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Factorization of Functions in Weighted Bergman Spaces

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Abstract. We consider spaces $A^{p,\varphi}$ of analytic functions on the unit disc which are in L^p with respect to a measure of the form $\varphi(r)drd\theta$, where φ is "submultiplicative". We show that these spaces are Möbius invariant and that if $f \in A^{p,\varphi}$ one can factor out some or all of its zeros in a standard bounded way; also one can represent f as a product of two functions in $A^{2p,\varphi}$. Finally, we show that our methods cannot be extended to the case of φ not submultiplicative.

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1. – Introduction

Let φ be a decreasing radial function on the unit disc $U \subset \mathbb{C}$ such that $\lim_{r\to 1^-} \varphi(r) = 0$. We consider the weighted Bergman spaces of analytic functions on U satisfying

$$\|f\|_{p,\varphi}^p = \int_U |f(z)|^p \varphi(z) dA(z) < \infty, \quad 0 < p < \infty,$$

$$\|f\|_{\infty,\varphi} = \sup_{z\in U} |f(z)|\varphi(z) < \infty,$$

where $dA = \frac{1}{\pi} r dr d\theta$. We shall also refer to the related Lebesgue spaces $L^{p,\varphi}$.

Our purpose is to show that for a large class of weights, namely, those which are "submultiplicative", we can generalize the results of [1] and [2] on the factorization of functions in such spaces. The outline of the paper is as follows: in Section 2 we define and develop basic properties of submultiplicative (s.m.) weights. We also show that when φ is s.m. $A^{p,\varphi}$ is closed under composition with Möbius automorphisms of the disc, and prove a partial converse to this result. In Section 3 we present our main theorems to the effect that when φ is s.m. one can factor out some or all of the zeros of $f \in A^{p,\varphi}$ in a standard bounded fashion, and one can also represent f as the product of two functions

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in $A^{2p,\varphi}$. We also generalize this fact. In Section 4 we show that if we allow φ to decline faster than any s.m. function then the above methods break down, and so submultiplicativity is a natural barrier for our approach.

2. – Submultiplicative weights

We consider spaces $A^{p,\varphi}$ where φ is a decreasing radial function satisfying $\lim_{r\to 1^-}\varphi(r)=0$. It is convenient to associate with φ the function

$$F(r)=F_\varphi(r)=\varphi(1-r^2)\,.$$

Thus F is increasing on $(0,1]$ and $\lim_{r\to 0^+}F(r)=0$.

Definition 2.1. φ (or F) is called submultiplicative (s.m.) if for some $C>1$

$$\ell(C)=\ell_F(C)=\sup_{0<r\leq 1/C}\frac{F(Cr)}{F(r)}<\infty\,. \tag{2.2}$$

Lemma 2.3. *If a weight φ (or F) satisfies (2.2) for some $C>1$ then it actually satisfies the same relation for all $x>1$, and there exist $M, m>0$ such that*

$$\ell_F(x)\leq Mx^m; \qquad x>1 \tag{2.4}$$

$$F(x)\geq M^{-1}F(1)x^m; \quad 0<x<1\,. \tag{2.5}$$

Proof. First consider $x\in(1,C]$. Then since F is increasing, whenever $0<r\leq 1/C$

$$\frac{F(xr)}{F(r)}\leq\frac{F(Cr)}{F(r)}\leq\ell(C)\,,$$

and if $1/C<r\leq 1/x$

$$\frac{F(xr)}{F(r)}\leq\frac{F(1)}{F(1/C)}\leq\ell(C)\,.$$

Now if $x\in(C,\infty)$ we can write $x=C^\alpha$, $\alpha>1$. Letting $n=[\alpha]+1$ we find that for all $r\in(0,1/C^n]$

$$\frac{F(xr)}{F(r)}\leq\frac{F(C^nr)}{F(r)}\leq[\ell(C)]^n\leq\ell(C)x^m\,,$$

where $m=\ln(\ell(C))/\ln C$. If $r\in(1/C^n,1/x)]$ then

$$\frac{F(xr)}{F(r)}\leq\frac{F(1)}{F(1/C^n)}\leq[\ell(C)]^n\leq\ell(C)x^m\,.$$

Putting together the above estimates, we obtain (2.4). (2.5) follows from the fact that if $0 < x < 1$

$$F(1)/F(x) \leq \ell\left(\frac{1}{x}\right) \leq Mx^{-m}. \qquad \square$$

We remark that for our purposes it is sufficient to demand that for some r_0, $0 < r_0 \leq 1$, and some $C > 1$

$$\sup_{0<r<r_0/C} \frac{F(Cr)}{F(r)} < \infty.$$

If so, we can modify F to be constantly $F(r_0)$ on $[r_0, 1]$, and then it will fulfill condition (2.2). Since this corresponds to modifying φ on $[0, 1-r_0^2]$, it induces an equivalent norm on $A^{p,\varphi}$.

The simplest examples of s.m. weights are the standard weights $\varphi_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > 0$. Here $F(r) = r^\alpha$ and of course $F(Cr) = F(C)F(r)$ for all C and r. As another example we can take

$$F(r) = r^\alpha \left(\log\frac{1}{r}\right)^\beta; \quad \alpha, \beta > 0.$$

Thus if $0 < x < 1$ and $y > 1$

$$F(xy) = x^\alpha y^\alpha \left(\log\frac{1}{x} + \log\frac{1}{y}\right)^\beta < \left(x^\alpha \log^\beta \frac{1}{x}\right) y^\alpha = F(x)\ell(y).$$

Similarly, one can verify that functions of the form

$$F(r) = r^\alpha \left[\log^+ \log^+ \dots \log^+ \frac{1}{r}\right]^\beta$$

satisfy (2.2).

We turn to the question of Möbius invariance of the spaces $A^{p,\varphi}$.

DEFINITION 2.6. For $|a| < 1$ we denote

$$T_a(z) = \frac{a-z}{1-\bar{a}z}, \quad (\text{so } T_a^{-1} = T_a); \quad B_a(z) = \frac{a}{|a|}T_a(z).$$

We say that a space of functions B on U is Möbius invariant if it is closed under composition with the automorphisms of U T_a, i.e., if

$$f \in B \Rightarrow f \circ T_a \in B \quad \text{for all} \quad a \in U.$$

The following proposition is probably known, but we include its simple proof since we have not seen it in the literature. However, we note that for the case $A^{p,\varphi}$ when $p = 2$ a stronger and more general result was proved in [4] and [5].

PROPOSITION 2.7. *If φ is a s.m. weight then $A^{p,\varphi}$ is Möbius invariant for $0 < p < \infty$. If for some $C > 1$*

$$\lim_{r\to 0}\frac{F_\varphi(Cr)}{F_\varphi(r)} = \infty, \tag{2.8}$$

then $A^{p,\varphi}$ is not Möbius invariant.

PROOF. Since φ is s.m. and decreasing, for all $a, z \in U$ (with $F = F_\varphi$, $\ell = \ell_\varphi$ as in (2.2))

$$\begin{aligned}\varphi(T_a(z)) = F(1-|T_a(z)|^2) &= F\left(\frac{(1-|a|^2)(1-|z|^2)|}{|1-\bar{a}z|^2}\right)\\ &\le F\left(\frac{2(1-|z|^2)}{1-|a|}\right) \le \ell\left(\frac{2}{1-|a|}\right)\varphi(z).\end{aligned} \tag{2.9}$$

Therefore if $f \in A^{p,\varphi}$, $0 < p < \infty$, and $|a| < 1$

$$\begin{aligned}\|f\circ T_a\|_{p,\varphi}^p &= \int_U |f(T_a(z))|^p\varphi(z)dA(z)\\ &= \int_U |f(z)|^p\varphi(T_a(z))|T_a'(z)|^2 dA(z)\\ &\le \ell\left(\frac{2}{1-|a|}\right)\cdot\frac{4}{(1-|a|)^2}\|f\|_{p,\varphi}^p.\end{aligned} \tag{2.10}$$

So $A^{p,\varphi}$ is Möbius invariant.

In the converse direction, we note that if $\frac{1}{1-|a|} > C$ then for all $z \in U$ in a neighborhood of $a/|a|$

$$\frac{1-|T_a(z)|^2}{1-|z|^2} = \frac{1-|a|^2}{|1-\bar{a}z|^2} > \frac{1}{1-|a|} > C.$$

It follows from (2.8) that

$$\lim_{z\to a/|a|}\frac{\varphi(T_a(z))}{\varphi(z)} = \infty.$$

By the change of variables formula, as in (2.10), we see that $A^{p,\varphi}$ is Möbius invariant if and only if $f \in A^{p,\varphi}$ implies that $f \in A^{p,\varphi h}$ where

$$h(z) = \frac{\varphi(T_a(z))}{\varphi(z)}.$$

Therefore the following lemma will complete the proof of our proposition.

LEMMA 2.11. *Let $h(z)$ be positive and measurable on U. If for some $\xi \in \partial U$ $\lim_{z\to\xi} h(z) = \infty$, then $A^{p,\varphi} \not\subset A^{p,\varphi h}$, $0 < p < \infty$.*

PROOF. Without loss of generality we take $\xi = 1$. If the lemma is false then by the closed graph theorem, the inclusion mapping from $A^{p,\varphi}$ to $A^{p,\varphi h}$ must be bounded; call its norm M. By hypothesis, for some $\varepsilon > 0$

$$h(z) > 4M^p \quad \text{in} \quad N_\varepsilon = \{z \in U : |z-1| < \varepsilon\}. \tag{2.12}$$

Now consider the peak function

$$f(z) = \frac{1.4\varepsilon}{1+\varepsilon-z}$$

which satisfies $|f(z)| \le q < 1$ in $U \setminus N_\varepsilon$, whereas inside N_ε $|f(z)| > 1$ on a "large" subset. It follows easily that for m sufficiently large the function

$$g(z) = f^m(z)$$

satisfies

$$\int_{N_\varepsilon} |g(z)|^p \varphi(z) dA(z) > \frac{1}{2} \int_U |g(z)|^p \varphi(z) dA(z).$$

By (2.12)

$$\begin{aligned}\int_U |g(z)|^p \varphi(z) h(z) dA(z) &\ge \int_{N_\varepsilon} |g(z)|^p \varphi(z) h(z) dA(z)\\ &\ge 4M^p \int_{N_\varepsilon} |g(z)|^p \varphi(z) dA \ge 2M^p \int_U |g(z)|^p \varphi(z)\end{aligned}$$

which is contrary to our hypothesis. □

3. – Factorization theorems

The following lemma and theorem generalize results from Section 7 in [1] and extend them to the case $p = \infty$.

LEMMA 3.1. *Let φ be a s.m. weight, and let $f \in A^{p,\varphi}$, $0 < p < \infty$. Denote by $\{z_k\}$ the zero set of f, with each zero repeated according to its multiplicity, and define T_{z_k} as in (2.6). Then the function*

$$f^*(z) = \frac{|f(z)|}{\prod_{k=1}^{\infty} |T_{z_k}(z)|(2 - |T_{z_k}(z)|)}$$

belongs to $L^{p,\varphi}$ and $\|f^*\|_{p,\varphi} \le C(p,\varphi)\|f\|_{p,\varphi}$ where $C(p,\varphi)$ is a constant depending only on p and φ.

PROOF. Since φ is s.m., inequality (2.5) implies that $\varphi(z) \geq C(1-|z|)^\alpha$ for some $\alpha > 0$. In particular, $A^{p,\varphi}$ is contained in $A^{p,\alpha}$ as defined in [1]. Now from Theorem 7.6 in that paper we conclude that if $f(0) \neq 0$

$$\log \frac{|f(0)|}{\prod_{k=1}^{\infty} |z_k|(2-|z_k|)} = \int_U \frac{1}{(2-|z|)^2} \log|f(z)| \frac{dA(z)}{|z|}.$$

Now we observe that both $dA(z)$ and $\frac{dA(z)}{|z|(2-|z|)^2}$ are unit measures on U. Therefore a calculation based on the fact that $\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$ is an increasing function of r yields that $\int_U \frac{1}{(2-|z|)^2} \log|f(z)|\frac{dA(z)}{|z|} \leq \int_U \log|f(z)|dA(z)$.

In light of Proposition 2.7 we can replace f by $f(T_w(z)) \in A^{p,\varphi}$. For any $w \in U$ such that $f(w) \neq 0$, we find that

$$\log f^*(w) = \log \frac{|f(w)|}{\prod_{k=1}^{\infty} |T_{z_k}(w)|(2-|T_{z_k}(w)|)} \leq \int_U \log|f(T_w(z)|dA(z).$$

Now note that for any $z, w \in U$

$$\varphi(w) = F(1-|w|^2) = F\left((1-|T_w(z)|^2)\frac{|1-\overline{w}z|^2}{1-|z|^2}\right) \leq \varphi(T_w(z))\ell\left(\frac{4}{1-|z|^2}\right).$$

Thus, if $w \in U$ and $f(w) \neq 0$

$$\begin{aligned} \log[f^*(w)^p\varphi(w)] \leq \int_U \Bigg[& \log(|f(T_w(z))|^p\varphi(T_w(z))) \\ & + \log\ell\left(\frac{4}{1-|z|^2}\right)\Bigg] dA(z). \end{aligned} \tag{3.2}$$

In view of (2.4) we have

$$\ell(x) \leq Mx^m,$$

so we obtain the inequality

$$\begin{aligned} & \int_U \left[\log\ell\left(\frac{4}{1-|z|^2}\right) + \log\frac{1}{1-|z|^2}\right] dA(z) \\ & \leq \log M + m\log 4 + (m+1) \equiv R. \end{aligned} \tag{3.3}$$

This together with (3.2) gives the pointwise estimate

$$\log|f^*(w)^p\varphi(w)| \leq R + \int_U \log[|f(T_w(z))|^p\varphi(T_w(z))(1-|z|^2)]dA(z).$$

Exponentiating and applying Jensen's inequality, we have

$$\begin{aligned} f^*(w)^p\varphi(w) &\le e^R \int_U |f(T_w(z))|^p \varphi(T_w(z))(1-|z|^2)\,dA(z) \\ &= e^R \int_U |f(z)|^p \varphi(z)|T_w'(z)|^2(1-|T_w(z)|^2)\,dA(z)\,. \end{aligned}$$

Now note that

$$\int_U |T_w'(z)|^2(1-|T_w(z)|^2)\,dA(w) = \int_U \frac{(1-|w|^2)^3(1-|z|^2)}{|1-\overline{w}z|^6}\,dA(w)$$

which is uniformly bounded, say by S, for all $z \in U$ (by Lemma 4.22 in [6]). It follows that

$$\|f^*\|_{p,\varphi} \le S^{1/p} \exp\left[\frac{1}{p}(\log M + m\log 4 + (m+1))\right] \|f\|_{p,\varphi}\,. \tag{3.4}$$

This completes the proof of the lemma. □

THEOREM 3.5. *Let φ be a s.m. weight; let $f \in A^{p,\varphi}$ $0 < p \le \infty$, and let $\{a_k\}$ be an arbitrary subset of the zero set of f. Define*

$$h(z) = \frac{f(z)}{\prod_{k=1}^{\infty} B_{a_k}(z)(2-B_{a_k}(z))} \qquad (\textit{as in } (2.6)).$$

Then $h \in A^{p,\varphi}$ and $\|h\|_{p,\varphi} \le C(p,\varphi)\|f\|_{p,\varphi}$ where C depends only on p and φ. In particular, every subset of an $A^{p,\varphi}$ zero set is also an $A^{p,\varphi}$ zero set.

PROOF. The convergence of the product defining h is a simple consequence of the condition $\sum(1-|a_k|)^2 < \infty$, which is equivalent to the convergence of the product defining $f^*(0)$ in Lemma 3.1. Now let $\{z_k\}$ denote the full zero set of f. Noting that $x(2-x) < 1$ for $0 < x < 1$, we have for all $z \in U$

$$|h(z)| \le \frac{|f(z)|}{\prod_{k=1}^{\infty} |B_{a_k}(z)|(2-|B_{a_k}(z)|)} \le \frac{|f(z)|}{\prod_{k=1}^{\infty} |T_{z_k}(z)|(2-|T_{z_k}(z)|)} = f^*(z)\,,$$

as defined in Lemma 3.1. Thus for $0 < p < \infty$,

$$\|h\|_{p,\varphi} \le \|f^*\|_{p,\varphi} \le C(p,\varphi)\|f\|_{p,\varphi}\,.$$

For $p = \infty$ we can use the fact that

$$\begin{aligned} \|h\|_{\infty,\varphi} &= \lim_{p\to\infty} \|h\varphi\|_{L^p(dA)} = \lim_{p\to\infty} \|h\|_{p,\varphi^p} \\ &\le \sup_p C(p,\varphi^p)\|f\|_{p,\varphi^p} \le \sup_p C(p,\varphi^p)\|f\|_{\infty,\varphi}\,. \end{aligned}$$

Thus it suffices to show that the numbers $C(p, \varphi^p)$ are bounded as $p \to \infty$. To that end we note that if the function F_φ associated with φ satisfies $F_\varphi(Cr) \leq F_\varphi(r)\ell(C)$, then for $p > 0$ $F_{\varphi^p} = (F_\varphi)^p$ so

$$F_{\varphi^p}(Cr) = [F_\varphi(Cr)]^p \leq F_{\varphi^p}(r)\ell^p(C).$$

Thus if $\ell_\varphi(x) \leq Mx^m$ (as in (2.4)) we have

$$\ell_{\varphi^p}(x) \leq M^p x^{mp}$$

and we can estimate the constant $C(p, \varphi^p)$ as in (3.4); namely

$$C(p, \varphi^p) \leq (S)^{1/p} \exp\left[\frac{1}{p}(\log M^p + mp\log 4 + (mp+1))\right]$$

which evidently is bounded as $p \to \infty$. □

Lemma 3.6. *Let φ be a s.m. weight and let $f \in A^{p,\varphi}$, $0 < p < \infty$. Let $\{z_k\}$ denote the zero set of f, and let $q > p$. Then the function*

$$g(z) = |f(z)|^p \prod_{k=1}^{\infty} \frac{\left(1 - \frac{p}{q}\right) + \frac{p}{q}|T_{z_k}(z)|^q}{|T_{z_k}(z)|^p}$$

belongs to $L^{1,\varphi}$ and $\|g\|_{1,\varphi} \leq C(p,q)\|f\|^p_{p,\varphi}$.

Proof. By formula (2.10) of [2], with n replaced by $\frac{q}{p}$, and assuming $f(0) \neq 0$, we have

$$\log|f(0)|^p + \sum_{k=1}^{\infty} \log\left[\frac{1 - \frac{p}{q} + \frac{p}{q}|z_k|^q}{|z_k|^p}\right] = \int_U \log|f(z)|^p du,$$

where du is a probability measure defined there. Now we can proceed exactly as in Lemma 3.1 to obtain the desired result. □

Theorem 3.7. *Let φ be a s.m. weight and let $f \in A^{p,\varphi}$, $0 < p < \infty$. Let $p_1, \ldots, p_n > 0$ be numbers such that*

$$\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i}.$$

Then there exist functions $f_i \in A^{p_i,\varphi}$, $i = 1, 2, \ldots, n$, such that

$$f = \prod_{i=1}^{n} f_i \quad \text{and} \quad \sum_{i=1}^{n} \|f_i\|^{p_i}_{p_i,\varphi} \leq C\|f\|^p_{p,\varphi} \tag{3.8}$$

where C depends only on $p_1, \ldots, p_n$.

PROOF. We follow the lines of proof in [2]. As a first case consider f in the dense subset of $A^{p,\varphi}$ consisting of functions having only finitely many zeros, $\{z_k\}_{k=1}^m$. Letting B represent the finite Blaschke product corresponding to these zeros we propose to factor $B = \prod_{i=1}^n B^{(i)}$, and then to choose $f_i = (\frac{f}{B})^{p/p_i} B^{(i)}$, $i = 1, 2, \ldots, n$.

The $B^{(i)}$ are chosen probabilistically; namely for a given i, $B^{(i)}$ will contain each factor B_{z_k} in B with probability p/p_i. If so, for each $z \in U$ the expected value of $|f_i(z)|^{p_i}$ is

$$E[|f_i(z)|^{p_i}] = \left|\frac{f(z)}{B(z)}\right|^p \prod_{k=1}^m \left(1 - \frac{p}{p_i} + \frac{p}{p_i}|T_{z_k}(z)|^{p_i}\right)$$
$$= |f(z)|^p \prod_{k=1}^m \frac{\left(1 - \frac{p}{p_i}\right) + \frac{p}{p_i}|T_{z_k}(z)|^{p_i}}{|T_{z_k}(z)|^p}.$$

Integrating with respect to $\varphi(z)dA(z)$ and applying Lemma 3.6 we conclude that

$$E\left[\|f_i\|_{p_i,\varphi}^{p_i}\right] \leq C(p_i, \varphi)\|f\|_{p,\varphi}^p.$$

Since each random factor of f has an appropriately bounded norm, we conclude that there exists a concrete factorization of f as in (3.8). For f having infinitely many zeros, we first choose a sequence $f_n \to f$ in $A^{p,\varphi}$ where each f_n has finitely many zeros. Factoring each f_n as above, we can select subsequences of the factors which approach a bounded factorization of f. □

4. – Limits of applicability of the factorization

The key to Theorem 3.5 above was Lemma 3.1 to the effect that if $f \in A^{p,\varphi}$ then the operation

$$f(z) \to f^*(z) = \frac{|f(z)|}{\prod_{f(z_k)=0} |T_{z_k}(z)|(2 - |T_{z_k}(z)|)} \tag{4.1}$$

is bounded in the $L^{p,\varphi}$ norm. The proof relied on the fact that φ was a s.m. weight. In this section we show that when φ is not s.m., (4.1) is generally unbounded. This is perhaps to be expected in light of the breakdown of conformal invariance noted in Proposition 2.7. Our theorem will be proved for φ (not s.m.) satisfying a certain normalization which we now describe.

DEFINITION 4.2. For $j = 1, 2, \ldots$ let $r_j = \exp(-2^{-j}) \cong 1 - 2^{-j}$. We say that a decreasing function $\varphi(r)$ is a normal weight if $\log \varphi(r)$ is a linear function of $\log r$ (i.e., $\varphi(r) = Mr^m$) on each interval $[r_j, r_{j+1}]$.

THEOREM 4.3. *Let $\varphi(r)$ be a normal weight function for which the numbers*

$$\frac{\varphi(r_j)}{\varphi(r_{j+1})} \quad \left(\cong \frac{F(2^{-j})}{F(2^{-j-1})}\right) \quad \text{increase without bound.} \tag{4.4}$$

In particular, φ is not s.m. Then the operation (4.1) *does not map $A^{p,\varphi}$ into $L^{p,\varphi}$.*

PROOF. We define

$$K(r) = \frac{1}{p} \log \frac{1}{\varphi(r)} \tag{4.5}$$

and note that the normality of φ together with (4.4) implies that K is an admissible rapidly growing function, as defined in [3], page 146. Thus by Theorem 3 of that paper we can construct a function f analytic in U such that $f(0) = 1$ and

$$\log|f(z)| \leq K(|z|) + O(1), \quad z \in U. \tag{4.6}$$

For $0 < r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta = \sum_{\substack{|z_k|\leq r\\ f(z_k)=0}} \frac{r}{|z_k|} = K(r) + O(1). \tag{4.7}$$

By the construction in [3], f has $2^j n_j$ zeros evenly spaced on the circle $|z| = r_j$, where

$$\begin{aligned} n_j &= 2K(r_{j+1}) - 3K(r_j) + K(r_{j-1}) + O(1) \\ &= 2[K(r_{j+1}) - K(r_j)] - [K(r_j) - K(r_{j-1})] + O(1). \end{aligned}$$

In view of (4.5), (4.4) is equivalent to the statement

$$K(r_{j+1}) - K(r_j) \quad \text{increases without bound.}$$

It then follows that for j large, the n_j are increasing and they tend to ∞.

Now we note that since f has $2^j n_j$ zeros evenly spaced on the circle $|z| = r_j$, if

$$r_j \leq |z| \leq r_{j+1}$$

the disc $\{w : |T_w(z)| \leq 2/3\}$ contains a fixed portion of the circle $|z| = r_j$, and therefore contains at least cn_j zeros of f, where c depends only on f, and not on j. This implies that in (4.1)

$$f^*(z) \geq \left(\frac{9}{8}\right)^{cn_j} |f(z)|$$

whenever $r_j \le |z| < r_{j+1}$. We define

$$P(r) = \left(\frac{9}{8}\right)^{cn_j}; \quad r_j \le |z| < r_{j+1}, \tag{4.8}$$

so $P(r)$ increases as $r \to 1$, $\lim_{r\to 1} P(r) = \infty$, and $f^*(z) \ge P(|z|)|f(z)|$ for all $z \in U$.

Next we propose to construct a function $\psi(r)$, $0 < r < 1$, with the following properties:

$$\psi(r) \quad \text{is increasing, but} \quad \sup_j \frac{\psi(r_{j+1})}{\psi(r_j)} < \infty. \tag{4.9}$$

$$\int_0^1 \psi(r)dr < \infty \quad \text{but} \quad \int_0^1 \psi(r)P(r)dr = \infty. \tag{4.10}$$

Before carrying out the construction, we show how it leads to the conclusion of the theorem. Specifically, in view of (4.9), Theorem 2 of [3] enables us to construct an analytic function H in U such that $H(1) = 0$,

$$\log|H(z)| \le \frac{1}{p}\log\psi(|z|) + O(1) \quad \text{and}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|H(re^{i\theta})|d\theta = \sum_{\substack{|z_k|<r \\ H(z_k)=0}} \log\frac{r}{|z_k|} = \frac{1}{p}\log\psi(r) + O(1), \tag{4.11}$$

$$0 < r < 1.$$

Combining these inequalities with (4.5) and (4.7) we find that the function $Q = fH$ satisfies

$$\frac{1}{p}\log\left(\frac{\psi(r)}{\varphi(r)}\right) + O(1) = \frac{1}{2\pi}\int_0^{2\pi} \log|Q(re^{i\theta})|d\theta.$$

Now multiply by p and apply Jensen's inequality to obtain that for $0 < r < 1$

$$C_1\psi(r) \le \frac{1}{2\pi}\int_0^{2\pi} |Q(re^{i\theta})|^p\varphi(r)d\theta \le C_2\psi(r), \tag{4.12}$$

where the last inequality follows from (4.6) and (4.11). This inequality together with (4.10) proves that $Q \in A^{p,\varphi}$. However, from (4.8) we deduce that Q^* (as in (4.1)) satisfies $Q^*(z) \ge P(|z|)|Q(z)|$ so that (4.10) and (4.12) together imply that $Q^* \notin L_{p,\varphi}$; and this is the desired conclusion of Theorem 4.3.

It remains only to construct ψ satisfying (4.9) and (4.10). To that end we first choose a subsequence of $\{r_j\}$, $\{r_{j_k}\}$, such that for each k, $P(r_{j_k}) \ge 2^k$.

Then we define ψ to have a constant value ψ_j on each interval $[r_j, r_{j+1})$ as follows: first on the subsequence r_{j_k} define

$$\text{(4.13)} \qquad \begin{aligned} \psi_{j_k} &= \frac{2^{-k}}{2^{-j_k} - 2^{-j_{k+1}}} \\ \text{so} \quad \frac{2^{-k-1}}{r_{j_{k+1}} - r_{j_k}} &\le \psi_{j_k} \le \frac{2^{-k}}{r_{j_{k+1}} - r_{j_k}}, \end{aligned}$$

and note that these ψ_{j_k} increase with k. In order to define ψ between $r_{j_{k-1}}$ and r_{j_k} we first choose an integer $n \ge 0$ such that

$$4^n < \frac{\psi_{j_k}}{\psi_{j_{k-1}}} \le 4^{n+1}.$$

This implies that there are more than n intervals $[r_j, r_{j+1}]$ between $r_{j_{k-1}}$ and r_{j_k} so we can "count backward" defining

$$\psi_{j_k - \ell} = 4^{-\ell}\psi_{j_k}; \;\; \ell = 1, 2 \ldots n,$$

and

$$\psi_j = \psi_{j_{k-1}}; \;\; j_{k-1} \le j < j_k - n.$$

Thus ψ increases and satisfies $\psi(r_{j+1})/\psi(r_j) \le 4$, giving (4.9). Also

$$\begin{aligned} \int_{r_{j_{k-1}}}^{r_{j_k}} \psi(r)dr &= \psi_{j_{k-1}}(r_{j_k - n} - r_{j_{k-1}}) + \sum_{\ell=0}^{n} 4^{-\ell}\psi_{j_k}(r_{j_k-\ell} - r_{j_k-\ell-1}) \\ &\le \psi_{j_{k-1}} \cdot 2[r_{j_{k-1}+1} - r_{j_{k-1}}] + \sum_{\ell=0}^{n} 4^{-\ell}\psi_{j_k} \cdot 2^{\ell}(r_{j_k+1} - r_{j_k}) \\ &\le 2 \cdot 2^{-k+1} + 2 \cdot 2^{-k} = 6 \cdot 2^{-k}. \end{aligned}$$

Therefore $\int_0^1 \psi(r)dr < \infty$.

However by (4.13)

$$\int_0^1 \psi(r)P(r)dr \ge \sum_{k=1}^{\infty} \int_{r_{j_k}}^{r_{j_{k+1}}} \psi(r)P(r)dr \ge \sum_{k=1}^{\infty} 2^k \cdot 2^{-k-1} = \infty,$$

fulfilling condition (4.10). This completes the proof of the theorem. □

We remark that by a similar argument we can show that Lemma 3.6 is no longer valid for φ as in Theorem 4.3.

REFERENCES

[1] C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693-710.

[2] C. Horowitz, *Factorization theorem for functions in the Bergman spaces*, Duke Math. J. **44** (1977), 201-213.

[3] C. Horowitz, *Zero sets and radial zero sets in some function spaces*, J. Analyse Math. **65** (1995), 145-159.

[4] T. L. Kriete, B. D. MacCluer, *Composition operators on large weighted Bergman spaces*, Indiana Univ. Math. J. **41** (1992), no. 3, 755-788.

[5] T. L. Kriete, *Kernel functions and composition operators in weighted Bergman spaces*, Contemp. Math. **213** (1998), 73-91.

[6] K. Zhu, "Operator Theory in Function Spaces", Marcel, Dekker, 1990.

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