Ziv Ran

Semiregularity, obstructions and deformations of Hodge classes


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Abstract. We show that the deformation theory of a pair \((X, \eta)\), where \(X\) is a compact Kähler manifold and \(\eta\) is a \(\(p, p\)\) class on \(X\), is controlled by a certain sheaf \(\ell_{\eta}\) of differential graded Lie algebra on \(X\); consequently, we show that relative obstructions to deforming a pair \((X, Y)\), where \(Y\) is a codimension-\(p\) submanifold of \(X\), relative to deforming \(X\) so that the fundamental class of \(Y\) remains of type \(\(p, p\)\), (in particular, deformations of \(Y\) fixing \(X\)) lie in the kernel of the semiregularity map \(\pi_1 : H^1(N_{Y/X}) \to H^{p+1, p-1}(X)\) of Bloch et al. We also give a number of extensions and applications of this result.


To a codimension-\(p\) embedding \(Y \subset X\) of compact complex manifolds one may associate at least 3 deformation problems: deforming \(Y\), fixing \(X\) (the local Hilbert scheme); deforming the pair \((Y, X)\); deforming \(X\) so that the cohomology class \(\eta = [Y] \in H^{2p}(X)\) maintains a given Hodge level \(q \leq p\). These problems- obviously interrelated- are all influenced by Hodge theory, via the so-called semiregularity map (which has antecedents in Severi and was more recently considered by Kodaira-Spencer, Mumford, Bloch . . . )

\[
\pi_1 : H^1(N) \to H^{p+1, p-1}(X), \quad N = \text{normal bundle}.
\]

Roughly speaking obstructions which a priori lie in \(H^1(N)\) are actually in \(\ker(\pi_1)\). Thus, e.g. a lower bound on the rank of \(\pi_1\) yields estimates on the dimension of deformation spaces etc. The precise statement is as follows.

Theorem 0. Let \(X\) be a compact complex manifold and \(Y \subset X\) a connected submanifold of codimension \(p\), with normal bundle \(N\), and fundamental class \(\eta = [Y] \in H^p(\Omega^p_X)\). Let \(\pi_1 : H^1(N) \to H^{p+1}(\Omega^{p-1}_X)\) be the semi-regularity map (reviewed below). Then
(i) obstructions to deforming \( Y \) in \( X \) lie in \( \ker \pi_1 \);
(ii) if moreover \( X \) is Kählerian then obstructions to deforming the pair \((X, Y)\), relative to deforming \( X \) so that \( \eta \in H^{2p}(X) \) remains of type \((p, p)\), lie in \( \ker \pi_1 \).

[In more detail, (ii) means: given an artin local \( \mathbb{C} \)-algebra \((S, m)\), an ideal \( I \subseteq S \) with \( mI = 0 \), a deformation \( \alpha \) of \((X, Y)\) over \( S/I \), a deformation \( \alpha' \) of \( X \) over \( S \), which induces the same deformation as \( \alpha \) over \( S/I \) and in which the (Gauss-Manin) flat lift of \( \eta \) has Hodge level \( p \), obstructions to lifting \( \alpha \) over \( S \) lie in \( \ker(\pi_1) \otimes I \).]

Theorem 0 was in essence proven by Bloch [B] for the case of deformations over an artin ring of the form \( \mathbb{C}[\varepsilon]/(\varepsilon^n) \), however neither the result nor the proof yield the general artin local case. In the present generality Theorem 0 was first stated in [RO] where the argument was based on the notion of “canonical element” controlling a deformation (see [R3] for a development of the theory and required properties of canonical elements).

The main purpose of this paper is to develop some methods pertaining to the interplay of “canonical” or “Lie-theoretic” deformation theory and Hodge theory and apply them to a proof of Theorem 0; the proof of part (i) in particular is short and essentially self-contained. A central role in these methods is played by a certain differential graded Lie algebra \( \mathcal{L} = \mathcal{L}_\eta \) which, as we prove with the method of [R3], controls deformations of \( X \) in which a given class \( \eta \) maintains a given Hodge level. Modulo this fact (which moreover is unnecessary for part (i)), the proof of Theorem 0 is quite simple and conceptual: indeed it boils down to constructing a Lie homomorphism \( \pi : N[-1] \to \mathcal{L}_\eta \) (“sheaf-theoretic semiregularity”) and realizing \( \pi_1 \) as the cohomology map induced by \( \pi \). Following the proof we present some applications to deformations of maps and integral curves on K3 surfaces. See [RO] for other applications.

As our foundational reference for (Lie algebra-controlled) deformation theory, we shall use [R2], [R3]; however for the proof of part (i) (which is already sufficient for most applications), essentially any reference, e.g. [GM], will do.

**Proof of Theorem.** Let \( T = T_X \) and \( T' \subset T \) be the subsheaf of vector fields tangent to \( Y \) along \( Y \), i.e. preserving the ideal sheaf \( \mathcal{I}_Y \). We identify the normal sheaf \( N \) with the complex in degrees \(-1, 0\)

\[
T' \xrightarrow{id} T
\]

and endow \( N[-1] \) with a structure of DGLA sheaf given by

\[
[a] : T' \times T' \to T', \quad \frac{1}{2}[a] : T' \times T \to T.
\]

(the \( 1/2 \) factor is needed to make \( id \) a Lie derivation). We thus have an exact triangle of DGLA’s

\[
N[-1] \to T' \to T \to
\]
(i.e. $N[-1]$ is a Lie ideal in $T'$), and these control, respectively, the deformations of $Y$ fixing $X$, of the pair $(X, Y)$, of $X$ (see e.g. [R2], [R3] for more details on this).

On the other hand, to any class $\eta \in H^q(\Omega^p)$ we may associate a DGLA $\mathcal{L} = \mathcal{L}_\eta$ as follows:

$$\mathcal{L}^{-q} = \Omega^{p-1} \to \mathcal{L}^0 = T,$$

differential = interior multiplication by $\eta$, bracket = usual one on $T$, Lie derivative $T \times \Omega^{p-1} \to \Omega^{p-1}$, zero otherwise.

More concretely, we may represent $\mathcal{L}^0$ (resp. $\mathcal{L}^{-q}$) by the Cech complex of $T$ (resp. of $\Omega^{p-1}$ shifted $q$ places to the left). Thus we have an exact triangle of DGLA's

$$\Omega^{p-1}[q - 1] \to \mathcal{L} \to T \to$$

with $\Omega^{p-1}[q - 1]$ an abelian ideal in $\mathcal{L}$.

By the local cohomology description of $\eta = [Y]$ given, e.g. in [B] it follows directly that interior multiplication by $\eta$ vanishes (in the derived category) on $T' \subset T$, and consequently we have a commutative diagram of exact triangles

$$\begin{array}{cccc}
N[-1] & \to & T' & \to & T \\
\pi \downarrow & & \pi' \downarrow & & \parallel \\
\Omega^{p-1}[q - 1] & \to & \mathcal{L} & \to & T \\
\end{array}
$$

$\pi_1 = H^2(\pi)$ may be taken as the definition of $\pi_1$ but it is immediate that this definition coincides with the one given in in [B]. It may be noted that $H^1(\pi)$ is none other than the infinitesimal Abel-Jacobi map associated to $Y$. Now we shall prove below that, for $X$ Kählerian, $\mathcal{L}$ controls precisely the deformations of $X$ where $\eta$ remains of type $(p, p)$. Given this, the Theorem follows immediately from (1): indeed by any general theory (e.g. [GM], [R2]), obstructions are induced by Lie bracket and lie in $H^2$ of the controlling Lie algebra and thus relative obstructions as in the Theorem lie in $\ker H^2(\pi') = \ker(\pi_1)$. \( \Box \)

Note that for the purpose of part (i) the interpretation of $\mathcal{L}$ is irrelevant, so this part does not require the Kählerian hypothesis (nor for that matter any of the rest of the paper).

It remains to establish the deformation-theoretic significance of $\mathcal{L}$. Precisely, we will show the $\mathcal{L}$ controls deformations $\hat{X}/S$ plus Čech cochains

$$\omega \in \check{C}^q(\Omega^{p-1}_{\hat{X}/S}),$$

$$\delta(\omega) + \tilde{\eta} \in \check{C}^q(F^p\Omega_{\hat{X}/S}),$$

where $\tilde{\eta} = \text{constant lift of } \eta$, modulo coboundaries $\omega = \delta(\tau)$.

To this end we first review the universal variation of Hodge structure associated to $X$, as developed in [R3], which will provide us with an explicit representative for the GM-constant lift of a cohomology class on $X$. Consider
the following double complex $J_m(T, \Omega)$ on $X < m \times X$ in bidegrees $[0, n] \times [-m, 0]$:

\[
\begin{array}{cccccc}
\mathcal{O} & \rightarrow & \ldots & \rightarrow & \Omega^p & \rightarrow & \ldots & \rightarrow & \Omega^n \\
\uparrow & & & & \uparrow & & & & \\
\rightarrow & T \times \Omega^p & \rightarrow & \ldots & & & & \\
\uparrow & & & & & & & & \\
\vdots & & & & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccc}
\lambda^m T \otimes \mathcal{O} & \rightarrow & \ldots & \rightarrow & \lambda^m T \otimes \Omega^p & \rightarrow & \ldots & \rightarrow & \lambda^m T \otimes \Omega^n \\
\end{array}
\]

with horizontal arrows induced by exterior derivative and vertical arrows of the form

\[
v_1 \times \ldots \times v_k \times \omega \mapsto \pm \sum_{i=1}^k v_1 \times \ldots \times \hat{v}_i \times \ldots \times v_k \times L_{v_j}(\omega)
\]

\[
\pm \sum_{i<j} (-1)^{i+j} v_1 \times \ldots \times \hat{v}_i \\
\times \ldots \times \hat{v}_j \times \ldots \times v_k \times [v_i, v_j] \times \omega
\]

(i.e. $J_m(T, \Omega)$ is just the standard complex for $\Omega^*$ as $T$-module, with variables separated.) As explained in [R3], the De Rham cohomology $H_{DR}(X_m/R_m)$ of the universal $m$-th order deformation $X_m/R_m$ of $X$, together with its Hodge filtration (i.e. the universal $m$-th order VHS associated to $X$) is obtained by applying a pure linear algebra construction to a suitable Kunneth component $H^0_0(J_m(T, \Omega^*))$ of the cohomology of $J_m(T, \Omega)$, (i.e. the one mapping to $(H^0(J_m(T))) \oplus \mathcal{C}) \otimes H^r(X)$ under the quasi-isomorphism $\mathcal{C} \rightarrow \Omega^*$), so one might as well work with the latter group directly. Thanks to Cartan’s formula for Lie derivative, the complex $J_m(T, \Omega^*)$ is “split”, i.e. isomorphic to the complex with the same entries and trivial action of $T$ on $\Omega^*$, the isomorphism in question being assembled from $\pm$ interior multiplication maps

\[
M_{i,k,p} : \lambda^k T \otimes \Omega^p \rightarrow \lambda^{k-i} T \otimes \Omega^{p-i}, \quad i \geq 0
\]

\[
v_1 \times \ldots \times v_k \times \omega \mapsto \sum \pm v_1 \times \ldots \times \hat{v}_j \times \ldots \times \hat{v}_j \times \ldots \times v_k \\
\times i(v_{j1} \wedge \ldots \wedge v_{jk})(\omega) \\
\mapsto 0, \quad i > \min(k, p)
\]

The induced map on cohomology is the Gauss-Manin isomorphism

\[
G : (\mathcal{C} \oplus H^0(J_m(T))) \otimes H^r(X) \rightarrow H^0_0(J_m(T, \Omega^*)).
\]
The “constant lift” of a class \( \eta \in H^r(X) \) is simply the map

\[
G_\eta : C \oplus H^0(J_m(T)) \rightarrow H^{0,r}(J_m(T, \Omega))
\]

given by \( G(\cdot \otimes \eta) \). More explicitly on Čech cohomology, \( G_\eta \) on \( H^0(J_m(T)) \) may be described as follows. We may represent an element \( v \in H^0(J_m(T)) \) by \( (v_1, \ldots, v_m) \) where

\[
v_m \in S^m(\tilde{\mathcal{Z}}^1(T)) \subset \tilde{\mathcal{Z}}^m(\lambda^m(T)) \]

\[
v_i \in S^i(\tilde{\mathcal{Z}}^1(T)) \subset \tilde{\mathcal{Z}}^i(\lambda^i(T)), \quad 1 \leq i < m,
\]

\[
\delta(v_i) = \pm b(v_{i+1}), \quad 1 \leq i < m
\]

\[
\delta(v_m) = 0,
\]

\( b \) being the map induced by bracket. On the other hand \( X \) being Kähler \( \eta \in H^{p,q}(X) \) may be represented by a Čech cocycle with values in the sheaf \( \tilde{\mathcal{O}}^p \) of closed \( p \)-forms (which in effect means choosing a lift of \( \eta \) to \( F^p H^{p+q}(X) \)), and \( G_\eta(v) \) may be represented by

\[
\begin{array}{cccc}
p - m & & p & \\
\hline
M_{m,m,p}(v_m \times \eta) & 0 & 0 \\
M_{m-1,m,p}(v_m \times \eta) & v_1 \times \eta & & \\
\vdots & \vdots & \vdots & \\
M_{2,m,p}(v_m \times \eta) & M_{1,m-1,p}(v_{m-1} \times \eta) & v_{m-2} \times \eta & \\
M_{1,m,p}(v_m \times \eta) & v_{m-1} \times \eta & v_m \times \eta & \ldots
\end{array}
\]

We are now in position to consider the obstruction to the constant lift \( G_\eta(v) \) having Hodge level \( p \) (in cohomology). Thus consider what hypercoboundary would push \( G_\eta(v) \) into \( J_m(T, F^p\Omega^*) \), i.e. kill all terms off the \( p \)-th column. Working from the bottom up, starting in position \( (p - 1, -m + 1) \) we require first a cochain

\[
\omega_{m-1} \in S^{m-1}(\tilde{\mathcal{Z}}^1(T)) \otimes \tilde{\mathcal{G}}^q(\Omega^{p-1}),
\]

\[
\delta(\omega_{m-1}) = M_{1,m,p}(v_m \times \eta).
\]

Clearly the latter right-hand side is a cocycle, so the obstruction to \( \omega_{m-1} \) existing is in \( S^{m-1}(H^1(T)) \otimes H^{q+1}(\Omega^{p-1}) \). Note that once \( \omega_{m-1} \) exists, we have

\[
\delta(M_{m,m-1,p-1}(\omega_{m-1})) = M_{k+1,m,p}(v_m \times \eta),
\]
all other terms along the bottom diagonal, i.e. in position \((p - k - 1, -m + k + 1), k = 0, \ldots, m - 1\), can be killed too. Next, to kill off the term in position \((p - 1, -m + 2)\) requires a cochain

\[
\begin{align*}
\omega_{m-2} &\in S^{m-2}(\tilde{Z}^1(T)) \otimes C^q(\Omega^{p-1}), \\
\delta(\omega_{m-2}) &= M_{1,m,p}(v_{m-1} \times \eta) + d(M_{1,m-1,p-1}(\omega_{m-1})) + L(\omega_{m-1})
\end{align*}
\]

where \(d\) and \(L\) denote the horizontal and vertical differentials in the complex \(J_m(T, \Omega^*)\). Again it is easy to see the latter right-hand side is a cocycle. So the obstruction to \(\omega_{m-2}\) existing (provided \(\omega_{m-1}\) does) is in \(S^{m-2}H^1(T) \otimes H^{q+1}(\Omega^{p-1})\); and again once \(\omega_{m-2}\) exists all elements in the diagonal \(\{(a, b), a + b = p - m + 1, a < p\}\) may be killed too. We continue in this way up to the 0-th row where what is required is

\[
\begin{align*}
\omega_0 &\in C^q(\Omega^{p-1}), \\
\delta(\omega_0) &= M_{1,1,p}(v_1 \times \eta) + d(M_{1,1,p-1}(\omega_1)) + L(\omega_1).
\end{align*}
\]

Turning to the algebra \(\mathcal{L}\) and its deformation theory, we claim that the obstructions are the same as for keeping \(G_q(v)\) of level \(p\), which it suffices to show in the universal situation. For the first-order case this is clear: given \(v \in H^1(T)\), the data required to lift \(v\) to \(H^1(\mathcal{L})\) is precisely a cochain \(\omega_1 \in C^q(\Omega^{p-1})\) with \(\delta(\omega_1) = i(v)(\eta)\), same obstruction as for \(G_q(v)\) to be of level \(p\). Next we turn to the second-order case. The complex \(J_2(\mathcal{L})\) takes the form

\[
\begin{align*}
T &\rightarrow \Omega^{p-1}[q] \\
\uparrow &\quad \uparrow \\
\lambda^2 T &\rightarrow T \otimes \Omega^{p-1}[q] \rightarrow \sigma^2 \Omega^{p-1}[q]
\end{align*}
\]

Given \(v \in H^0(J_2(T))\), the assumption \(G_q(v)\) is of level \(p\) to first order means that writing

\[
v = (v_2, v_1), \ v_i \in S^i(\tilde{Z}^1(T)), \ b(v_2) = \delta(v_1),
\]

we have some \(\omega'_1 \in C^q(\Omega^{p-1}) \otimes \tilde{Z}^1(T)\) with

\[
M_{1,2,p}(v_2 \otimes \eta) = \delta(\omega'_1).
\]

To lift this data to \(H^0(J_2(\mathcal{L}))\) requires precisely

\[
\begin{align*}
\omega'_0 &\in \tilde{C}^q(\Omega^{p-1}), \\
\delta(\omega'_0) &= L(\omega_1) + i(v_1)(\eta).
\end{align*}
\]
In other words, the obstruction is \([L(\omega_1) + i(\nu_1)(\eta)] \in H^{q+1}(\Omega^{p-1})\). On the other hand as we saw above (5) the obstruction to \(G_p(v)\) being of level \(p\) is

\([L(\omega_1) + i(\nu_1)(\eta) + d(M_{1,1,p-1}(\omega_1))] \in H^{q+1}(\Omega^{p-1})\).

Now \(X\) being Kähler, we have \([d(M_{1,1,p-1}(\omega_1))]) = 0\), so the two obstructions coincide.

In the general \(m\)-th order case, the situation is similar: given \(v \in H^0(J_m(T))\) plus data \(\omega_{m-1}, \ldots, \omega_0\) making \(G_p(v)\) of level \(p\), the data required to lift \(v\) to \(H^0(J_m(\mathcal{L}))\) consists of cochains \(\omega'_{m-1}, \ldots, \omega'_0\) with

\[\delta(\omega'_i) = \delta(\omega_i) + (d - \text{exact cocycle}),\]

so again the obstructions are the same. \(\square\)

We conclude with a brief partial treatment of semiregularity for maps, insofar as results follow from the above. Let

\[f: Y \to X\]

be a generically finite map of compact Kähler manifolds of dimensions \(n - p, n\), and let \(\tilde{Y} \subset Y \times X\) be the graph of \(f\). Assuming, say, that

\[(6) \quad h^1(\mathcal{O}_X) = h^0(T_X) = 0,\]

it is well known that deformations of \(Y \times X\) are of the form \(Y' \times X'\) with \(Y', X'\) deformations of \(Y, X\), respectively, and it follows easily that deformations of the triple \((f, Y, X)\) correspond bijectively with deformations of the pair \((Y \times X, \tilde{Y})\), hence the above results apply. Note that

\[N_{\tilde{Y}} \simeq f^*T_X,\]

while \(\bar{\eta} = [\tilde{Y}] \in H^p(\Omega^Z_{Y \times X}) \subset H^{2h}(Y \times X)\) is “the same” as the pullback map

\[\bar{\eta}^* = f^*: H^\bullet(X) \to H^\bullet(Y)\]

or its dual, the Gysin map

\[\bar{\eta}_* = f_*H^\bullet(Y) \to H^\bullet + p(X),\]

and \(\bar{\eta}\) being of type \((n, n)\) means \(\bar{\eta}^*\) preserves Hodge level or \(\bar{\eta}_*\) raises Hodge level by \(\leq p\), so we conclude that
COROLLARY 2. Assuming (6), obstructions to deforming \( f \), relative to deforming \( X, Y \) so that \( \eta_* \) raises Hodge level by at most \( p \), lie in

\[
\ker \tilde{\pi}_1 : H^1(f^*T_X) \to H^{n+1,n-1}(X \times Y).
\]

Consider next the case of deformations of \( f \) with \( X \) fixed. As is well known \([AC]\), these are controlled by the normal sheaf \( N_f \), which fits in an exact diagram (identifying \( T_{X \times Y} = p_1^*T_Y \otimes p_2^*T_X \)):

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to p_1^*T_Y(\tilde{Y}) & T_{X \times Y}' & T_{X,f} & 0 \\
\downarrow & \downarrow & \downarrow \\
(7) & 0 \to p_1^*T_Y & p_1^*T_Y \oplus p_2^*T_X & p_2^*T_X & 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to T_Y & f^*T_X & N_f & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

Again \( N_f[-1] \) forms a DGLA sheaf on \( Y \) and obstructions to deforming \( (Y, f) \) are in \( H^1(N_f) \) and come from the bracket map \( N_f \times N_f \to N_f[1] \).

On the other hand, \( H^{n-1,n+1}(Y \times X) \) has as one Künneth component

\[
H^{n-p,n-p}(Y) \otimes H^{p-1,p+1}(X) \cong H^{p-1,p+1}(X)
\]

and by its definition the semiregularity map for \( \tilde{Y} \) factors

\[
H^1(f^*T_X) \xrightarrow{\tilde{\pi}_1} H^{n-1,n+1}(Y \times X) \\
\downarrow \\
H^1(N_f) \xrightarrow{\pi_{1,f}} H^{p-1,p+1}(X).
\]

where \( \pi_{1,f} \) is induced by interior multiplication by the component of \([\tilde{Y}]\) in \( H^{n-p,n-p}(Y) \otimes H^{p-p}(X) \) so we conclude (note this does not use assumption (6)):

COROLLARY 3. Obstructions to deforming \( (f, Y) \), fixing \( X \), relative to deforming \( Y \) so that \( \tilde{\eta}_* \) raises Hodge level by \( \leq p \), lie in \( \ker \pi_{1,f} \).

Note that there are many cases, e.g. \( Y \) is a curve and \( h^{n-1}(O_X) = 0 \), where the cohomological condition on \( \tilde{\eta} \) is vacuous, for then \( \tilde{\eta}_* \) is simply given by

\[
[Y]_Y \mapsto [Y]_X = \eta \\
[pt]_Y \mapsto [pt]_X.
\]
In particular, suppose $Y$ is a smooth connected curve of genus $g$ and $X$ is a $K3$ surface. Then from (7) we get a nonzero map

$$ N_f \longrightarrow N_{f/\text{tor}} \longrightarrow K_Y $$

and the semiregularity map factors through $H^1(K_Y) \to H^2(\mathcal{O}_Y)$ Serre dual to $H^0(K_X) \to H^0(K_X|_Y) = H^0(\mathcal{O}_Y)$, which map is clearly nonzero, hence so is $\pi_{1,f}$ because $H^1(N_f) \to H^1(K_Y)$ is surjective, $Y$ being a curve. On the other hand $c_1(N_f) \sim K_Y$, so $\chi(N_f) = g - 1$. We conclude then that the deformation space of $(f, Y)$ is at least $g$-dimensional.

Now suppose in addition that $f$ is of degree 1 to its image $\bar{Y}$. It is then clear that unobstructed deformations of $(f, Y)$ must move $\bar{Y}$, hence must project injectively to $H^0(N_{f/\text{tor}})$, and since $N_{f/\text{tor}}$ is a subsheaf of $K_Y$ with quotient $=\text{tor}$ (supported exactly on the ramification locus of $f$), its $h^0$ is $< g$ unless $g = 0$ or $\text{tor} = 0$; and if $\text{tor} = 0$, i.e. $f$ is unramified, then $N_f \simeq K_Y$ so $\pi_{1,f}$ is injective and $(f, Y)$ is unobstructed. So putting things together we conclude

**Corollary 4.** On a $K3$ surface, the locus of integral curves of geometric genus $g > 0$ is generically reduced, purely $g$-dimensional (or empty), and smooth at any immersed curve.

**Appendix: The semiregularity homomorphism**

In the course of the proof of Part (i) of the Theorem, we implicitly alluded to the fact that the semiregularity map $\pi$ is a Lie homomorphism, which implies that so is $\pi'$. As this may not be generally known, we include a proof for completeness.

First we recall the local fundamental class and semi-regularity map. Take an acyclic cover $\mathcal{U} = (U_a)$ of an open subset $U \subset X$ and let $Y \cap U_a$ be defined by $F_a = (f_a^1, \ldots, f_a^p) = (0)$, and set

$$ \text{dlog} F_a = \text{dlog} f_a^1 \wedge \ldots \wedge \text{dlog} f_a^p = \frac{d f_a^1 \wedge \ldots \wedge d f_a^p}{f_a^1 \cdot \ldots \cdot f_a^p}. $$

This yields a cocycle in $\check{Z}^{p-1}(U_a \setminus Y, \Omega^p)(\Omega^p = \Omega_X^p)$, for the open cover $(D_{i,a} = U_a - (f_a^i = 0))$, whence a class

$$ \bar{\eta}_a \in H_{Y \cap U_a}^0(\Omega^p) = H_{Y \cap U_a}^0(\Omega^p|_U), $$

and these glue together to yield

$$ \bar{\eta}_U \in H_{Y \subset U}^0(U, \Omega^p|_U). $$
which maps to the fundamental class

$$\eta_U = [Y \cap U] \in H^0(U, \Omega^p[p]).$$

(More pedantically, one computes $H^{p-1}(U - Y, \Omega^p)$ from the Čech bicomplex $(\tilde{C}^\cdot, \delta_1, \delta_2)$ associated to the biindexed cover $(D_{i,a})$ of $U - Y$. Writing on

$$F_\beta = A F_\alpha$$

for a suitable matrix $A$ expressed as a product of elementary matrices, it is easy to see that

$$\delta_2(d\log F_\alpha) = d\log F_\alpha - d\log F_\beta$$

is a sum of terms with $< p$ distinct $f'_a$ in the denominator, so by elementary properties of local cohomology there is an explicit $(p-2,1)$-cochain $G_{a\beta}$ with

$$\delta_1(G_{a\beta}) = d\log F_\alpha - d\log F_\beta,$$

and $(d\log F_\alpha, G_{a\beta})$ is a bicocycle representing a class in $H^{p-1}(U - Y, \Omega^p)$ whose image is $\tilde{\eta}_U$.)

Now by a similar remark about denominators, note that $\tilde{\eta}_a$ is killed by any function vanishing on $Y \cap U_a$; likewise, if $v_a \in \Gamma(U_a, T')$, the interior product

$$i(v_a)(d\log F_\alpha) = \sum (-1)^i \frac{v_a(f'_a)}{f'_a} d\log f^1_a \wedge \ldots \wedge d\log f^i_a \wedge \ldots \wedge d\log f^p_a$$

has vanishing cohomology class $i(v_a)(\tilde{\eta}_a)$. Using the Čech bicomplex above these statements may be extended to $U$ and hence globalised: thus the arrows given by (interior) multiplication by $\eta$

$$\mathcal{I}_Y \to \Omega^p[p]$$

$$T' \to \Omega^{p-1}[p]$$

vanish in the derived category; in particular interior multiplication by $\eta$ descends to a map (‘sheaf-theoretic semi-regularity’)

$$\pi : N \to \Omega^{p-1}[p]$$

which induced the (cohomological) semi-regularity

$$\pi_1 = H^1(\pi) : H^1(N) \to H^{p+1}(\Omega^{p-1}),$$

as well as the infinitesimal Abel-Jacobi map

$$H^0(\pi) : H^0(N) \to H^p(\Omega^{p-1}).$$

Now we come to the crux of the (semi-regularity) matter:
LEMMA 1. The composite

\[ N \times N \xrightarrow{[1]} N[1] \xrightarrow{\pi} \Omega^{p-1}[p+1] \]

vanishes in the derived category; in other words, \( \pi \) is a Lie homomorphism in the derived category.

PROOF. First a calculus observation: if \( \omega \) is a closed \( p \)-form and \( x, y \) vector fields on a manifold then (check!)

\[ i([x, y])(\omega) = L_x(i(y)\omega) - L_y(i(x)\omega) - d(i(x \wedge y)\omega). \]

Now take sections \( v' = (v'_a), \ v'' = (v''_a) \in \Gamma(U, N) \). So

\[ [v', v''] = ([t'_{a\beta}v''_a] - [t''_{a\beta}, v'_a]). \]

As \( t'_{a\beta}, t''_{a\beta} \in T'(U_a \cap U_\beta) \), note that the cohomology classes corresponding to \( i(t'_{a\beta} \wedge v''_a)(d\log F_a), i(t_{a\beta} \wedge v'_a)(d\log F_a) \) vanish, hence \( \pi([v', v'']) \) is represented by

\[ L'_{a\beta} (i(v''_a)d\log F_a) + L''_{a\beta} (i(v'_a)d\log F_a) - L''_{a\beta} (i(t'_{a\beta})d\log F_a) \]

\[ -L'_{a\beta} (i(t''_{a\beta})d\log F_a) + L_{a\beta} (i(v'_a)d\log F_a) + L_{a\beta} (i(v''_a)d\log F_a) \]

Now consider the diagram

\[
\begin{array}{ccc}
N \times N & \to & T'[1] \otimes \Omega^p[p] \oplus \Omega^p[p] \otimes T'[1] \to T[1] \otimes \Omega^p[p] \oplus \Omega^p[p] \otimes T[1] \\
\downarrow & & \downarrow \\
N[1] & \to & \Omega^{p-1}[p+1] \\
\end{array}
\]

where the top left arrow is given by \( \partial \times \pi \oplus \pi \times \partial \), \( \partial : N \to T'[1] \) the natural map. We have just proven that the left square commutes while the right one does obviously. Clearly the top arrows compose to zero because \( N \to T'[1] \to T[1] \) do. Hence the composite \( N \times N \to N[1] \to \Omega^{p-1}[p+1] \) vanishes, as claimed.

REFERENCES


Mathematics Department
University of California
Riverside CA 92521 USA
ziv@math.ucr.edu