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Abstract. We consider some weakly nonlinear elliptic equations on the whole space and use local and global bifurcations methods to construct solutions periodic in one variable and decaying in the other variables. We then use analyticity techniques to prove there are many subharmonic bifurcations from these solutions.


In this paper, we construct positive bounded solutions of

\[ -\Delta u = f(u) \quad \text{on } \mathbb{R}^n \]

which are decay to zero in some variables and are periodic in the remaining variables. We also show that there are frequently solutions of this type with large supremum (and with rapid local changes in magnitude).

These solutions are of interest for a number of reasons. Firstly little is known about the structure of the positive bounded solutions on $\mathbb{R}^n$. We provide more examples. Note that solutions on $\mathbb{R}^n$ are also of interest because they arise in a limiting problem for equations on bounded domains as the diffusion goes to zero (and hence arise in populations models and combustion theory, just to give two examples). Secondly, these examples are of interest in connection with understanding the limits of results such as those of Zou [37] on the positive bounded solutions of $-\Delta u = u^p$ on $\mathbb{R}^n$ for $p$ supercritical. The author became interested in this problem (for $u^p$) because it arises as a limiting problem for studying the asymptotic behaviour of the branches of

\[ -\Delta u = \lambda f(u) \quad \text{in } D \\
\]  
\[ u > 0 \quad \text{in } D \\
\]  
\[ u = 0 \quad \text{on } \partial D \]

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when $f$ is superlinear (and in fact supercritical) and $D$ is a bounded domain. This is discussed in [13]. In [13], the results assume symmetry of the domain and to try to avoid this, we need to understand more general positive bounded solutions of $-\Delta u = u^p$ on $\mathbb{R}^n$.

Thirdly, our results can be used to study the problem $-\epsilon^2 \Delta u = f(u)$ on rectangular domains (in $\mathbb{R}^n$) with Neumann boundary conditions. Our methods can be used to construct solutions with a sharp peak on a planar hypersurface but at the same time the solutions have large variation along the hypersurface. These seem to be the first such solutions of this type.

Fourthly, our solutions show the limits of some as yet unpublished work of Busca and Felmer on solutions on $\mathbb{R}^n$ which decay in some variables.

We now discuss in more detail what we prove. Firstly, by a bifurcation argument we construct solutions periodic in one variable and decaying in the others. We also show that for positive nonlinearities, these are much more difficult to construct. We show they cannot occur if $n = 2, 3$. On the other hand for some supercritical nonlinearities which are a small perturbation with compact support (away from zero) of $u^p$ with $p = \frac{n+1}{n-3}$ and $n > 3$ we construct such solutions. These seem of interest in connection with Zou’s results. Here we have to be very careful in the choice of spaces.

For some nonlinearities (including $u^p - u$), we use Fredholm degree theory to construct a global branch. This gives the solutions with large variation claimed earlier. We also show that in many supercritical cases with $n = 3$ this branch oscillates (as in [13]) as it becomes unbounded. This ensures non-uniqueness for many $\lambda$.

Finally, for certain real analytic nonlinearities and $n = 3$, we show that there are many bifurcations off the primary branch to solutions with large minimal period. These solutions decrease and increase many times as we move across the period. This distinguishes them from the primary branch of solutions. To construct these, we use the techniques from Buffoni, Dancer and Toland [3], [4]. In particular, we obtain a band spectrum theory for operators $-\Delta + (a(x', x_n) + 1)I$ on $L^2(\mathbb{R}^n)$ where $a$ is periodic in $x_n$ and decays in $x'$. (Here we are writing $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$.) This appears of independent interest.

In Section 1, we do the local bifurcation theory, and in Section 2, we do global bifurcation. In Section 3, we construct the band theory while in Section 4, we prove the subharmonic bifurcation.

1. Solutions by local bifurcation

We consider the equation

\[ -\Delta u = u^p - u \]
on \( \mathbb{R}^n \), through the methods hold for more general nonlinearities. We prove the existence of positive solutions which are periodic in some variables and decay to zero in the others. Later in the section we discuss other nonlinearities. For simplicity for the moment we look for solutions which decay in all but one of the variables. Thus we look for positive bounded solutions of the form \( u(x', x_n) \) where \( u \) decay in \( x' \) and are \( 2\pi \alpha \) periodic in \( x_n \) (but are not constant in \( x_n \)).

By an obvious rescaling this is equivalent to looking for solutions of

\[
-\Delta u = \lambda (u^p - u)
\]

on \( \mathbb{R}^n \) which decay in \( x' \) and are \( 2\pi \) periodic in \( x_n \) where \( \lambda > 0 \). We will obtain these solutions by a secondary bifurcation from the family of solution \( \bar{u}_0(\lambda)(x') = u_0(\lambda^{\frac{1}{p}}x') \) where \( u_0 \) is the unique decaying radial solution of \( -\Delta u = u^p - u \) on \( \mathbb{R}^{n-1} \). Here we must assume \( 2 \leq p < \frac{n+1}{n-1} \) \((p \geq 2 \text{ if } n \leq 3)\) to ensure the existence of \( u_0 \). We need \( p \geq 2 \) (rather than \( p > 1 \)) for purely technical reasons which can be avoided as we indicate at the end of the proof.

Let \( Y = \mathbb{R}^{n-1} \times [-\pi, \pi] \). We will use the space \( T = \{ u \in L^\infty(Y) : u \text{ is continuous, } u(x', -\pi) = u(x', \pi), u \to 0 \text{ as } |x'| \to \infty \text{ uniformly in } x_n, u \text{ is even} \} \).

Note that \( T \) is a closed subspace of \( L^\infty(Y) \). It is easy to see that the map \( u \to u^p \) is a \( C^2 \) mapping on \( T \). (This uses \( p \geq 2 \)). We extend our nonlinearity for \( u < 0 \) to keep it \( C^2 \). Moreover if \( \gamma > 0 \), \((-\Delta + \gamma I)^{-1} \) maps \( T \) into itself. (More precisely, we mean that for each \( f \in T \), the equation \( -\Delta u + \gamma u = f \) has a unique solution in \( W^{2,p}_0(Y) \cap T \) satisfying the extra boundary condition \( \frac{\partial u}{\partial x_n}(x', -\pi) = \frac{\partial u}{\partial x_n}(x', \pi) \).) To prove uniqueness, we use the maximum principle and think of solutions as defined on \( \mathbb{R}^n \) and periodic in one variable (and thus they and their differences achieve their maximum). To obtain existence we first choose \( g \) a function of \( |x'| \) only so \( g \in T \) and \( |f| \leq g \).

We find a radial function \( \tilde{u}_0 \in C_0(\mathbb{R}^{n-1}) \) (i.e. vanishing at infinity) such that \(-\Delta \tilde{u}_0 + \gamma \tilde{u}_0 = g \) where \( \Delta \) is the Laplacian in the \( x' \) variables. It is well known and easy to prove that \( \tilde{u}_0 \) exists though it is difficult to find a good reference. (For example, one first assumes \( g \) has compact support and solve our equation on the ball \( B_k \) with Dirichlet boundary conditions to obtain a solution \( \tilde{u}_k \), extend \( \tilde{u}_k \) to be zero outside \( B_k \) and prove \( \tilde{u}_k \) converge uniformly to \( \tilde{u}_0 \). To prove the uniform convergence we use the natural uniform (in \( k \)) \( L^2 \) estimates for the Dirichlet problem on \( B_k \). To prove the result for general \( g \) one approximates \( g \) by functions of compact support and use natural \( L^\infty \) estimates.)

We then solve our equation \(-\Delta u_k + \gamma u_k = f \) on \( Y_k = \{ x \in Y : |x'| \leq k \} \) with periodic boundary conditions on \( x_n = \pm \pi \) and Dirichlet boundary conditions on \( |x'| = k \). (Alternatively we could think on this as solving a Dirichlet problem on the “strip” \( B_k \times R \).) We can easily use the maximum principle to prove \(-u_0(x') \leq u_k(x', x_n) \leq u_0(x') \) on \( Y_k \) and then we can easily use a diagonalization argument to obtain a solution of our equation on \( Y \). We can also easily using the maximum principle to prove that \((-\Delta + \gamma I)^{-1} \) (with the boundary conditions) is bounded on \( T \) (for the sup norm) with normal at
most $\gamma^{-1}$. We can think of $-\Delta + \gamma I$ is a closed operator on $T$ with domain $D = \{u \in T \cap W^{2,p}_0(Y) : \frac{\partial u}{\partial x_n}(x', -\pi) = \frac{\partial u}{\partial y_n}(x', \pi), \Delta u \in T\}$. (Here we could also think of functions as defined on $\mathbb{R}^n$ and $2\pi$ periodic in $x_n$ rather than the strip. Note that this implies $\frac{\partial u}{\partial x_n}$ is $2\pi$ periodic in $x_n$.) If $\tilde{g} \in T$, it is easy to check the map $u \mapsto \tilde{g}u$ is relatively compact as a map of $\tilde{D}$ with the graph norm to $D$ and hence the linear map $u \mapsto -\Delta u + \gamma u + \tilde{g}u$ (with domain $\tilde{D}$) is Fredholm of index zero on $T$ when $\gamma > 0$.

We now need to consider the spectrum of the operator $Z(\lambda) = -\Delta h - \lambda(pu_0^{p-1} - 1)h$ on $T$ with domain $\tilde{D}$. First note that $Z(\lambda) - BI$ is a relatively compact perturbation of $-\Delta + (\lambda - B)I$ and hence is Fredholm of index zero for $B < \lambda$ and in particular for $B \leq 0$. We can calculate the spectrum by separating variables. Before doing this, note that the operator $-\Delta - (pu_0^{p-1} - 1)I$ on the even functions in $C_0(\mathbb{R}^{n-1})$ has spectrum consisting of a simple negative eigenvalue $\alpha_1$ and all of the rest of the spectrum is real and lies in $\beta \geq \mu > 0$. (Very similar arguments appear in [8], p. 965-966, and p. 970-971 or [9].) By a rescaling and the separation of variables, one finds the spectrum of $Z(\lambda)$ consists of simple eigenvalues $\alpha_1 \lambda + n^2$ for $n \geq 0, n$ an integer, and the rest of the spectrum is real and lies in $\beta > \mu$. Hence a simple eigenvalue crosses zero when $\lambda$ crosses $-\alpha_1^{-1} = \lambda^*$.

We now prove there is a Crandall-Rabinowitz bifurcation at $\lambda = \lambda^*$ off the primary branch $\tilde{u}_0(\lambda)$. We have shown above that the operators are Fredholm of index zero nearby so the only thing to check is that there is a strict crossing of eigenvalues across zero as we move along the branch. In fact this is a consequence of the result that $\alpha_1 \lambda + 1$ has non-zero derivation in $\lambda$ at $\lambda^*$. Formally, to apply Crandall-Rabinowitz, we consider $H : \tilde{D} \times R \rightarrow T$ where $H(u, \lambda) = -\Delta - \lambda f(u)$ (where $f(u) = u^p - u$ and $\tilde{D}$ is given the graph norm for $-\Delta + I$). We then apply Theorem 1.7 in [6] to the map $F(u, \lambda) = H(u + \tilde{u}_0(\lambda), \lambda)$. The transversality condition they require is equivalent to our comment above that the derivative of the eigenvalue $\alpha_1 \lambda + 1$ has non-zero derivative in $\lambda$ at $\lambda^*$ (cp. the proof of the first part of Theorem 1.16 in the later paper of Crandall and Rabinowitz [7].)

Hence we have a second curve $(u, \lambda)$ of solutions of our equation in $T$ branching off the original curve $\tilde{u}_0(\lambda)$ at $\lambda = \lambda^*$. It remains to prove they are positive solutions. If not there must exist an open set $S$ in $\mathbb{R}^n$ such that $u < 0$ on $S$, $u = 0$ on $\partial S$ and $f'(u(x)) \leq -\frac{1}{2}$ on $S$. (Remember that $f'(0) = -1, u(x) \geq -\varepsilon$ on $\mathbb{R}^n$ by continuity and $u$ can only be negative where $u(\lambda^*)$ is small.) Here once again it is convenient to think of solutions as defined on $\mathbb{R}^n$ (and periodic in $x_n$). Hence $-u$ on $S$ is a bounded positive solution of $\Delta w = a(x)w$ on $S$, $w = 0$ on $\partial S$ where $a(x) \geq \tau > 0$ on $S$ such that $w$ is $2\pi$ periodic in $x_n$ and $w \rightarrow 0$ as $|x'| \rightarrow \infty$ uniformly in $x_n$. This contradicts the maximum principle since $w$ achieves its maximum.

We have proved the following theorem.

**Theorem 1.** Assume that $2 \leq p < \frac{n+1}{n-3}$ ($p \geq 2$ if $n \geq 3$). Then there is a curve of positive solutions of (2) in $L^\infty(\mathbb{R}^n)$ which are bounded, periodic in $x_n$ (but not
constant in $x_n$ and decay to zero uniformly in $x_n$ as $|x'| \to \infty$ and which bifurcate from the trivial solution $u_0$.

**REMARKS.**

1. With a little more care we find that we really only need $f$ is $C^1$. To prove this, one reduces our problem locally to a one dimensional bifurcation equation by a Liapounov-Schmitt reduction and uses a degree argument. This enables us to replace $p \geq 2$ by $p > 1$ which is more natural especially for large $n$. (We could also use degree theory for Fredholm maps as in Section 2.) However, in the $C^2$ case, we know that the bifurcating solutions form a smooth curve.

2. Our argument clearly works for much more general nonlinearities. We need a nonlinearity $f(u)$ with $f(0) = 0$, $f'(0) < 0$ (to obtain the Fredholm condition) such that the problem $-\Delta u = f(u)$ has a radial solution which is non-degenerate in the space of radial functions. Note that it follows automatically that the there must be a negative eigenvalue (as in [8]) and that it is possible to use transversality (as in [33] and the end of Section 4 here) to prove that the non-degeneracy condition holds for “most” $f$. Note that we do not need to assume uniqueness.

3. Note that, if $p \geq \frac{n+1}{n-3}$, one can prove that $-\Delta u = u^p - u$ has no positive solution $u_0(x')$ such that $u_0(x') \to 0$ as $|x'| \to \infty$. To prove this one first proves the decay is exponential and then uses the Pokojaev identity. This is discussed in more detail in Section 4.

4. Our techniques can be used to look for solutions $u(x', \vec{x})$ where $u$ decays in $x'$ and $u$ is periodic in $\vec{x}$ (where $\vec{x} \in \mathbb{R}^k$ with $k > 1$). To apply our techniques directly, we need to choose the ratio of the periods to retain simplicity. This can be avoided by using a Liapounov-Schmidt reduction and the variational structure (and ideas of Rabinowitz in [29]). A similar result is proved near the end of Section 4 (in a more complicated case). Note that the solutions we construct are all radial in $x'$. This is not an accident as we will see later.

5. One might try to obtain further solutions by looking at $\lambda$ so that $-\alpha_1 \lambda$ is the square of an integer greater than 1. However, by a careful bifurcation analysis one can prove that these solutions are simply a rescaling and translation of the solutions already obtained. In some cases, we obtain many more solutions later by secondary bifurcations.

It might be asked whether we always have solutions of this type.

**PROPOSITION 1.** Assume that $f$ is continuous, $f(0) = 0$ and $f(y) \geq 0$ if $y \geq 0$. If $n > 3$, assume that $f(y) > 0$ for $y > 0$, $f(y) \geq \mu y^r$ for small positive $y$ where $\mu > 0$ and $1 \leq r \leq \frac{n-1}{n-3}$ ($r \geq 1$ if $n = 2, 3$). Then the equation $-\Delta u = f(u)$ has no positive bounded solution which is $2\pi$ periodic in $x_n$ and decays to zero as $|x'| \to \infty$ uniform in $x_n$.

**PROOF.** Let $w(x') = \int_{-\pi}^{\pi} u \, dx_n$. Then $-\Delta' w = \int_{-\pi}^{\pi} f(u) \, dx_n$ and $w \to 0$ as $|x'| \to \infty$. If $n = 2$ or $3$, $-\Delta w \geq 0$ on $\mathbb{R}^{n-1}$ and hence by well-known
results (cp. [20]), \( w \) is constant. Since \( w \to 0 \) as \( |x'| \to \infty, w \equiv 0 \). This is impossible since \( u > 0 \). If \( n > 3, f(y) \geq \hat{\mu} y \) on \([0, \|u\|_\infty]\) and hence by Jensen's inequality (cp. [21, p. 202]) \(-\Delta w \geq \hat{\mu} w\) where \( \hat{\mu} > 0 \). We then obtain a contradiction by using this differential inequality as in Toland [34] or Gidas [7] and that \( r \leq \frac{n-1}{n-3} \).

**Remark.** We can use analogous techniques if \( u \) is periodic in more than one variable. It would be interesting to weaken the upper inequality on \( r \).

On the other hand, there are examples with \( n \geq 4 \) and \( f(y) > 0 \) for \( y > 0 \) where there are solutions of our type.

**Theorem 2.** If \( 4 \leq n < 7 \) and \( p^* = \frac{n+1}{n-3} \) there is a non-linearity \( f(y) = y^{p^*} + r(y) \) such that \( f(y) > 0 \) for \( y > 0 \), \( f \) is \( C^1 \), \( r \) has compact support and \( 0 \) is not in the support of \( r \) such that the equation \(-\Delta u = f(u)\) has a curve of positive solutions which are periodic in \( x_n \) (but not constant in \( x_n \)) and decay to zero as \( |x'| \to \infty \) uniformly in \( x_n \).

**Remark.** We assume \( n < 7 \) purely for simplicity. Note that our \( p^* \) is the critical exponent in dimension \( n - 1 \).

Here it seems convenient to use a different space \( L^{\infty,p^*} \) which is the measurable functions on \( Y \) for which \( \sup_{x_n} (u(x', x_n))_{p^*, \mathbb{R}^{n-1}} \) is finite. This is easily seen to be a norm (denoted by \( \| \cdot \|_{\infty,p^*} \)). The space is complete (cp. [28]).

The construction is in two steps. We first construct a radial solution \( u_0 \) of an equation \(-\Delta u = u^{p^*} + r(u) \) in \( D^{1,2}(\mathbb{R}^{n-1}) \) where \( r \) is \( C^2 \) and has compact support (not containing zero) such that \( u_0 \) is non-degenerate in the space of even functions in \( D^{1,2}(\mathbb{R}^{n}) \). Here \( D^{1,2}(\mathbb{R}^{n-1}) \) is the closure of \( C_0^\infty(\mathbb{R}^{n-1}) \) in the norm \( \| \nabla u \|_2 \). This is by a local perturbation of a critical manifold. Note that we cannot simply use the nonlinearity \( u^{p^*} \) because the extra symmetry ensures the non-degeneracy fails for the nonlinearity \( u^{p^*} \) in the space of even functions. Secondly, we obtain the solutions we require by modifying our earlier argument. We actually will do the second step first. This is the more technical step. Note that our nonlinearity behaves like \( u^{p^*} \) if \( u \) is small or \( u \) is large. Note that \( n < 7 \) ensures \( y^{p^*} \) is \( C^2 \) on \( \mathbb{R} \). (We extend our nonlinearity on \( \mathbb{R} \) to be even.) These solutions we construct are of interest in connection with the results of Zou [37] on positive solutions of \(-\Delta u = u^p \).

**Proof of Step 2.** Firstly, we modify our nonlinearity \( y^{p^*} + r(y) \) for \( |y| \geq \|u_0\|_{\infty}+1 \) so that the new nonlinearity \( g(y) \) is \( C^2 \) on \( \mathbb{R} \) and \( g \) is constant for large \( y \). This simplifies the proof though it could be avoided. We now consider the equation

\[
-\Delta u = \lambda g(u)
\]

on \( \mathbb{R}^n \) on \( L^{\infty,p^*}(D) \). It is easy to see that the curve \( \tilde{u}_0(\lambda) = u_0(\lambda^{\frac{1}{2}}x) \) is a \( C^2 \) family of solutions. As earlier, our proof is by a Crandall-Rabinowitz
bifurcation. The argument is very similar to earlier. The only points we have to worry about is the smoothness of the nonlinearity in our space, the separation of variables to justify the crossing and the Fredholm condition. The main difficulty is the last of these. We need to be more precise on our spaces. We will prove $I - (\Delta)^{-1}(\lambda g'(\tilde{u}_0(\lambda)))I$ is Fredholm of index zero as a map of $T = L^{\infty,p^*}(\mathbb{R}^n)$ into itself. To do this we write $\lambda g'(\tilde{u}(\lambda)) = a(x') + b(x')$ where $a$ has compact support and $-\Delta - bI$ is coercive on $D^{1,2}(\mathbb{R}^{n-1})$. We explain this later in the proof of Step 2. We then consider the problem

$$ -\Delta u - b(x')u = h $$

where we look for a solution $u$ in $T$, $u$ is $2\pi$ periodic in $x_n$ (more precisely $u(x', \pi) = u(x', -\pi))$ and $-\Delta - bI$ is coercive on $T = L^{\infty,q^*}$. We then consider the problem where we look for a solution $u$ in $T$, $u$ is $2\pi$ periodic in $x_n$ (more precisely $u(x', \pi) = u(x', -\pi)$) and $h \in L^{\infty,q^*}$, where $\frac{1}{q^*} + \frac{1}{p^*} = 1$. To solve this equation, we first solve the corresponding equation on $B_r \times \mathbb{R}$ with solutions zero on $|x'| = r$ and $2\pi$ periodic in $x_n$ and $h \in L^{\infty}(B_r \times [-\pi, \pi])$. It is easy to prove this truncated problem has a unique weak solution $L_r h$ in $W^{1,2}(B_r \times [-\pi, \pi])$ satisfying the boundary conditions (by using coercivity). Since $-\Delta - b(x')I$ is coercive in $D^{1,2}(\mathbb{R}^{n-1})$, we easily see that this solution is non-negative if $h$ is non-negative. (Simply multiply the equation by $u$.)

Now $-h_1 \leq h \leq h_1$ where $h_1(x') = \sup_{x_n}|h(x', x_n)|$. Then $|L_r h| \leq L_r h_1$ by positivity. Since $L_r h_1$ is independent of $x_n$ (and thus $L_r h_1$ is a solution of a Dirichlet problem in one dimension lower). By standard estimates and the Sobolev embedding theorem (as in [19]), $\|L_r h_1\|_{p^*} \leq k_1 \|\nabla (L_r h_1)\|_2 \leq K \|h_1\|_{q^*}$ where $K$ and $k_1$ are independent of $r$. (This uses that $W^{1,2}(B_r) \subseteq D^{1,2}(\mathbb{R}^{n-1})$).

Hence we see that $\|L_r h\|_{p^*, \infty} \leq K_1 \|h_1\|_{q^*} \leq K \|h\|_{\infty, q^*}$ where $K_1$ is independent of $r$. Now by multiplying the equation for $u$ by $u = L_r h$, we see that

$$ \int_{B_r \times [-\pi, \pi]} |\nabla u|^2 - b(x')u^2 = \int_{B_r \times [-\pi, \pi]} hu \leq \|h\|_{\infty, q^*} \|u\|_{\infty, p^*} $$

(by our estimate above for $L_r h$). On the other hand by our coercivity assumption on $b$, the left hand side is at least $k_1 \int_{B_r \times [-\pi, \pi]} |\nabla u|^2$ where $k_1 > 0$ and is independent of $r$. Hence $\|\nabla u\|_2 \leq K_3 \|h\|_{\infty, q^*}$ where $K_3$ is independent of $r$. By using truncations and passing to the limit, we can delete the assumption that $h$ is bounded.

We can now pass to the limit as $r$ tends to infinity and obtain a solution of (4) in $T \cap D'([\mathbb{R}^{n-1} \times [-\pi, \pi])$. Here $D'$ is the closure in the norm $\|\nabla u\|_2$ of the smooth functions in $\mathbb{R}^{n-1} \times [-\pi, \pi]$ which have compact support in $x'$ and are $2\pi$ periodic in $x_n$. By multiplying by $u^-$, and using the coercivity we can easily check the solution is unique (and positivity is preserved by the solution operator.)

We now prove the operator $Z = -\Delta - a(x')I - b(x')I$ is Fredholm as a map of $\tilde{T} = \{u \in T \cap D'([\mathbb{R}^{n-1} \times [-\pi, \pi]) : \Delta u - bu \in L^{\infty,q^*}\}$ with the graph norm into
This is a little complicated because it does not seem obvious that the map $u \mapsto a(x')u$ is a relatively compact perturbation. Hence we proceed indirectly. In the space $\mathcal{D}'$, we can easily calculate the kernel of $-\Delta - a(x')I - b(x')I$ by separation of variables and find it is finite dimensional. (Note that the Fourier series expansion converges in $\mathcal{D}'$.) We now prove the range $\mathcal{R}$ of $Z$ is closed. Once we do this, our operator is semi-Fredholm and by using the deformation $a \mapsto ta$ we deduce that the index is zero who proves our claim. It suffices to prove the estimate $\| - \Delta u - (a + b)u \|_{q^*} \geq K\| u \|_{\mathcal{F}}^2$ if $u \in \mathcal{F}$ (and $\| u \|_{\mathcal{F}}$ is the graph norm on $\mathcal{F}$) and $u$ is in a closed complement $M$ to the kernel in $\mathcal{F}$. If not, there exists $\bar{u}_n \in M$, $\| \bar{u}_n \|_{\mathcal{F}}^2 = 1$, $-\Delta \bar{u}_n - (a + b)\bar{u}_n = f_n \rightarrow 0$ in $L^{\infty,q^*}$.

We write $u_n = v_n + w_n$ where $w_n = (-\Delta - bI)^{-1}f_n \rightarrow 0$ in $\mathcal{F}$ as $n \rightarrow \infty$ by our earlier results. Now

$$-\Delta v_n - bw_n = av_n + aw_n.$$  

By induction, $v_n = ((-\Delta - bI)^{-1}a)^k v_n + r_{n,k}$ where $r_{n,k} \rightarrow 0$ in $\mathcal{F}$ as $n$ tends to infinity. Now by bootstrapping on compact sets (starting from $v_n$ converges in $L^2$ on compact sets and using $a$ has compact support), we find $((-\Delta - bI)^{-1}a)^k v_n$ converges in $L^\infty$ on compact sets. Hence by our formula for $v_n$, $v_n$ converges in $L^{p^*,\infty}$ on compact subsets of $D$. Hence since $a$ has compact support, we see $a(v_n + v_n)$ converges in $L^{\infty,q^*}$ and hence, by (5), $v_n$ (and thus $\bar{u}_n$) converges to $z$ in $\mathcal{F}$. Hence we see $z \in M$ and $z$ is in the kernel. This is a contradiction and we have proved our claim.

As we commented above the separation of variables is justified because any element of the kernel is in $\mathcal{D}'$ where the eigenfunction expansions most converge. (One easily checks kernel elements in $\mathcal{D}'$ also belong to $T$.) Thus the spectral conditions are much as before. Once again, we need to work in the space of even functions. We need to explain one more point here. We see from [2] or by a similar argument to that in [8] that the operator $-\Delta - g'(u_0)I$ on $L^2(\mathbb{R}^{a-1})$ has exactly one eigenvalue (where the operator is Fredholm) and this eigenvalue is simple (as in [8], p. 966-967). When we separate variables, this is the one which generate the crucial eigenvalue crossing zero. Thus our argument in the proof of Theorem 1 generalizes to this case.

We still have to obtain our decomposition of $g'(u)$. By Holder’s inequality and the Sobolev embedding theorem

$$\int_{\mathbb{R}^{a-1}\setminus B_s} |g'(\bar{u})|^2 \leq K\| g'(\bar{u}) \|_{L^{2(a-1)}(\mathbb{R}^{a-1}\setminus B_s)} \| \nabla h \|^2_2,$$

$$\leq \frac{1}{4} \| \nabla h \|^2_2$$

if $s$ is large since $|g'(y)| \leq K|y|^{p^*-1}$ on $\mathbb{R}$ ensures $g'(\bar{u}) \in L^{1/2}(\mathbb{R}^{a-1})$. Hence we see that if we choose $X$ a smooth function such that $X = 1$ if $|x| \geq 2s$, $X = 0$ if $|x| < s$ and $0 \leq X \leq 1$ and if we set $b(x') = X(x')g'(\bar{u}_0(x'))$, then $-\Delta - bI$ is coercive on $\mathcal{D}^{1,2}(\mathbb{R}^{a-1})$ and $a$ has compact support, as required.
To check the differentiability, it suffices to prove that the map \( u \to g(u) \) is \( C^2 \) as a map of \( L^{p*,\infty} \) to \( L^{q*,\infty} \) (by our regularity for \( (\Delta - (a + b)I)^{-1} \)). This is a slight modification of results in [35] for maps in \( L^p \) spaces. Hence, to prove that \( g \) is \( C^1 \), it suffices to prove \((u, h) \to g'(u)h \) is continuous as a map of \( L^{p*,\infty} \) into \( B(L^{p*,\infty}, L^{q*,\infty}) \) (cp. [26] or [35]). Here \( B(X, Y) \) denotes the continuous linear operator from \( X \) to \( Y \) with the usual norm. (It is easy to check the Gateaux differentiability of \( g \).) By Holder's inequality, it suffices to prove the map \( u \to g'(u) \) is continuous as a map of \( L^{p*,\infty} \) into \( L^{r*,\infty} \) where \( s^* = \frac{1}{2}(n - 1) \). Since \( |g'(y)| \leq K|y|^{\frac{n}{4(n-3)}} \) on \( \mathbb{R} \), it is easy to prove that \( g'(u) \in L^{s*,\infty} \) if \( u \in L^{p*,\infty} \) and we can easily prove continuity by using Fatou's lemma and our growth condition. (Similar arguments also appear in [35].) Similarly, we can prove the map \( u \to g'(u) \) is continuous as a map of \( L^{p*,\infty} \) into \( L^{s*,\infty} \) where \( r^* = \frac{2n-2}{7-n} \). Here we use that \( 4 \leq n < 7 \) to ensure \( r^* \geq 1 \) and use that \( |g''(y)| \leq K|y|^{\frac{7-n(n-3)}{n}} \) on \( \mathbb{R} \). Much as before it easily from Holder's inequality follows that the map \( u \to g'(u)hk \) is continuous as a map of \( L^{p*,\infty} \) into the continuous bilinear maps of \( L^{p*,\infty} \) into \( L^{q*,\infty} \) (with the natural norm) (which is the same as \( B(L^{p*,\infty}, B(L^{p*,\infty}, L^{q*,\infty})) \)) cp. [5], p. 26). Hence we see that the map \( u \to g'(u) \) considered as a map of \( L^{p*,\infty} \) into \( B(L^{p*,\infty}, L^{q*,\infty}) \) will be \( C^1 \) if it is Gateaux differentiable. By Holder's inequality again, it suffices to prove the map \( u \to g'(u) \) is Gateaux differentiable as a map of \( L^{p*,\infty} \) to \( L^{r*,\infty} \). As before, this is easy to prove. This completes Step 2.

It remains to obtain Step 1, that is, to modify \( g \) to obtain a non-degenerate solution. We look at a nonlinearity \( g(y) = y^{p*} + \varepsilon(Br_1(y) - r_2(y)) \) where \( r_1 \) and \( r_2 \) are \( C^1 \) of compact support and support away from zero and we choose a particular positive solution \( \tilde{u}_0 \) of \( -\Delta u = u^{p*} \). We will work in the space of even functions in \( D^{1,2}(\mathbb{R}^{n-1}) \). By well known results (cp. [2] or [31]) the positive solutions of \( -\Delta u = u^{p*} \) in this space are a one-dimensional manifold (due to the symmetry \( \alpha \to \alpha^{2/(p-1)}u(ax') \) and the kernel of the linearization at \( \tilde{u}_0 \) is one dimensional spanned by \( w = 2/(p^* - 1)\tilde{u}_0 + x' \cdot \nabla \tilde{u}_0 \). We will use results on the persistence of solutions when a smooth manifold of solutions is perturbed. It is proved in Dancer [14], [15] or Ambrosetti et al. [1] (by a modified Liapounov Schmidt reduction and by looking at the dominating term of the reduced equation) that we obtain a solution of the perturbed equation near \( u(\alpha_0) \equiv \alpha_0^{2/(p^*-1)}\tilde{u}_0(\alpha_0 x') \) for all small \( \varepsilon \) if \( \alpha_0 \) is a non-degenerate critical point of the reduced functional

\[
G(\alpha) = \int_{\mathbb{R}^{n-1}} \phi(u(\alpha)) dx
\]

where \( \phi : \mathbb{R} \to \mathbb{R} \) and \( \phi'(y) = Br(y) - r_2(y) \). Moreover, the perturbed solution is non-degenerate. The technical conditions in [14] on [1] are very easily checked. (Note that the theory in [14] assumes compactness of the group of symmetries but this does not matter as we work purely locally and the group action is easily seen to be locally proper.) Thus, we need to choose \( B, r_1, r_2 \) so that \( G'(1) = 0, G''(1) \neq 0 \). We choose \( r_1(y) = q_1(y^+)^{q_1}, r_2(y) = q_2(y^+)^{q_2} \)
where \( q_1 \) and \( q_2 \) are both large. Note that these do not satisfy our original conditions on \( g_1 \) and \( g_2 \) but we will rectify this in a moment. Then

\[
G(\alpha) = \int_{\mathbb{R}^n-1} B(\alpha^{2/(p+1)} u_0(\alpha x'))^{q_1} - (\alpha^{2/(p+1)} u_0(\alpha x'))^{q_2} \, dx \\
= \alpha^{-(N-1)} (B \alpha^{2q_1/(p+1)} A_1 - \alpha^{2q_2/(p+1)} A_2)
\]

where \( A_1, A_2 > 0 \) (by the change of variable \( y = \alpha x' \)). If \( q_1 \neq q_2 \), it is then an easy computation to check that, if \( B \) is chosen suitably, \( G'(1) = 0 \) and \( G''(1) \neq 0 \). If we choose \( q_1, q_2 > \frac{2(a-1)}{n-3} \) (so there are no convergence problems), truncate \( r_1 \) and \( r_2 \) for \(|y| > \|u_0\|_\infty + 1 \) so their support is bounded and then replace \( r_1 \) and \( r_2 \) by \( r_1(\delta + y), r_2(\delta + y) \) where \( \delta \) is small and negative, then it is easy to see that our assumptions continue to hold and we have the required example.

**Remark.** As before, we can remove the condition that \( n < 7 \) will a little more care and we can produce solutions that are periodic in several variables by similar arguments.

### 2. Global theory

In this section, we show that our bifurcating branches frequently persist globally and become unbounded.

Let \( \mathcal{D}_2 = \{(u, \lambda) \in \mathcal{D} \times [0, \infty) : u \text{ is positive}, -\Delta u = f(u), u \neq u_0(\lambda)\} \cup \{(0, \lambda_*)\} \) where \( \mathcal{D}, \lambda_* \) and \( u_0(\lambda) \) were defined in Section 1.

**Theorem 3.** Assume that \( f \) is \( C^1 \), \( f(0) = 0, f'(0) < 0 \), \( f \) has a unique positive zero which is non-degenerate (on \( \mathbb{R} \)) and the equation \(-\Delta u = f(u)\) has a unique even positive solution \( u_0 \) on \( \mathbb{R}^{n-1} \) in \( C_0(\mathbb{R}^{n-1}) \) which is non-degenerate in the space of even functions in \( C_0(\mathbb{R}^{n-1}) \). Then the component of \( \mathcal{D}_2 \) containing \((0, \lambda_*)\) is unbounded in \( L^\infty(D) \times \mathbb{R} \).

**Remark.** For example as in [8], we could take \( f(y) = y^p - y \) where \( 1 < p \) and \( p < \frac{a+1}{n-3} \) if \( n > 3 \).

**Proof.** We will in fact prove a stronger result. We will look only at solutions on \( S_1 = \mathbb{R}^{n-1} \times [0, \pi] \) which satisfy Neumann boundary conditions on \( x_n = 0, \pi \) because these can then be reflected to obtain 2\( \pi \) periodic solutions. We will also restrict ourselves to solutions which are functions of \(|x'|\) and \( x_n \) only. It can be shown by a modified sliding plane argument that this is in fact not a restriction. (In other words, all positive solutions are functions of \(|x'|\) and \( x_n \).) This will appear separately. Let \( X = \{u \in L^\infty(S_1) : u \text{ is continuous}, u = u(|x'|, x_n), u(x', x_n) \to 0 \text{ as } |x'| \to \infty \text{ uniformly in } x_n \} \) and let \( A(u, \lambda) = (-\Delta + I)^{-1}(\lambda f(u) + u) \) on \( X \times \mathbb{R} \).
Here we will use the degree theory for $C^1$ Fredholm maps $f : X \to X$ which have the property that $I - tf'(x)$ is Fredholm for $0 \leq t \leq 1$. Then the degree of these can be defined on open subset $U$ such that $I - f$ is proper on $U$ and the degree is homotopy invariant within this class. See for example Isnard [22] or [11] in the $C^2$ case. Since a simple eigenvalue crosses zero as $\lambda$ crosses $\lambda_*$ along $u = \bar{u}_0(\lambda)$, the formula for the degree of a non-degenerate fixed point in [11] shows that index $(I - A (\lambda, \lambda), \bar{u}_0(\lambda))$ changes sign as $\lambda$ crosses $\lambda_*$. Thus we have another proof of bifurcation. Note that the condition $I - tf'(u)$ is Fredholm for $0 \leq t \leq 1$ is easily obtained by similar arguments to those in the proof of Theorem 1. Hence we can very easily modify the proof of Theorem 1 in [11] to prove that the component $D$ containing $(\bar{u}_0(\lambda_*), \lambda_*)$ of the closure in $X \times \mathbb{R}$ of

$$D_1 = \{ (u, \lambda) \in X \times [0, \infty) : u = A(u, \lambda), u \text{ a positive solution}, u \neq \bar{u}_0(\lambda) \}$$

is such that $D$ is either non-compact in $X \times \mathbb{R}$ or it contains $\bar{u}_0(\lambda)$, $\hat{\lambda}$ where $\hat{\lambda} \neq \lambda_*$. Note that points $(0, \lambda)$ or $(u, 0)$ are not in the closure of $D_1$, as is easily proved. We now prove that the former alternative holds. To prove this, we first show that if $(u, \lambda) \in D$, $\frac{\partial u}{\partial x_n}$ has fixed sign on the interior of $S_1$ (and $x_i > 0$ if $i < n$). This is by continuity arguments. First assume $i = n$. Then $\frac{\partial u}{\partial x_n}$ is a solution of $-\Delta h = \lambda f'(u)h$, $h = 0$ on $x_n = 0, \pi$ and it is easily to be in $L^2(S_1)$ (since $u$ decays exponentially).

We consider the eigenvalue problem

$$(6) \quad -\Delta h = \lambda f'(u)h + \alpha h \text{ in } S_1, \quad h = 0 \text{ on } x_n = 0, \pi, \quad h \in L^2(S_1).$$

Since $f'(0) < 0$, we easily see that this operator is Fredholm in $L^2(S_1)$ if $\alpha < 0$.

We need the following lemma. We prove this after the proof of Theorem 3.

**Lemma 1.** If $(u, \lambda) \in D$, the smallest eigenvalue $B(u, \lambda)$ of (6) exists, $B(u, \lambda) \leq 0$, $B(u, \lambda)$ is simple, the eigenfunction corresponding to $B(u, \lambda)$ is positive on int $S_1$ and $B(u, \lambda)$ is the only eigenvalue which has a non-negative eigenfunction.

Now suppose $(u, \lambda) \in D$ and $\frac{\partial u}{\partial x_n} < 0$ for $x_n \in (0, \pi)$ (or $> 0$ for $x_n \in (0, \pi)$). Since $\pm \frac{\partial u}{\partial x_n}$ is a positive eigenfunction corresponding to the eigenvalue zero, by Lemma 1, $B(u, \lambda) = 0$. If $(u_m, \lambda_m) \in D$ is close to $(u, \lambda)$ in $L^\infty(S_1) \times \mathbb{R}$, then by continuous dependence of eigenvalues, $B(u_m, \lambda_m)$ must be small. However since $B(u, \lambda)$ is simple, continuous dependence ensures that the only eigenvalue of (6) for $u = u_m$, $\lambda = \lambda_m$ near zero must be the least eigenvalue. Moreover zero is an eigenvalue as before with eigenfunction $\pm \frac{\partial u_m}{\partial x_n}$. Hence $B(u_m, \lambda_m) = 0$ and $\pm \frac{\partial u_m}{\partial x_n} > 0$ on $S_1$. Now, for $\lambda_m$ near $\lambda_*$ and $u_m$ close up to $\bar{u}_0(\lambda_*)$, $B(u_m, \lambda_m)$ is close to the principal eigenvalue of $-\Delta h - \lambda_0 \bar{f}'(u_0(\lambda_*)h)$ on $S_1$ with Dirichlet boundary conditions. By separation of variables (as before), this eigenvalue is zero and the eigenfunction is positive. Hence we can use the
argument we have just used to deduce $B(u_m, \lambda_m) = 0$ and $\frac{\partial u_m}{\partial x_n}$ has fixed sign on $S_1$. Hence by connectedness, we see $\pm \frac{\partial u}{\partial x_i} < 0$ on $S_1$ if $(u, \lambda)$ belongs to any component $D_2$ of $D_1$ with $(\tilde{u}_0(\lambda_*), \lambda_*)$ in its closure. We can use a similar argument to prove that $\frac{\partial u}{\partial x_i} < 0$ if $(u, \lambda) \in D_2, x_i > 0$ and $i < n$ by considering the same eigenvalue problem on $S_4 = \{ x \in S_1 : x_i > 0 \}$ with Neumann boundary conditions on the boundary $\partial S_4$. Note that here we use that $\tilde{u}_0(\lambda_*)$ decreases in $x_i$ for $x_i > 0$.

We now show that our branch cannot return to $(\tilde{u}_0(\tilde{\lambda}), \tilde{\lambda})$ where $\tilde{\lambda} \neq \lambda_*$. If this occurs, by Ward [36], Theorem 110, there is a connected subset $D_3$ of $D_1$ with $(\tilde{u}_0(\lambda_*), \lambda_*)$ and $(\tilde{u}_0(\tilde{\lambda}), \tilde{\lambda})$ where $\tilde{\lambda} \neq \lambda_*$ in its closure. Thus, by the previous paragraph, $\frac{\partial u}{\partial x_n}$ has fixed sign on int $S_1$ if $(u, \lambda) \in D_3$. Now if $(u_m, \lambda_m) \in D_4$ converge to $(\tilde{u}_0(\tilde{\lambda}), \tilde{\lambda})$ in $X$ as $m \to \infty$, it is easy to see that a subsequence of $\frac{\partial u_m}{\partial x_n}$ (normalized) will converge in $X$ to a solution of $-\Delta h - \tilde{\lambda}f'(\tilde{u}_0(\tilde{\lambda}))h = 0$ in $S_1$, $h = 0$ on $\partial S_1$, $h \in L^2(S_1), h > 0$. Note that because our limit operator is Fredholm our eigenfunctions cannot become non-compact. Since we can separate variables, it is easy to see that this is impossible. (This uses that $-\Delta u = f'(u_0)u$ on $\mathbb{R}^{n-1}$ has a unique eigenvalue to which there corresponds a positive eigenfunction.)

It now follows that $\frac{\partial u}{\partial x_n}$ has fixed sign in int $S_1$ if $(u, \lambda) \in D$ and $(u, \lambda) \neq (\tilde{u}_0(\lambda_*), \lambda_*)$.

We will complete our proof of a global branch if we preclude a bounded sequence $(u_m, \lambda_m) \in D$ being non-compact. It is easy to see from standard estimates that the $u_m$ are bounded in $C^1$ and hence no non-compactness of $\{u_m\}$ can occur on bounded subsets of $S_1$. It follows easily by a simple diagonalization argument that $D$ is compact if it is bounded in $X'$ and given $\epsilon > 0$ there is an $a > 0$ such that $u_m(x', x_n) \leq \epsilon$ if $(u_m, \lambda_m) \in D$, $0 \leq x_n \leq \pi$ and if $|x'| \geq a$. Hence, if compactness fails and $D$ is bounded, there exists $(x'_m, x''_m) \in S_1$, $(u_m, \lambda_m) \in D$ and $\epsilon_1 > 0$ such that $|x'_m| \to \infty$ as $m \to \infty$ and $u_m(x'_m, x''_m) = \epsilon_1$ for all $m$. Now $u_m$ is a solution of $-\Delta u - \tilde{\lambda}f(u_m) = 0$. Here $r = |x'|$. We look at $\tilde{u}_m(s, x_n) = u_m(s + |x'_m|, x_n)$. Thus $\tilde{u}_m$ is defined on $[-|x'_m|, \infty) \times [0, \pi)$ and $\tilde{u}_m(0, x''_m) = \epsilon_1$. Now standard estimates applied on compact sets imply that $\{\tilde{u}_m\}$ is bounded in $C^1$. Thus by a standard convergence argument a subsequence of $\tilde{u}_m$ will converge on compact sets of $(-\infty, \infty) \times [0, \pi)$ to a solution $\tilde{u}$ of $-\Delta \tilde{u} = \tilde{\lambda}f(\tilde{u})$ which is decreasing in $x_1$ and monotone in $x_2$ if $0 \leq x_2 \leq \pi$, and $\tilde{u}$ satisfies Neumann boundary conditions on $x_2 = 0, \pi$. (It is decreasing in $x_1$ because $\tilde{u}_m$ is decreasing in $r$ for $r \geq 0$). Note that the term $\frac{1}{r} \frac{\partial u_m}{\partial r}$ vanishes in the limit because $\nabla u_m$ is bounded and $s + |x'_m|$ is large if $s$ is bounded. Moreover $\tilde{u}(0, \tilde{x}_n) = \epsilon_1$ where $\tilde{x}_n$ is the limit of a subsequence of $\{x''_m\}_{m=1}^\infty$. Note that by the decreasing properties, we can choose $\epsilon_1$ to be small. This ensures $\tilde{u}$ must depend on $x_1$. To see this, we note that solutions independent of $x_1$ are non-negative solutions $\tilde{u}$ of

$$-u'' = \tilde{\lambda}f(u)$$
satisfying Neumann boundary conditions on \( x_2 = 0, \pi \). By the maximum principle, either \( \tilde{u} \equiv 0 \) or \( \|\tilde{u}\|_\infty \geq s_0 \) where \( s_0 = \inf \{t > 0, f(t) > 0\} > 0 \). In the latter case, by the Harnack inequality, \( \inf \tilde{u} \) has a positive lower bound which is impossible since \( \varepsilon_1 \) is small. Note here that \( \tilde{\lambda} \) is bounded since \( D \) is bounded and \( \tilde{\lambda} \) is bounded away from zero since our earlier Dirichlet boundary problem for \( \frac{\partial \tilde{u}}{\partial x_n} \) implies a positive lower bound for \( \lambda_m \) (and hence for \( \tilde{\lambda} \)). Hence \( \tilde{u} \) depends on \( x_1 \). It is then easy and standard to check that \( \tilde{u}_\pm = \lim_{x_1 \to \pm \infty} \tilde{u}(x_1, x_2) \) are solutions of (7) (since \( \tilde{u} \) is decreasing in \( x_1 \)), \( \|\tilde{u}_+\|_\infty \leq \varepsilon_1 \) (and hence \( \tilde{u}_+ \equiv 0 \)) and \( \|\tilde{u}_-\|_\infty \geq \varepsilon_1 \). Now by a slight variant of the theory in Sections 1-2 of [10] (in particular p. 8 and p. 14), \( \tilde{u}_\pm \) must be stable or neutrally stable solutions of (7) (for the natural parabolic) and must have the same energy. (The theory in [10] is for Dirichlet boundary conditions rather than Neumann boundary conditions.) This ensures \( \tilde{u}_- \) is not identically equal to \( s_0 \) (on either count). Moreover, if \( \tilde{u}_- \) depends on \( x_2 \), it is unstable. This follows because \( \frac{\partial \tilde{u}_-}{\partial x_2} \) is a fixed sign solution with eigenvalue zero of the Dirichlet eigenvalue problem (for \( \alpha \)) - \( v'' = \lambda f'(\tilde{u}_-) v + \alpha v, v(\pi) = v(0) = 0 \). It then follows easily from the variational characterization of eigenvalues that the first eigenvalue for the same eigenvalue problem with Neumann boundary conditions is negative and hence \( \tilde{u}_- \) is unstable. Hence if \( f \) has only one positive zero, there is no suitable candidate for \( \tilde{u}_- \) and hence \( \tilde{u} \) does not exist. Hence the component must be unbounded as required. This proves Theorem 3.

**Proof of Lemma 1.** If \( (u, \lambda) \in D, \frac{\partial u}{\partial x_n} \) is an eigenfunction of (6) corresponding to the eigenvalue zero. Hence \( B(u, \lambda) \leq 0 \). Since (6) is Fredholm for \( \alpha \leq 0 \), it is easy to see that \( Q(h) = \int_{\mathbb{R}^{n-1}} |\nabla f|^2 - \lambda f' \) achieves its infimum on \( W^{1,2}(S_1) \) subject to the constraint \( \|h\|_2 = 1 \). This infimum is \( B(u, \lambda) \). If \( h \) minimizes \( Q(h) \) subject to the constraint so does \( |h| \) and hence \( |h| \) is an eigenfunction of (6) corresponding to the eigenvalue \( B(u, \lambda) \). By Harnack's inequality, \( |h| > 0 \) in \( \text{int} \ S_1 \). This is only possible if \( h > 0 \) in \( S_1 \), or \( h < 0 \) in \( \text{int} S_1 \). The orthogonality of eigenfunctions then ensures that the eigenspace corresponding to \( B(u, \lambda) \) is one-dimensional and \( B(u, \lambda) \) is the only eigenvalue to which there corresponds a non-negative eigenfunction. This completes the proof.

Note that, since \( (u(x', -x_n), \lambda) \) is a solution of (1) when \( (u(x', x_n), \lambda) \) is, we can assume without loss of generality that \( \frac{\partial u}{\partial x_n} > 0 \) on \( \text{int} S_1 \) along our unbounded branch. It is easy to that the parts \( D^+ \) and \( D^- \) of \( D \) corresponding to \( \frac{\partial u}{\partial x_n} > 0 \) and \( \frac{\partial u}{\partial x_n} < 0 \) have closures only intersecting at \( (\bar{u}_0(\lambda_*), \lambda_* ) \).

Now assume our nonlinearity is real analytic on \((-\gamma, \infty)\) where \( \gamma > 0 \). (For example, if \( n = 3 \), we could use \( u^p - u \) for \( p \in \mathbb{Z}, p \geq 2 \).) In this case it is not difficult to prove the map \( u \rightarrow f(u) \) is real analytic on \( X \) and \( \bar{u}_0 \) is real analytic. Hence, as in [13], the local Crandall Rabinowitz bifurcation for analytic maps applies and the bifurcating curve \( (u(\alpha), \lambda(\alpha)) \) when they bifurcate from \( (\bar{u}_0(\lambda_*), \lambda_* ) \) has the property that \( \lambda(\alpha) \) and \( u(\alpha) \) are real analytic in \( \alpha \). Hence either \( \lambda(\alpha) \) is constant near zero (and the bifurcation is locally purely vertical) or \( \lambda'(\alpha) \neq 0 \) for small non-zero \( \alpha \) in which case Theorem 1.17 in [6]
implies that the natural linearized equation in X at \((u(\alpha), \lambda(\alpha))\) is invertible for small positive \(\alpha\). In the former case, there is an unbounded curve set in \(\tilde{Y} = \{(u, \lambda) \in D : \lambda = \lambda_*\} \) which contains \((\tilde{u}_0(\lambda_*), \lambda_*\)\). This is proved by using the Lemma 8 in [12] to prove that \(Y\) contains a closed one-dimensional real analytic set which contains \((u(\alpha), \lambda_*)\) for small \(\alpha\) (and which necessarily intersects closed bounded sets in compact sets). Now it is well known that a one-dimensional real analytic set has only isolated singular points and an even number of branches meeting at each singular point (cp. [4], appendix, or [12]). Hence we see that \(Y \cap D^+\) is a graph with every vertex but one (corresponding to \((\tilde{u}_0(\lambda_*), \lambda_*\)) has an even number of edges meeting at a vertex. Hence, the result follows as in [12] by elementary combinatorics. Thus there are many solutions of our equation, in particular at least one will \(\|u\|_\infty = r\) for any \(r > \|\tilde{u}_0(\lambda_*)\|_\infty\). We will prove in Section 4 that frequently these solutions have quite sharp peaks at their maximum if \(r\) is large (by blowing up). In the latter case, we will prove in certain circumstances in Section 4 that there are many secondary bifurcations to solutions \(2\pi k\) periodic in \(x_n\) and hence in either case there are many distinct solutions periodic in \(x_n\) and decaying in \(x'\). (A variant of our arguments in Section 4 also implies that the former case does not hold if \(1 < p < (n + 2)/(n - 2)^{-1}\).)

When the branch does not start off vertical, we can argue as in Section 2 and the appendix of [4] to deduce that there is a natural unbounded analytic arc \(D'\) in \(D\) (that is with only isolated self intersections) containing \((\tilde{u}_0(\lambda_*), \lambda_*)\) in its closure such that the map \(h \mapsto -\Delta h - \lambda f(u)h\) is invertible for all points \((u, \lambda)\) of \(D'\) except at isolated points. We can then refine \(D'\) by removing "loops" to obtain an unbounded arc in \(D\) with no self-intersections containing \((\tilde{u}_0(\lambda_*), \lambda_*)\) in its closure such that the map \(h \mapsto -\Delta h - \lambda f'(u)h\) is invertible except at isolated points of \(D'\). Note that this curve need not be analytic. We refer to \(D'\) as the good branch.

We have established the following theorem.

**Theorem 4.** Assume that the assumptions of Theorem 3 hold and \(f : \mathbb{R} \rightarrow \mathbb{R}\) is real analytic on \((-\gamma, \infty)\) where \(\gamma > 0\). Then one of the following two possibilities hold. Either

(i) there is an unbounded curve of positive solutions of \(-\Delta u = \lambda_* f(u)\) in \(D^+ \cap \{\lambda = \lambda_*\}\) bifurcating from \((\tilde{u}_0(\lambda_*), \lambda_*\)

or

(ii) there is an unbounded arc \(D'\) in \(D^+\) with no self intersection bifurcating from \((\tilde{u}_0(\lambda_*), \lambda_*\)) such that except at isolated points of \(D'\), the map \(h \mapsto -\Delta h - \lambda f'(u)h\) is invertible on \(X\).

### 3. – Band theory

In this section we construct a band theory for the spectrum of certain linear operators on \(\mathbb{R}^n\). This is motivated by the theory for Hill's equation [3] and [30].
We need this work for the subharmonic bifurcation theory in Section 4.

To construct our band theory, we will use a variant of ideas in [4] and [30], Section XIII.16. We consider the problem

\[-\Delta u + (1 + a(x', x_n))u = \alpha u\]

on \(L^2(\mathbb{R}^n)\) where \(a\) is \(2\pi\) periodic in \(x_n\) and decays to zero as \(|x'| \to \infty\) uniformly in \(x_n\) (and \(a\) is continuous). Let \(W\) denotes the operator on the left hand side with domain \(W^{2,2}(\mathbb{R}^n)\). To study this problem we use Fourier transform in \(x_n\). Now \(a = \sum_{k=-\infty}^{\infty} a_k(x')e^{ikx_n}\) where the series converges in \(L^2\).

We write \(u = \int_{-\infty}^{\infty} \tilde{u}(\lambda, x')e^{ikx_n}dx_n\) where \(\tilde{u}(\lambda, x') \in L^2(\mathbb{R}^n)\).

Taking the Fourier transforms we find

\[-\Delta' u + (\lambda^2 + 1)u + \sum_{k=\infty} a_k u_{\lambda-k} = \alpha u\]

As usual, we then split the \(\lambda's\) into equivalence classes parameterized by \(\tau \in [0, 1)\). (The equivalence class is \([\tau + k : k \in Z]\)).

Hence we have for fixed \(\tau \in [0, 1)\) an infinite discrete system of equations (parametrized by \(k\)). We prove this is a Fredholm system for \(\alpha < 1\) on the space of functions for which \(\sum_{k=-\infty}^{\infty} \|u_{\tau-k}\|_{L^2(\mathbb{R}^{n-1})}^2\) is finite, that is, \(\ell^2(L^2(\mathbb{R}^{n-1})) \equiv \mathbb{Z}\) (The operator is closed for the natural definition of its domain.) If the \(a_k\) were all zero, we have a diagonal system and the result is easy.

We have the system of equations

\[-\Delta' u_{\tau+k} + (k + \tau^2 + 1)u_{\tau+k} + \sum_{q=-\infty}^{\infty} a_q(x') u_{\tau+k-q} = \alpha u_{\tau+k}\]

for \(k = -\infty, \cdots, \infty\).

Denote the operator on the left hand side by \(A(\tau)\). To study some of the properties of this system and to check the infinite sum term makes sense, it is convenient to proceed indirectly. We consider the eigenvalue problem on the strip \(S = \mathbb{R}^{n-1} \times [-\pi, \pi]\) where our space is \(L^2(S)\)

\[-\Delta u + (1 + a(x', x_n))u = \alpha u\] on \(S\)

\(e^{-i\tau x_n} u\) is \(2\pi\) periodic in \(x_n\).

(This gives two boundary conditions relating \(u\) on \(x_n = -\pi\) and \(x_n = \pi\).) This problem is easily seen to be self-adjoint with domain \(u \in W^{2,2}(S)\): \(u\) satisfies the boundary conditions. (To prove this, it is easiest to use that the \(a\) term is a bounded perturbation and to use Fourier decompositions.) We can write

\[u = e^{i\tau x_n} \sum_{k=-\infty}^{\infty} s_k(x') e^{ikx_n}\]
Note that \( u \rightarrow a(x', x_n)u \) is clearly bounded on \( L^2(S) \) and we see that the infinite sum term in (8) is just this term written in the other notation (which ensures the infinite sum term makes sense in (8) suitably interpreted). Now, if we use the above Fourier decomposition for \( u \), it is easily seen that (9) becomes (8) and that (8) and (9) are equivalent. (Remember \( u \in L^2[-\pi, \pi] \iff \hat{u} \in \ell^2(\mathbb{Z}) \). This ensures that the left hand side of (8) defines a closed operator (with the obvious domain). Now \( u \rightarrow a(x', x_n)u \) is relatively compact (as a map of the domain of (9) into \( L^2(S) \)). This is easy to see since we can write \( a = a_1 + a_2 \) where \( a_1 = 0 \) if \( |x'| \geq K \) and \( |a_2| \leq \varepsilon \) on \( S \). (The map \( u \rightarrow a_1 u \) is then relatively compact because sequences bounded in the graph norm of (9) are relatively compact when restricted to \( S \cap \{(x', x_n) : |x'| \leq K_1\} \). Now, it is easily proved by a Fourier decomposition that \(-\Delta + (1 - \alpha)I\) on \( L^2(S) \) with our periodic boundary conditions is invertible if \( \alpha < 1 \). Hence (9) is Fredholm if \( \alpha < 1 \).

Note that this is best possible since it is not hard to show \([1, \infty)\) is in the essential spectrum of \( A(\tau) \). Moreover, it is easy to see \( A(\tau) \) is self adjoint and bounded below on \( \mathbb{Z} \). (Remember that the map \( u \rightarrow au \) is bounded on \( L^2(S) \).) Note that \( A(\tau) \) continues to be defined for complex \( \tau \). Note that the problem (9) is often called the Bloch function decomposition.

Let \( \lambda_i(\tau) \) denote the ordered (increasing) eigenvalues of \( A(\tau) \) counting multiplicity where we take \( \lambda_i(\tau) \) to be 1 if there are fewer than \((i - 1)\) eigenvalues of \( A(\tau) \) counting multiplicity less than 1.

We next prove \( A \) depends analytically on \( \tau \) in the sense of Kato [23], p. 366. It suffices to prove this when \( a = 0 \) since the \( u \rightarrow au \) term is bounded and independent of \( \tau \) (cp. [23], p. 367). We use the representation (8) and note that, since \( a = 0 \), our system is diagonal in \( k \). By Theorem VII.1.2.3 in [23], it suffices to prove \((A(\tau) + \gamma I)^{-1}\) analytic in \( \tau \) for fixed \( \gamma \). Since \((A(\tau) + \gamma I)^{-1}\) is diagonal in \( k \), Theorem 3.3.12 and the remark after it in [23] imply that it suffices to prove that each of the operators for fixed \( k \) which make up \((A(\tau) + \gamma I)^{-1}\) are analytic in \( \tau \). This is easy to prove. Hence our claim for \( A(\tau) \) follows.

We now show \( \lambda_i(\tau) \) is never constant on any interval \((\tau_0 - \delta, \tau_0 + \delta)\) of the real axis where \( \lambda_i(\tau) < 1 \). Here \( 0 < \tau_0 < 1 \).

If \( \lambda_i(\tau) \) were constant on this interval, then by the analyticity of \( A(\tau) \), we see by Theorem VII.1.9 in [23], that \( \lambda_i(\tau_0) \) is an eigenvalue of \( A(\tau) \) on an open neighbourhood of \( G = \{\tau + is : \tau_0 - \delta < \tau < \tau_0 + \delta, s \geq 0\} \) in the complex plane. Here to apply the result we need to prove that \( A(\tau) - \lambda_i(\tau_0)I \) is Fredholm on \( G \). We will prove this in a moment. Note that Kato assumes a compactness condition but an examination of his proof shows that the Fredholm condition suffices. Hence we will have a contradiction if we prove the Fredholm condition and prove that \( \lambda_i(\tau_0) \) is not an eigenvalue of \( A(\tau_0 + is) \) for large positive \( s \). For the first result, it suffices as before to prove \( A(\tau + is) - B I \) is Fredholm if \( B < 1 \) and \( s > 0 \). As before, it suffices to prove this when \( a = 0 \) (since the map \( u \rightarrow au \) is relatively compact). Since our system is then diagonal in \( k \), it suffices to prove each individual operator is Fredholm and the
operators are invertible for $|k|$ large with uniformly bounded inverse. The first of these is easy to check and the second is clear when we note

$$(\tau + is + k)^2 = (\tau + k)^2 - s^2 + 2i\tau(\tau + k)$$

Since the real part is large positive for $|k|$ large, it is easy to see that the inverses are small for $|k|$ large (since $\text{Re}(A_k(\tau)u, u) \geq ((\tau + k)^2 - s^2 + 1)(u, u)$). Here $A_k(\tau)$ is the $k$th component of $A(\tau)$. Lastly to prove that $\lambda_0(\tau_0)$ is not an eigenvalue if $s$ is large it suffices to prove $(\tilde{A}(\tau_0 + is) - \lambda_0(\tau_0)I)^{-1}$ has small norm if $s$ is large. Here $A(\tau) = \tilde{A}(\tau) + T$ where $T$ is the bounded operator induced by $a$. This follows since we can write the equation

$$\tilde{A}(\tau_0 + is)u + Tu = \lambda_0(\tau_0)u \quad \text{as} \quad u = -\tilde{A}(\tau_0 + is) - \lambda_0(\tau_0)I^{-1}Tu.$$

Now, once again, $\tilde{A}(\tau_0 + is)$ is diagonal in $k$ and so it suffices to prove the inverse of each component is uniformly (in $k$) small if $s$ is large. Since $\tau_0 + k \neq 0$ for all $k$ (remember that $0 < \tau_0 < 1$) and is bounded below in absolute value, our estimate follows from (10) since, for a self adjoint operator $C$, $\|C + iB\|^{-1} \leq \frac{1}{|B|}$ is $B$ is real and non-zero. Hence we see $\lambda_i(\tau)$ is not constant on any non-trivial subinterval of $[0, 1)$.

Our arguments imply a little more. In fact $\lambda_i$ has isolated zeros. This follows since Kato’s theorem quoted in the previous paragraph implies either that $A(\tau)$ is invertible except at isolated points or $A(\tau)$ fails to be invertible at all points near $[0, 1)$. Our argument in the previous paragraph shows the second case does not occur.

Next we show $\lambda_i(\tau) \rightarrow 1$ as $i \rightarrow \infty$ uniformly in $\tau$. Since $\lambda_i(\tau) \rightarrow 1$ as $i \rightarrow \infty$ for each $i$ and $\lambda_i(\tau)$ is continuous in $\tau$ (including at $\tau = 1$) this follows by a simple contradiction argument.

**Theorem 5.** $\sigma(W) \cap (-\infty, 1) = \cup_{i=1}^{\infty} (\lambda_i([0, 1]) \cap (-\infty, 1))$. Moreover each $\lambda_i$ has only isolated zeros and is not a constant function.

**Remark.** This is a partial band description of $\sigma(W)$.

**Proof.** Since $\lambda_i(\tau) \rightarrow 1$ as $i \rightarrow \infty$ uniformly in $\tau$, the right hand side is closed in $(-\infty, 1)$. To prove this result, we use the Fourier transform decomposition in $x_n$. In this space we can write our operator as earlier as an uncountable direct sum (technically a direct integral as in [30]) of operators (8) (parameterized by $\tau \in [0, 1]$). If $\alpha$ is not a member of the right hand side of the equation in Theorem 5, and $\alpha < 1$, there exists $\tilde{k} > 0$ such that $|\alpha - \lambda_i(\tau)| \geq \tilde{k}$ for all $i$ and all $\tau$ in $[0, 1]$. Here we are using that $\lambda_i(\tau)$ is continuous in $\tau$ and $\lambda_i(\tau) \rightarrow 1$ as $i \rightarrow \infty$ uniformly in $\tau$. Hence $\|(A(\tau) - \alpha I)^{-1}\|_2 \leq \tilde{k}^{-1}$ for all $\tau$ by self adjointness and hence putting these inverses together, we obtain a bounded inverse of $W - \alpha I$ of norm at most $\tilde{k}^{-1}$. (More formally, we can use direct integral theory and Theorem XIII.85 (d) in [30].) On the other hand if $\alpha = \lambda_i(\tau)$, we can obtain the converse by applying the same theorem again. Alternatively and this is useful later, we can obtain approximate eigenvectors
in $L^2(\mathbb{R}^n)$ by choosing an eigenvector $\{\phi_j(x')\}_{j=1}^{\infty}$ of $A(\tau)$ corresponding to the eigenvalue $\lambda_i(\tau)$ and then choosing $\psi_i$ smooth of $L^2$ norm 1 where Fourier transform has support close to $\{j + \tau\}$ and then using $\sum_{j=1}^{\infty} \phi_j(x') \psi_i(x_n)$ as the approximate eigenvector. (One could avoid convergence questions by taking large finite sums.) This proves Theorem 5.

By using Fourier series rather than Fourier transforms, one can similarly prove that the operator $W_m = -\Delta - (a + 1)I$ on $L^2(\mathbb{R}^{n-1} \times [-m\pi, m\pi])$ with periodic boundary conditions on $x_n = \pm m\pi$ has spectrum in $(-\infty, 1)$.

Thus we see $1 \in \sigma(W)$ is the closure of the union of $(-\infty, 1)$ (since $\lambda_i$ is continuous).

**Remark 1.** In the next section, we need to prove that each of our bands $\lambda_i([0, 1])$ becomes negative as a parameter varies. To prove this, we use the representation (9) of $A(\tau)$. By our earlier remarks, $\{\lambda_i(\tau)\}_{i=1}^{\infty}$ are simply the eigenvalues less than 1 of the self-adjoint problem $-\Delta u + (1 + a(x', x_n))u = au$ on $L^2(\mathbb{R}^{n-1} \times [-\pi, \pi])$ with the boundary conditions that $e^{-i\tau x_n}u$ is $2\pi$ periodic in $x_n$. If we can find $m$ smooth orthogonal functions $\{\phi_i\}_{i=1}^{m}$ of compact support in $\mathbb{R}^{n-1} \times (-\pi, \pi)$ so that $\int_{\mathbb{S}}(\nabla \phi_i)^2 - a(\phi_i)^2 < 0$ for each $i$, then since $\phi_i$ are suitable test functions for every $\tau$ it follows from the variational characterization of eigenvalues that $\lambda_i(\tau) < 0$ if $\tau \in [0, 1), i = 1, \cdots, m$. This will be useful in Section 4.

**Remark 2.** There are a number of variants. We could clearly replace $-\Delta + (1 + a)I$ by $-\Delta + (B + a)I$ where $B > 0$. We have analogous theories if we restrict to functions radial in $x'$ (if $a$ is radial in $x'$) or as in [3] restrict to functions even (if $a$ is even) or functions radial in $x'$ and even in $x_n$ (if $a$ is both radial in $x'$ and even in $x_n$). The proofs need very few modifications.

**Remark 3.** In fact, the spectrum in $(-\infty, 1)$ is unchanged by working in the space of even functions. We explain this. Remembering that a closed self adjoint operator on a Hilbert space is invertible if it is onto and noting that, if $W(u) - \alpha u = h$ has a solution for every $h$ in $L^2(\mathbb{R}^n)$, then it has an even solution for every $h$ in $L^2(\mathbb{R}^n)$ (since, if $W(u) - \alpha u = h, \frac{1}{2}(u(x) + u(-x))$ is also a solution), we see that the spectrum can only decrease by going to the space of even functions. Conversely, since the spectrum is closed, it suffices to prove that if $\alpha = \lambda_i(\tau) < 1$ where $0 < \tau < 1, \tau \neq \frac{1}{2}$, then $\alpha$ is an approximate eigenvalue in the space of even functions. If $u_m(x)$ are approximate eigenvectors corresponding to $\alpha$ on the whole space, $\frac{1}{2}(u_m(x) + u_m(-x))$ is an even approximate eigenvector unless $u_m(x) + u_m(-x) \rightarrow 0$ in $L^2(\mathbb{R}^n)$ as $m \rightarrow \infty$, that is unless $u_m$ are asymptotically odd. We show that this does not happen which will complete the proof. By our construction of approximate...
eigenvectors in the proof of Theorem 5, this is valid unless the eigenvector \( v_i \) of (9) corresponding to the eigenvalue \( \alpha = \lambda_i(\tau) \) is an odd function of \( x \). Since \( \{\tau + k\}_{k \in \mathbb{Z}} \) does not intersect \( \{-\tau + k\}_{k \in \mathbb{Z}} \), that this is not the case is obvious from the Fourier expansion of \( v_i(x) \), that is, \( e^{ixn} \sum_{k=0}^{\infty} s_k(x') e^{ikxn_n} \). A similar result holds if we restrict to the functions radial in \( x' \).

**Remark 4.** We can use standard regularity theory to show that the spectrum of our operators in \((-\infty, 1)\) on strips is unchanged if we replace \( X \) by the continuous functions in \( L^\infty(\mathbb{R}^{n-1} \times [-q\pi, q\pi]) \) that decay to zero as \( |x'| \to \infty \) uniformly in \( x_n \). (Note that as in [10] eigenfunctions corresponding to eigenvalues less than 1 will decay exponentially in \( x' \).)

### 4. - Subharmonic bifurcations

In this section, we obtain many subharmonic bifurcations to solutions of large minimal period in certain cases where the nonlinearity grows so as to be subcritical in dimension \( n - 1 \) but supercritical in dimension \( n \). We also obtain more information about the global branch in more general cases.

We assume throughout the section that \( f \) satisfies the assumptions at the beginning of Section 2, that \( y^{1-p} f'(y) \to B \in (0, \infty) \) as \( y \to \infty \) where \( p > 1 \), that \( f \) is real analytic on \((-\gamma, \infty)\) where \( \gamma > 0 \) and that the local bifurcating branch at \((\bar{u}_0(\lambda_\ast), \lambda_\ast)\) is not purely vertical, that is \( \lambda \) is not locally constant on the branch near the bifurcation point. We also assume that \( n \geq 3 \) and \( \frac{n+2}{n-2} < p \leq \frac{n-1}{n-3} \), though some results hold without this. (If \( n = 3 \), the second inequality is to be interpreted as \( p < \infty \).) Finally, we assume that

\[
(12) \quad \frac{1}{2} n^{-1} (n-2) y f'(y) - F(y) > 0 \text{ on } (0, \infty),
\]

(or more generally \( -\Delta u = f(u) \) has no positive radial solution on \( \mathbb{R}^n \) decaying to zero as \( r = \|x\| \) tends to infinity). (Note that this assumption implies \( p \geq (n + 2)/(n - 2) \).) Here \( F(y) = \int_0^y f(s) ds \). We will show later that there are many examples where these conditions are satisfied and indeed the not purely vertical condition is "generic". Our main interest is the case where \( n = 3 \) and indeed our main results are only proved if \( n = 3 \).

We will modify the theory in Buffoni, Dancer and Toland [4]. We prove that the branch oscillates as it becomes unbounded and there are many secondary bifurcations off this branch. We eventually will assume that \( n = 3 \) but we study more general \( n \) as long as we can.

We will need the following lemma.

**Lemma 2.** Assume that \( -\Delta u = f(u) \) on \( \mathbb{R}^n \) where \( u \) is bounded and \( f \) satisfies the assumption at the beginning of this section and \( u \) is monotone in \( x_n \) for \( x_n \geq 0 \) and not constant for \( x_n \geq 0 \). Finally assume that \( \lim_{x_n \to \infty} u(x', x_n) \equiv h \) decays to zero as \( |x'| \to \infty \). Then \( h \equiv 0 \).
**Proof.** This is an easy modification of the proof on p. 965 of [8] if we note that since $\frac{\partial u}{\partial x_n}$ has fixed sign for $x_n \geq 0$ and $\frac{\partial u}{\partial x_n}$ solves $-\Delta v = f'(u)v$, the weak maximum principle ensures that $\frac{\partial u}{\partial x_n}(x', x_n) \neq 0$ at any point of $x_n > 0$.

**Remark.** This holds much more generally and does not use any growth condition on $f$ at infinity.

We first consider the behaviour of the global branch of Section 2. Most of this does not use the analyticity. It is easy to see that solutions in $X$ (where $X$ was defined in Section 2) are after reflection the same as the even positive solutions on $\mathbb{R}^{n-1} \times [-\pi, \pi]$ which are radial in $x'$ and decay to zero uniformly in $x_n$ as $|x'| \to \infty$. In this section it is convenient to think of our functions as defined on $\mathbb{R}^{n-1} \times [-\pi, \pi]$. By the decreasing properties of our solutions proved in Section 2, we may assume $(u, \lambda) \in D^+$ attains its global maximum at $(0, 0)$. We know the branch $D^+$ in $X$ is unbounded. Hence there exists $(u_m, \lambda_m) \in D^+$ such that

(i) $\|u_m\|_{\infty} \to \infty$ or
(ii) $\|u_m\|_{\infty} \leq K$ and $\lambda_m \to \infty$ as $m \to \infty$.

We first eliminate possibility (ii). Note that $u_m(0, 0)$ is the maximum of $u_m$ (and 0 if $|x_n| \leq \pi$). By a standard rescaling and limit argument, a subsequence of the $u_m$ (which is $u_m$ rescaled) will converge uniformly on compact sets to a positive solution of

$$(-\Delta u = f(u))$$

on $\mathbb{R}^n$ which is decreasing in $x_i$ for $x_i \geq 0$ and even in each $x_i$. (It is easy to show that $\|u_m\|_{\infty}$ has a positive lower bound since $f(\|u_m\|_{\infty}) \geq 0$ by considering where $u_m$ has its maximum.) Now, we can easily follow the arguments in the proof of Lemmas 2-4 in [8] to show that $u \equiv \hat{a}$ or $u$ is the unique positive decaying (to zero) solution of (13) or is a decaying (to zero) solution of fewer variables. (Note that this part of the arguments in [8] did not use $p$ was subcritical.) Here we use part of the proof of Lemma 1 in [8] to prove $u \geq \hat{a}, u \neq \hat{a}$ is impossible. Since $u = u(|x'|, x_n)$, $u$ can only be a function of fewer variables if $u = u(x_n)$ or $u = u_0(|x'|)$. Suppose $u = u(x_n)$. In this case, since $\tilde{u}_m$ decays in $x'$ while $u$ does not, there must exist $r_m$ large with $u_m(r_m, 0) = \epsilon_1$ where $\epsilon_1$ is small and fixed. ($r_m \to \infty$ as $m \to \infty$.) Much as in the proof of Lemma 1 in [8] to prove $u \geq \hat{a}, u \neq \hat{a}$ is impossible. Since $u = u(|x'|, x_n)$, $u$ can only be a function of fewer variables if $u = u(x_n)$ or $u = u_0(|x'|)$. Suppose $u = u(x_n)$. In this case, since $\tilde{u}_m$ decays in $x'$ while $u$ does not, there must exist $r_m$ large with $u_m(r_m, 0) = \epsilon_1$ where $\epsilon_1$ is small and fixed. ($r_m \to \infty$ as $m \to \infty$.) Much as in the proof of Lemma 1 in [8] to prove $u \geq \hat{a}, u \neq \hat{a}$ is impossible. Since $u = u(|x'|, x_n)$, $u$ can only be a function of fewer variables if $u = u(x_n)$ or $u = u_0(|x'|)$. Suppose $u = u(x_n)$. In this case, since $\tilde{u}_m$ decays in $x'$ while $u$ does not, there must exist $r_m$ large with $u_m(r_m, 0) = \epsilon_1$ where $\epsilon_1$ is small and fixed. ($r_m \to \infty$ as $m \to \infty$.) Much as in the proof of Lemma 1 in [8] to prove $u \geq \hat{a}, u \neq \hat{a}$ is impossible. Since $u = u(|x'|, x_n)$, $u$ can only be a function of fewer variables if $u = u(x_n)$ or $u = u_0(|x'|)$. Suppose $u = u(x_n)$.
subharmonic on $R^2$ and non-constant which is impossible. Thus this case does not occur.

We now show the case that $u = u_0(|x'|)$ can not occur. Here $u_0$ decays in $x'$. Now, in this case, there is a positive solution of $-\Delta h_m = f'(u_m)h_m$ on $R^{n-1} \times [0, \pi \lambda_m^{-1}]$ with Dirichlet boundary conditions on the boundary and $\|h_m\|_\infty = 1$. (Now $u_m$ is $-\frac{\partial u_m}{\partial x_n}$ rescaled.) Now $u_m \rightarrow u_0(|x'|)$ on compact sets. If $u_m \rightarrow u_0(|x'|)$ uniformly, we have by a translation to where $h_m$ has it maximum and a limiting argument, a positive bounded solution of

$$-\Delta h = f'(u_0(|x'|))h$$

or $-\Delta h = f'(0)h$ on $R^{n-1} \times [0, \infty)$ or $R^{n-1} \times (-\infty, 0)$ or $R^n$. (In the first two cases with Dirichlet boundary conditions on $x_n = 0$.) Now the problem $-\Delta' h - f'(u_0(|x'|))h = \alpha h$ on $R^{n-1}$ has a negative eigenvalue $\alpha$ corresponding to an exponential decaying positive radial eigenfunction $\phi_0$. (cp. the proof on p. 966-967 in [8].) Much as in [8], if we multiply (14) by $\phi_0$ and integrate over $R^{n-1}$ using the exponential decay of $\phi_0$ and its derivative, we easily find that $w(x_n') = \int_{R^{n-1}} h\phi_0 dx'$ satisfies $w'' = \tilde{\alpha} w$, $w$ is non-trivial (and $w(0) = 0$ if $w$ is defined only on a half space). Since $\tilde{\alpha} < 0$, we find by explicitly solving this equation, this case does not occur. The cases where the equation is $-\Delta h = f'(0)h$ on $R^n$ or a half space and $h$ is positive and bounded (and $h = 0$ on the boundary of the half space) are also impossible by using a slight generalisation of Remark 2 in of Dancer [16] to reduce the half space case to the full space case and by taking radial averages in the case of $R^n$ (which reduces our problem to a standard ordinary differential equation).

Next we consider the case that $u_m$ does not converge uniformly to $u_0(|x'|)$.

We can use a similar argument to the previous paragraph if $u_m$ does not converge uniformly (in $m$ and $x_n$) to zero as $|x'| \rightarrow \infty$. (By a translation and limiting argument, we would obtain a “homoclinic” solution as before. This is impossible.) By the decreasing properties and the uniform convergence on compact sets, $u_m(x', x_n) \leq u_0(|x'|) + \epsilon$ for $x'$ in a compact set for large $m$. Thus the lack of uniform convergence must be from below. Hence if there is not uniform convergence, $u_m(x'_m, x_{mn}) \leq u_0(|x'_m|) - \epsilon$ for all $m$ (for suitable $x'_m, x_{mn}$). By our comments above, $|x'_m|$ is bounded and $|x_{mn}| \rightarrow \infty$ as $m \rightarrow \infty$ (after taking subsequences). By decreasing $x_{mn}$, we can assume $2\epsilon \leq \lim inf_{m \rightarrow \infty} u_0(|x'_m|)$.

Thus after a shift of origin in $x_n$ and a limiting argument we obtain a positive bounded solution $\tilde{u}$ of $-\Delta' u = f'(u)$ on $R^n$ such that $\tilde{u}(x', 0) = u_0(|x'|) - \epsilon$, $\tilde{u}$ decreasing in $x_n$ (or even in $x_n$ and increasing in $x_n$ for $x_n \geq 0$), $\tilde{u}$ is radial in $x'$ and decreasing in $|x'|$ and $\tilde{u}(x', x_n) \leq u_0(|x'|)$. The last result follows from the uniform convergence from above. If $\tilde{u}$ is strictly decreasing in $x_n$, it is easy to see as before (or see p. 965 of [8]) that $\lim_{x_n \rightarrow \pm \infty} \tilde{u}(x', x_n)$ are distinct solutions of $-\Delta' u = f'(u)$ on $R^{n-1}$ which are less than or equal to $u_0$. Hence one must be positive. This contradicts Lemma 2. If $\tilde{u}$ is independent of $x_n$, $\tilde{u} = u_0$ by the assumed uniqueness which contradicts $\tilde{u}(x') = u_0(|x'|) - \epsilon$. If $\tilde{u}$ is even in $x_n$ and increasing in $x_n$ for $x_n \geq 0$, we see as above that $u$
must depend on \( x_n \). Then \( \lim_{n \to \infty} \tilde{a}(x', x_n) \) must be a positive solution of 
\(-\Delta u = f(u) \) which is below \( u_0 \). Once again this contradicts Lemma 2.

There remains the possibility that \( u \to 0 \) as \( \|x\| \to \infty \) and \(-\Delta u = f(u) \)
on \( \mathbb{R}^n \). Since \( f(0) = 0 \) and \( f'(0) < 0 \), we can easily deduce \( u \) decays exponentially (cp. Gidas, Ni and Nirenberg [18]). Thus we can easily obtain a contradiction from the Pokojajv identity (cp. [24], Remark 2.4) and our assumption on (12).

Hence our branch becomes unbounded by \( \|u\|_\infty \) becoming unbounded for
\((u, \lambda) \in D' \). We analyze the behaviour at infinity. Assume \((u_m, \lambda_m) \in D'^+ \) and
\( \|u_m\|_\infty \to \infty \) as \( m \to \infty \). By our decreasing properties \( u_m(0, 0) = \|u_m\|_\infty \).
Since \( y^{-p} f(y) \to \hat{a} \) as \( y \to \infty \), a rather standard blowing up argument
shows that either \( \lambda_m(\|u_m\|_\infty)^{p-1} \) stays bounded or we have a bounded positive
even solution of \(-\Delta u = u^p \) on \( \mathbb{R}^n \) such that \( u \) is decreasing in \( x_i \) for \( x_i > 0 \).
In the latter case, we can argue as in [13] to deduce that \(-\Delta - \lambda_m f'(u_m) I \) on
\( \mathbb{R}^{n-1} \times [-\pi, \pi] \) with our boundary conditions has many negative eigenvalues
 corresponding to even eigenfunctions radial in \( x' \) and decaying in \( x' \) for \( m \) large. (Once again this uses our assumption on \( p \).) It remains to consider the
possibility that \( \lambda_m(\|u_m\|_\infty)^{p-1} \) stays bounded. We show that this case does not
occur. Firstly, suppose \( \lambda_m(\|u_m\|_\infty)^{p-1} \to 0 \) as \( m \to \infty \). Now \(-\frac{\partial u_m}{\partial x_n} \) is a
solution of \(-\Delta h = \lambda_m f'(u_m) h \) in \( L^2(S_1) \) with Dirichlet boundary conditions
on \( \partial S_1 \). Here \( S_1 = \mathbb{R}^{n-1} \times [-\pi, \pi] \). This is impossible since our assumptions
imply \( \lambda_m f'(u_m) \to 0 \) uniformly on \( S_1 \) and since \( \int_{S_1} |\nabla h|^2 \geq k \int h^2 \) if \( h \in W^{1,2}(S_1) \),
as is easily proved by separation of variables. Here \( k > 0 \). If
\( \lambda_m(\|u_m\|_\infty)^{p-1} \to a \in (0, \infty) \) as \( m \to \infty \), \( \frac{u_m}{\|u_m\|_\infty} \) rescaled must converge
to a positive solution of \(-\Delta u = u^p \) on the infinite step \( \mathbb{R}^{n-1} \times [-c, c] \) with
Neumann boundary conditions. We can extend this solution to be periodic in
\( x_n \) and hence we can apply results in Proposition 1 to obtain a contradiction if
\( n = 3 \). It is here we need \( n = 3 \). We have proved the following theorem.

**Theorem 6.** Assume that the conditions of Theorem 3 hold, \( n = 3 \), \( p > 5 \), (12)
holds and \( y^{1-p} f'(y) \to B \in (0, \infty) \) as \( y \to \infty \). Then the connected branch
\( D^+ \) of solutions becomes unbounded by \( \|u\|_\infty \) and \( \lambda(\|u\|_\infty)^{p-1} \) tending to infinity.
Moreover as this occurs the number of negative eigenvalues \( \alpha \) of \(-\Delta h - \lambda f'(u) h = ah \) in \( X \) (counting multiplicity) tends to infinity.

**Remark 1.** If \( 1 < p < \frac{n+2}{n-2} \), an examination of our arguments show that
\( \|u_m\|_\infty \) stays bounded (and hence \( \lambda_m \to \infty \) as \( m \to \infty \) and \( u_m(\lambda_m^{-1/2} x) \) converges uniformly to the unique positive decaying solution of \(-\Delta u = f(u) \) on
\( \mathbb{R}^n \) as \( m \to \infty \)) By Theorem 4, it follows that if \( f \) is real analytic, then, in
this case, the bifurcation is not vertical when it bifurcates from \( (\tilde{u}_0(\lambda*), \lambda*) \).
2. If \( f \) is real analytic on \((-\gamma, \infty) \) the assumptions of Theorem 6 hold and the
branch \( D^+ \) does not bifurcate vertically at \( (\tilde{u}_0(\lambda_*), \lambda_*) \), then we can combine
Theorems 4 and 6 and the methods in [13] to show that the good branch \( D' \)
must have infinitely many bifurcations or changes of direction as it becomes
unbounded. In particular, uniqueness fails for some \( \lambda \).
Henceforth we assume $n = 3$ and $f$ is real analytic. We now show if the branch does not start out vertical when it branches at $\tilde{u}_0(\lambda_*)$ there is much branching to higher harmonics (in $x_n$) of arbitrary high order. More precisely, we mean to solutions of large minimal period in $x_n$. As in [4], the idea is simple. In the space $X_k$ of $2k\pi$ periodic (in $x_n$) even functions which tend to zero as $|x'| \to \infty$ uniformly in $x_n$, an eigenvalue crosses zero as we move along a branch and it crosses zero where zero is not an eigenvalue in $X = X_1$. (Here we extend a solution $u$ by periodicity.) Hence under some minor technical conditions there will be bifurcation of solutions which are in $X_k$ but not in $X_1$. These new solutions will not be the old solutions rescaled and tend to have large minimal period in $x_n$. The main difficulty is to indeed prove that an eigenvalue crosses zero in $X_k$ as we move along the branch (in fact for all large $k$) without there being a crossing in $X_1$. There is where we use the theory of Section 3. We will work with the good sub branch $D'$ constructed at the end of Section 2.

More formally, let $X_k = \{ u(x', x_n) \in L^\infty(\mathbb{R}^n) : u$ is continuous and even, $u$ is $2k\pi$ periodic in $x_n$, $u$ is radial in $x'$ and $u(x', x_n) \to 0$ as $|x'| \to \infty$ uniformly in $x_n \}$. Note that $X_1 = X$.

Suppose that $(u_m, \lambda_m)$ are an unbounded sequence on $D'$. We consider the band spectrum in the sense of Section 3 of the operator $-\Delta h - \lambda_m f'(u_m) h$ in $(-\infty, -\lambda_m f'(0))$ on the space of even functions in $L^2(\mathbb{R}^n)$ which are radial in $x'$. Note that we use here Remarks 1-4 at the end of Section 3 to ensure that the spectrum of $W_q$ is the same in many different spaces. Let $\{ \lambda_i^m(\tau) : 0 \leq \tau \leq 1 \}_{i=1}^t$ denote the bands. Note that by Remark 3 at the end of Section 3 working in a space of even functions does not affect the bands. We prove that, given $k > 0$, $\lambda_i^m(\tau) < 0$ for $0 \leq \tau < 1, 1 \leq i \leq k$ if $m$ is large. This means that $k$ of the bands have crossed zero if $m$ is large. By Remark 1 at the end of Section 3, to prove this it suffices to find $k$ orthogonal radial function $\phi_1, \ldots, \phi_k$ with compact support in $\mathbb{R}^{n-1} \times (-\pi, \pi)$, such that $\int_{\mathbb{R}^n} |\nabla \phi_i|^2 - \lambda_m f'(u_m) \phi_i^2 < 0$ for $1 \leq i \leq k$ and large $m$. We know from Theorem 6 and its proof that as the branch $D'$ becomes unbounded we have for $(u_m, \lambda_m)$ on the branch, $\lambda_m \|u_m\|_{L^p(\mathbb{R}^n)}^{p-1} \to C \in (0, \infty)$ as $m \to \infty$ while $\|u_m\|_{L^\infty(\mathbb{R}^n)}$ rescaled converges uniformly on compact sets to a decaying positive solution $\tilde{w}$ of $-\Delta \tilde{w} = \tilde{w}$ on $\mathbb{R}^n$, $\tilde{w}(0) = 1$ such that $\tilde{w}$ is radial in $x'$ and $\tilde{w}$ is decreasing in $x_i$ for $x_i > 0$. By the argument on p. 7 of [13], there exist $k$ radial orthogonal functions $h_i$ of compact support such that $\int_{\mathbb{R}^n} \frac{1}{2} |\nabla h_i|^2 - p \tilde{w}^{p-1} h_i^2 < 0$ on $\mathbb{T} = (\text{span} \{ h_i \})\setminus \{0\}$. (This uses our assumption on $p$.) It is then easy to prove (cp. [13]) that these functions rescaled are the required functions $\phi_i$ and we have proved our claim.

Next, we need to consider the bands at the bifurcation point $(\tilde{u}_0(\lambda_*), \lambda_*)$.

This is much easier because we can separate variables. If $\tilde{\lambda}_i, i = 1, \ldots, t$ (where $t$ could be infinity), are the eigenvalues less than $-\lambda_* f'(0)$ of $-\Delta h = \lambda_* f'(\tilde{u}_0(\lambda_*)) h + \alpha h$ on $\mathbb{R}^{n-1}$ with radial eigenfunctions counting multiplicity, it is easy to check that at $(\tilde{u}_0(\lambda_*), \lambda_*)$, $\lambda_i(\tau) = \inf \{ \tilde{\lambda}_i + \tau^2, 1 \}$ or $\lambda_{i,k}(\tau) = \inf \{ \tilde{\lambda}_i + (k \pm \tau)^2, 1 \}$. Note that it is easier to use the Block function decomposition here (that is (9)) and that unlike earlier the bands are not numbered to be increasing in $i$. Here $k$ is a positive integer. Note that only finitely many bands
touch or go below zero. Hence whole bands must move through zero as we move along the branch \( D' \) as we move from the bifurcation point to infinity.

We now prove that, if \( q \) is a large prime, there are points \((\tilde{u}, \tilde{\lambda})\) on \( D''\) where the Morse index of \(-\Delta h - \lambda f(u)h\) in \( X_q \) changes as \((u, h)\) crosses \((u, h)\) along the arc \( D' \) but \(-\Delta h - \tilde{\lambda} f'(\tilde{u})h\) is invertible in \( X_1 \). Moreover, we prove that, at all points close to \((\tilde{u}, \tilde{\lambda})\) on \( D'\), but not \((\tilde{u}, \tilde{\lambda})\), \(-\Delta h - \lambda f'(u)h\) is invertible in \( X_q \). Define the Morse index \( m(u, \lambda, q) \) to be the number of negative eigenvalues of \( -\Delta - \lambda f'(u)I \) counting multiplicity on \( X_q \). The main part of the proof is a counting argument as in [4]. At \((\tilde{u}_0(\lambda*), \lambda*)\), only a finite number \( k \) of bands can intersect \((-\infty, 0)\) and hence, by (11), \( m(\tilde{u}_0(\lambda*), \lambda*, q) \) is at most \( kq \) for large prime \( q \). On the other hand, as we move to infinity along \( D' \) many bands totally cross zero. Hence, if we choose a point \((u_1, \lambda_1)\) on \( D' \) where \( rk \) bands lie totally below zero, \( m(u_1, \lambda_1, q) \geq rq \) (again by (11)). On \( D'\), there are only finitely many points \( d_1, \ldots, d_r \) on the arc \( D' \) between \((\mu_0(\lambda*), \lambda*)\) and \((u_1, \lambda_1)\) where \(-\Delta - \lambda f'(u)I \) fails to be invertible on \( X \) (by Theorem 4). Consider one of these points, say \( d_1 \). For this point, only finitely many bands \( \lambda_i(\tau), i = 1, \ldots, \tilde{p}, \) can intersect zero (since \( \lambda_i(\tau) \to -\lambda f'(0) \) as \( \tau \to \infty \) uniformly in \( \tau \)). Now, for band \( i \), by Theorem 5, \( \lambda_i \) has only finitely many zeros. Hence, by (11), we see there is a bound independent of \( q \) for the multiplicity of the eigenvalue zero of \(-\Delta - \lambda f'(u)I \) at \( d_1 \) in \( X_q \). Since there are only a finite number of \( d_i \)'s and since the change of the Morse index between \((\tilde{u}_0(\lambda*), \lambda*)\) and \((u_1, \lambda_1)\) is of order \( q \), we see that for large \( q \), there must be other points on \( D' \) between \((\tilde{u}_0(\lambda*), \lambda*)\) and \((u_1, \lambda_1)\) where the Morse index changes for all large \( q \).

It remains to check the invertibility on \( X_q \) except at isolated points of \( D' \). Note that in \( X_q \) it is still a non-vertical Crandall-Rabinowitz bifurcation at \((\tilde{u}_0(\lambda*), \lambda*)\) and hence as in the proof of Theorem 4 (ii) invertibility holds at all points in \( D' \) near \((\tilde{u}_0(\lambda*), \lambda*)\) but not \((\tilde{u}_0(\lambda*), \lambda*)\). In the proof of Theorem 4 (b) in Section 2, we first constructed an unbounded curve \( \tilde{D} \) in \( D^+ \) containing \((\tilde{u}_0(\lambda*), \lambda*)\) such that invertibility (in \( X_1 \)) holds except at isolated points. (This curve may have self intersections.) Since we have already proved invertibility in \( X_q \) holds near \((\tilde{u}_0(\lambda*), \lambda*)\), we can use analyticity (technically Theorem VII.1.2 in [23]), to prove that \(-\Delta - \lambda f'(u)I \) is invertible on \( X_q \) except at isolated points. Since our good branch \( D' \) is a subset of \( \tilde{D} \), our claim follows.

If we choose \((u_2, \lambda_2) \in D' \) between \((u_1, \lambda_1)\) and \((\mu_0(\lambda*), \lambda*)\) where \(-\Delta - \lambda f'(u_2)I \) is invertible on \( X_1 \) but \( m(u, \lambda, q) \) changes at \((u_2, \lambda_2)\), we prove there is a bifurcation to solutions of \(-\Delta u = \lambda f(u) \) in \( X_q \setminus X_1 \). To do this, we modify \( f \) smoothly for \( y > \|u_2\|_\infty + 1 \) so that \( f \) is bounded for large \( y \). We now apply rather standard bifurcation theorems to \(-\Delta u = \lambda f(u) \) written in weak form on the space \( \hat{T} = \{ u \in W^{1,2}(S_q) : u \text{ is even, } u(x', -q\pi) = u(x', q\pi) \} \). Here \( S_q = \mathbb{R}^{d-1} \times [-q\pi, q\pi] \). Note that the condition on the boundary is in the trace sense. Our equation is easily seen to have a variational structure. It is easy to see that written in the weak form we obtain a good weak equation.
Theorem 7. Assume that the assumptions of Theorem 6 hold, that \( f \) is real analytic on \((-\gamma, \infty)\) (where \( \gamma > 0 \)) and the bifurcation in \( X_1 \) at \((\bar{\lambda}_0(\lambda_\ast), \lambda_\ast)\) is not purely vertical. Then for all large prime \( q \) there is secondary bifurcation to obtain positive solution of \(-\Delta u = \lambda f(u)\) on \( \mathbb{R}^n \) of minimal period \( q\pi \) in \( x_n \) and decaying to zero uniformly in \( x_n \) as \( |x'| \to \infty \).

Remark. Note that we in fact prove that there are many of those secondary bifurcations on a compact part of the branch \( D' \) (even for a fixed large \( q \)). The solutions we obtain are not the original solutions in \( X_1 \) rescaled because they have minimal period \( q\pi \) in \( x_n \) and on \( x' = 0 \) they increase and decrease at least \( q \) times from \( x_n = 0 \) to \( x_n = q\pi \).

Lastly, for this section, we need to prove that there are examples satisfying Theorem 7. The difficulty is in proving the bifurcation is not vertical. We suspect this is true for \( f(y) = y^p - y \) (for appropriate \( p \)) but we are unable to prove this. We prove it is true for \( f_\epsilon(y) = y^p - y + \epsilon h(y) \) where \( \epsilon \) is small, \( h \) is a suitable real analytic function, which is rather flat at the origin and decays rapidly at infinity. It is easy to see that the other conditions are satisfied and the question is whether we can choose \( \epsilon \) and \( h \) so that the bifurcation is not vertical. (To prove the uniqueness of the decaying radial solution on \( \mathbb{R}^{n-1} \) one can easily use the implicit function theorem to obtain local uniqueness. One then uses
very similar arguments to those in the proof of the compactness of bounded sets of $D^+$ in the proof of Theorem 3 to prove the compactness of the set of solutions uniformly in $\varepsilon$ for small $\varepsilon$). We in fact use a transversality argument and prove that it is generically true. We work in the space $X_1$. Suppose by way of contradiction that the bifurcation off $(\bar{u}_0(\lambda_+), \lambda_+)$ is always purely vertical. By the proof of the Crandall-Rabinowitz theorem in [6], one easily sees that there is a neighbourhood $U$ of $(\bar{u}_0(\lambda_+), \lambda_+)$ in $X_1 \times \mathbb{R}$ such that the solutions of $-\Delta u = \lambda f(u)$ in $U$ which depend on $x_n$ (for $|\varepsilon| \leq \delta$ and $h$ bounded in $C^2$ on compact sets) are part of the curve $\{(u(\alpha), \lambda(\alpha)) : |\alpha| \leq \mu_1\}$ where $\lambda, u$ depend continuously on $\alpha, \varepsilon, h$ (for $h$ given the $C^2$ norm on compact sets). Note that $(u(0), \lambda(0))$ also depends on $\varepsilon$ and $h$. Since we are assuming the branch is always vertical, Theorem 4 (i) implies that $\lambda$ is constant for $|\alpha| \leq \mu_1$. Moreover, by the proof of Theorem 1.17 in [6], $-\Delta - \lambda(\alpha)(\partial^2 f(u(\alpha)))I$ will have only a one-dimensional kernel on $X_1$ spanned by $k(\alpha)$ for $|\alpha| \leq \mu_1$ (uniformly in $\varepsilon, h$). Choose $\alpha_0$ with $0 < \alpha_0 < \frac{1}{2}\mu_1$. We will use a transversality argument to prove that the kernel of this operator is zero for some small $\varepsilon$ and some $h$ which gives a contradiction to the verticality. We consider a fixed $h$ (to be chosen later) and consider the map on $X_1$

$$F(u, t) = -\Delta u - \lambda_+(t)(f(u) - th(u))$$

defined for $t$ small and $u$ near $\bar{u} = u(\alpha_0)$. Here $\lambda_+(t)$ is the bifurcation point which depends analytically on $t$. We prove that the total derivative of $F$ is onto at $(\bar{u}, 0)$ and hence by transversality as in Saut and Temam [33], for most small $t$, zero is a regular value of the map $u \rightarrow F(u, t)$ when $F(u, t) = 0$ and $u$ is near $\bar{u}$ and hence the bifurcation cannot be vertical there which contradicts what we have already proved. Now by our earlier comments and self-adjointness, it is easy to see the $u$ derivative of $F$ at $(\bar{u}, 0)$ has range of codimension 1 and the range is orthogonal to the kernel vector $k(\alpha_0)$. Hence we see that the total derivative is onto if and only if the $t$ derivative at $t = 0$ of $\lambda_+(t)(f(u(\alpha_0)) - th(u(\alpha_0)))$ has a non-zero component in the direction of $k(\alpha_0)$, that is,

$$\lambda_+(0)(u(\alpha_0), k(\alpha_0)) < -\lambda_+(0)(h(u(\alpha_0)), k(\alpha_0)) \neq 0$$

where $(,)\,$ denotes the usual scalar product on $L^2(\mathbb{R}^{n-1} \times [-\pi, \pi])$. Hence we need to choose $h$ so that (15) holds.

By our earlier construction (in particular, the decreasing properties along the branch), $u(\alpha_0)$ only takes its maximum in the strip at $(0, 0)$. We choose $h$ smooth so that $h(y)$ is zero near $y = 0, 0 \leq h \leq 1, h(u(\alpha_0)(0, 0)) = 1$ and $h$ has support close to $u(\alpha_0)(0, 0)$ and not containing $u(0)(0, 0)$. This is possible since $u(\alpha)(0, 0)$ strictly increases in $\alpha$ for small positive $\alpha$. This follows since

$$\frac{\partial u}{\partial \alpha} \big|_{\alpha=0}(0, 0) = k(0)(0, 0) = \phi_0(x') \cos x_n$$
(where the last equation follows from a separation of variables and $\phi_0$ is the positive eigenfunction corresponding to the least eigenvalue of $-\Delta h - f'(u_0)h \equiv \tau h$ on $L^2(\mathbb{R}^{n-1})$. Hence $h(u(\alpha_0)(x', x_n)) = 0$ on the strip except close to $(0, 0)$. Hence, if $k(\alpha_0)(0, 0) \neq 0$,

\begin{equation}
(h(u(\alpha_0), k(\alpha_0)) \neq 0
\end{equation}

if the support of $h$ is very close to $u(\alpha_0)(0, 0)$. Note that, if $\alpha_0$ is small, $k(\alpha_0)$ is close to $k(0)$ (by continuous dependence since $u(\alpha_0)$ is close to $\bar{u}_0(\lambda_*)$), and hence $k(\alpha_0)(0, 0) \neq 0$. Hence (16) follows. Moreover, $h(u(\alpha)) \equiv 0$ if $\alpha$ is very close to zero by our choice of the support of $h$. Since we see that $\bar{u}_0(\lambda)$ and $\lambda_*$ are not changed by the term $t h(u)$ for small $t$ (for this choice of $h$), $\lambda_0'(0) = 0$. Thus (15) holds. Finally to obtain a real analytic $h$, choose $\gamma_2 > u(\alpha_0)(0, 0)$ and choose $r$ an integer with $r > 2$. Then $y^{-r} h(y)$ is $C^2$ on $[0, \gamma_2]$ and hence we can approximate it uniformly in $C^2$ on $[0, \gamma_2]$ by polynomials $z_m(y)$. Then $z_m(y) = y^r z_m(y) \exp(-m^{-1} y^2)$ is real analytic with fast decay at zero and infinity and uniformly approximates $h$ on $[0, \gamma_2]$ in the $C^2$ norm. We then easily see that $(z_m(u(\alpha_0)), k(\alpha_0))$ is close to $(h(u(\alpha_0)), k(\alpha_0))$ for large $m$. (Remember that $k(\alpha_0)$ is in $L^1$ since it decays exponentially in $x'$ on the strip, as in [10].) It is also easy but tedious to check that $\lambda_*'(0)$ depends continuously on $h$ (if $h$ is given the $C^2$ norm), Hence $\lambda_*'(0)$ is small if $h = \bar{z}_m$ and (15) holds for $h = \bar{z}_m$ for large $m$ (since $\lambda_*'(0) = 0$ for our original $h$). Hence we have proved transversality and our claim follows.

REFERENCES

NEW SOLUTIONS OF EQUATIONS ON $\mathbb{R}^d$


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