{O}(\overline{D})}(\Gamma)$, and hence, in view of (2.8), we achieve the desired conclusion. $\Box$

**REMARKS.** (i) In connection with the assumption of Theorem 3, we point out that, if $\mathcal{M}$ is an orientable topological 3-manifold with boundary, such that $H_1(\mathcal{M}, \mathbb{Z}) = 0$, then it follows that the boundary $\partial \mathcal{M}$ of $\mathcal{M}$ is a union of topological 2-spheres. Indeed, the vanishing of the homology group $H_1(\mathcal{M}, \mathbb{Z})$ implies that also the cohomology group $H^1(\mathcal{M}, \mathbb{Z})$ is null (recall that $H^q(\cdot, \mathbb{Z})$ is isomorphic to $\text{Hom}_\mathbb{Z}(H_q(\cdot, \mathbb{Z}), \mathbb{Z})$, provided $H_{q-1}(\cdot, \mathbb{Z})$ is a free $\mathbb{Z}$-module). Then, by the Poincaré duality for compact manifolds with boundary (see [8, Proposition 9.1]), also the relative homology group $H_2(\mathcal{M}, \partial \mathcal{M}; \mathbb{Z})$ is null. By the exact sequence of relative homology

$$\cdots \to H_2(\mathcal{M}, \partial \mathcal{M}; \mathbb{Z}) \to H_1(b \mathcal{M}, \mathbb{Z}) \to H_1(\mathcal{M}, \mathbb{Z}) \to \cdots,$$

it follows that $H_1(b \mathcal{M}, \mathbb{Z}) = 0$. This implies first that the connected components of $b \mathcal{M}$ are orientable (see [8, Proposition 2.12]) and then, being orientable compact surfaces of genus zero, that these connected components are topological 2-spheres.
It is simple to show that Theorem 3 does not extend to higher dimensions. Consider in \( \mathbb{C}^r \) for \( r \geq 3 \), the open unit ball \( B \) and in \( S^{2r-1} = bB \) the two disjoint closed semi-2-spheres

\[
\Sigma_1^2 = \{ z \in S^{2r-1} : |z_1|^2 + (i z_2)^2 = 1, \ \Re z_2 \geq 0, \ \Im z_2 = 0, z_3 = \cdots = z_r = 0 \},
\]

\[
\Sigma_2^2 = \{ z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \ \Re z_{r-1} = 0, \ \Im z_{r-1} \geq 0, (\Im z_{r-1})^2 + |z_r|^2 = 1 \},
\]

and put \( K = \Sigma_1^2 \cup \Sigma_2^2 \). It is evident that \( K \) verifies the assumption of Theorem 3. On the other hand, since the intersection \( h_{\mathcal{O}(bB)}(\Sigma_1^2) \cap h_{\mathcal{O}(bB)}(\Sigma_2^2) \) is nonempty, as it contains at least the origin, it is trivially not true that every \( f \in \mathcal{O}(K) \) may have a holomorphic extension to a neighborhood of \( h_{\mathcal{O}(bB)}(K) \). In the preceding counterexample \( K \) is disconnected, but this does not affect its validity, since also in Theorem 3 \( K \) is allowed to be disconnected. On the other hand in Section 6 we shall be able to show a less trivial counterexample in which \( K \) is connected.

6. – Non-extendability to higher dimensions

In the first place we state the weaker extension theorems that generalize Theorem 1 and Theorem 2 to the setting of a Stein manifold \( M' \) of dimension \( r \geq 2 \), rather than \( r = 2 \). In view of the remarks at the ends of Section 2, Section 3 and Section 4, we have:

**Theorem 4.** Let \( D \subset M' \) be an open domain of holomorphy, whose boundary \( bD \) is a real hypersurface of class \( C^2 \), strictly Levi-convex with respect to \( D \). Put \( \Omega = M' \setminus \overline{D} \). Then the three sets \( M' \setminus h_{\mathcal{O}D}^\infty(\overline{D}), \Omega \cup [h_{\mathcal{O}D}(bD) \setminus h_{\mathcal{O}D}^\infty(bD)] \) and \( h_{\mathcal{O}M'^2}(\Omega) \setminus h_{\mathcal{O}D}^\infty(bD) \) are a same open subset of \( M' \), \( \tilde{\Omega} \), say, such that the restriction map \( \mathcal{O}(\tilde{\Omega}) \rightarrow \mathcal{O}(\Omega) \) is bijective.

**Theorem 5.** Let \( D \) be as in Theorem 4. Let \( K \) be an arbitrary compact subset of \( bD \), and put \( \tilde{K} = h_{\mathcal{O}D}(bD) \setminus h_{\mathcal{O}D}(bD \setminus K) \). Then \( \tilde{K} \) is a compact set containing \( K \), such that the restriction map \( \mathcal{O}(\tilde{K}) \rightarrow \mathcal{O}(K) \) is bijective. Consequently, \( \tilde{K} \) is also equal to the set \( h_{\mathcal{O}D}(K) \setminus h_{\mathcal{O}D}(bD \setminus K) \).

Furthermore, if \( H^{r-1}(K, \mathcal{F}) = 0 \), for every coherent analytic sheaf \( \mathcal{F} \), on \( K \), then \( \tilde{K} = K \).

Now we wish to show that for \( r \geq 3 \) the open set \( \tilde{\Omega} \) of Theorem 4 may not be Stein, as well as the compact set \( \tilde{K} \) of Theorem 5 may not be a Stein compactum.

Preliminarily, consider a \( C^2 \)-bounded strongly pseudoconvex domain \( \mathcal{D} \subset \subset \mathbb{C}^r \) and a compact set \( \mathcal{K} \subset b\mathcal{D} \). Let us push \( b\mathcal{D} \) away from \( \mathcal{D} \) by a small \( C^2 \)
perturbation which leaves $\mathcal{R}$ fixed pointwise, so as to obtain a Stein domain, call it $M'$, with $\overline{D} \setminus \mathcal{R} \subset M'$ and $bM' \cap \overline{D} = \mathcal{R}$. We may consider $\mathcal{D}$ as an unbounded open domain of holomorphy in the Stein manifold $M'$. Then we change the notations, so that $D$ denotes the domain $\mathcal{D}$ when it is regarded as a domain in $M'$ rather than in $\mathbb{C}^r$, whereas $bD$ and $\overline{D}$ denote the boundary and the closure of $D$ in $M'$. Then $b\mathcal{D} = bD \cup \mathcal{R}$ and $\overline{\mathcal{D}} = \overline{D} \cup \mathcal{R}$. We claim that

\[(6.1) \quad h_{\mathcal{O}(\overline{D})}(\overline{D}) = h_{\mathcal{O}(\overline{D})}(\mathcal{R}).\]

As a matter of fact, consider the open sets $D_n$, $n \in \mathbb{N}$ defined at the beginning of Section 2. It is evident that, for each $n \in \mathbb{N}$, $h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus D_n) \subset h_{\mathcal{O}(\overline{D})}(\overline{\mathcal{D}} \setminus \overline{D}_n)$, whereas the local maximum modulus principle implies that $h_{\mathcal{O}(\overline{D})}(\mathcal{R}) \cap \overline{D}_n \subset h_{\mathcal{O}(\overline{D})}(bD_n \cap h_{\mathcal{O}(\overline{D})}(\mathcal{R})) \subset h_{\mathcal{O}(\overline{D})}(\overline{D} \setminus \overline{D}_n)$. Hence, making the intersections for all $n \in \mathbb{N}$ gives the two inclusions $h_{\mathcal{O}(\overline{D})}(\overline{D}) \subset h_{\mathcal{O}(\overline{D})}(\mathcal{R})$ and $h_{\mathcal{O}(\overline{D})}(\mathcal{R}) \setminus \mathcal{R} \subset h_{\mathcal{O}(\overline{D})}(\overline{D})$ and (6.1) follows at once.

That being stated, to produce an example, for $r \geq 3$, in which $\tilde{\Omega} = M' \setminus h_{\mathcal{O}(\overline{D})}(\overline{D})$ is not Stein, it suffices to consider the preceding construction, taking as $\mathcal{R}$ the intersection of $b\mathcal{D}$ with any complex-analytic subvariety $V$ of $\mathbb{C}^r$, of codimension $q$ in the range $2 \leq q \leq r - 1$, passing through $D$: since in this case $h_{\mathcal{O}(\overline{D})}(\mathcal{R}) = \mathcal{R} \cup (V \cap \mathcal{D})$, it follows, in view of (6.1), that $\tilde{\Omega} = M' \setminus V$, which is not a Stein manifold. Moreover, if we choose a suitably small open neighborhood $U$ of the variety $V$, also the interior of $M' \setminus U$ is not Stein, and hence the compact set $\overline{D} \setminus U$ is not a Stein compactum. We can take as $U$ a Stein open neighborhood of $V$ which is Runge in $\mathbb{C}^r$, so as to have $U \cap \overline{D} = h_{\mathcal{O}(\overline{D})}(U \cap b\mathcal{D})$. Then the compact set $K = bD \setminus U$ verifies $\overline{K} = \overline{D} \setminus U$, thus providing an example, for $r \geq 3$, of a compact set $K \subset bD$ such that $\overline{K}$ is not a Stein compactum.

Next we show that for $r \geq 3$ the envelope of holomorphy of $\Omega$ (which, by Theorem 3, coincides with the envelope of holomorphy of $\tilde{\Omega}$) may be multi-sheeted. Indeed Chirka and Stout [7, 4.5] exhibited a $C^\infty$-bounded strongly pseudoconvex domain $\mathcal{D} \subset \subset \mathbb{C}^{2m}$, $m \geq 2$, a compact set $\mathcal{R} \subset b\mathcal{D}$ (with $b\mathcal{D} \setminus \mathcal{R}$ being connected) and a function $f \in \mathcal{O}(\overline{D}) \setminus h_{\mathcal{O}(\overline{D})}(\mathcal{R})$ in such a way that $f$ can be continued analytically in the sense of Weierstrass to the whole $\overline{D} \setminus \mathcal{R}$, so as to give rise to two different determinations at each point of $h_{\mathcal{O}(\overline{D})}(\mathcal{R}) \setminus \mathcal{R}$. Therefore, by applying in this case the procedure described above, we can obtain at once, on account of (6.1), a counterexample to $E(\Omega)$ being single-sheeted, valid for all even dimensions $\geq 4$. On the other hand, we can obtain also a counterexample valid for all dimensions $\geq 3$, rather than only for the even ones, by modifying in a suitable manner the construction of Chirka and Stout. For the convenience of the reader we give a complete description of the modified construction, parallel to the description of the original construction given in [7, 4.5]. Consider in $\mathbb{C}^r$, $r \geq 3$, the open unit ball $\mathbb{B}$, and in $S^{2r-1} = b\mathbb{B}$ the two
disjoint closed 2-spheres

\[ S_1^2 = \{ z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 = 1, \Im z_2 = 0, z_3 = \cdots = z_r = 0 \}, \]
\[ S_2^2 = \{ z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \Re z_{r-1} = 0, (\Im z_{r-1})^2 + |z_r|^2 = 1 \}. \]

Let \( \Gamma_1 \) and \( \Gamma_2 \) be connected open neighborhoods, in \( S^{2r-1} \), of \( S_1^2 \) and \( S_2^2 \), respectively, such that \( \Gamma_1 \cap \Gamma_2 = \emptyset \), and put \( \mathcal{R} = S^{2r-1} \setminus (\Gamma_1 \cup \Gamma_2) \). Then let \( \gamma \) be a smooth arc in \((\mathbb{C}' \setminus \mathcal{B}) \cup \{(1, 0, \ldots, 0), (0, \ldots, 0, 1)\}\), which connects the points \((1, 0, \ldots, 0)\) and \((0, \ldots, 0, 1)\), is orthogonal to \( b\mathcal{B} \) at these points, and verifies the following two conditions: a) if \( \phi \) is the function on \( \mathbb{C}' \) given by \( \phi(z_1, \ldots, z_r) = z_1 - z_r \), then \( \gamma_1 = \phi(\gamma) \) is a smooth arc in the upper half plane \( \Pi \subset \mathbb{C} \), which connects the points 1 and \(-1\); b) the point \( 2i \) belongs to the relatively compact component of \( \mathcal{R} \). Since \( |z_1 - z_r| \leq 1 \) on \( S_1^2 \cup S_2^2 \), we may assume that \( \Gamma_1 \) and \( \Gamma_2 \) have been chosen so small that \( |z_1 - z_r| < 2 \) on \( \Gamma_1 \cup \Gamma_2 \). Hence we can define a continuous argument of \( z_1 - z_r - 2i \) on \( \Gamma_1 \cup \Gamma_2 \cup \gamma \) which takes values in the interval \((-\pi, 0)\) on \( \Gamma_1 \) and takes values in the interval \((\pi, 2\pi)\) on \( \Gamma_2 \). Consequently, the function \( f \) defined by

\[ (6.2) \quad f(z_1, \ldots, z_r) = (z_1 - z_r - 2i) \frac{1}{2} = \sqrt{|z_1 - z_r - 2i|^2}, \]

with the above mentioned argument function, is holomorphic on a neighborhood of \( \Gamma_1 \cup \Gamma_2 \cup \gamma \), and \( \Im f < 0 \) on \( \Gamma_1 \), \( \Im f > 0 \) on \( \Gamma_2 \). The envelope of holomorphy of \( \Gamma_1 \) contains the compact 3-ball \( \overline{B}_1^3 = \{ z \in S^{2r-1} : |z_1|^2 + (\Re z_2)^2 \leq 1, \Im z_2 = 0, z_3 = \cdots = z_r = 0 \} \), and the envelope of holomorphy of \( \Gamma_2 \) contains the compact 3-ball \( \overline{B}_2^3 = \{ z \in S^{2r-1} : z_1 = \cdots = z_{r-2} = 0, \Re z_{r-1} = 0, (\Im z_{r-1})^2 + |z_r|^2 \leq 1 \} \).

Therefore the function \( f \) extends holomorphically into a neighborhood of \( \overline{B}_1^3 \) and into a neighborhood of \( \overline{B}_2^3 \) with different values: \( \Im f < 0 \) on \( \overline{B}_1^3 \) and \( \Im f > 0 \) on \( \overline{B}_2^3 \). It follows that the domain of holomorphy of \( f \) has two different sheets at least on a neighborhood of \( \overline{B}_1^3 \cap \overline{B}_2^3 \). Finally, given a small neighborhood \( V \) of \( \gamma \) in \( \mathbb{C}' \), such that the above function \( f \) is holomorphic on \( V \) and \( V \cap \mathcal{R} = \emptyset \), consider a \( \mathcal{C}^\infty \)-bounded strongly pseudoconvex domain \( \mathcal{D} \) with \( \mathcal{B} \cup \gamma \subset \mathcal{D} \subset \mathbb{C} \cup V \). Then \( \mathcal{R} \) is a compact subset of \( b\mathcal{D} \) such that \( b\mathcal{D} \setminus \mathcal{R} \) is connected and \( f \) is a function holomorphic on a neighborhood of \( b\mathcal{D} \setminus \mathcal{R} \) whose domain of holomorphy is not single-sheeted. It follows that, by applying in this case the procedure described above, we can obtain, on account of (6.1), a counterexample to \( \Omega \) being single-sheeted which is valid for all dimensions \( \geq 3 \).

We conclude the paper by providing a counterexample to the possibility of extending Theorem 3 to higher dimensions, in which, unlike in the final remark of Section 5, the compact set \( K \) is connected. By perturbing slightly the arc \( \gamma \) of the preceding construction, we can find a smooth arc \( \gamma' \subset \mathbb{C}' \), contained in a neighborhood of \( \Gamma_1 \cup \Gamma_2 \cup \gamma \) where the function \( f \) of (6.2) is single-valued, which connects two points \( p_1, p_2 \in b\mathcal{B} \setminus (S_1^2 \cup S_2^2) \), close to \((1, 0, \ldots, 0), (0, \ldots, 0, 1)\), respectively, which is orthogonal to \( b\mathcal{B} \) at these points and is
ENVELOPES OF HOLOMORPHY IN $\mathbb{C}^2$

contained in $(\mathbb{C}^r \setminus \overline{B}) \supset \{p_1, p_2\}$. Then, given a small neighborhood $V'$ of $\gamma'$ in $\mathbb{C}^r$, such that $f$ is holomorphic on $V'$ and $V' \cap (S_1^2 \cup S_2^2) = \emptyset$, consider a $C^\infty$-bounded strongly pseudoconvex domain $D$ with $B \cup \gamma' \subset D \subset B \cup V'$, analogous to the preceding domain $D$. Then $S_1^2 \cup S_2^2 \subset bD$, and we can find a smooth arc $\gamma'' \subset \partial D \setminus (S_1^2 \cup S_2^2) \cup \{(1,0,\ldots,0), (0,\ldots,0,1)\}$, joining the points $(1,0,\ldots,0)$ and $(0,\ldots,0,1)$, such that $f$ is single-valued on a neighborhood of $S_2^2 \cup \gamma''$. Now consider again the two closed semi-2-spheres of the final remark of Section 5, and put $K = \Sigma_1^2 \cup \Sigma_2^2 \cup \gamma''$. It is evident that $K$ verifies the assumption of Theorem 3 and is connected; however the domain of holomorphy of the function $f$ is not single-sheeted over a neighborhood of the origin, and consequently $E(K) \neq h_{\partial(D)}(K)$.

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