FRANCESCO MADDALENA
SERGIO SOLIMINI

Concentration and flatness properties of the singular set of bisected balls


<http://www.numdam.org/item?id=ASNSP_2001_4_30_3-4_623_0>
Concentration and Flatness Properties of the
Singular Set of Bisected Balls

FRANCESCO MADDALENA – SERGIO SOLIMINI

Abstract. In this paper we exploit the relation between the notion, introduced in [22], called splitness and the notion introduced in [9] (see [18]) called concentration. An immediate consequence is that the so called bisection property proved in [22] for the minimizers of functionals with free discontinuities implies the concentration property which in turn (see [16]) gives the lower semicontinuity of the Hausdorff measure with respect to the Hausdorff distance.

Mathematics Subject Classification (2000): 49K99 (primary), 49Q20 (secondary).

Contents

1 Notation and main results 625
2 Geometric preliminaries 627
3 Bisection properties 630
4 Flatness of minimal sets 634
5 Proof of the main results via Coarea Formula 635
6 Discrete geometric tools 638
7 Discrete bisection 640
8 Flatness of minimal punchings 647
9 Similarity arguments 649
10 Proof of Theorem 1.3 655
11 Proof of the main results through similarity arguments 657

Introduction

The aim of this paper consists in establishing some properties of the minimizers of the functionals with free discontinuities which are useful for some approaches to both the existence and the regularity theory. In spite of this fact, the content of this paper can be set at a higher level of generality and we shall never need to mention any functional in the following. Therefore we shall not introduce the reader to this subject but we shall refer him to the existent literature (see the papers [2], [3], [4], [9], [10], [11], [12], [13], [20] and the books [6], [18]).

The existence approach introduced for the case of planar domains in [9] makes use of a property called Concentration Property which was proved therein in two dimensions. On the other hand, in [22] a related property called Bisection Property has been established in any dimension. We shall show here that the first property is actually implied by the second one, this will allow the extension of the approach of [9] which will be made in [16], where the results in this paper find a first application. Moreover in [17] other aspects, based on the notion of flatness, of the arguments which we are going to develop here are used in order to produce some regularity theorems, which improve and are based on some previous ones due to L. Ambrosio, N. Fusco and D. Pallara [5].

We shall give two different proofs of the main results of the paper, a more technical but faster one based on the use of Coarea Formula and a more elementary but longer one based on a discrete technique which also gives a further result needed in some applications ([17]).

The paper is organized as follows. In Section 1 we shall introduce the basic notation and we shall state the main final results. Section 2, which is essentially descriptive, introduces the main geometric tools. Section 3 is devoted to the formal proof of a very intuitive assertion, which will be after employed and which, roughly speaking, states that the most convenient way to split a ball $B$ in parts with measure less than or equal to one half of the total measure consists in dividing the ball exactly in two parts. In Section 4 it is shown that the sets with almost minimal measure which divide the ball in two connected components with almost equal measure have a very small flatness, namely they are almost completely close to a suitable hyperplane through the center of the ball. In Section 5 we shall prove the main results by using the Coarea Formula. In Section 6 we state a discrete version of the geometric properties introduced in Section 2 and, in particular, the notion of some specific coverings, called punchings, which will be used to pass to a discretization of the properties. Sections 7 and 8 respectively are the discrete counterpart of Sections 3 and 4. In Section 9 we shall introduce some similarity arguments which will be needed for the Flatness Theorem which will then be deduced in the subsequent Section 10 and finally used in Section 11 for the alternative proof of the main results.

The results in this paper have been announced in May 1997 at the Meeting on Free Discontinuity Problems, held in Cortona and, in a more detailed form, in
October of the same year at the Meeting on Differential Equations and Calculus of Variations held at Isola d'Elba. Subsequently, in a paper of S. Rigot ([21]) a result on the existence of big $C^1$-pieces of $K$ (see also [17]) has shown in particular the concentration property for minima of the Mumford-Shah functional, giving a different proof of the main application of this paper. The arguments of [21] only need to be used in a simpler form to this aim and lead back the concentration property of the minimizers to the uniform rectifiability estimates established by G. David and S. Semmes in [11]. We are grateful to Luigi Ambrosio for the indication of the reference [7] and to Guy David for several discussions on this topic.

1. – Notation and main results

Let $X$ be a subset of $\mathbb{R}^N$, we shall denote by $\mathcal{H}^\alpha(X)$ the Hausdorff measure of dimension $\alpha$ of $X$ and by $|X|$ its Lebesgue measure. $1_X$ represents the characteristic function of $X$. If $u$ is any real function defined on a subset $X$ of $\mathbb{R}^N$, then $\text{osc}_X u$ will represent the oscillation of a function $u$ on $X$, that is $\text{osc}_X u = \sup_X u - \inf_X u$. For $\varepsilon > 0$, the closed $\varepsilon$-neighborhood of $X$ is

$$X(\varepsilon) = \{ x \in \mathbb{R}^N \mid d(x, X) \leq \varepsilon \}.$$ 

Given a subset $X \subset \mathbb{R}^N$, we shall say that $\mathcal{N}$ is an $\varepsilon$-net of $X$ if $\mathcal{N}$ is a finite set such that every point in $X$ has a distance less than $\varepsilon$ to some point of $\mathcal{N}$. The open ball with center $x \in \mathbb{R}^N$ and radius $r$, $0 < r < \infty$, will be denoted by $B(x, r)$. $B_1$ will represent the unitary ball $B(0, 1)$ of $\mathbb{R}^N$. For $\lambda \in \mathbb{R}$, we shall use the notation $\lambda B = B(x, \lambda r)$ when $B = B(x, r)$. For every $k \in \mathbb{N}$, $b_k$ denotes the measure of the unit ball of $\mathbb{R}^k$, thus $kb_k$ is the $(k-1)$-dimensional measure of its boundary.

The main results stated in the present paper make use of the notion of $\varepsilon$-splitness of a ball $B$, introduced in [22], which will be formally defined in Section 2. Roughly speaking, if $K$ is any closed subset of $B$, the $\varepsilon$-splitness of $B$ requires the existence of a function $u$ having a suitably small gradient on $B \setminus K$ and a large oscillation, quantified in terms of $\varepsilon$, on all the subsets of $B$ with a measure greater than $(\frac{1}{2} + \varepsilon)|B|$. Furthermore, the results concern some geometric properties of the set $K$, namely the $\varepsilon$-closeness to a disk, which will be introduced in Section 2, the flatness $A_K(B)$ and the tilt $T_K(B)$, which are well known in literature and will be recalled in the same section. The main results are the following.

**Theorem 1.1 (Concentration).** For every $\varepsilon > 0$ there exists $\delta$ such that, if $K$ is a closed subset of a ball $B$ of radius $R$ and $B$ is $\delta$-split by $K$, then

$$\mathcal{H}^{N-1}(K) \geq (1 - \varepsilon)b_{N-1}R^{N-1}.$$ 

(1.1)
THEOREM 1.2 (Flatness - Tilt). For every \( \epsilon > 0 \) there exists \( \delta \) such that, if a ball \( B \) of radius \( R \) is \( \delta \)-split by \( K \) and
\[
\mathcal{H}^{N-1}(K) \leq (1+\delta)B_{n-1}R^{N-1},
\]
then the set \( K \) is \( \epsilon \)-close to a disk \( D \) of \( B \) and \( A_K(B) < \epsilon, T_K(B) < \epsilon \).

We shall give two different proofs of the above theorems. The first one will be based on the use of Coarea Formula in order to convert the splitness assumption in a real disconnection condition by filling the holes of \( K \) with a suitable small level set. The second proof will be deduced by a generalization of Theorem 1.2, namely Theorem 1.3 below which turns out to be useful for further developments of the regularity properties ([17]). The gain of generality consists in the replacement of the absolute minimality assumption (1.2) with a relative minimality condition (with respect to big enough sub-balls) which we are going to introduce.

We shall fix \( \epsilon' > 0 \) and, consequently, \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \) in the following way:
\[
t = \min \left\{ \frac{\epsilon'}{5^{N+1}}, \frac{1}{4} \left( \left( \frac{1+\epsilon'}{1+\frac{2\epsilon'}{3}} \right)^{N-1} - 1 \right) \right\}
\]
and \( n \) is the least integer such that
\[
(1+3t)^{N-1} + 2^{2N-1}n^{-1} < \frac{1+\epsilon'}{1+\frac{\epsilon'}{2}}
\]
holds. So \( t \) and \( n \) are defined as functions of \( \epsilon' \). When \( \epsilon' \) is given, for any \( \delta > 0 \), \( t \) and \( n \) as above, for a given ball \( B \) of radius \( R \) and a closed subset \( K \subset B \), we introduce the following minimality condition.

For every \( \delta \)-split ball \( B' \subset B \) of radius \( r \geq t^8 R \):
\[
(M_0) \quad R^{1-N}\mathcal{H}^{N-1}(K) \leq (1+\delta)^{1-N}\mathcal{H}^{N-1}(K \cap B').
\]

THEOREM 1.3 (Flatness). For every \( \epsilon, \delta_1 > 0 \), there exist \( \epsilon' \) (only depending on \( \epsilon \)) and \( \delta_2 \) such that, if \( B \) is \( \delta_2 \)-split by \( K \) and satisfies \((M_0)\), then \( K \) is \( \epsilon \)-close to some hyperplane \( P \) through the center of \( B \).

COROLLARY 1.4. For every \( \epsilon, \delta_1, c_0 > 0 \), there exist \( \epsilon' \) (only depending on \( \epsilon \)) and \( \delta_2 \) such that, if \( B \) is \( \delta_2 \)-split by \( K \), satisfies \((M_0)\) and \( \mathcal{H}^{N-1}(K \cap B) \leq c_0 R^{N-1} \), then \( A_K(B) < \epsilon \).

The key idea of the proof relies in passing from the splitness of the ball \( B \) to a real disconnection property, through a sort of discretization of the problem. Such a discretization is achieved by substituting the set \( K \) with a finite set of small balls, which we shall call punching. Subsequently, a similarity argument, linking the configuration of \( K \) in \( B \) with that of its trace on the balls of the punching and based on \((M_0)\), will allow the proof of Theorem 1.3 and, consequently a longer but more elementary alternative proof of the main results.
2. - Geometric preliminaries

We are going to set some geometric tools which will turn out to be useful in characterizing some properties, in terms of mean densities, of the singular set $K$ of a given real function.

Let $P$ denote any affine $(N - 1)$-dimensional subspace of $\mathbb{R}^N$. Let $K$ be a closed subset of a given ball $B$ with radius $R$, if $P$ is a hyperplane through the center of $B$, then $D = P \cap B$ is a disk of $B$ and, by denoting by $p$ the orthogonal projection onto $P$, we give the following definition.

**DEFINITION 2.1.** The set $K$ is said to be $\varepsilon$-close to $D$ if

\begin{align}
\mathcal{H}^{N-1}(K \setminus P(\varepsilon R)) &\leq \varepsilon \mathcal{H}^{N-1}(K), \\
\mathcal{H}^{N-1}(D \setminus p(K)) &\leq \varepsilon R^{N-1}.
\end{align}

Let $X \subset \mathbb{R}^N$ and $u \in L^1_{\text{loc}}(X)$ be a real function. We shall denote by $\|u\|_{2N}^2$ the least constant $c$ such that the following weak summability condition of $\nabla u$ holds.

\[\text{(WS)} \quad \text{For every ball } B \subset \mathbb{R}^N : \int_B |\nabla u| \leq c|B|^{\frac{2N-1}{2N}},\]

where $\nabla u$ is defined in the sense of distributions on $\hat{X}$ and it is assumed to be extended by zero on the rest of $\mathbb{R}^N$. The above condition states that the distributional gradient $\nabla u$, extended by zero on $\mathbb{R}^N \setminus \hat{X}$, belongs to the Morrey Space $L^{1, \frac{2N-1}{2}}$. Let $B = B(x, r)$ be a given ball, $K \subset B$ any closed subset.

**DEFINITION 2.2.** Let $\varepsilon > 0$ be given. The ball $B$ is said to be $\varepsilon$-split by a function $u : B \rightarrow \mathbb{R}$ if it does not contain any subset $\bar{B}$ such that

\begin{align}
|\bar{B}| &\geq \left(\frac{1}{2} + \varepsilon\right)|B|, \\
\text{osc}_{\bar{B}} u &\leq \varepsilon^{-1}r^{\frac{1}{2}}.
\end{align}

**DEFINITION 2.3.** The ball $B$ is said to be $\varepsilon$-split by a closed set $K$ if there exists $u : B \rightarrow \mathbb{R}$, satisfying (WS) on $B \setminus K$ with $\|u\|_{2N}^2 = 1$, such that $B$ is $\varepsilon$-split by $u$.

We shall refer to any subset $\tilde{B}$ satisfying (2.7) and (2.8) as to a **region with ordinary oscillation**. In other words, for every real function $u$ with $\|u\|_{2N}^2 \leq 1$ we can find a subset with ordinary oscillation in a ball $B$ if and only if $B$ is not $\varepsilon$-split. We remark that if $u$ satisfies (WS) on $B \setminus K$ and $B$ is $\varepsilon$-split by $u$, then $B$ is $\varepsilon'$-split by $K$, with $\varepsilon' = \max\{\varepsilon, \varepsilon \|u\|_{2N}\}$. Let $B$ be a given ball.
with radius $R$, for every $\varepsilon > 0$ we shall denote by $S_\varepsilon(B)$ the set of all closed subsets $K$ of $B$ such that $B$ is $\varepsilon$-split by $K$. We put
\begin{align*}
s_\varepsilon(B) &= \inf_{K \in S_\varepsilon(B)} \mathcal{H}^{N-1}(K), \\
\sigma_\varepsilon &= s_\varepsilon(B_1)
\end{align*}
and
\begin{align*}
s_0 &= \lim_{\varepsilon \to 0} s(\varepsilon).
\end{align*}
We note that $s(\varepsilon)$ is a monotone decreasing function. By rescaling, we can easily see that for every $R > 0$
\begin{align*}
s_\varepsilon(B) &= R^{N-1}s(\varepsilon).
\end{align*}
The above quantities will be compared with the analogous ones related to disconnection conditions.

**Definition 2.4.** Let $\varepsilon > 0$ be given. The ball $B$ is said to be $\varepsilon$-disconnected by a set $K \subset B$ if there does not exist any connected component of $B \setminus K$ with measure greater than or equal to $(\frac{1}{2} + \varepsilon)|B|$.

We shall denote by $D_\varepsilon(B)$ the set of all closed subsets $K$ of $B$ such that $B$ is $\varepsilon$-disconnected by $K$. We put
\begin{align*}
\sigma_\varepsilon(B) &= \inf_{K \in D_\varepsilon(B)} \mathcal{H}^{N-1}(K), \\
\sigma_\varepsilon &= \sigma_\varepsilon(B_1)
\end{align*}
and
\begin{align*}
\sigma_0 &= \lim_{\varepsilon \to 0} \sigma(\varepsilon).
\end{align*}
By rescaling, we get for every $R > 0$
\begin{align*}
\sigma_\varepsilon(B) &= R^{N-1}\sigma(\varepsilon).
\end{align*}
We remark that if $B$ is $\varepsilon$-disconnected by $K$, then it is also $\varepsilon$-split by $K$, thus
\begin{align*}
D_\varepsilon(B) \subset S_\varepsilon(B)
\end{align*}
(indeed it can be seen by taking a constant function $u$ on the connected components of $B \setminus K$). Therefore, for every $\varepsilon > 0$ we have
\begin{align*}
(2.9) & \quad s_\varepsilon(B) \leq \sigma_\varepsilon(B), \\
(2.10) & \quad s_0 \leq \sigma_0 \leq b_{N-1}.
\end{align*}
The last inequality is really an equality (see [7]), nevertheless in order to get this information we need to prove a very intuitive property, which constitutes the subject of the next section.
If \( B' \subset B \) is any ball and \( K \in \mathcal{D}_E(B) \), we define in the usual way the
restriction \( K' \) of \( K \) to \( B' \). An extension \( K'' \) of a given \( K \in \mathcal{D}_E(B) \) to a ball
\( B'' \supset B \) with radius \( R'' > R \) is defined as any set containing \( K \) and such that
any two different connected components of \( B \setminus K \) are never contained in the
same connected component of \( B'' \setminus K'' \), namely they are not merged in the
bigger ball. We notice that such operations do not keep the property of being
in \( \mathcal{D}_E \) because they do not preserve the estimate of the relative measure of the
connected components with respect to the measure of the new ball. Such an
estimate is just what we have to prove if we want to claim that \( K' \in \mathcal{D}_E(B') \)
or \( K'' \in \mathcal{D}_E(B'') \).

Let \( B \) be any ball, with radius \( R \). By denoting with \( \rho \) the supremum
among the radii of the balls contained in \( B \), which are \( \varepsilon \)-split by \( K \), we shall
call the ratio \( v_b = \frac{\rho}{R} \) bisection factor. It represents the scale transition needed
to reach an \( \varepsilon \)-split ball in \( B \); obviously \( v_b = v_b(B, K, \varepsilon) \). We shall say that a
closed subset \( K \) of an open set \( \Omega \subset \mathbb{R}^N \) satisfies the Bisection Property on \( \Omega \)
when for every \( \varepsilon > 0 \) there exists \( \alpha(\varepsilon) \) such that, for every ball \( B \subset \Omega \) centered
on \( K \), with radius \( R \leq 1 \), the lower estimate \( v_b(B, K, \varepsilon) \geq \alpha(\varepsilon) \) holds.

Given a ball \( B \) centered on \( K \), with radius \( R \), we say that the set \( K \) is
\( \varepsilon \)-concentrated on \( B \) if the mean density of \( K \) on \( B \) is bigger than \( 1 - \varepsilon \),
namely
\[
\mathcal{H}^{N-1}(K \cap B) > (1 - \varepsilon)b_{N-1}R^{N-1}.
\]
Given any ball \( B \), with radius \( R \), we denote with \( v_c = v_c(B, K, \varepsilon) \) the concentration factor,
that is the ratio between the supremum among the radii of the
balls contained in \( B \) on which \( K \) is \( \varepsilon \)-concentrated and \( R \). We shall say that a
closed subset \( K \) of an open set \( \Omega \subset \mathbb{R}^N \) satisfies the Concentration Property
on \( \Omega \) when for every \( \varepsilon > 0 \) there exists \( \alpha(\varepsilon) \) such that, for every ball \( B \subset \Omega \) centered
on \( K \), with radius \( R \leq 1 \), the lower estimate \( v_c(B, K, \varepsilon) \geq \alpha(\varepsilon) \) holds.

We remark that Theorem 1.1 allows us to claim the following statement.

**Corollary 2.1.** For every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) \) such that, for every \( B \)
and \( K \), \( v_c(\varepsilon) \geq v_b(\delta) \).

In particular, in [22] it is proved that condition (WS) is satisfied by a
minimum \((u, K)\) of the \( N \)-dimensional Mumford-Shah functional. So the bi-
section property proved therein with respect to all the minimal pairs \((u, K)\)
falls in the definition given above. Therefore, by Corollary 2.1 we can as-
sert that the concentration property easily follows. Indeed, for every ball,
\( v_c(\varepsilon) \geq v_b(\delta) \geq \alpha(\delta(\varepsilon)) \). The concentration property was introduced for the
first time in [9], its importance relies on the circumstance that the Hausdorff
measure is lower semicontinuous along sequences of sets uniformly enjoying
such a property and converging for the Hausdorff distance (see [9] and [18,
chap. 10]). Furthermore, Theorem 1.2 already provides a reinforcement to the
uniform concentration property, indeed we can claim that, at a scale bounded
below, there exists a ball contained in \( B \) where the density of \( K \) is greater or
equal to \( 1 + \varepsilon \), or \( K \) is \( \varepsilon \)-close to a suitable hyperplane, while Theorem 1.3
provides the tool towards further reinforcements (see [17] or [21] under a different approach). As in the two dimensional case, this argument allows to prove directly the existence of a closed set minimizing the Mumford-Shah functional. For arbitrary dimension this theme is fully developed in [16] where the results of the present paper find a first application.

Now we introduce the following quantities which play a key role in studying the regularity of the set $K$ in the ball $B = B(x, r)$ (see [5] and [6] for more details).

**Scaled flatness**

$$A_K(B) = A_K(x, r) = r^{-N-1} \min_{P \in A} \int_{B \cap K} d^2(y, P) d\mathcal{H}^{N-1}(y),$$

where $A$ denotes the set of the affine hyperplanes in $\mathbb{R}^N$. $A_K(B)$ measures the scaled flatness of $K$ into the ball $B$.

**Scaled tilt**

$$T_K(B, T) = T_K(x, r, T) = r^{1-N} \int_{B \cap K} \| S_y - T \|^2 d\mathcal{H}^{N-1}(y),$$

where $T$ is a given hyperplane of $\mathbb{R}^N$ and $\| S_y - T \|$ denotes the distance of the directions of the approximate tangent space $S_y$ to a rectifiable set $K$ at $y$ from $T$. It measures the oscillation of $S_y$ in $B$ with respect to a given hyperplane. We will use the same notation also in the case of an unrectifiable set $K$, by assuming the integrand equal to one when $S_y$ is not defined. Moreover, we put

$$T_K(B) = \inf_{T \in A} T_K(B, T).$$

3. **Bisection properties**

The aim of the definition of $D_x(B)$ is to get a disconnection of the ball $B$ in parts whose measures are not much greater than one half of the measure of $B$. Although it is evident that the way to reach such a disconnection through a set with minimal measure is to create exactly two connected components, the rigorous proof of this fact is not completely obvious. The crucial point consists in showing that, if a set has almost minimal measure in $D_x(B)$, then at least one of the connected components has a measure close to $\frac{1}{2} |B|$, as stated in the following theorem. The present section is all devoted to the proof of Theorem 3.1, so, if the reader believes the assertion true, he can skip to the next section.
THEOREM 3.1. For every \( \varepsilon > 0 \) there exists \( \delta \) such that, if \( B \) is a ball of radius \( R \) and \( K \in \mathcal{D}_3(B) \) with \( \mathcal{H}^{N-1}(K) \leq (\sigma_0 + \delta)R^{N-1} \), then there exists a connected component \( C \) of \( B \setminus K \), such that \( |C| > (\frac{1}{2} - \varepsilon)|B| \).

The above property is obviously scale invariant, so we can work on a fixed ball \( B \) of radius \( R \). In order to prove Theorem 3.1 by contradiction, we assume that a constant \( \gamma < \frac{1}{2} \) can be fixed in such a way that, for \( \delta \) arbitrarily small, there exists a set \( K \in \mathcal{D}_3(B) \) with \( \mathcal{H}^{N-1}(K) < (\sigma_0 + \delta)R^{N-1} \) such that, for every \( C \) connected component of \( B \setminus K \), we have \( |C| < \gamma|B| \). Therefore, by continuity, we can fix \( R' \) and \( R'' \) depending on \( \gamma \) (but constant with respect to \( \delta \)), with \( R' < R < R'' \), such that, if \( B' \) and \( B'' \) are any pair of balls, \( B' \subset B \subset B'' \), of radius respectively greater than or equal to \( R' \) and less than or equal to \( R'' \), \( K' \) is the restriction of \( K \) to \( B' \) and \( K'' \) is any extension of \( K' \) from \( B' \) to \( B'' \), then \( K' \) and \( K'' \) are respectively in \( \mathcal{D}_3(B') \) and \( \mathcal{D}_3(B'') \).

For \( \rho > 0 \) let \( B(\rho) = \frac{\rho}{R}B \) be the ball with the same center of \( B \) and radius \( \rho \); we define the following density function

\[
(3.11) \quad d(\rho) = \frac{\mathcal{H}^{N-2}(K \cap \partial B(\rho))}{\rho^{N-2}}.
\]

In order to simplify the computations, we will work up to quantities which are infinitesimal as \( \delta \) tends to zero. To this aim, we will denote by \( h(\delta) \) any function which tends to zero as \( \delta \) tends to zero, and we shall omit to specify it more. So, in general, different occurrences of the symbol \( h(\delta) \) will stand for different quantities with the common feature of being infinitesimal, analogously to the Landau symbols \( O \) and \( o \), which we are also going to use. Our first aim consists in proving some lemmas whose final goal relies in stating that, modulo an infinitesimal error, the density \( d(\rho) \) is a constant function for \( R' \leq \rho \leq R \). This result will represent the main ingredient used in the proof of Theorem 3.1 and can be stated as follows.

**Lemma 3.1.** There exists a positive constant \( \bar{d} \), such that

\[
(3.12) \quad \int_{R''}^{R} |d(\rho) - \bar{d}|\rho^{N-2}d\rho = h(\delta),
\]

\[
(3.13) \quad \mathcal{H}^{N-1}\left(K \setminus \frac{R'}{R}B\right) = \bar{d}\int_{R''}^{R} \rho^{N-2}d\rho + h(\delta).
\]

We shall give the proof of Lemma 3.1 after some preliminary steps.

**Lemma 3.2.** For every \( r \in [R', R] \) let \( B' = \frac{r}{R}B \), then the following inequality holds.

\[
(3.14) \quad \mathcal{H}^{N-1}(K \setminus B') \leq \sigma_0(R^{N-1} - r^{N-1}) + h(\delta).
\]

**Proof.** As we have already noted, \( K' \) is in \( \mathcal{D}_3(B') \), so we have by definition

\[
(3.15) \quad \mathcal{H}^{N-1}(K') \geq \sigma(\delta)r^{N-1} = (\sigma_0 + h(\delta))r^{N-1}.
\]

Moreover, by hypothesis we know that

\[
(3.16) \quad \mathcal{H}^{N-1}(K) \leq (\sigma_0 + \delta)R^{N-1} = (\sigma_0 + h(\delta))R^{N-1}.
\]

By combining (3.15) and (3.16), we get (3.14). \( \Box \)
By (3.14) we obtain a first integral estimate on \( d(\rho) \), indeed for every \( r \in [R', R] \) the following inequality holds

\[
\int_{R'}^{R} d(\rho) \rho^{N-2} d\rho \leq \mathcal{H}^{N-1}(K \setminus B') \leq \sigma_0(R^{N-1} - r^{N-1}) + h(\delta).
\]

Now we want to get a uniform lower estimate on \( d(\rho) \) which matches (3.17) and shows that the two inequalities are actually equalities. To this aim, we are going to consider for a fixed \( r \in [R', R] \) the restriction \( K' \) of \( K \) to the concentric ball \( B' \) to \( B \) of radius \( r \) and then the radial extension \( K'' \) of \( K' \) to the ball \( B'' \) of radius \( R'' \). To this aim we set

\[
\bar{d} = \inf_{\rho \in [R', R]} d(\rho)
\]

and we take \( \bar{\rho} \in [R', R] \) such that

\[
d(\bar{\rho}) \leq (1 + \delta) \bar{d}.
\]

**Lemma 3.3.** The following lower bound holds true

\[
\bar{d} \geq \sigma_0(N - 1) + h(\delta).
\]

**Proof.** If \( B' \) is the ball with the same center of \( B \) and radius \( \bar{\rho} \), by taking \( K'' \) as above, we get

\[
\mathcal{H}^{N-1}(K'') = \mathcal{H}^{N-1}(K) - \mathcal{H}^{N-1}(K \setminus B') + d(\bar{\rho}) \int_{\bar{\rho}}^{R''} \rho^{N-2} d\rho.
\]

So

\[
\mathcal{H}^{N-1}(K'') \leq \mathcal{H}^{N-1}(K) - \int_{\bar{\rho}}^{R} d(\rho) \rho^{N-2} d\rho + d(\bar{\rho}) \int_{\bar{\rho}}^{R''} \rho^{N-2} d\rho
\]

\[
\leq (\sigma_0 + \delta) R^{N-1} - \int_{\bar{\rho}}^{R} \rho^{N-2} d\rho + (1 + \delta) \bar{d} \int_{\bar{\rho}}^{R''} \rho^{N-2} d\rho
\]

\[
= (\sigma_0 + \delta) R^{N-1} + \bar{d} \int_{\bar{\rho}}^{R''} \rho^{N-2} d\rho + h(\delta).
\]

Since \( K'' \) is in \( D_3(B'') \) and therefore

\[
\mathcal{H}^{N-1}(K'') \geq \sigma(\delta)(R'')^{N-1} = (\sigma_0 + h(\delta))(R'')^{N-1},
\]

by (3.21) and (3.22) we finally get

\[
\bar{d} \geq \left( \int_{\bar{\rho}}^{R''} \rho^{N-2} d\rho \right)^{-1} \left( (\sigma_0 + h(\delta))(R''^{N-1} - (\sigma_0 + \delta) R^{N-1}) + h(\delta) \right)
\]

\[
= \sigma_0(N - 1) + h(\delta).
\]

**Proof of Lemma 3.1.** By integrating (3.19) we get

\[
\int_{R'}^{R} \bar{d} \rho^{N-2} d\rho \geq \sigma_0(R^{N-1} - (R')^{N-1}) + h(\delta),
\]

so, by using (3.17) for \( r = R' \), since \( d(\rho) - \bar{d} \geq 0 \), we finally have (3.12). So the inequalities in (3.17) are equalities and the first one of them, combined with (3.12) gives (3.13). \( \square \)
PROOF OF THEOREM 3.1. We assume by contradiction that the thesis is false, so we can use all the construction made in this section. We set $R^* = R + R'$ and for every $y \in \partial B$ we denote by $B^y$ the ball with radius $R^*$ contained in $B$ and tangent in $y$ to $\partial B$, whereas $B^*$ is the ball with the same center of $B$ and radius $R^*$. We set

$$p(x) = \int_{\partial B_R} 1_{B^y}(x) d\mathcal{H}^{N-1}(y) \leq 1.$$  

Because of the radial symmetry of the mappings $1_{B^*}$ and $p$, we can deal with $1_{B^*}(\rho)$ and $p(\rho)$. We can claim that

$$\int_{\mathbb{R}^N} p(x) dx = |B^*|$$

and therefore

$$\int_{\mathbb{R}^N} (1_{B^*} - p) dx = 0, \text{ namely } \int_0^\infty (1_{B^*} - p)(\rho) \rho^{N-1} d\rho = 0.$$  

We note that, since

$$(1_{B^*} - p)(\rho) \begin{cases} = 0 & \text{if } 0 < \rho < R' \\ > 0 & \text{if } R' < \rho < R^* \\ < 0 & \text{if } R^* < \rho < R, \end{cases}$$

then we can find a positive constant $\bar{c}$ such that

$$\int_0^R (1_{B^*} - p)(\rho) \rho^{N-2} d\rho > \bar{c}.$$  

Since $1_{B^*} = p$ for $\rho \leq R'$,

$$\int_{R'}^R p(\rho) \rho^{N-2} d\rho \leq \int_{R'}^{R^*} \rho^{N-2} d\rho - \bar{c}.$$  

Now we are going to estimate the mean trace $t_m$, defined as

$$t_m = \int_{\partial B} \mathcal{H}^{N-1}(K \cap B^y) d\mathcal{H}^{N-1}(y) = \int_{\partial B} \left( \int_K 1_{B^y} d\mathcal{H}^{N-1} \right) d\mathcal{H}^{N-1}(y) = \int_K p \ d\mathcal{H}^{N-1}$$

and to this aim we evaluate $t_m$ by using (3.12) and (3.13).

$$t_m = \int_{K \cap B'} p \ d\mathcal{H}^{N-1} + \int_{K \setminus B'} p \ d\mathcal{H}^{N-1} = \mathcal{H}^{N-1}(K \cap B') + \int_{R'}^{R} \bar{d} \rho^{N-2} p(\rho) d\rho + h(\delta).$$
By (3.12), (3.13), (3.28) and (3.19) we get

\[ t_m \leq \mathcal{H}^{N-1}(K \cap B') + \int_{R^*} \overline{d} \rho^{N-2} d \rho + h(\delta) - \overline{d} \overline{c} \]

\[ = \mathcal{H}^{N-1}(K \cap B') + \int_{R^*} \overline{d} \rho^{N-2} d \rho - \int_{R^*} \overline{d} \rho^{N-2} d \rho + h(\delta) - \overline{d} \overline{c} \]

\[ = \mathcal{H}^{N-1}(K \cap B) - \frac{\overline{d}}{N-1} (R^{N-1} - (R^*)^{N-1}) + h(\delta) - \overline{d} \overline{c} \]

\[ \leq \sigma_0 R^{N-1} - \sigma_0(R^{N-1} - (R^*)^{N-1}) + h(\delta) - \overline{c} \sigma_0 (N - 1) \]

\[ < \sigma(\delta)(R^*)^{N-1} , \]

for a small \( \delta \). Then we can find \( y \in \partial B \) such that \( \mathcal{H}^{N-1}(K \cap B^y) < \sigma(\delta)(R^*)^{N-1} \), which leads to a contradiction, since \( K \cap B^y \in \mathcal{D}_\delta(B^y) \). 

\[ \square \]

4. – Flatness of minimal sets

In this section we will show how, thanks to the property by which the set with minimal measure which splits a ball \( B \) of radius \( R \) in two connected components of equal measure is a disk, we can claim that \( \sigma_0 = b_{N-1} \) and the sets in \( \mathcal{D}_\delta(B) \) with almost minimal measure are close to a hyperplane through the center of \( B \). More precisely, we shall prove the following theorem.

**Theorem 4.1.** \( \sigma_0 = b_{N-1} \) and for every \( \varepsilon > 0 \) there exists \( \delta \) such that every set \( K \in \mathcal{D}_\delta(B) \), with \( \mathcal{H}^{N-1}(K) \leq (\sigma_0 + \delta)R^{N-1} = (b_{N-1} + \delta)R^{N-1} \), turns out to be \( \varepsilon \)-close to some disk through the center of \( B \).

**Proof.** Let \( B \) be a given ball and let \((K_n)_{n \in \mathbb{N}}\) be a sequence of sets such that, for every \( n \in \mathbb{N} \), \( \mathcal{H}^{N-1}(K_n) \leq (\sigma_0 + \frac{1}{n})R^{N-1} \leq (b_{N-1} + \frac{1}{n})R^{N-1} \) and let \( C_n \) be the connected component of \( B \), relative to \( K_n \), having maximal measure. By Theorem 3.1 we get \( |C_n| \to \frac{1}{2}|B| \) as \( n \to \infty \), furthermore we note that \((C_n)_{n \in \mathbb{N}}\) is a sequence of sets with finite perimeter in \( B \) (Caccioppoli sets), that is \( 1_{C_n} \in BV(B) \) for every \( n \), \( BV(B) \) denoting the space of functions of bounded variation defined on \( B \) (see [14], [15]). Since the perimeters of the sets \( C_n \) are bounded, \((1_{C_n})_{n \in \mathbb{N}}\) is a bounded sequence in \( BV(B) \) and, by the compactness theorem on BV functions ([15, Theorem 1.19]), we know that there exists a subsequence \((1_{C_{n_k}})_{k \in \mathbb{N}}\), by which we shall replace the whole \((1_{C_n})_{n \in \mathbb{N}}\), converging in \( L^1(B) \) to a function which can be identified as the characteristic function of a measurable set \( \overline{C} \). Then

\[ |\overline{C}| = \lim_{n} |C_n| = \frac{1}{2}|B| . \]
By virtue of a suitable relative isoperimetric inequality (see [7, Theorem 18.3.4]), we can say that (4.29) implies

\[(4.30) \quad P(\overline{C}, B) \geq b_{N-1} R^{N-1},\]

where \(P(A, B) = |\partial A|_B\) denotes the perimeter of \(A\) in \(B\) and, if the equality holds, that the reduced boundary \(\partial^*\overline{C}\) can be identified with a disk of \(B\). The reverse inequality follows by semicontinuity and enables us to conclude that \(\partial^*\overline{C}\) is a disk and that \(a_0 = b_{N-1}\). The closeness of \(K_n\) to \(\partial C\) follows then by simple arguments (further details will be given in Section 8 in the more complicated discrete case).

5. Proof of the main results via Coarea Formula

It is well known ([19, Theorem 3.5.2]) that if \(\Omega \subset \mathbb{R}^N\) is connected and suitably regular and \(u : \Omega \to \mathbb{R}\) satisfies condition (WS), then \(u\) is Hölder continuous with index \(\frac{1}{2}\). Therefore, for \(u : B \to \mathbb{R}\) satisfying (WS) on \(B \setminus K\), the condition that \(B\) is \(\varepsilon\)-split by \(u\) leads to guess that the singular set \(K\) should disconnect \(B\) in at least two parts of almost equal measure. This is not true, indeed one can easily give examples of sets \(K\) which split \(B\), leaving it connected. On the other hand, what really happens is not so far from a disconnection, analogously to the situation considered in [11] about the failure of Poincaré Inequality. Indeed we shall prove that the sets \(K\) of minimal measure, among those which split \(B\), must be close to some disk of \(B\), in the sense of Definition 2.1. The main argument used to prove this result is Coarea Formula, whose application requires a preliminary regularization result which is proved in the following lemma.

According to Definition 2.3, if \(K \in S_s(B)\) we can find a function \(u\) splitting \(B\) and such that \(\|u\|_{2N} = 1\) which implies, in particular, that

\[(5.31) \quad \int_{B \setminus K} |\nabla u| \leq R^{N-\frac{1}{2}}\]

and this is the only summability property which we are going to use in this section.

**Lemma 5.1.** Let \(\varepsilon > 0\) be given and \(B\) be \(\varepsilon'\)-split by \(K\) for some \(\varepsilon' < \varepsilon\). Then there exists a function \(u \in C^\infty(B \setminus K)\), satisfying (5.31) and such that \(B\) is \(\varepsilon\)-split by \(u\).

**Proof.** Since \(B\) is \(\varepsilon'\)-split by \(K\) then there exists a function \(u' \in L^1_{\text{loc}}(B \setminus K)\) as in Definition 2.2. By truncating \(u'\) and applying Meyers-Serrin theorem ([1, Theorem 3.16]) we can take a sequence \((\varphi_n)_{n \in \mathbb{N}}\) with \(\varphi_n \in C^\infty(B \setminus K)\), such that \(\varphi_n \to u'\) in \(L^1_{\text{loc}}(B \setminus K)\) and \(\nabla \varphi_n \to \nabla u'\) in \(L^1(B \setminus K)\). For every \(\lambda \in ]0, 1[\) and
for \( n \in \mathbb{N} \) large enough, \( \lambda \varphi_n \) satisfies (5.31). Now we show that for every fixed \( \varepsilon > \varepsilon' \) and for \( \lambda \) close enough to 1, \( \lambda \varphi_n \) cannot have an ordinary oscillation region. Indeed, let us suppose that, given \( \varepsilon > 0 \), for every \( n \in \mathbb{N} \) there exists \( \tilde{B}_n \) such that \( |\tilde{B}_n| \geq \left( \frac{1}{2} + \varepsilon \right) |B| \) and \( \text{osc}_{\tilde{B}_n} \lambda \varphi_n \leq \varepsilon^{-1} R^{\frac{1}{2}} \). Now, since \( \varphi_n \to u' \) in \( L^1_{\text{loc}}(B \setminus K) \) and \( B \setminus K \) is a bounded open set, then \( \varphi_n \) converges in measure to \( u' \), so, by fixing a convenient value of \( \lambda \) and by setting for every \( n \)

\[
X_n = \left\{ x \in B \setminus K \mid |\varphi_n(x) - u'(x)| > \frac{1}{2} \left( \frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon} \right) R^{\frac{1}{2}} \right\},
\]

for \( n \) large enough we get \( |X_n| < (\varepsilon - \varepsilon')|B| \). For such a value of \( n \) we take \( \widetilde{B} = \tilde{B}_n \setminus X_n \), then

\[
|\widetilde{B}| \geq |\tilde{B}_n| - |X_n| \\
\geq \left( \frac{1}{2} + \varepsilon \right) |B| - (\varepsilon - \varepsilon')|B| \\
= \left( \frac{1}{2} + \varepsilon' \right) |B|.
\]

Moreover, for every \( x, y \in \widetilde{B} \) we have \( x, y \notin X_n \) and so

\[
|\varphi_n(x) - u'(x)| + |\varphi_n(y) - u'(y)| \leq \left( \frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon} \right) R^{\frac{1}{2}}.
\]

Then

\[
|u'(x) - u'(y)| \leq |\varphi_n(x) - \varphi_n(y)| + \left( \frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon} \right) R^{\frac{1}{2}} \\
\leq \frac{\lambda^{-1}}{\varepsilon} R^{\frac{1}{2}} + \left( \frac{1}{\varepsilon'} - \frac{\lambda^{-1}}{\varepsilon} \right) R^{\frac{4}{2}} \\
= \frac{\lambda^{-1}}{\varepsilon} R^{\frac{1}{2}},
\]

which is a contradiction since \( u' \) splits \( B \). Therefore, for \( n \) large enough \( \lambda \varphi_n \) realizes the function \( u \) as required in the thesis. \( \Box \)

The previous result enables us to take, after a slight modification of \( \varepsilon \), the function \( u \) splitting \( B \) with the smoothness needed to apply Coarea Formula which will be employed in the next lemma.

**Lemma 5.2.** For every \( \varepsilon' < \varepsilon \) and for every \( K \in S_{\varepsilon'}(B) \) there exists a closed set \( H \) such that \( \mathcal{H}^{N-1}(H) \leq 4 \varepsilon R^{N-1} \) and \( K \cup H \in \mathcal{D}_{\varepsilon}(B) \).
PROOF. Let $u$ be as in Lemma 5.1, for every $I \in \mathbb{R}$ we introduce the following notation.

- $X_I = \{ x \in B \setminus K \mid u(x) = I \}$,
- $S_I = \{ x \in B \setminus K \mid u(x) < I \}$,
- $S^I = \{ x \in B \setminus K \mid u(x) > I \}$.

We set

$$
\bar{l} = \sup \left\{ I \in \mathbb{R} \mid |S_I| < \frac{1}{2}|B| \right\}
$$

and we see that $\bar{l} \in \mathbb{R}$. Then we take two open contiguous intervals $I_1, I_2 \in \mathbb{R}$ each one of measure $\frac{1}{2\varepsilon} R^{\frac{1}{2}}$ and with $\bar{l}$ as common extremum. By applying Coarea Formula, from (5.31) we get the following estimate for $i = 1, 2$.

$$
\int_{I_i} \mathcal{H}^{N-1}(X_i) dl \leq 2\varepsilon R^{-\frac{1}{2}} \int_B \| \nabla u \| \leq 2\varepsilon R^{N-1}.
$$

Therefore there exist $l_1 \in I_1$ and $l_2 \in I_2$ such that $\mathcal{H}^{N-1}(X_{l_i}) \leq 2\varepsilon R^{N-1}$ for $i = 1, 2$. Let $H = X_{l_1} \cup X_{l_2}$, then

$$
\mathcal{H}^{N-1}(H) \leq 4\varepsilon R^{N-1}.
$$

By observing that each connected component $C$ of $B \setminus (K \cup H)$ is contained in one of the sets $S_{l_1}, S_{l_2}$ and $S_{l_1} \cap S_{l_2}$, we get one of the following estimates on its measure:

$$
|C| \leq |S_{l_1}| \leq \frac{1}{2}|B| \quad \text{(since } l_1 < \bar{l}),
$$

$$
|C| \leq |S_{l_2}| \leq \frac{1}{2}|B| \quad \text{(since } l_2 > \bar{l})
$$

or

$$
|C| \leq |S_{l_1} \cap S_{l_2}| \leq \left( \frac{1}{2} + \varepsilon \right) |B| \quad \text{(since } \text{osc}_{S_{l_1} \cap S_{l_2}} u \leq \frac{1}{\varepsilon} R^{\frac{1}{2}} \text{ and } u \text{ splits } B).
$$

So we can conclude that $K \cup H \in \mathcal{D}_e(B)$. \hfill \Box

The following assertions are simple consequences of the above result and of Theorem 4.1.

**Corollary 5.1.** For every $\varepsilon > \varepsilon' > 0$

$$
\sigma(\varepsilon') \geq \sigma(\varepsilon) - 4\varepsilon.
$$

**Corollary 5.2.**

$$
\sigma_0 = \sigma_0 = b_{N-1}.
$$

**Proof of Theorem 1.1.** Given $\varepsilon > 0$, we take $\delta > 0$ such that

$$
\sigma(\delta) \geq (1 - \varepsilon)\sigma_0 = (1 - \varepsilon)b_{N-1}
$$

and so the thesis is proved. \hfill \Box
Proof of Theorem 1.2. Let \( \varepsilon > 0 \) be given. We fix \( \delta \) as in Theorem 4.1 and we fix \( K \in S_{\frac{\varepsilon}{3}}(B) \) such that \( \mathcal{H}^{N-1}(K) \leq (\sigma_0 + \frac{\varepsilon}{2}) R^{N-1} \). By Lemma 5.2 we take a closed set \( H \) such that \( \mathcal{H}^{N-1}(H) \leq \frac{4}{3} \delta \) and \( K \cup H \in D_\delta(B) \). Obviously

\[
\mathcal{H}^{N-1}(K \cup H) \leq (b_{N-1} + \delta) R^{N-1}.
\]

Because of the \( \varepsilon \)-closeness of \( K \) to a disk \( D \), guaranteed by Theorem 4.1, we have that

\[
A_K(B) \leq (\varepsilon^2 + \varepsilon) R^{1-N} \mathcal{H}^{N-1}(K) \leq (\varepsilon^2 + \varepsilon) (b_{N-1} + \delta).
\]

We can always take \( D \) such that the orthogonal projection of the irregular part of \( K \) has null \( (N-1) \) measure. Then from

\[
\mathcal{H}^{N-1}(K) \leq (b_{N-1} + \delta) R^{N-1} \leq \mathcal{H}^{N-1}(\rho(K)) + (\varepsilon + \delta) R^{N-1},
\]

the smallness of \( T_K(B) \) follows. By the arbitrariness of \( \varepsilon \) and \( \delta \) we deduce the thesis. \( \square \)

6. - Discrete geometric tools

Now we are going to introduce a tool which will turn out to be useful in the subsequent developments, it will allow a sort of discretization of the main problems discussed in this paper. Roughly speaking, we will enclose the singular set \( K \subset B \) in a finite set made of small balls \( B_i \subset B \). This set, which will be named punching, has, for some aims, nicer properties than the set \( K \) because the measure properties of the last one can be controlled by the analogous one of the punchings through a similarity argument. Let \( B \subset \mathbb{R}^N \) be a given ball with radius \( R \), we take a finite set \( \mathcal{B} \), made of balls \( B_i \) with radius \( r_i \), and define the measure of \( B \) as

\[
\mu(B) = \sum_i r_i^{N-1}.
\]

DEFINITION 6.1. For every \( \varepsilon > 0 \), \( B \) will be named \( \varepsilon \)-punching, or punching with a thickness less than or equal to \( \varepsilon \), if it is made of balls \( B_i \) with radius \( r_i \) such that \( \forall i, r_i \leq \varepsilon R \) and such that all the connected components of \( B \setminus \bigcup_{B_i \in B} B_i \) have a measure less than or equal to \( (\frac{1}{2} + \varepsilon)|B| \).

We shall denote by \( \mathcal{P}_\varepsilon(B) \) the set of all the \( \varepsilon \)-punchings of \( B \), which is, roughly speaking, the discrete counterpart of the set \( D_\varepsilon \) introduced above. Analogously, beside \( \sigma_\varepsilon \), we introduce the following notation.

\[
m_\varepsilon(B) = \inf_{B \in \mathcal{P}_\varepsilon(B)} \mu(B),
\]
and, by similarity, 
\[ m_\varepsilon(B) = m(\varepsilon)R^{N-1}, \]
where we have set 
\[ m(\varepsilon) = m_\varepsilon(B_1). \]
Finally, we note that \( m(\varepsilon) \) is a monotone decreasing function; we set
\[ m_0 = \lim_{\varepsilon \to 0} m(\varepsilon) = \sup m(\varepsilon) \leq 1, \]
as one can easily check by covering a disk with suitably small balls. We finally give the suitable version of Definition 2.1 whose notation is kept.

**Definition 6.2.** The punching \( B \) is said to be \( \varepsilon \)-close to \( D \) if
\[
\mu(\mathcal{B} \setminus P(\varepsilon R)) \leq \varepsilon \mu(\mathcal{B}),
\]
\[
\mathcal{H}^{N-1} \left( D \setminus \bigcup_{B' \in \mathcal{B}} B' \right) \leq \varepsilon R^{N-1}.
\]

If \( r_1 < r_2 \), we shall denote by \( A_{r_2}^{r_1} \) the annulus of \( B \) enclosed between the radii \( r_1 \) and \( r_2 \) and, if \( B \) is any collection of balls, we shall use the notation
\[ \mu_{r_1}^{r_2}(B) = \mu(\{ B' \in B \mid B' \cap A_{r_2}^{r_1} \neq \emptyset \}). \]

We shall make use of two main operations on the punchings, namely restrictions and extensions which we are going to define. If \( B' \subset B \) is any ball and \( B \in \mathcal{P}_\delta(B) \), we define the restriction \( B' \) of \( B \) to \( B' \) as
\[ B' = \{ B_i \in B \mid B_i \cap B' \neq \emptyset \}. \]

A \( \delta \)-extension \( B'' \) of a given \( B \in \mathcal{P}_\delta(B) \) to a ball \( B'' \supset B \) with radius \( R'' > R \) is defined as a set of balls obtained by adding new balls \( B_i \) with a radius \( r_i \) such that \( \sup r_i \leq \delta R'' \) and such that two different connected components of \( B \setminus \bigcup_{B_i \in B} B_i \) are not merged in the largest ball and so they are contained in two different connected components of \( B'' \setminus \bigcup_{B_i \in B''} B_i \). We point out that, as happens in the continuous case, that, in general, restrictions and extensions of punchings are not still punchings.

A particular extension, which will be employed in Section 3, is the one which we shall call extension by reflection which allows to define a \( \lambda \delta \)-extension of a \( \delta \)-punching \( B \) from \( B \) to \( \lambda B \) (\( \lambda > 1 \)), by using the balls which intersect \( A_{\lambda-1}^R \). More precisely, let \( B_i = B(x_i, r_i) \) the balls such that \( B_i \cap A_{\lambda-1}^R \neq \emptyset \), the extension by reflection \( B'' \) of \( B \) is defined by adding to \( B \) the balls \( B(y_i, \lambda^2 r_i) \), with \( y_i = \frac{x_i R^2}{|x_i|^2} \). The measure of the new set of balls is given by \( \mu(B) + \lambda^{2(N-1)} \mu_{\lambda-1}^R(B) \). We remark that such a construction prevents merging.
of connected components. Such an extension can be iterated up to $\lambda^2 R$, but in the next step, in order to prevent merging, it is not necessary to multiply once more by $\lambda^2$ the radii, it suffices to add balls with center $\lambda^2 x_i$ and radius $\lambda^2 r_i$ as in the previous extension. Thus we get a $\delta$-extension of $B$. With this strategy we can iterate up to the ball $\lambda^n B$ and the contribution of the new balls inherent to $A_{\lambda^{i-1} R}$ is estimated by $\lambda^{(i+1)(N-1)} \mu^R B (B)$, (the term $+1$ being necessary only if $i$ is odd). By iterating the construction, we get a $\lambda^2 \delta$-punching of any ball $B'' \supset B$, with the same center of $B$. In the following, we shall refer to such an extension as to the radial extension of $B$ to $B''$ from the annulus $A_{\lambda^{n-1} R}^R$. This is at most a $\lambda^2 \delta$-extension of $B$ and is actually a $\delta$-extension when $R'' = \lambda^n R$, with $n$ even.

The following definition states, roughly speaking, that a set $K$ is well packed in the balls of a punching $B$ if $B$ has a small measure and if the part of $K$ left out of the balls of $B$ or in a few of them is small.

**Definition 6.3.** We shall say that a $\varepsilon$-punching $B$ of $B$ with radius $R$ is a $\varepsilon$-packing of a closed subset $K \subset B$, if

$$\mu(B) \leq (1 + \varepsilon) R^{N-1}$$

and, if $B'$ is any subset of $B$ such that $\mu(B) < 2 \mu(B')$, then

$$\mathcal{H}^{N-1} \left( K \setminus \bigcup_{B \in B'} B \right) \leq \left( \varepsilon + R^{1-N} (\mu(B) - \mu(B')) \right) \mathcal{H}^{N-1}(K).$$

### 7. Discrete bisection

This section contains the discrete version of the arguments in Section 3 which has been fully developed for the reader's convenience. Indeed, even if the arguments are constructed in the same way, some extra difficulties due to the discrete formulation must be taken into account.

**Theorem 7.1.** For every $\varepsilon > 0$ there exists $\delta$ such that, if $B$ is a ball of radius $R$ and $B \in P_\delta(B)$ with $\mu(B) \leq (m_0 + \delta) R^{N-1}$, then there exists a connected component $C$ of $B \setminus \bigcup_{B \in B'} B$, such that $|C| > (\frac{1}{2} - \varepsilon)|B|$. 

The same remarks in the beginning of Section 3 can be made also here. The above property is obviously scale invariant, so we can work on a fixed ball $B$. In order to prove Theorem 7.1 by contradiction, we assume that a constant $\gamma > \frac{1}{2}$ can be fixed in such a way that, for $\delta$ arbitrarily small, there exists a $\delta$-punching $B$ with $\mu(B) < (m_0 + \delta) R^{N-1}$ such that, for every $C$ connected component of $B \setminus \bigcup_{B \in B'} B$, we have $|C| < \gamma |B|$. Therefore, by continuity,
we can fix $R'$ and $R''$ depending on $\gamma$ (but constant with respect to $\delta$), with $R' < R < R''$, such that, if $B'$ and $B''$ are any pair of balls, $B' \subseteq B \subseteq B''$, of radius respectively greater or equal to $R' - 2\delta R$ and less than or equal to $R''$, $B'$ is the restriction of $B$ to $B'$ and $B''$ is any $2\delta$-extension of $B'$ from $B'$ to $B''$, then $B'$ and $B''$ are $2\delta$-punchings respectively of $B'$ and $B''$. Now we are going to introduce some functions which will be employed in the following. Let $\rho > 0$, we put $S_{\rho} = A^{\rho}_{\rho(1+\sqrt{\delta})^{-1}}$, moreover for every $B_i \in B$ we define

$$c_i(\rho) = \begin{cases} 1 & \text{if } B_i \cap S_{\rho} \neq \emptyset \\ 0 & \text{if } B_i \cap S_{\rho} = \emptyset. \end{cases}$$

We will denote by $f(\rho)$ the measure $\mu^\rho_{\rho(1+\sqrt{\delta})^{-1}}(B)$, normalized by $\sqrt{\delta}$, namely, if $n = \text{card}(B)$,

$$f(\rho) = \frac{1}{\sqrt{\delta}} \mu^\rho_{\rho(1+\sqrt{\delta})^{-1}}(B) = \frac{1}{\sqrt{\delta}} \sum_{i=1}^{n} c_i(\rho) \rho_i^{N-1},$$

so we can define the following density function

$$d(\rho) = \frac{f(\rho)}{\rho^{N-1}}.$$

Throughout this section we will use the same notation $h(\delta)$ as in Section 3 to denote infinitesimal quantities as $\delta$ tends to zero. The analogous of Lemma 3.1 is the following result which also takes into account that $d(\rho)$ needs now to be estimated also for $\rho \geq R$.

**Lemma 7.1.** There exists a positive constant $\overline{d}$, such that

$$\int_{R'}^{R} |d(\rho) - \overline{d}| \rho^{N-2} \rho \, d\rho + \int_{R}^{\infty} d(\rho) \rho^{N-2} d\rho = h(\delta).$$

We shall give the proof of Lemma 7.1 after some preliminary steps which include the analogous ones of those in Section 3 and some more ones due to the necessity of justifying some arguments which become less obvious in the discrete formulation. This is the case of the two following lemmas.

**Lemma 7.2.** Let $\delta R \leq R_0 \leq R$ be given and let $B_0$ be the ball with the same center of $B$ and radius $R_0$. If $i$ is such that $B_i \not\subseteq B_0$, then

$$\frac{1}{\sqrt{\delta}} \int_{R_0}^{\infty} c_i(\rho) \frac{d\rho}{\rho} = 1 + h(\delta).$$

**Proof.** Let $\overline{\rho}_1$ and $\overline{\rho}_2$ be respectively the smallest and the greatest values of $|x|$, for $x \in B_i$. By setting $\rho_1 = \max(\overline{\rho}_1, R_0)$ and $\rho_2 = \overline{\rho}_2(1 + \sqrt{\delta})$, we have

$$\int_{R_0}^{\infty} c_i(\rho) \frac{d\rho}{\rho} = \log \rho_2 - \log \rho_1 = \log \left( \frac{\rho_2}{\rho_1} \right) - \log \left( \frac{\rho_1}{\overline{\rho}_1} \right) + \log(1 + \sqrt{\delta})$$

$$= \log \left( \frac{\rho_2}{\rho_1} \right) - \log \left( \frac{\rho_1}{\overline{\rho}_1} \right) + \sqrt{\delta} + o(\sqrt{\delta}).$$
Moreover we have

\[ 1 \leq \frac{\rho_1}{\rho_1} \leq \frac{\rho_2}{\rho_1} \leq \frac{1}{1 - 2\delta \delta^{-\frac{1}{3}}} = 1 + O(\delta^{\frac{2}{3}}) \]

and therefore

\[ \log \left( \frac{\rho_2}{\rho_1} \right) = O(\delta^{\frac{2}{3}}), \quad \log \left( \frac{\rho_1}{\rho_1} \right) = O(\delta^{\frac{2}{3}}), \]

then the thesis is proved. \( \square \)

**LEMMA 7.3.** For every \( r \in [R', R] \) the following equality holds.

\[ \int_r^\infty d(\rho)\rho^{N-2}d\rho = (1 + h(\delta))\mu_r^R(B) = \mu_r^R(B) + h(\delta). \]  

**PROOF.** We note that, by (7.39), (7.38) and (7.41),

\[
\int_r^\infty d(\rho)\rho^{N-2}d\rho = \int_r^\infty \frac{f(\rho)}{\rho}d\rho = \sum_i \frac{1}{\sqrt{\delta}} \left( \int_r^\infty c_i(\rho)\rho d\rho \right) r_i^{N-1} = \sum_i (1 + h(\delta)) r_i^{N-1} = (1 + h(\delta)) \mu_r^R(B),
\]

where the sum is extended to the balls \( B_i \in B \) which intersect \( A_r^R \), since for all the other indexes \( c_i(\rho) = 0 \) for \( \rho \geq r \). \( \square \)

An upper bound on the mean value of \( d(\rho) \) can now be easily deduced by estimating \( \mu_r^R(B) \), leading to the analogous of Lemma 3.2.

**LEMMA 7.4.** For every \( r \in [R', R] \) the following inequality holds.

\[ \mu_r^R(B) \leq m_0(R^{N-1} - r^{N-1}) + h(\delta). \]

**PROOF.** Let \( B' \) be the ball with the same center of \( B \) and radius equal to \( r - 2\delta R \geq R' - 2\delta R \) and \( B' \) the restriction of \( B \) to \( B' \). As we have already noted, \( B' \) is a \( 2\delta \)-punching of \( B' \). Since \( \mu(B') \geq \mu(B) - \mu_r^R(B) \), we have by definition

\[ \mu(B) - \mu_r^R(B) \geq m(2\delta)(r - 2\delta R)^{N-1} = (m_0 + h(\delta))r^{N-1}. \]

Moreover, by hypothesis we know that

\[ \mu(B) \leq (m_0 + \delta)R^{N-1} = (m_0 + h(\delta))R^{N-1}. \]

By combining (7.44) and (7.45), we get (7.43). \( \square \)
By (7.42) and (7.43) we obtain a first integral estimate on \(d(\rho)\), indeed for every \(r \in [R', R]\) the following inequality holds (see (3.17)).

\[
\int_r^\infty d(\rho) \rho^{N-2} d\rho \leq m_0(R^{N-1} - r^{N-1}) + h(\delta).
\]

Now we want to get a uniform lower estimate on \(d(\rho)\) which matches (7.46) and, to this aim, we are going to consider for a fixed \(r \in [R', R]\) the restriction \(B'\) of \(B\) to the concentric ball to \(B\) of radius \(r\) and then the radial extension \(B''\) of \(B'\) to the ball \(B''\) of radius \(R''\) from the annulus \(S_r\). The aim of the next lemma is to prove an estimate on the measure of this new set of balls \(B''\).

**Lemma 7.5.** For a given \(r \in [R', R]\), let \(B''\) the above introduced radial extension of \(B'\) to the ball \(B''\). Then, the following estimate holds

\[
\mu_r^{B''}(B'') \leq d(r) \left( \int_r^{R''} \rho^{N-2} d\rho + h(\delta) \right).
\]

**Proof.** Since \(B''\) is defined as the radial extension of \(B'\) from its measure can be conveniently evaluated by adding the contributions related to a generic annulus \(A_{\gamma(r,1+\sqrt{\delta})}^{r(1+\sqrt{\delta})-1}\) which have been computed in the end of Section 6. We will denote by \(\mu_i^*\) the measure of the balls competing the \(i\)-th annulus. For \(i = 1\) we have

\[
\mu_1^* = (1 + \sqrt{\delta})^{2(N-1)} \mu_{r(1+\sqrt{\delta})-1}(B) = (1 + \sqrt{\delta})^{2(N-1)} \sqrt{\delta} f(r) = (1 + \sqrt{\delta})^{2(N-1)} \rho^{N-1} \int_r^{r(1+\sqrt{\delta})} \rho^{N-2} d\rho.
\]

In the second annulus, by scaling the balls which intersect \(S_r\) we find

\[
\mu_2^* \leq (1 + \sqrt{\delta})^{2(N-1)} \sqrt{\delta} f(r) = \sqrt{\delta} d(r) ((1 + \sqrt{\delta})^2)^{N-1}
\]

\[
\leq (1 + \sqrt{\delta})^{N-1} d(r) \int_{r(1+\sqrt{\delta})}^{r(1+\sqrt{\delta})^2} \rho^{N-2} d\rho.
\]

By iterating the argument, for every \(i\) we get

\[
\mu_i^* \leq (1 + \sqrt{\delta})^{2(N-1)} d(r) \int_{r(1+\sqrt{\delta})^i}^{r(1+\sqrt{\delta})^{i-1}} \rho^{N-2} d\rho,
\]

where the term 2 appearing in the first exponent can be omitted when \(i\) is even. Summing up with respect to \(i\), we get (7.47). \(\square\)
In the next lemma we shall prove the desired uniform lower estimate on the density function \( d(\rho) \). To this aim we set
\[
\overline{d} = \inf_{\rho \in [R', R]} d(\rho)
\]
and we take \( \bar{\rho} \in [R', R] \) such that
\[
(7.51) \quad d(\bar{\rho}) \leq (1 + \delta)\overline{d}.
\]
We can now state the corresponding case of Lemma 3.3.

**Lemma 7.6.** The following lower bound holds
\[
(7.52) \quad \overline{d} \geq m_0(N - 1) + h(\delta).
\]

**Proof.** By taking \( B'' \) as in the previous lemma for \( r = \bar{\rho} \), we get
\[
(7.53) \quad \mu(B'') \leq \mu(B) = \mu_{\bar{\rho}}^R(B) + \mu_{\bar{\rho}}^{R''}(B'').
\]
Since by (7.42)
\[
\mu_{\bar{\rho}}^R(B) = \int_{\bar{\rho}}^\infty d(\rho)\rho^{N-2}d\rho + h(\delta),
\]
by (7.47), for \( r = \bar{\rho} \), and by (7.51), we deduce
\[
\mu(B'') \leq \mu(B) - \int_{\bar{\rho}}^\infty d(\rho)\rho^{N-2}d\rho + d(\bar{\rho})\int_{\bar{\rho}}^{R''} \rho^{N-2}d\rho + h(\delta)
\]
\[
(7.54) \quad \leq (m_0 + \delta)R^{N-1} - \int_{\bar{\rho}}^R \rho^{N-2}d\rho + (1 + \delta)\overline{d} \int_{\bar{\rho}}^{R''} \rho^{N-2}d\rho + h(\delta)
\]
\[
= (m_0 + \delta)R^{N-1} + \bar{d} \int_{\bar{\rho}}^{R''} \rho^{N-2}d\rho + h(\delta).
\]
Since \( B'' \) is a \( 2\delta \)-punching of \( B'' \) and therefore
\[
(7.55) \quad \mu(B'') \geq m(2\delta)(R'')^{N-1} = (m_0 + h(\delta))(R'')^{N-1},
\]
by (3.21) and (3.22) we finally get
\[
(7.56) \quad \overline{d} \geq \left( \int_{R''} \rho^{N-2}d\rho \right)^{-1} (m_0 + h(\delta))(R'')^{N-1} - (m_0 + \delta)R^{N-1}) + h(\delta)
\]
\[
= m_0(N - 1) + h(\delta).
\]

**Proof of Lemma 7.1.** By integrating (7.52) we get
\[
\int_{R'}^{R} \overline{d} \rho^{N-2}d\rho \geq m_0(R^{N-1} - (R')^{N-1}) + h(\delta),
\]
so, by using (7.46) for \( r = R' \), since \( d(\rho) - \overline{d} \geq 0 \), we finally have (7.40). □
PROOF OF THEOREM 7.1. We assume by contradiction that the thesis is false, so we can use all the construction made in this section. We set $R^* = \frac{R+R'}{2}$ and for every $y \in \partial B$ we denote by $B^y$ the ball with radius $R^*$ contained in $B$ and tangent in $y$ to $\partial B$, whereas $B^*$ is the ball with the same center of $B$ and radius $R^*$. For $\lambda \geq 0$, let $B^y_\lambda$ be the ball with the same center of $B^y$ and radius $R^* + \lambda R$. For $x \in B$ let us introduce the function

$$p_\lambda(x) = \int_{\partial B_R} 1_{B^y_\lambda}(x)d\mathcal{H}^{N-1}(y) \leq 1.$$  

Because of the radial symmetry of the mappings $1_{B^*}$ and $p_\lambda$, we can deal with $1_{B^*}(\rho)$ and $p_\lambda(\rho)$. We can claim that

$$\int_{R^N} p_0(x)dx = |B^*|$$

and therefore

$$\int_{R^N} (1_{B^*} - p_0)dx = 0,$$  namely

$$\int_0^\infty (1_{B^*} - p_0)(\rho)\rho^{N-1}d\rho = 0.$$

We note that, since

$$(1_{B^*} - p_0)(\rho) \begin{cases} = 0 & \text{if } 0 < \rho < R' \\ > 0 & \text{if } R' < \rho < R^* \\ < 0 & \text{if } R^* < \rho < R, \end{cases}$$

then we can find a positive constant $c$ such that

$$\int_{R^N} (1_{B^*} - p_0)(\rho)\rho^{N-2}d\rho > 2\bar{c}$$

and, subsequently, for $\lambda$ sufficiently small

$$\int_0^\infty (1_{B^*} - p_\lambda)(\rho)\rho^{N-2}d\rho \geq \bar{c}$$

and so, since $1_{B^*} = p_\lambda$ for $\rho \leq R'$,

$$\int_{R'}^\infty p_\lambda(\rho)\rho^{N-2}d\rho \leq \int_{R'}^{R^*} \rho^{N-2}d\rho - \bar{c}.$$

We fix $\lambda > 0$ in such a way to have (7.62) and we shall take a $\delta$-punching $B$ with $\sqrt{\delta} < \frac{\lambda}{3}$. By (7.40) and (7.62) we get

$$\int_{R'}^\infty d(\rho)p_\lambda(\rho)\rho^{N-2}d\rho = \bar{d} \int_{R'}^\infty \rho^{N-2}p_\lambda(\rho)d\rho + h(\delta)$$

$$\leq \bar{d} \int_{R'}^{R^*} \rho^{N-2}d\rho - \bar{c} \bar{d} + h(\delta).$$
For every \( y \in \partial B \) let \( B^y \subset B \) be the restriction of the punching to the ball \( B^y \) and \( B^* \subset B \) be the restriction to \( B^* \). Now our aim is to estimate

\[
\int_{\partial B} \mu(B^y)d\mathcal{H}^{N-1}
\]

and we shall begin by computing the mean value of \( \mu^R(B^y) \), since the balls which do not intersect \( A^R \) belong to all the punchings \( B^y \). So we shall take into account in the following sums only indexes \( i \) such that \( B_i \not\subset B^y \), where \( B' \) is the ball with the same center of \( B \) and radius \( R' \). For each \( i \) we set

\[
c_i^*(y) = \begin{cases} 
1 & \text{if } B_i \subset B^y \\
0 & \text{if } B_i \not\subset B^y
\end{cases}
\]

then

\[
(7.64) \quad \int_{\partial B} \mu^R(B^y)d\mathcal{H}^{N-1} = \sum_i \left( \int_{\partial B} c_i^*(y)d\mathcal{H}^{N-1} \right) r_i^{N-1}.
\]

Let \( B_i(\sqrt{\delta}R) \) be the closed \( \sqrt{\delta}R \)-neighborhood of \( B_i \). We denote by \( z_i \) a point of \( B_i(\sqrt{\delta}R) \) having maximum modulus and we put \( \rho_i = |z_i| \). We see that, if \( c_i^*(y) \neq 0 \), then, for every \( x \in B_i(\sqrt{\delta}R) \), \( x \) belongs to the \((2r_i + \sqrt{\delta}R)\)-neighborhood of \( B' \), with \( 2r_i + \sqrt{\delta}R \leq \lambda R \), so \( 1_{B'}(x) = 1 \). Thus, we can claim that

\[
\int_{\partial B} c_i^*(y)d\mathcal{H}^{N-1} \leq \int_{\partial B} 1_{B_i^*(z_i)}d\mathcal{H}^{N-1} = p_\lambda(\rho_i).
\]

Since when \( c_i(\rho) \neq 0 \) we have \( \rho \leq \rho_i \) and consequently \( p_\lambda(\rho) \geq p_\lambda(\rho_i) \), then by (7.41) we have

\[
(7.65) \quad \int_{\partial B} c_i^*(y)d\mathcal{H}^{N-1} \leq p_\lambda(\rho_i) \leq \frac{1}{\sqrt{\delta}} \int_{R'} \rho_\lambda(\rho) c_i(\rho) \frac{d\rho}{\rho} + p_\lambda(\rho_i) h(\delta)
\]

\[
\leq \frac{1}{\sqrt{\delta}} \int_{R'} \rho_\lambda(\rho) c_i(\rho) \frac{d\rho}{\rho} + h(\delta).
\]

Therefore, by combining (7.64) and (7.65) and by using (7.38), (7.39), (7.40) and (7.63), we can establish the following estimate

\[
(7.66) \quad \int_{\partial B} \mu^R(B^y)d\mathcal{H}^{N-1} \leq \sum_i \left( \frac{1}{\sqrt{\delta}} \int_{R'} \rho_\lambda(\rho) c_i(\rho) \frac{d\rho}{\rho} + h(\delta) \right) r_i^{N-1}
\]

\[
\leq \int_{R'} \rho_\lambda(\rho) f(\rho) \frac{d\rho}{\rho} + h(\delta)
\]

\[
= \int_{R'} d(\rho) \rho^{N-2} \rho_\lambda(\rho) f(\rho) \frac{d\rho}{\rho} + h(\delta)
\]

\[
\leq \int_{R^*} d(\rho) \rho^{N-2} d\rho - c \lambda + h(\delta).
\]
For every $y \in \partial B$, we can pass from $\mu^R_{R_t}(B^y)$ to $\mu(B^y)$ by adding the measure of the set of balls of $B$ contained in $B'$, namely $\mu(B) - \mu^R_{R_t}(B)$. So from (7.66) and by Lemma 7.3 we get
\[
\int_{\partial B} \mu(B^y) d\mathcal{H}^{N-1} \leq \mu(B) - \mu^R_{R_t}(B) + \int_{R^*}^{\infty} d(\rho) \rho^{N-2} d\rho - \varepsilon d + h(\delta)
\]
\[
\leq (m_0 + \delta)R^{N-1} - \int_{R^*}^{\infty} d(\rho) \rho^{N-2} d\rho - \varepsilon d + h(\delta).
\]
Finally, by (7.40) and Lemma 7.6 we conclude
\[
\int_{\partial B} \mu(B^y) d\mathcal{H}^{N-1} \leq (m_0 + \delta)R^{N-1} - R^{N-1} - (R^*)^{N-1} - \varepsilon d + h(\delta)
\]
\[
\leq m_0 (R^*)^{N-1} - \varepsilon d + h(\delta) < m(2\delta)(R^*)^{N-1},
\]
for a small $\delta$. Then we can find $y \in \partial B$ such that $\mu(B^y) < m(2\delta)(R^*)^{N-1}$, which leads to a contradiction, since $B^y$ is a $2\delta$-punching of $B^y$. \[\Box\]

8. — Flatness of minimal punchings

The analogous of Theorem 4.1 holds for minimal punchings. More precisely we shall prove the following theorem.

**Theorem 8.1.** $m_0 = 1$ and for every $\varepsilon > 0$ there exists $\delta$ such that every $\delta$-punching of $B$, with measure less than or equal to $(m_0 + \delta)R^{N-1} = (1 + \delta)R^{N-1}$, turns out to be $\varepsilon$-close to some disk through the center of $B$.

The discrete formulation does not allow a proof as obvious as in Section 4. Indeed, before proving the main theorem, we set some preliminary results. Let $B$ be a given ball and let $(B_n)_{n \in \mathbb{N}}$ be a sequence of punchings of $B$ with infinitesimal thickness such that, for every $n \in \mathbb{N}$, $\mu_n(B) \leq (m_0 + 1/n)R^{N-1} \leq (1 + \frac{1}{n})R^{N-1}$ and let $C_n$ be the connected component of $B$, relative to $B_n$, having maximal measure. By Theorem 3.1 we get $|C_n| \to \frac{1}{2}|B|$ as $n \to \infty$, furthermore we note that $(C_n)_{n \in \mathbb{N}}$ is a sequence of sets with finite perimeter in $B$ (Caccioppoli sets), that is $1_{C_n} \in BV(B)$ for every $n$, $BV(B)$ denoting the space of functions of bounded variation defined on $B$ (see [14], [15]). Indeed $\partial C_n \cap B \subset \bigcup B_i \in B_n \partial B_i$. Then for every $n \in \mathbb{N}$

(8.67)
\[
P(C_n, B) \leq N b_n \left(1 + \frac{1}{n}\right) R^{N-1},
\]
where $P(A, B) = |D1_A|(B)$ denotes the perimeter of $A$ in $B$. Therefore $(1_{C_n})_{n \in \mathbb{N}}$ is a bounded sequence in $BV(B)$ and, by the compactness theorem on
BV functions ([15, Theorem 1.19]), we know that there exists a subsequence \((1_{C_n})_{n \in \mathbb{N}},\) by which we shall replace the whole \((1_{C_n})_{n \in \mathbb{N}},\) converging in \(L^1(B)\) to a function which can be identified as the characteristic function of a measurable set \(\overline{C}\). Then, as in Section 4, we have (4.29) and (4.30). We are now going to see that \(B \cap \partial C\) is a disk. This will show, in particular, that, for any given \(\varepsilon > 0\), \(B_n\) satisfies (6.35) for \(n\) large and for \(D = B \cap \partial \overline{C}\). Indeed, if \(x \in D\) and \(x \notin \bigcup_{B \in B_n} p(B)\), then the chord \(c_x\) through \(x\) and orthogonal to \(D\) is such that \(c_x \subset C_n\) or \(c_x \subset B \setminus C_n\). Therefore \(\left| \bigcup_{x} c_x \right| \) tends to zero, so \(\mathcal{H}^{N-1}(D \setminus \bigcup_{B \in B} p(B))\) tends to zero.

**Lemma 8.1.** Let \(\overline{C}\) be as above, then

\[
(8.68) \quad P(\overline{C}, B) \leq m_0 b_{N-1} R^{N-1} \leq b_{N-1} R^{N-1}.
\]

**Proof.** By the definition of perimeter we have

\[
P(\overline{C}, B) = \int_B |D1_{\overline{C}}| = \sup_{\varphi} \int_\overline{C} \text{div}\varphi \, dx,
\]

where the sup is taken on the set of the vector valued functions \(\varphi \in C_0^1(B, \mathbb{R}^N)\) such that \(|\varphi(x)| \leq 1\) for a.e. \(x\). So, let \(\varphi\) be any function as above, to prove the thesis we need to estimate \(\int_\overline{C} \text{div}\varphi \, dx\). To this aim, we begin by taking \(\varepsilon > 0\) and we note that, since \(\partial C_n \subset \bigcup_i \partial B_i\), then there exists a family of mutually disjoint borel measurable sets \(A_i \subset \partial B_i\) such that \(\partial C_n = \bigcup_i A_i\). Now, for every \(n \in \mathbb{N}\), let \(\varphi_i\) be a constant vector corresponding to a value assumed by \(\varphi\) on \(\partial B_i\) and let \(\nu\) be the normal field to \(\partial B_i\), we introduce the sets \(A_i^+ = \{ x \in \partial B_i : \varphi_i \cdot \nu > 0 \}\) which is a hemisphere and the corresponding disk \(D_i\). Thus, for every \(n\) such that \(\text{osc}_X \varphi \leq \varepsilon\) when \(\text{diam}(X) \leq \frac{2}{n}\), we can proceed to the following estimate in which we will use the divergence theorem.

\[
\int_{\partial C_n} \varphi \cdot \nu \mathcal{H}^{N-1} = \sum_i \int_{A_i} \varphi \cdot \nu \mathcal{H}^{N-1} \leq \sum_i \left( \int_{A_i} \varphi_i \cdot \nu \mathcal{H}^{N-1} + \int_{A_i} |\varphi - \varphi_i| |d\nu| \right)
\]

\[
\leq \sum_i \int_{A_i^+} \varphi_i \cdot \nu \mathcal{H}^{N-1} + \sum_i \int_{A_i} \varepsilon |d\nu| \leq \sum_i \int_{D_i^+} d\mathcal{H}^{N-1} + \varepsilon N b_N \sum_i r_i^{N-1} \leq (b_{N-1} + \varepsilon N b_N) \sum_i r_i^{N-1}.
\]

Then, since for every \(\varepsilon\) (8.69) is definitively true, we get

\[
\int_\overline{C} \text{div}\varphi \, dx = \lim_n \int_{\partial C_n} \varphi \cdot \nu \mathcal{H}^{N-1} \leq \lim_n (b_{N-1} + \varepsilon N b_N) \left( m_0 + \frac{1}{n} \right) R^{N-1}
\]

and, by the arbitrariness of \(\varepsilon\), we get the thesis. \(\square\)
We know that $C$ satisfies the inequality (4.30) and that, if the equality holds, that the reduced boundary $\partial^* \overline{C}$ can be identified with a disk of $B$. The reverse inequality has been just proved in the previous lemma which enables us to conclude that $\partial^* \overline{C}$ is a disk and that $m_0 = 1$. Now we are in a position to prove the main theorem of this section.

**Proof of Theorem 8.1.** Let us suppose that there exists $\bar{\varepsilon} > 0$ such that for every $n$ there exists a punching $B_n$ with measure less than or equal to $(1 + \frac{1}{n})R^{N-1}$ such that, for every hyperplane $P$ through the center of $B$, $B_n$ is not $\bar{\varepsilon}$-close to $D = P \cap B$. Now, since $(B_n)_{n \in \mathbb{N}}$ has infinitesimal thickness, the conclusion reached above allows us to claim that the boundary in $B$ of $C$ coincides with a disk $D = P \cap B$. Since $B_n$ is not $\varepsilon$-close to $D$ and (6.35) has already been checked, then there exists a finite set $E'_n$ of balls $B_i \in B_n$, with a measure greater than $\varepsilon \mu(B) \geq \varepsilon (1 - \bar{\varepsilon}b_{N-1}^1) R^{N-1}$, such that $\bigcup_{B_i \in E'_n} B_i \cap D(\bar{\varepsilon} R) = \emptyset$. Thus, we can repeat the same computations of (8.69), by taking $\text{sp}_{\varepsilon}^w C$ This leads to the conclusion $P(C, B) \leq (b_{N-1} - \bar{\varepsilon}) R^{N-1}$, which is a contradiction. 

### 9. – Similarity arguments

The main results of this section are summarized in the following theorem.

**Theorem 9.1.** For every $\varepsilon > 0$ and $\delta_1 > 0$ there exist $\varepsilon'$ (defined in Section 1 and depending on $\varepsilon$) and $\delta_2$ such that, if $B$ is by the set $K$ and condition (M$_{\delta_1}$) (depending on $\varepsilon'$) holds, then there exists a $\varepsilon$-packing of $K$.

This will allow us, in the next section, to take advantage of the results established so far and in particular Theorem 8.1. The proof of Theorem 9.1 will follow from some results which we are going to establish in the following lemmas. If $B$ is any set of balls, we will use the notation $\mu'_B = \mathcal{H}^{N-1}(K \setminus \bigcup_{B \in B} B)$.

**Lemma 9.1.** Let $\varepsilon'$ > 0 be fixed (and so $t$ and $n$ are also fixed). Let $B$ be a ball with radius $R$ such that (M$_{\delta_1}$) holds. If $\tilde{B}$ is any set of disjoint balls contained in $B$ which are $\delta_1$-split by $K$ and have a radius bigger or equal to $t^n r$, then

$$
\mu(\tilde{B}) \leq R^{N-1} \left( 1 + \frac{\varepsilon'}{2} \right) \left( 1 - \frac{\mu_B'}{\mathcal{H}^{N-1}(K)} \right).
$$

**Proof.** For every $B' \in \tilde{B}$, with radius $r$, we obtain by (M$_{\delta_1}$)

$$
r^{N-1} \leq \left( 1 + \frac{\varepsilon'}{2} \right) \frac{\mathcal{H}^{N-1}(K \cap B')}{{\mathcal{H}^{N-1}(K)}} R^{N-1}.
$$
Since the balls of $\widetilde{B}$ are disjoint, we get
\[ \mu(\widetilde{B}) = \sum r^{N-1} \left( 1 + \frac{\varepsilon'}{2} \right) \left( 1 - \frac{\mu'_{B^i}}{\mathcal{H}^{N-1}(K)} \right). \]

The proof of Theorem 9.1 can be easily deduced from the following Lemma which will be proved in the sequel.

**Lemma 9.2.** For every $\varepsilon' > 0$ and $\delta_1 > 0$ there exists $\delta_2$ such that, if $B$ is any $\delta_2$-split ball satisfying $(M_{\delta_1})$, then there exists $B$, $\varepsilon'$-punching of $B$ such that, if $B'$ is any subset of $B$ such that $\mu(B') \geq \frac{1}{2} \mu(B)$, then
\[ \mu(B') \leq (1 + \varepsilon') R^{N-1} \left( 1 - \frac{\mu'_{B^i}}{\mathcal{H}^{N-1}(K)} \right). \]

**Proof of Theorem 9.1.** Let $\varepsilon > 0$ and $\delta_1 > 0$ be given. We fix $\varepsilon' \leq \varepsilon$ such that
\[ 1 - \frac{m(\varepsilon')}{1 + \varepsilon'} < \varepsilon \]
holds, as we can do since the left hand side tends to zero as $\varepsilon'$ tend to zero. We can also take $\delta_2$ small enough as in Lemma 9.2 so that, if $B$ is $\delta_2$-split and enjoys $(M_{\delta_1})$, we can find a $\varepsilon'$-punching (and so $\varepsilon$-punching, since $\varepsilon' \leq \varepsilon$) $B$ which satisfies (9.72). Then (6.36) trivially follows from (9.72) for $B' = B$. In order to prove (6.37), we observe that, since, by definition, $\mu(B) \geq m(\varepsilon') R^{N-1}$ and $2 \mu(B') \geq \mu(B)$, then (9.72) and (9.73) give
\[ \frac{\mu'_{B'}}{\mathcal{H}^{N-1}(K)} \leq 1 - \frac{\mu(B')}{(1 + \varepsilon') R^{N-1}} \leq 1 - \frac{m(\varepsilon')}{(1 + \varepsilon') R^{N-1}} \leq \varepsilon + \frac{\mu(B) - \mu(B')}{R^{N-1}}. \]
By the definition of $\mu'_{B'}$, we get (6.37).

In order to prove Lemma 9.2, we need to construct $B$. To this aim, let $\varepsilon'$ (and consequently $t$, $n$) and $\delta_1$ be fixed and $B$ be any ball which satisfies $(M_{\delta_1})$. Let $B_1$ be a set, maximal by inclusion, of pairwise disjoint $\delta_1$-split balls contained in $B$ and radius $t R$. In the same way, we define $B_2$ by taking disjoint $\delta_1$-split balls having radius $t^2 R$ and disjoint from the former ones, thus recursively, for $k = 1 \cdots n$, we define $B_k$ as a maximal set of disjoint balls of radius $t^k R$. In order to simplify the notation, we put $\mu_k = \mu(B_k) = \text{card}(B_k)(t^k R)^{N-1}$, $\mu = \sum_k \mu_k$, $\mu' = \mu'_{\bigcup B_k}$. Lemma 9.1 applies in particular to all the subsets $\widetilde{B}$ of $\bigcup B_k$. By (9.70) for $\widetilde{B} = \bigcup B_k$, we deduce that $\mu \leq R^{N-1}(1 + \frac{\varepsilon'}{2})$ and so there exists an index $i$ which singles out a scale to which, roughly speaking, we find few balls, namely their total measure is such that
\[ \mu_i = \mu(B_i) \leq \frac{\mu}{n} \leq \frac{R^{N-1}(1 + \frac{\varepsilon'}{2})}{n}. \]
By fixing $i$ as above, we introduce the following sets

$$B^1 = \bigcup_{k \neq i} \bigcup_{B' \in B_k} ((1 + 3t)B'), \quad B^2 = \bigcup_{B' \in B_i} \{4B'\},$$

$$B = B^1 \cup B^2,$$

$$U = \bigcup_{B' \in B} B'.$$

So, every ball $B_j \in B$ is obtained, by definition, by suitably increasing the radius of a ball $B' \in \bigcup_k B_k$. We shall refer to $B'$ as to the ball corresponding to $B_j$.

**Lemma 9.3.** Let $i$ be fixed as above and $B' \subset B$ be a ball of radius $r = t^i R\,$, such that $2B' \not\subset U$. Then $B'$ cannot be $\delta_1$-split.

**Proof.** Fix any ball $B''$ in $\bigcup_{k \neq i} B_k$. By construction, the $3t^i R$-neighborhood of $B''$ is contained in a ball $B''' \in B$. By assumption $2B' \not\subset B'''$, so there exists $z \in 2B'$ such that $d(z, B'') \geq 3t^i R$. So, if $\bar{x}$ is the center of $B'$, by the triangular inequality we deduce $d(\bar{x}, B'') \geq t^i R$. This implies $B' \cap B'' = \emptyset$ and by the maximality of $B_i$ we get the conclusion. \(\square\)

Now, we can show that $B$ is as required in Lemma 9.2.

**Lemma 9.4.** Let $B$ be the above defined set and let $B' \subset B$ be any subset such that $\mu(B') \geq \frac{1}{2} \mu(B)$, then (9.72) holds.

**Proof.** Let $\bar{B} \subset \bigcup_k B_k$ be the set of the balls in $\bigcup_k B_k$ corresponding to those belonging to $B'$. We set $\bar{B} = \bar{B}_1 \cup \bar{B}_2$, with $\bar{B}_1 = \bar{B} \setminus B_i$ and $\bar{B}_2 = \bar{B} \cap B_i$. We shall denote by $r$ the radius of the generic ball $B'$ belonging to $\bar{B}$. Firstly, we remark that

$$\sum_{B' \in \bar{B}_2} r^{N-1} \leq \frac{2\mu(\bar{B})}{n},$$

because, otherwise, from $2\mu(B') \geq \mu(B)$ we get $2\mu(\bar{B}) \geq \mu$ and therefore

$$\sum_{B' \in \bar{B}_2} r^{N-1} \leq \frac{\mu}{n} \leq \frac{2\mu(\bar{B})}{n}.$$

Thus, from (1.4) and (9.70), we can establish the following inequalities

$$\mu(B') = \sum_{B' \in B_1} ((1 + 3t)r)^{N-1} + \sum_{B' \in B_2} (4r)^{N-1}$$

$$\leq \left( (1 + 3t)^{N-1} + \frac{2^{2N-1}}{n} \right) \mu(\bar{B}) \leq \frac{1 + \epsilon'}{1 + \frac{\epsilon'}{2}} \mu(\bar{B})$$

$$\leq (1 + \epsilon') R^{N-1} \left( 1 - \frac{\mu(B)}{\mu(\bar{B})} \right),$$

since $\mu'_{B'} \leq \mu'_{B}$, by construction. \(\square\)
What we still have to show for ending the proof of Lemma 9.2 is that, if \( B \) is furthermore \( \delta_2 \)-split, with a suitable value of \( \delta_2 \), the above introduced set \( B \) actually is a \( \varepsilon' \)-punching of \( B \). To this aim we need to estimate the measure of the connected components of \( B \setminus U \). Firstly, we note that there exists \( k \in \mathbb{N} \), with \( k = k(\delta_1) \), such that, for every pair of balls which are not \( \delta_1 \)-split, with radius \( r \) and with the centers at a distance less than \( \frac{r}{2} \), then each ordinary oscillation region of the first one has a nonempty intersection with each ordinary oscillation region of the other one. Now, put \( \eta = \frac{4R}{28} \), we take an \( \eta \)-net \( \mathcal{N} \) of the ball \( B^1 \) with the same center as \( B \) and radius \( (1 - \varepsilon')R \), such that \( \omega = \text{card}(\mathcal{N}) \) is of the same order as \( \left( \frac{1}{2} \right)^N \). Thus \( \omega \) is a function of \( (\delta_1, \varepsilon, n) \) and so \( \omega = \omega(\delta_1, \varepsilon') \). We define \( \pi \) as the minimal distance projection from \( B \) to \( B^1 \). Let \( C \) be any connected component of \( B \setminus U \), by introducing the set

\[
\mathcal{N}' = \{ x \in \mathcal{N} : |d(x, \pi(C))| \leq \eta \},
\]

we see that \( \pi(C) \subset \bigcup_{x \in \mathcal{N}'} B(x, t^i R) \). Let \( S \) be the union of all the \( \delta_1 \)-split balls contained in \( B \), with radius \( r \leq t^i R \) and \( C' = C \setminus (S \cup K) \).

**Lemma 9.5.** Let \( x, y \) be in \( N' \), then there exists a finite chain \( \{x_1, x_2, \ldots, x_m\} \subset N' \) such that \( x_1 = x \), \( x_m = y \) and, for every \( i < m \), \( d(x_i, x_{i+1}) < 2\eta \) (obviously, we can take \( m \leq \omega(\delta_1, \varepsilon') \)).

**Proof.** Fix \( X \in N' \) and let \( N'' \) be the set of \( y \in N' \) such that the thesis holds. We note that, if \( z \in N' \) and \( d(z, N'') \leq 2\eta \), then \( z \in N'' \). Moreover \( N'' \neq \emptyset \), because \( x \in N'' \). We are going to show that the hypothesis \( N' \setminus N'' \neq \emptyset \) leads to claim that \( d(N'', N' \setminus N'') \leq 2\eta \) and this is a contradiction to what we have stated above. The equality \( N' = N'' \) will follow and the thesis will be proved. Let us introduce the sets

\[
A' = \{ x \in \pi(C) : |d(x, N'')| \leq \eta \}
\]

and, analogously,

\[
A'' = \{ x \in \pi(C) : |d(x, N' \setminus N'')| \leq \eta \}.
\]

Now, by the definition of \( N' \), we get that \( A' \cup A'' = \pi(C) \), while the condition \( N'' \neq \emptyset \) implies \( A' \neq \emptyset \) and the condition \( N' \setminus N'' \neq \emptyset \) implies \( A'' \neq \emptyset \). The sets \( A' \) and \( A'' \) are closed. Since \( \pi(C) \) is connected, then there exists \( z \in A' \cap A'' \). Therefore, we have that \( d(z, N') \leq \eta \) and \( d(z, N'') \leq \eta \), then the claim.

**Lemma 9.6.** There exists \( c = c(\delta_1, \varepsilon') \) such that

\[
\text{osc}_{C'U} \leq c\sqrt{R}.
\]

**Proof.** Fix \( \bar{x} \) and \( \bar{y} \) in \( C' \). Let \( \lambda < 1 \) be suitably chosen and let \( (B_h)_{h \in \mathbb{N}} \) be a monotone decreasing sequence of balls, contained in \( B \), with radius \( \lambda^{h+1} R \),
such that $B_0 = B(\pi(x), t'R)$ and for every $h \in \mathbb{N}$, $\bar{x} \in B_h$, i.e. $\{\bar{x}\} = \bigcap_h B_h$.

Since $\bar{x} \notin S$ and the radii are less than or equal to $t'R$, by the definition of $S$ these balls cannot be $\delta_1$-split, then each one of them has an ordinary oscillation region $\tilde{B}_h$. Thus, by choosing $\lambda$ suitably close to 1, we can claim that, for every $h \in \mathbb{N}$, $\tilde{B}_h \cap \tilde{B}_{h+1} \neq \emptyset$. Obviously $\lambda = \lambda(\delta_1)$. We fix an arbitrary point $z_0 \in \tilde{B}_0$ and for every $h \in \mathbb{N}$, $h \geq 1$, we take $z_h \in \tilde{B}_{h-1} \cap \tilde{B}_h$; since $z_h$ and $z_{h+1}$ both belong to $\tilde{B}_h$, we get the following estimate.

$$|u(z_h) - u(z_{h+1})| \leq \text{osc}_{\tilde{B}_h} u \leq \delta_1^{-1} \sqrt{\lambda^h t'R}.$$  
(9.75)

Since $z_h \to \bar{x}$ and $\bar{x} \notin K$, then we have

$$|u(z_0) - u(\bar{x})| \leq \sum_{h=0}^{\infty} |u(z_h) - u(z_{h+1})| \leq \delta_1^{-1} \sqrt{t'R} \sum_{h=0}^{\infty} \sqrt{\lambda^h}.$$  
(9.76)

By repeating the above argument, we can find $w_0$ belonging to the ordinary oscillation region of $B(\pi(y), t'R)$, such that

$$|u(w_0) - u(y)| \leq \delta_1^{-1} \sqrt{t'R} \sum_{h=0}^{\infty} \sqrt{\lambda^h}.$$  
(9.77)

By definition of $N'$, we can find $x, y \in N'$ such that both $d(x, \pi(\bar{x}))$ and $d(y, \pi(\bar{y}))$ are less than or equal to $\eta$. Furthermore, we can find a finite chain $\{x_1, x_2, \ldots, x_m\} \subset N'$ such that $x_1 = x$, $x_m = y$, as in Lemma 9.5, also we put $x_0 = \pi(x)$, $x_{m+1} = \pi(y)$. We claim that for every $j = 0, \ldots, m + 1$, the ball $B(x_j, t'R)$ is not $\delta_1$-split. Indeed $B(x_j, t'R) \cap \pi(C) \supset B(x_j, \eta) \cap \pi(C) \neq \emptyset$. Therefore $B(x_j, 2t'R) \cap C \neq \emptyset$ and so $B(x_j, 2t'R) \not\subset U$ and the claim follows from Lemma 9.3. Thus let $\tilde{A}_j$ be an ordinary oscillation region of $B(x_j, t'R)$. Since, for every $j$, $d(x_j, x_{j+1}) \leq 2\eta$, then we have that, for $j = 0, \ldots, m$, $\tilde{A}_j \cap \tilde{A}_{j+1} \neq \emptyset$. Let us fix $v_j \in \tilde{A}_j \cap \tilde{A}_{j+1}$. Obviously, $\tilde{A}_0 = \tilde{B}_0$ and we can take $z_0 = v_0$. Likewise, we can take $v_{m+1} = w_0$. Therefore, for every $j = 1, \ldots, m + 1$, $v_j$ and $v_{j-1}$ are both by construction in $\tilde{A}_j$. Thus

$$|u(v_j) - u(v_{j-1})| \leq \text{osc}_{\tilde{A}_j} u \leq \delta_1^{-1} \sqrt{t'R}.$$  
(9.78)

By the triangular inequality, we get

$$|u(z_0) - u(w_0)| = \sum_{j=1}^{m+1} |u(v_j) - u(v_{j-1})| \leq (m + 1)\delta_1^{-1} \sqrt{t'R} \leq ((\omega(\delta_1, \epsilon') + 1)\delta_1^{-1} \sqrt{t'R}.$$  
(9.79)
Finally, by combining the above inequalities, with (9.76) and (9.77), we can easily deduce the following estimate which concludes the proof.

\[ |u(\overline{x}) - u(\overline{y})| \leq |u(\overline{x}) - u(z_0)| + |u(z_0) - u(w_0)| + |u(w_0) - u(\overline{y})| \]

(9.80)

\[ \leq \delta_1^{-1} \sqrt{t^i} \left( \omega(\delta_1, \varepsilon') + 1 + 2 \sum_{h=0}^{\infty} \sqrt{\lambda^h} \right) \sqrt{R} = c(\delta_1, \varepsilon') \sqrt{R}, \]

provided we take \( c(\delta_1, \varepsilon') = \delta_1^{-1} \sqrt{t^i} \left( \omega(\delta_1, \varepsilon') + 1 + 2 \sum_{h=0}^{\infty} \sqrt{\lambda^h} \right). \)

If we fix \( \delta_2 < c(\delta_1, \varepsilon')^{-1} \) and we take \( B \) \( \delta_2 \)-split, then we get from the above lemma

(9.81)

\[ |C'| < \left( \frac{1}{2} + \delta_2 \right) |B|. \]

**Proof of Lemma 9.2.** Let \( B \) and \( K \), closed subset of \( B \), be given, we fix \( \delta_2 \) as above and we assume that \( B \) is \( \delta_2 \)-split by \( K \) and satisfies (\( M_{\delta_1} \)). Since \( B \) has a thickness less than or equal to \( 4tR \) and so, by (1.3), less than or equal to \( \varepsilon'R \), in order to prove that \( B \) is a \( \varepsilon' \)-punching of \( B \), it remains to show that the measure of any connected component \( C \) of \( B \setminus \bigcup_{B' \in B} B' \) is less than or equal to \( \left( \frac{1}{2} + \varepsilon' \right) |B| \). To this aim, in view of (9.81), we just need to estimate \( |S| \). By a Vitali covering type argument, we can say that

\[ S \subset \bigcup_{j \in J} 5B_j, \]

where \( J \) is a set of indexes and the sets \( B_j \) are mutually disjoint \( \delta_1 \)-split balls with radius \( r_j \geq t^n r \), by construction. By Lemma 9.1 we find

(9.82)

\[ \sum_{j \in J} r_j^{N-1} \leq \left( 1 + \frac{\varepsilon'}{2} \right) R^{N-1} \leq 2R^{N-1}. \]

Therefore

(9.83)

\[ |S| \leq 5^N \sum_{j \in J} |B_j| \leq 5^N b_N \sup_{j \in J} \sum_{j \in J} r_j^{N-1} \leq 2 \cdot 5^N t b_N R^N. \]

Then, by (9.81) and (9.83), we deduce that

(9.84)

\[ |C| \leq \left( \frac{1}{2} + \delta_2 \right) |B| + |S| \leq \left( \frac{1}{2} + \delta_2 \right) |B| + 2 \cdot 5^N t |B|. \]

By also taking, as allowed by (1.3),

\[ \delta_2 < \varepsilon' - 2 \cdot 5^N t, \]

we get \( |C| \leq \left( \frac{1}{2} + \varepsilon' \right) |B| \), which, combined with (9.74) shows that \( B \) is an \( \varepsilon' \)-punching of \( B \). Finally, (9.72) follows from Lemma 9.4. \( \square \)
10. – Proof of Theorem 1.3

We begin this section by showing how Theorem 8.1 combines with Theorem 9.1.

**Lemma 10.1.** For every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $K$ has a $\delta$-packing $B$, then there exists a hyperplane $P$ through the center of $B$ such that (2.5) holds.

**Proof.** The assertion trivially follows from Theorem 8.1 (whose use is allowed by (6.36)) and from (6.37).

Let $B = B(x, R)$ and let $D$ be a disk through the center of $B$. Given $\varepsilon > 0$, we take the two disks of $B$, $D_1$ and $D_2$, obtained by cutting $B$ with two hyperplanes parallel to $D$ and distant $\varepsilon$ from $D$. Thus, the ball $B$ turns out to be divided in three regions, namely the set enclosed between $D_1$ and $D_2$, which will be denoted by $F_\varepsilon$, and the remaining two regions $E_1$ and $E_2$, symmetric with respect to $D$. Let $S$ be a subset of $E_1 \cup E_2$ and let $p$ denote the orthogonal projection of $B$ on the hyperplane containing $D_1$. Let $K \subset B$ be a closed subset, $u : B \to \mathbb{R}$ satisfying condition (WS) on $B \setminus K$ with $\| u \|_{2}^{*} = 1$. We introduce the notation $A = D_1 \setminus p(K \cup S)$ and $a = R^{1-N} H^{N-1}(A)$, and we claim the following statement.

**Lemma 10.2.** If $a \neq 0$ there exists a chord $C$ of $B$, orthogonal to $D$, such that $C \cap (S \cup K) = \emptyset$, on which the following estimate holds

$$
\int_{C \cap F_\varepsilon} | \nabla u | \leq c a^{-1} \sqrt{\varepsilon R},
$$

where $c$ denotes a positive constant only depending on the dimension $N$.

**Proof.** Let $N_\varepsilon$ be an $\varepsilon R$-net of $D$, then there exists a positive constant $c$, only depending on $N$, such that $\text{card}(N_\varepsilon) \leq c e^{1-N}$. Then, by observing that

$$
F_\varepsilon \subset \bigcup_{y \in N_\varepsilon} B(y, 2\varepsilon R)
$$

and by using (WS), we can establish the following estimate

$$
\int_{F_\varepsilon} | \nabla u | \leq \sum_{y \in N_\varepsilon} \int_{B(y, 2\varepsilon R)} | \nabla u | \leq \text{card}(N_\varepsilon)(2\varepsilon R)^{N-\frac{1}{2}}
\leq c e^{1-N}(2\varepsilon R)^{N-\frac{1}{2}} \leq c \sqrt{\varepsilon R} R^{N-1}.
$$

Thus, by Fubini theorem and (10.86), we can easily deduce the assertion. □
LEMMA 10.3. For every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $B$ is $\delta$-split by $K$, $K$ has a $\delta$-packing and $P$ is as in Lemma 10.1, the following estimate holds.

\[
\mathcal{H}^{N-1}(p(K)) > (1 - \varepsilon)b_{N-1}R^{N-1}.
\]

PROOF. Given $\varepsilon > 0$, by Lemma 10.1, if $\delta$ is suitably chosen and if $K$ has a $\delta$-packing, we can find a hyperplane $P$ such that, with the notation introduced in Lemma 10.2,

\[
\mathcal{H}^{N-1}(K \setminus F_\varepsilon) < \varepsilon R^{N-1}.
\]

Thus we assume the above construction with $D = P \cap B$ and we take a function $u$ which satisfies (WS) with $\|u\|_{L^\infty} = 1$ on $B \setminus K$ and which splits $B$. By (WS) and by [22, Theorem 6.1] we know that there exists a neighborhood $S$ of $K \setminus F_\varepsilon$ and a positive constant $c$ such that $u$ is Hölder continuous with index $\frac{1}{2}$ on each of the two sets $E_1 \setminus S$ and $E_2 \setminus S$ with a given Hölder norm $c_H$ and such that

\[
\mathcal{H}^{N-1}(\partial S) \leq c\mathcal{H}^{N-1}(K \setminus F_\varepsilon) \leq c\varepsilon R^{N-1}.
\]

Let $\sigma$ be the jump of $u$ between $E_1$ and $E_2$, defined by $\sigma = R^{-\frac{1}{2}}(\inf_{E_2 \setminus S}u - \sup_{E_1 \setminus S}u)$, under a suitable choice of the indexes. By taking the chord $C$ found in Lemma 10.2, by (10.85) we can state

\[
\sigma \leq R^{-\frac{1}{2}} \int_{C \cap F_\varepsilon} |\nabla u| \leq ca^{-1}\sqrt{\varepsilon}.
\]

Since $B$ is $\delta$-split, by the Hölder continuity of $u$ on the sets $E_i$ we get $\sigma \geq \frac{1}{2}\delta^{-1}$ and therefore the above inequality gives

\[
a \leq \frac{c\sqrt{\varepsilon}}{\sigma} \leq c\sqrt{\varepsilon}\delta.
\]

By recalling the definition of $A$ and by combining (10.88) with (10.89), we get

\[
\mathcal{H}^{N-1}(p(K)) \geq \mathcal{H}^{N-1}(D_1) - \mathcal{H}^{N-1}(p(S)) - \mathcal{H}^{N-1}(A)
\]

\[
= \mathcal{H}^{N-1}(D_1) - \mathcal{H}^{N-1}(\partial S) - aR^{N-1}
\]

\[
\geq \mathcal{H}^{N-1}(D_1) - c\varepsilon R^{N-1} - c\sqrt{\varepsilon}\delta R^{N-1}.
\]

Now, since $\mathcal{H}^{N-1}(D \setminus D_1) \leq c\varepsilon^2 R^{N-1}$, by the arbitrariness of $\varepsilon$, from (10.90) we get the thesis.

COROLLARY 10.1. For every $\varepsilon > 0$ there exists $\delta$ such that, if $K$ has a $\delta$-packing and $B$ is $\delta$-split, then there exists a hyperplane $P$ through the center of $B$ such that $K$ is $\varepsilon$-close to $P$.\qed
11. – Proof of the main results through similarity arguments

Theorem 1.3 allows a more elementary proof of Theorem 1.1 and Theorem 1.2 (of course the results in Section 5 will not be considered to be known in this last section).

Lemma 11.1. For every \( \varepsilon > 0 \) there exists \( \delta \) such that, for every \( K \in S_{\delta}(B) \) satisfying (1.2), \( K \) is \( \varepsilon \)-close to some hyperplane \( P \) through the center of \( B \).

Proof. Given \( \varepsilon > 0 \), we fix \( \varepsilon' \) as given by Theorem 1.3 (\( \varepsilon' \) just depends on \( \varepsilon \)). Subsequently, we fix \( \delta_1 \) such that
\[
\frac{s_0 + \delta_1}{s(\delta_1)} < 1 + \frac{\varepsilon'}{2},
\]
as we can do since \( \frac{s_0 + \delta_1}{s(\delta_1)} \to 1 \) as \( \delta_1 \to 0 \). Then we fix \( \delta_2 \) as in Theorem 1.3, and we take \( \delta \leq \min\{\delta_1, \delta_2\} \). We observe that (1.2) implies (M1). Indeed, if a ball \( B' \) with radius \( r \leq R \) is \( \delta_1 \)-split, we have
\[
R^{1-N} \mathcal{H}^{N-1}(K) \leq s_0 + \delta \leq s_0 + \delta_1
\]
\[
\leq (s_0 + \delta_1) \frac{\mathcal{H}^{N-1}(K \cap B')}{s(\delta_1) r^{N-1}}
\]
\[
< \left(1 + \frac{\varepsilon'}{2}\right) R^{1-N} \mathcal{H}^{N-1}(K \cap B').
\]
So, if \( K \) is as in the statement, from Theorem 1.3 we deduce the thesis. \( \square \)

Corollary 11.1.

(11.91) \( s_0 = b_{N-1} \).

Proof. Fix any \( \varepsilon > 0 \) and then a constant \( \delta \) as given by the previous lemma. We can always take \( K \) such that (1.2) holds and then we have from (2.5), which is guaranteed by the previous lemma,
\[
(b_{N-1} - \varepsilon) R^{N-1} \leq \mathcal{H}^{N-1}(p(K)) \leq \mathcal{H}^{N-1}(K) \leq (s_0 + \delta) R^{N-1}.
\]
Since we can take \( \delta \) and \( \varepsilon \) arbitrarily small in the above inequality, we get \( s_0 \geq b_{N-1} \). Since the reverse inequality of (11.91) has already been observed, the thesis follows. \( \square \)
PROOF OF THEOREM 1.1. The theorem is just a restatement of the last corollary (see Section 5).

PROOF OF THEOREM 1.2. Given $\varepsilon > 0$, by Lemma 11.1 we have $\delta > 0$ such that, for every $K \in \mathcal{S}_3(B)$ satisfying (1.2), $K$ is $\varepsilon$-close to some hyperplane $P$ through the center of $B$. Therefore, the estimates on $A_K(B)$ and $T_K(B)$ follow as in Section 5.

REFERENCES


Dipartimento Interuniversitario di Matematica
Via E. Orabona, 4
70125 Bari, Italy
solimini@dm.uniba.it