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Selfdual Einstein Hermitian Four-Manifolds

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Abstract. We provide a local classification of selfdual Einstein Riemannian four-manifolds admitting a positively oriented Hermitian structure and characterize those which carry a hyperhermitian, non-hyperkähler structure compatible with the negative orientation. We show that selfdual Einstein 4-manifolds obtained as quaternionic quotients of $\mathbb{H}P^2$ and $\mathbb{H}H^2$ are Hermitian.

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Introduction

This paper is concerned with oriented, four-dimensional Einstein manifolds which are *Hermitian*, i.e. admit a positively oriented (integrable) complex structure, and are *selfdual*, meaning that the anti-selfdual Weyl tensor W^- vanishes identically.

A Riemannian version of the celebrated Goldberg-Sachs theorem of General Relativity implies that a Riemannian Einstein 4-manifold locally admits a positively oriented Hermitian structure if and only if the *selfdual Weyl tensor* W^+ is *degenerate* [2], [11], [39], [43]; this means that at any point of M at least two of the three eigenvalues of W^+ coincide, when W^+ is viewed as a symmetric trace-free operator acting on the three-dimensional space of selfdual 2-forms at each point.

Riemannian Einstein 4-manifolds whose selfdual Weyl tensor W^+ is degenerate have been much studied by A. Derdziński; in particular, we know from [22] that, on each connected component, W^+ either vanishes identically or else has no zero, i.e. has exactly two distinct eigenvalues at any point: one of them, say λ, is simple and the other one is of multiplicity 2 (and therefore

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equals $-\frac{\lambda}{2}$ as W^+ is trace-free); in the latter case, any one of the two normalized generators of the simple eigenspace of W^+ is the Kähler form of an integrable Hermitian structure, say J, and the conformal metric $\bar{g} = |W^+|^{\frac{2}{3}} g$ is *Kähler* with respect to J; if, moreover, g is *selfdual* the simple eigenvalue λ of W^+ is constant (equivalently, the norm $|W^+|$ is constant) if and only if (M, g) is locally symmetric, i.e. a real or a complex space form.

From this and from general properties of the Bach tensor of Kähler surfaces, we eventually obtain a natural bijection between the following three classes of Riemannian 4-manifolds:

(1) selfdual Einstein 4-manifolds with degenerate selfdual Weyl tensor W^+, such that $|W^+|$ is not constant;
(2) selfdual Einstein Hermitian 4-manifolds which are neither conformally-flat nor Kähler;
(3) selfdual Kähler manifolds with nowhere vanishing and non-constant scalar curvature.

A precise statement and a proof are provided by Lemma 2 below. In this correspondence, Riemannian metrics are defined on the same manifold and belong to the same conformal class; we observe that each class is defined by an algebraic closed condition (the vanishing of some tensors) and an open genericity condition.

Since the compact case is completely understood —see [19] or [1], [7], [11], [22], [33] for a classification— the paper will concentrate on the local situation.

The first known examples of (non locally symmetric) selfdual Einstein Hermitian metrics have been metrics of cohomogeneity one under the isometric action of a four-dimensional Lie group. Einstein metrics which are of cohomogeneity one under the action of a four-dimensional Lie group are automatically Hermitian [22]. By using this remark, A. Derdziński constructed [21] a family selfdual Einstein Hermitian metrics of cohomogeneity one under the action of U(1,1) and U(2); this family actually includes (in a rather implicit way) the well-known *LeBrun-Pedersen metrics* [36], [40] which play an important rôle in Section 3 of this paper.

It is *a priori* far from obvious that there are any other examples of selfdual Einstein Hermitian 4-manifolds, since the conditions of being selfdual, Einstein and Hermitian constitute an over-determined second order PDE system for the metric g. We show however that there are actually many of them; more precisely, we classify all local solutions of this system and provide a simple, *explicit* (local) Ansatz for selfdual Einstein Hermitian 4-manifolds (see Theorem 2 and Lemma 3 for a precise statement).

An amazing, *a priori* unexpected fact comes out from the argument and explains *a posteriori* the integrability of the above mentioned system: *all selfdual Einstein Hermitian metrics admit a local isometric action of* $\mathbb{R}^2$ *with two-dimensional orbits* (Theorem 2 and Remark 3). In particular, these metrics locally fall into the more general context of selfdual metrics with torus action considered in [35] and, more recently, in [14], [18] (see also Remark 3 (ii)).

It turns out that the property of having more (local) symmetries than expected is actually shared by Kähler metrics with vanishing Bochner tensor in all dimensions, as shown in the recent work of R. Bryant [12] (see [12] for precise statements). Since the Bochner tensor of a Kähler manifold of real dimension four is the same as the anti-selfdual Weyl tensor W^- —so that Bochner-flat Kähler metrics are a natural generalization of selfdual Kähler metrics in higher dimensions— via the correspondence given by Lemma 2, Bryant's work provides an alternative approach to our classification presented in Section 2.

The paper is organized as follows:

Section 1 displays the background material; the notation closely follows our previous work [2] —with the exception that the definition of the Lee form is here slightly different from the one in [2]— and we refer the reader to [2] for more details and references.

Section 2.1 provides a complete description of (locally defined) selfdual Einstein Hermitian metrics of cohomogeneity one (Theorem 1). It turns out that they all admit a local isometric action (with three-dimensional orbits) of certain four-dimensional Lie groups, in such a way that the metrics can be put in a *diagonal* form; in other words, these are *biaxial diagonal Bianchi metrics of type A*, see e.g. [17], [46]. Theorem 1 relies on the fact that every (non locally symmetric) selfdual Einstein Hermitian metric (g, J) has a distinguished non-trivial Killing field, namely $K = J\mathrm{grad}_g(|W^+|^{-1/3})$, cf. [22]. Then, Jones-Tod reduction [48] with respect to K provides a three-dimensional space of *constant curvature*. The diagonal form of the metrics follows from [48] and [46] (a unified presentation for these cohomogeneity-one metrics also appears in [17]). To the best of our knowledge, apart from these metrics no other examples of selfdual Einstein Hermitian metrics were known in the literature (see however Section 4).

Section 2.2 is devoted to the generic case, when the metric is neither locally symmetric nor of cohomogeneity one. Our approach is similar to Armstrong's one in [3]: When considering the Einstein condition alone, the Riemannian Goldberg-Sachs theorem together with Derdziński's results reported above imply a number of relations for the 4-jet of an Einstein Hermitian metric (Section 2.1, Proposition 2); these happen to be the only obstructions for prolonging the 3-jet solutions of the problem to 4-jet and no further obstructions appear when reducing the equations for non-Kähler, non-anti-selfdual Hermitian Einstein 4-manifolds to a (simple) perturbated SU(∞)-Toda field equation [3], [42]. But if we insist that g be also *selfdual*, we find further relations for the 5-jet of the metric and we show that they have the form of an integrable closed Frobenius system of PDE's for the parameter space of the 4-jet of the metric. This implies the local existence of selfdual Einstein Hermitian metrics which are neither locally symmetric nor of cohomogeneity (Theorem 2). It turns out that this Frobenius system can be explicitly integrated (Lemma 3). We thus obtain a uniform local description for *all* selfdual Einstein Hermitian metrics in an explicit way.

Section 3 is devoted to the subclass of selfdual Einstein Hermitian metrics which admit a compatible, non-closed, anti-selfdual hypercomplex structure. This is the same, locally, as the class of selfdual Einstein Hermitian metrics which admit a non-closed Einstein-Weyl connection (see Section 1.2). From this viewpoint, it is a particular case of four-dimensional conformal metrics admitting two distinct Einstein-Weyl connections. In our case, one of them is the Levi-Civita connection of the Einstein metric, whereas the other one is *non-closed*, hence attached to a non-closed hyperhermitian structure (see Proposition 3). Recall that a conformal 4-manifold admitting two distinct *closed* Einstein-Weyl structures is necessarily conformally-flat (folklore), and that, conversely, a conformally-flat 4-manifold only admits closed Einstein-Weyl structures [23] (see also Proposition 3 and Corollary 1 below).

It turns out that selfdual Einstein Hermitian metrics which admit a compatible, non-closed, anti-selfdual hypercomplex structure, actually admit a second one and thus fall into the *bi-hypercomplex* situation described by Madsen in [38]; in particular, these metrics admit a local action of U(2), with three-dimensional orbits, and are diagonal Bianchi XI metrics, see Theorem 3 below.

A general description of (anti-selfdual) metrics admitting two distinct compatible hypercomplex structures appears in [15], whereas a family of selfdual Einstein metrics with compatible non-closed hyperhermitian structures, parameterized by holomorphic functions of one variable, has been constructed in [16].

In Section 4, we show that all anti-selfdual, Einstein four dimensional *orbifolds* obtained by a quaternionic reduction [24], [25], [26] from the eight-dimensional quaternionic-Kähler spaces $\mathbb{H}P^2$ and $\mathbb{H}H^2$ are Hermitian with respect to the non-standard orientation, hence are locally isomorphic to metrics described in Section 2. These orbifolds include the *weighted projective planes* $\mathbb{C}P^{[p_1,p_2,p_3]}$ for integers $0 < p_1 \leq p_2 \leq p_3$ satisfying $p_3 < p_1 + p_2$, cf. [26, Sec. 4]. On these orbifolds, Bryant has constructed Bochner-flat Kähler metrics with everywhere positive scalar curvature [12, Sec. 4.3], hence also selfdual, Einstein Hermitian metrics according to Lemma 2 below; in view of the results of Section 2, Galicki-Lawson's and Bryant's metrics agree locally.

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1. – Einstein metrics, Hermitian structures and Einstein-Weyl geometry in dimension 4

1.1. – Einstein metrics and compatible Hermitian structures

In the whole paper (M, g) denotes an oriented Riemannian four-dimensional manifold.

A specific feature of the four-dimensional Riemannian geometry is the splitting

$$AM = A^+M \oplus A^-M\,, \tag{1}$$

of the Lie algebra bundle, AM, of skew-symmetric endomorphisms of the tangent bundle, TM, into the direct sum of two Lie algebra subbundles, $A^{\pm}M$, derived from the Lie algebra splitting $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ of the orthogonal Lie algebra $\mathfrak{so}(4)$ into the direct sum of two copies of $\mathfrak{so}(3)$.

A similar decomposition occurs for the bundle $\Lambda^2 M$ of 2-forms

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M\,, \tag{2}$$

given by the spectral decomposition of the Hodge-star operator, $*$, whose restriction to $\Lambda^2 M$ is an involution; here, $\Lambda^{\pm} M$ is the eigen-subbundle for the eigenvalue $\pm$ of $*$.

Both decompositions are actually determined by the conformal metric $[g]$ only. When g is fixed, $\Lambda^2 M$ is identified to AM by setting: $\psi(X, Y) = g(\Psi(X), Y)$, for any Ψ in AM and any vector fields X, Y; then, we can arrange signs in (1) so that (1) and (2) are identified to each other. A similar decomposition and a similar identification occur for the bundle $\Lambda^2(TM)$ of bivectors.

Sections of $\Lambda^+ M$, resp. $\Lambda^- M$, are called *selfdual*, resp. *anti-selfdual*, and similarly for sections of AM or $\Lambda^2(TM)$.

In the sequel, the vector bundles AM, $\Lambda^2 M$ and $\Lambda^2(TM)$ will be freely identified to each other; similarly, the cotangent bundle T^*M will be freely identified to TM; when no confusion can arise, the inner product determined by g will be simply denoted by $(\cdot, \cdot)$; we adopt the convention that $(\Psi_1, \Psi_2) = -\frac{1}{2}\mathrm{tr}\,(\Psi_1 \circ \Psi_2)$, for sections of AM, and the corresponding convention for $\Lambda^2 M$ and $\Lambda^2(TM)$.

The Riemannian curvature, R, is defined by $R_{X,Y} = D^g_{[X,Y]} - [D^g_X, D^g_Y]$, where D^g denotes the Levi-Civita connection of g; R is thus an AM-valued 2-form, but will be rather considered as a section of the bundle $S^2(\Lambda^2 M)$ of symmetric endomorphisms of $\Lambda^2 M$.

The Weyl tensor, W, commutes with $*$ and, accordingly, splits as $W = W^+ + W^-$, where $W^{\pm} = \frac{1}{2}(W \pm W \circ *)$; W^+ is called the *selfdual Weyl tensor*; it acts trivially on $\Lambda^- M$ and will be considered in the sequel as a field of (symmetric, trace-free) endomorphisms of $\Lambda^+ M$; similarly, the *anti-selfdual Weyl tensor* W^- will be considered as a field of endomorphismes of $\Lambda^- M$.

The Ricci tensor, Ric, is the symmetric bilinear form defined by $\mathrm{Ric}(X, Y) = \mathrm{tr}\{Z \to R_{X,Z} Y\}$; alternatively, $\mathrm{Ric}(X, Y) = \sum_{i=1}^4 (R_{X,e_i} Y, e_i)$ for any g-orthonormal basis $\{e_i\}$. We then have $\mathrm{Ric} = \frac{s}{4} g + \mathrm{Ric}_0$, where s is the scalar curvature (= the trace of Ric with respect to g) and Ric_0 is the *trace-free Ricci tensor*. The latter can be made into a section of $S^2(\Lambda^2 M)$, then denoted by $\widetilde{\mathrm{Ric}_0}$, by putting $\widetilde{\mathrm{Ric}_0}(X \wedge Y) = \mathrm{Ric}_0(X) \wedge Y + X \wedge \mathrm{Ric}_0(Y)$.

It is readily checked that $\widetilde{\mathrm{Ric}_0}$ satisfies the first Bianchi identity, i.e. $\widetilde{\mathrm{Ric}_0}$ is a tensor of the same kind as R itself, as well as W^+ and W^-; moreover, $\widetilde{\mathrm{Ric}_0}$ anti-commutes with $*$, so that it can be viewed as a field of homomorphisms from $\Lambda^+ M$ into $\Lambda^- M$, or from $\Lambda^- M$ into $\Lambda^+ M$ (adjoint to each other); we eventually get the well-known Singer-Thorpe decomposition of R, see e.g. [6]:

$$R = \frac{s}{12} \mathrm{Id}_{|\Lambda^2 M} + \frac{1}{2} \widetilde{\mathrm{Ric}_0} + W^+ + W^- , \tag{3}$$

or, in a more pictorial way

$$R = \begin{pmatrix} W^+ + \frac{s}{12} \mathrm{Id}_{|\Lambda^+ M} & \frac{1}{2} \widetilde{\mathrm{Ric}_0}_{|\Lambda^- M} \\ \frac{1}{2} \widetilde{\mathrm{Ric}_0}_{|\Lambda^+ M} & W^- + \frac{s}{12} \mathrm{Id}_{|\Lambda^- M} \end{pmatrix} .$$

The metric g is *Einstein* if $\mathrm{Ric}_0 = 0$ (equivalently, g is Einstein if R commutes with $*$).

The metric g (or rather the conformal class $[g]$) is *selfdual* if $W^- = 0$; *anti-selfdual* if $W^+ = 0$.

An *almost-complex structure* J is a field of automorphisms of TM of square $-\mathrm{Id}|_{TM}$. An *integrable* almost-complex structure is simply called a *complex structure*.

In this paper, the metric g, or its conformal class $[g]$, is fixed and we only consider g-orthogonal almost-complex structures, i.e. almost-complex structure J satisfying the identity $g(JX, JY) = g(X, Y)$, so that the pair (g, J) is an *almost-hermitian structure*; then, the associated bilinear form, F, defined by $F(X, Y) = g(JX, Y)$ is a 2-form, called the *Kähler form*.

The pair (g, J) is *Hermitian* if J is integrable; it is *Kähler* if J is parallel with respect to the Levi-Civita connection D^g; if (g, J) is Kähler then J is integrable and F is closed; conversely, these two conditions together imply that (g, J) is Kähler.

A g-compatible almost-complex structure J is either a section of A^+M or a section of A^-M; it is called *positive*, or *selfdual*, in the former case, *negative*, or *anti-selfdual* in the latter case. Alternatively, the Kähler form is either selfdual or anti-selfdual. Conversely, any section Ψ of A^+M, resp. A^-M, such that $|\Psi|^2 = 2$, is a positive, resp. negative, g-orthogonal almost-complex structure. It follows that any non-vanishing section, Ψ, of A^+M —if any— determines a (positive) almost-complex structure J, defined by $J = \sqrt{2}\frac{\Psi}{|\Psi|}$ (similarly for non-vanishing sections of A^-M).

Whereas the existence of a (positive) g-orthogonal almost-complex structure is a purely topological problem, the similar issue for *complex* structures heavily depends on the geometry of g, and this dependence is essentially measured by the selfdual Weyl tensor W^+.

This assertion can be made more precise in the following way. Let's denote by $\lambda_+ \geq \lambda_0 \geq \lambda_-$ the eigenvalues of W^+ at some point x of M, and assume that W^+ does not vanish at x; equivalently, since W^+ is trace-free, assume that $\lambda_+ - \lambda_-$ is positive; we denote by F_+ an eigenform of W^+ with respect to λ_+, normalized by $|F_+|^2 = 2$; similarly, F_- denotes an eigenform of W^+ for λ_-, again normalized by $|F_-|^2 = 2$; the *roots*, P, of W^+ at x are then defined by $P = \frac{(\lambda_+-\lambda_0)^{1/2}}{(\lambda_+-\lambda_-)^{1/2}}F_- + \frac{(\lambda_0-\lambda_-)^{1/2}}{(\lambda_+-\lambda_-)^{1/2}}F_+$; it is easily checked that this expression actually determine *two* distinct pairs of conjugated roots in the generic case, when the eigenvalues are all distinct, and *one* pair in the degenerate case, when λ_0 is equal to either λ_+ or λ_-.

It is a basic fact that when J is a positive, g-orthogonal *complex* structure defined on M, the value of J at any point x where W^+ does not vanish must be equal to a root of W^+ at that point. This means that on the open subset of M where W^+ does not vanish, the conjugacy class of a positive, g-orthogonal complex structure —if any— is determined by g (in fact by $[g]$) with at most a twofold ambiguity.

On the other hand, it is an easy consequence of the integrability theorem Atiyah-Hitchin-Singer in [4] that A^+M can be locally trivialized by integrable (positive, g-orthogonal) almost-complex structures if and only if $[g]$ is anti-selfdual.

In the sequel, W^+ will be called *degenerate* at some point x if it has at most two distinct eigenvalues at that point. The terms *anti-selfdual* and *non-anti-selfdual* will be abbreviated as ASD and non-ASD respectively.

For a given non-ASD metric g it is a subtle question to decide whether the roots of W^+ actually provide complex structures (this is of course not true in general). The situation is quite different if g is Einstein. It is then settled by the following (weak) Riemannian version of the Goldberg-Sachs theorem, cf. [2], [22], [39], [43]:

Proposition 1. *Let (M, g) be an oriented Einstein 4-manifold; then the following three conditions are equivalent:*

(i) *W^+ is everywhere degenerate;*

(ii) *there exists a positive g-orthogonal complex structure in a neighbourhood of each point of M;*
(iii) (M, g) *is either ASD or W^+ has two and only two distinct eigenvalues at each point.*

A consequence of this proposition is that the selfdual Weyl tensor W^+ of a non-ASD Einstein Hermitian 4-manifold nowhere vanishes and has two distinct eigenvalues at any point, one simple, the other one of multiplicity 2; moreover, the Kähler form F is an eigenform of W^+ for the simple eigenvalue. Conversely, for any oriented, Einstein 4-manifold whose W^+ has only two distinct eigenvalues, the generator of the simple eigenspace of W^+ determines a (positive) Hermitian structure.

For any positive g-orthogonal almost-complex structure J, A^+M splits as follows:

$$A^+M = \mathbb{R} \cdot J \oplus A^{+,0}M\,, \tag{4}$$

where $\mathbb{R} \cdot J$ is the trivial subbundle generated by J and $A^{+,0}M$ is the orthogonal complement (equivalently, $A^{+,0}M$ is the subbundle of elements of A^+M that anticommute with J); $A^{+,0}M$ is a rank 2 vector bundle and will be also considered as a complex line bundle by putting $J\Phi = J \circ \Phi$. We have the corresponding decomposition

$$\Lambda^+M = \mathbb{R} \cdot F \oplus \Lambda^{+,0}M\,, \tag{5}$$

where $\Lambda^{+,0}M$ is the subbundle of J-anti-invariant 2-forms, i.e. 2-forms satisfying $\phi(JX, JY) = -\phi(X, Y)$; again, $\Lambda^{+,0}M$ is considered as a complex line bundle by putting $(J\phi)(X, Y) = -\phi(JX, Y) = -\phi(X, JY)$. As complex line bundles, both $A^{+,0}M$ and $\Lambda^{+,0}M$ are identified to the *anti-canonical bundle* $K^{-1}M = \Lambda^{0,2}M$ of the (almost-complex) manifold (M, J).

For an Einstein, Hermitian 4-manifold, the action of W^+ preserves the decompositions (4) and (5).

1.2. – The Lee form

The *Lee form* of an almost-hermitian structure (g, J) is the real 1-form, θ, defined by

$$\mathrm{d}F = -2\theta \wedge F\,; \tag{6}$$

equivalently, $\theta = -\frac{1}{2} J\, \delta F$, where δ denotes the co-differential with respect to g (here, and henceforth, the action of J on 1-forms is defined via the identification $T^*M \simeq TM$ given by the metric; we thus have $(J\alpha)(X) = -\alpha(JX)$, for any 1-form α). The reason for the choice of the factor -2 in (6) will be clear in the next subsection (notice that a different normalization was used in our previous work [2]).

When (g, J) is Hermitian, it is Kähler if and only if θ vanishes identically; it is conformally Kähler if and only if θ is exact, i.e. if $\theta = -\mathrm{d}\ln f$ for a positive smooth real function f (then, J is Kähler with respect to the conformal metric $g' = f^{-2} g$); it is locally conformally Kähler if and only if θ is closed, hence locally of the above type.

The Lee form clearly satisfies $(\mathrm{d}\theta, F) = 0$; this means that the selfdual part, $\mathrm{d}\theta^+$, of $\mathrm{d}\theta$ is a section of the rank 2 subbundle, $\Lambda^{+,0}M$.

In the Hermitian case, $\mathrm{d}\theta^+$ is an eigenform of W^+ for the mid-eigenvalue λ_0; moreover, $\lambda_0 = -\frac{\kappa}{12}$, where κ is the *conformal scalar curvature*, of which a more direct definition is given in the next subsection; κ is related to the (Riemannian) scalar curvature s by

$$\kappa = s + 6\,(\delta\theta - |\theta|^2)\,, \tag{7}$$

and we also have

$$\kappa = 3\,(W^+(F), F)\,, \tag{8}$$

see [51], [27]. Notice that, in the Hermitian case, the mid-eigenvalue λ_0 of W^+ is always a *smooth* function (this, however, is not true in general for the remaining two eigenvalues of W^+, λ_+ and λ_-, which are given by:

$$\lambda_\pm = \frac{1}{24}\kappa \pm \frac{1}{8}(\kappa^2 + 32|\mathrm{d}\theta^+|^2)^{\frac{1}{2}}\,,$$

cf. [2]).

It follows that for Hermitian 4-manifolds the following three conditions are equivalent (cf. [9], [2]):

(i) $\mathrm{d}\theta^+ = 0$;
(ii) W^+ is degenerate;
(iii) F is an eigenform of W^+.

In the latter case F is actually an eigenform for the simple eigenvalue of W^+, which is then equal to $\frac{\kappa}{6}$, also equal to λ_+ or λ_- according as κ is positive or negative. If, moreover, M is compact, any one of the above three conditions is equivalent to (g, J) being locally conformally Kähler; if, in addition, the first Betti number of M is even, (g, J) is then globally conformally Kähler [50].

By Proposition 1 we conclude that for every Einstein Hermitian 4-manifold, we have $\mathrm{d}\theta^+ = 0$, i.e. $\mathrm{d}\theta$ is selfdual. In fact, a stronger statement is true, see [2, Prop. 1] and [22, Prop. 4]:

PROPOSITION 2. *Let (M, g, J) be an Einstein, non-ASD Hermitian 4-manifold. Then the conformal scalar curvature κ nowhere vanishes and the Lee form θ is given by $\theta = \frac{1}{3}\mathrm{d}\ln|\kappa|$ (in particular, (g, J) is conformally Kähler).*

If, moreover, κ is not constant, i.e. if (g, J) is not Kähler, then $K = J\,\mathrm{grad}_g(\kappa^{-1/3})$ is a non-trivial Killing vector field with respect to g, holomorphic with respect to J.

1.3. – The canonical Weyl structure

Another specific feature of the four-dimensional geometry is that to each conformal Hermitian structure $([g], J)$ is canonically attached a unique *Weyl connection* D such that J is parallel with respect to D; in other words, any Hermitian structure is "Kähler" in the extended context of Weyl structures (of course, (g, J) is Kähler in the usual sense —the only one used in this paper— if and only if D is the Levi-Civita connection of some metric in the conformal class $[g]$).

Recall that, given a conformal metric $[g]$, a Weyl connection with respect to $[g]$ is a torsion-free linear connection, D, on M which preserves $[g]$; the latter condition can be reformulated as follows: for any metric g in $[g]$, there exists a real 1-form θ_g such that $Dg = -2\theta_g \otimes g$; θ_g is called the *Lee form* of D with respect to g; then, the Weyl connection D and the Levi-Civita connection D^g are related by $D = D^g + \tilde{\theta}_g$, meaning

$$D_X Y = D^g_X Y + \theta_g(X) Y + \theta_g(Y) X - g(X, Y)\, \theta_g^{\sharp_g}\,, \tag{9}$$

where $\theta_g^{\sharp_g}$ is the Riemannian dual of θ_g with respect to g. If $g' = f^{-2} g$ is another metric in $[g]$, then the Lee form, $\theta_{g'}$, of D with respect to g' is related to θ_g by $\theta_{g'} = \theta_g + \mathrm{d}\ln f$.

A Weyl connection D is the Levi-Civita connection of some metric in the conformal class $[g]$ if and only if its Lee form with respect to any metric g in $[g]$ is exact, i.e. $\theta_g = -\mathrm{d}\ln f$; then, $D = D^{f^{-2}g}$; such a Weyl connection is called *exact*. More generally, a Weyl connection is said to be *closed* if its Lee form with respect to any metric in $[g]$ is closed; then, D is locally of the above type, i.e. it is the Levi-Civita connection of a (locally defined) metric in $[g]$.

The definitions of the curvature R^D and the Ricci tensor Ric^D of a Weyl connection D are formally identical as the ones we gave for D^g (notice that the derivation of Ric^D from R^D requires no metric); however, R^D is now a $AM \oplus \mathbb{R}\,\mathrm{Id}|_{TM}$-valued 2-form, i.e. has a *scalar part* equal to $F^D \otimes \mathrm{Id}|_{TM}$, where the real 2-form F^D, the so-called *Faraday tensor* of the Weyl connection, is equal to $-\mathrm{d}\theta_g$ for any metric g in $[g]$; moreover, Ric^D is not symmetric in general: its skew-symmetric part is equal to $\frac{1}{2}F^D$; Ric^D is thus symmetric if and only if D is closed.

A Weyl connection D is called *Einstein-Weyl* if the symmetric, trace-free part of Ric^D vanishes with respect to any metric g in $[g]$; writing θ instead of θ_g this conditions reads

$$D^g\theta - \theta \otimes \theta + \frac{1}{4}(\delta\theta + |\theta|^2)\, g - \frac{1}{2}\mathrm{d}\theta - \frac{1}{2}\mathrm{Ric}_0 = 0\,, \tag{10}$$

see e.g. [28]; for a fixed metric g, (10) should be considered as an equation for the unknown 1-form θ.

The *conformal scalar curvature* of D with respect to g, denoted by κ_g, is the trace of Ric^D with respect to g; it is related to the (Riemannian) scalar curvature s by:

$$\kappa_g = s + 6\,(\delta\theta - |\theta|^2)\,, \tag{11}$$

see e.g. [28].

A key observation is that the Lee form θ of an almost-hermitian structure (g, J) is also the Lee form with respect to g of the Weyl connection canonically attached to the conformal almost-hermitian structure $([g], J)$; in other words, the Weyl connection D defined by $D = D^g + \tilde{\theta}$ is actually independent of g in its conformal class $[g]$. The Weyl connection D defined in this way is called the *canonical Weyl connection* of the (conformal) almost-hermitian structure $([g], J)$.

The scalar curvature κ_g of D with respect to g is called the *conformal scalar curvature* of (g, J); it coincides with the function κ introduced in the previous paragraph.

The canonical Weyl connection is an especially interesting object when J is integrable, because of the following lemma:

Lemma 1. (i) *J is integrable if and only if $DJ = 0$.*

(ii) *If J_1 and J_2 are two g-orthogonal complex structures, the corresponding canonical connections, D_1 and D_2, coincide if and only if the scalar product (J_1, J_2) is constant.*

Proof. (i) The condition $DJ = 0$ reads

$$D^g_X J = [X \wedge \theta, J]\,; \tag{12}$$

this identity is proved e.g. in [27], [51].

(ii) Let p denote the *angle function* of J_1 and J_2, defined by $p = -\frac{1}{4}\mathrm{tr}(J_1 \circ J_2) = \frac{1}{2}(J_1, J_2)$; we then have

$$J_1 \circ J_2 + J_2 \circ J_2 = -2p\,\mathrm{Id}|_{TM}\,. \tag{13}$$

Let θ_1 and θ_2 be the Lee forms of D_1, D_2; from (12) applied to J_1, we infer $(D^g J_1, J_2) = ([J_1, J_2]X, \theta_1)$; similarly, we have $(D^g J_2, J_1) = ([J_2, J_1]X, \theta_2)$; putting together these two identities, we get

$$\mathrm{d}p = -\frac{1}{2}[J_1, J_2](\theta_1 - \theta_2)\,. \tag{14}$$

This obviously implies $\mathrm{d}p = 0$ if $D_1 = D_2$; the converse is also true, as the commutator $[J_1, J_2]$ is invertible at each point where $J_2 \neq \pm J_1$. $\square$

1.4. – Hypercomplex and Einstein-Weyl structures

An *almost-hypercomplex structure* on M is the datum of three almost-complex structures, I_1, I_2, I_3, such that

$$I_1 \circ I_2 = -I_2 \circ I_1 = I_3 .$$

Since M is four-dimensional any almost-hypercomplex structure I_1, I_2, I_3 determines a conformal class $[g]$ with respect to which each I_i is orthogonal: $[g]$ is defined by decreeing that for any non-vanishing (local) vector field X the frame $X, I_1 X, I_2 X, I_3 X$ is (conformally) orthonormal; for any g in the conformal class defined in this way we thus get an *almost-hyperhermitian structure* (g, I_1, I_2, I_3); notice that the I_i's are pairwise orthogonal with respect to g, so that I_1, I_2, I_3 is a (normalized) direct orthonormal frame of A^+M; conversely, for a given Riemannian metric g any (normalized) direct orthonormal frame of A^+M can be thought of as an almost-hypercomplex structure and, together with g, form an almost-hyperhermitian structure.

An almost-hyperhermitian structure (g, I_1, I_2, I_3) is called *hyperhermitian* if all I_i's are integrable; it is called *hyperkähler* if the I_i's are all parallel with respect to the Levi-Civita connection D^g.

In the hyperhermitian case the canonical Weyl connections, D_1, D_2, D_3, of the almost-hermitian structures (g, I_1), (g, I_2), (g, I_3) are the same by Lemma 1; the common Weyl connection, D, is called the *canonical Weyl connection* of the hyperhermitian structure.

Conversely, the condition $D_1 = D_2 = D_3$ implies that (g, I_1, I_2, I_3) is hyperhermitian (this observation is due to S. Salamon and F. Battaglia, see e.g. [31]).

The canonical Weyl connection of a hyperhermitian structure (g, I_1, I_2, I_3) is closed if and only if I_1, I_2, I_3 is locally hyperkähler with respect to some (local) metric belonging to the conformal class $[g]$; for brevity, a hyperhermitian structure will be called *closed* or *non-closed* according as its canonical Weyl connection being closed or non-closed.

REMARK 1. In general, for any given hypercomplex structure I_1, I_2, I_3 on a n-dimensional manifold, there exists a *unique* torsion–free linear connection on M that preserves the I_i's, called the *Obata connection*; the canonical connection thus coincides with the Obata connection; for $n > 4$ however, there is no conformal metric canonically attached to I_1, I_2, I_3 and, in general, the Obata connection is not a Weyl connection.

If (g, I_1, I_2, I_3) is hyperhermitian, we have $DI_1 = DI_2 = DI_3 = 0$, where D is the canonical Weyl connection acting on sections of A^+M; it follows that the connection of A^+M induced by D is *flat*; conversely, if D is a Weyl connection, whose induced connection on A^+M is flat, then A^+M can be locally trivialized by a D-parallel (normalized) orthonormal frame I_1, I_2, I_3, which, together with g, constitute a hyperhermitian structure.

The curvature, R^{D,A^+M}, of the induced connection is given by

$$R^{D,A^+M}_{X,Y}\Psi = [R^D_{X,Y}, \Psi],$$

where $R^D_{X,Y}$ is understood as a field of endomorphisms of TM —more precisely a section of $AM \oplus \mathbb{R}\,\mathrm{Id}|_{TM}$— and $[R^D_{X,Y}, \Psi]$ is the commutator of $R^D_{X,Y}$ and Ψ; we easily infer that the vanishing of R^{D,A^+M} is equivalent to the following system of four conditions:

(1) $W^+ = 0$;
(2) $(F^D)^+ = 0$; if θ denotes the Lee form of D, this also reads $\mathrm{d}\theta^+ = 0$;
(3) D is Einstein-Weyl, i.e. the Lee form θ is a solution of (10);
(4) The scalar curvature of D vanishes identically; in view of (11), this condition reads

$$s = 6\,(-\delta\theta + |\theta|^2)\,. \tag{15}$$

It follows from the preceding discussion that for an ASD Riemannian 4-manifold the existence of a compatible hypercomplex structure is locally equivalent to the existence of an Einstein-Weyl connection satisfying the above conditions 2 and 4 (cf. [31] and [41]). In this correspondence, conformally hyperkähler structures correspond to closed Einstein-Weyl structures. The existence of a non-closed hyperhermitian structure is actually equivalent to the existence of a non-closed Einstein-Weyl connection, in view of the following result of D. Calderbank.

PROPOSITION 3 ([13]). *Let $(M, [g], D)$ be an anti-selfdual Einstein-Weyl* 4-*manifold. Then either D is closed, or else D satisfies the conditions* 2 *and* 4 *above, i.e. is the canonical Weyl connection of a hyperhermitian structure.*

In the case when M is compact, $d\theta^+ = 0$ implies $\mathrm{d}\theta = 0$, hence *any* hyperhermitian structure is locally conformally hyperkähler; a complete classification appears in [10].

2. – Selfdual Einstein Hermitian 4-manifolds

By Proposition 2, a Hermitian, Einstein 4-manifold whose selfdual Weyl tensor W^+ has constant eigenvalues is either anti-selfdual or Kähler-Einstein, cf. [22]. If, moreover, the metric g is selfdual, the latter happens precisely when g is locally symmetric, i.e. when (M, g) is a real or a complex space form, see [49]. More generally, a selfdual Einstein 4-manifold is locally symmetric if and only if W^+ is degenerate, with constant eigenvalues, cf. [22]. In the case, we have the following lemma:

LEMMA 2. *Non locally symmetric selfdual Einstein Hermitian metrics are in one-to-one correspondence with selfdual Kähler metrics of nowhere vanishing and non-constant scalar curvature.*

PROOF. Every selfdual Einstein Hermitian 4-manifold (M, g, J) of non-constant curvature is conformally related (via Proposition 2) to a selfdual Kähler metric $\bar{g}$ of nowhere vanishing scalar curvature. A selfdual Kähler metric is locally symmetric if and only if its scalar curvature is constant [22]; thus, one direction in the correspondence of Lemma 2 follows by observing that $\bar{g}$ is locally symmetric as soon as g is. Since the Bach tensor of a selfdual metric vanishes [30] (see also e.g. [13]), it follows from [22, Prop. 4] that any selfdual Kähler metric of nowhere vanishing scalar curvature gives rise to an Einstein Hermitian metric in the same conformal class. □

In the remainder of this section, (M, g, J) will be an Einstein, selfdual Hermitian 4-manifold, and we will assume that g is *not* locally symmetric; in particular, W^+ is degenerate, but its eigenvalues $\lambda, -\frac{\lambda}{2}$, as well as its norm $|W^+| = \sqrt{\frac{3}{2}}\,|\lambda|$, are not constant.

Since (M, g, J) is not Kähler (Proposition 2), by substituting to M the dense open subset where the Lee form θ does not vanish, we shall assume throughout this section that $D^g J$ nowhere vanishes, see (12).

For convenience, we choose a (local, normalized) orthonormal frame of $\Lambda^{+,0}M$ of the form $\{\phi, J\phi\}$, where $|\phi| = \sqrt{2}$; such a frame will be called a *gauge*. Then, the triple $\{F, \phi, J\phi\}$ is a (local, normalized) direct orthonormal frame of $\Lambda^+ M$.

Recall that by Proposition 1 we have

$$W^+(\psi) = -\frac{\kappa}{12}\psi\,, \tag{16}$$

for any section ψ of $\Lambda^{+,0}M$, whereas

$$W^+(F) = \frac{\kappa}{6}F\,. \tag{17}$$

With respect to the gauge $\{\phi, J\phi\}$, the covariant derivative $D^g F$ is written as

$$D^g F = \alpha \otimes \phi + J\alpha \otimes J\phi\,, \tag{18}$$

where

$$\alpha = \phi(J\theta)\,; \tag{19}$$

equivalently,

$$\phi = -\frac{1}{|\theta|^2}\big(\alpha \wedge J\theta + J\alpha \wedge \theta\big); \quad J\phi = \frac{1}{|\theta|^2}\big(\alpha \wedge \theta - J\alpha \wedge J\theta\big)\,. \tag{20}$$

We also have

$$D^g\phi = -\alpha \otimes F + \beta \otimes J\phi; \quad D^g(J\phi) = -J\alpha \otimes F - \beta \otimes \phi\,, \tag{21}$$

for some real 1-form β.

From (18), we infer

$$\begin{aligned}(D^g)^2|_{\Lambda^2 M}F &= (\mathrm{d}\alpha + J\alpha \wedge \beta) \otimes \phi + (\mathrm{d}(J\alpha) - \alpha \wedge \beta) \otimes J\phi \\ &= -R(J\phi) \otimes \phi + R(\phi) \otimes J\phi\,.\end{aligned}$$

Because of (16) this reduces to

$$\begin{cases} \mathrm{d}\alpha - \beta \wedge J\alpha = \dfrac{(\kappa - s)}{12} J\phi \\[2ex] \mathrm{d}(J\alpha) + \beta \wedge \alpha = -\dfrac{(\kappa - s)}{12}\phi\,. \end{cases} \tag{22}$$

Similarly, from (21) and (17) we infer the additional relation:

$$\mathrm{d}\beta + \alpha \wedge J\alpha = -\frac{(s + 2\kappa)}{12}F\,. \tag{23}$$

Notice that real 1-forms α and β are both *gauge dependent*; if

$$\phi' = (\cos\varphi)\phi + (\sin\varphi)J\phi$$

they transform to

$$\alpha' = (\cos\varphi)\alpha + (\sin\varphi)J\alpha; \quad \beta' = \beta + \mathrm{d}\varphi\,.$$

We next introduce real 1-forms $n_i, m_i, i = 1, 2$ by

$$D^g\theta = m_1 \otimes \theta + n_1 \otimes J\theta + m_2 \otimes \alpha + n_2 \otimes J\alpha\,. \tag{24}$$

By using (18) and (20) we derive

$$\begin{aligned} D^g(J\theta) &= -n_1 \otimes \theta + m_1 \otimes J\theta - (n_2 + J\alpha) \otimes \alpha + (m_2 + \alpha) \otimes J\alpha\,; \\ D^g\alpha &= -m_2 \otimes \theta + (n_2 + J\alpha) \otimes J\theta + m_1 \otimes \alpha - (n_1 - \beta) \otimes J\alpha\,; \\ D^g(J\alpha) &= -n_2 \otimes \theta - (m_2 + \alpha) \otimes J\theta + (n_1 - \beta) \otimes \alpha + m_1 \otimes J\alpha\,. \end{aligned} \tag{25}$$

A straightforward computation, using identities (22) and the fact that the vector field $K = (\kappa^{-\frac{1}{3}}J\theta)^{\sharp_g}$, the dual of $\kappa^{-\frac{1}{3}}J\theta$, is Killing (see Proposition 2), gives the following expressions for m_i and n_i:

$$\begin{cases} m_1 = m_0 + \left(p - \dfrac{(\kappa - s)}{24|\theta|^2} + \dfrac{1}{2}\right)\theta \\[2ex] n_1 = Jm_0 + \left(p - \dfrac{(\kappa - s)}{24|\theta|^2} - \frac{1}{2}\right)J\theta \\[2ex] m_2 = J\phi(m_0) - \left(p + \dfrac{(\kappa - s)}{24|\theta|^2} + \dfrac{1}{2}\right)\alpha \\[2ex] n_2 = -\phi(m_0) - \left(p + \dfrac{(\kappa - s)}{24|\theta|^2} + \dfrac{1}{2}\right)J\alpha\,, \end{cases} \tag{26}$$

where p is a smooth function, and m_0 is a 1-form which belongs to the distribution $\mathcal{D}^\perp = \mathrm{span}\{\alpha, J\alpha\}$, the orthogonal complement of $\mathcal{D} = \mathrm{span}\{\theta, J\theta\}$.

Since $m_1 = \mathrm{d}\ln|\theta|$, the 1-form m_0 is nothing else than the projection of $\mathrm{d}\ln|\theta|$ to the subbundle $\mathcal{D}^\perp$. Moreover, with respect to any gauge $\{\phi, J\phi\}$, we write

$$m_0 = q\alpha + rJ\alpha\,, \tag{27}$$

for some smooth functions q and r.

In view of (12), identities (24) and (26) are conditions on the 2-jet of J. Since J is completely determined by W^+ (see Proposition 1), these are the conditions on the 4-jet of the metric referred to in the introduction.

This completes the analysis of the Einstein condition and we are now going to see how the vanishing of W^- interacts on further jets of g.

For that, we introduce the "mirror frame" of $\Lambda^- M$:

$$\bar{F} = -F + \frac{2}{|\theta|^2}\theta \wedge J\theta; \quad \bar{\phi} = \phi + \frac{2}{|\theta|^2}J\alpha \wedge \theta\,;$$
$$I\bar{\phi} = J\phi + \frac{2}{|\theta|^2}J\alpha \wedge J\theta\,;$$

here, I is the *negative* almost-hermitian structure of which the anti-selfdual 2-form $\bar{F}$ is the Kähler form, equal to J on $\mathcal{D}$ and to $-J$ on $\mathcal{D}^\perp$. By (25) and the fact that $\theta = \frac{\mathrm{d}\kappa}{3\kappa}$, we obtain the following expression for the covariant derivative of the Killing vector field $K = (\kappa^{-\frac{1}{3}}J\theta)^{\sharp_g}$

$$D^g K = \kappa^{-\frac{1}{3}}|\theta|^2\left(q\bar{\phi} - rI\bar{\phi} - \left(p - \frac{1}{2}\right)\bar{F} + \frac{(\kappa - s)}{24|\theta|^2}F\right)\,. \tag{28}$$

Moreover, since K is Killing, we have

$$D^g_X \Psi = R(K, X)\,, \tag{29}$$

where $\Psi = D^g K$.

By considering the ASD parts of both sides of (29), we infer that the condition $W^- = 0$ is equivalent to

$$D^g(\Psi^-) = \frac{s}{24}(\bar{\phi}(K) \otimes \bar{\phi} + I\bar{\phi}(K) \otimes I\bar{\phi} + IK \otimes \bar{F})\,, \tag{30}$$

where

$$\Psi^- = \kappa^{-\frac{1}{3}}|\theta|^2\left(q\bar{\phi} - rI\bar{\phi} - \left(p - \frac{1}{2}\right)\bar{F}\right)$$

is the ASD part of $\Psi = D^g K$, see (28). Furthermore, by (24) and (25) one gets

$$\begin{aligned} D^g \bar{F} &= -(2m_2 + \alpha) \otimes \bar{\phi} + (2Jm_2 + J\alpha) \otimes I\bar{\phi}\,; \\ D^g \bar{\phi} &= (2m_2 + \alpha) \otimes \bar{F} + (2n_1 - \beta) \otimes I\bar{\phi}\,; \\ D^g I\bar{\phi} &= -(2Jm_2 + J\alpha) \otimes \bar{F} - (2n_1 - \beta) \otimes \bar{\phi}\,. \end{aligned} \tag{31}$$

Keeping in mind that $\theta = \frac{\mathrm{d}\kappa}{3\kappa}$ and $m_1 = \mathrm{d}\ln|\theta|$, (30) then reduces to

$$\begin{aligned} \mathrm{d}p = &-\left(p - \frac{1}{2}\right)(2m_1 - \theta) + q(m_2 + \alpha) \\ &+ r(Jm_2 + J\alpha) - \frac{s}{24|\theta|^2}\theta \end{aligned} \tag{32}$$

$$\begin{aligned} \mathrm{d}q = &-\left(p - \frac{1}{2}\right)(m_2 + \alpha) - q(2m_1 - \theta) \\ &- r(2n_1 - \beta) - \frac{s}{24|\theta|^2}\alpha \end{aligned} \tag{33}$$

$$\begin{aligned} \mathrm{d}r = &-\left(p - \frac{1}{2}\right)(Jm_2 + J\alpha) + q(2n_1 - \beta) \\ &- r(2m_1 - \theta) - \frac{s}{24|\theta|^2}J\alpha\,. \end{aligned} \tag{34}$$

Now, taking into account (22) and (23), (32)-(34) constitute a closed differential system that a selfdual Einstein Hermitian metric must satisfy; by (22), (23), (25) and (26) one can directly check that the integrability conditions $\mathrm{d}(\mathrm{d}p) = \mathrm{d}(\mathrm{d}q) = \mathrm{d}(\mathrm{d}r) = 0$ are satisfied. This is a first evidence that the existence of selfdual Einstein Hermitian metrics with prescribed 4-jet at a given point can be expected. To carry out this program explicitly, we first consider the case when $q \equiv 0, r \equiv 0$ and show that it precisely corresponds to *selfdual Einstein Hermitian metrics of cohomogeneity one*.

2.1. – Selfdual Einstein Hermitian metrics of cohomogeneity one

A Riemannian 4-manifold (M, g) is said to be (locally) *of cohomogeneity one* if it admits a (local) isometric action of a Lie group G, with three-dimensional orbits. The manifold M is then locally a product

$$M \cong (t_1, t_2) \times G/H\,.$$

The metric g descends to a left invariant metric $h(t)$ on each orbit $\{t\} \times G/H$, and, by an appropriate choice of the parameter t, can be written as

$$g = dt \otimes dt + h(t)\,.$$

If, moreover, (M, g) is Einstein and selfdual, and G is at least of dimension four, then, according to a result of A. Derdziński [22], the selfdual Weyl tensor W^+ of g is everywhere degenerate, and g is Hermitian with respect to some invariant complex structure.

Here is a way of constructing such metrics, all belonging to the class of *diagonal Bianchi metrics of type A* (see e.g. [46]). Let $\widetilde{G}$ be one of the following six three-dimensional Lie groups: $\mathbb{R}^3$, Nil^3, Sol^3, $\mathrm{Isom}(\mathbb{R}^2)$, SU(1,1) or SU(2); let H be a discrete subgroup of $\widetilde{G}$ and consider the family of diagonal metrics $h(t)$ of the form

$$h(t) = A(t)\sigma_1^2 + B(t)\sigma_2^2 + C(t)\sigma_3^2\,, \tag{35}$$

defined on $\widetilde{G}/H$, where A, B, C are positive smooth functions, and σ_i are the standard left invariant generators of the corresponding Lie algebras; we thus have

$$d\sigma_1 = \epsilon_1\sigma_2 \wedge \sigma_3;\;\; d\sigma_2 = -\epsilon_2\sigma_1 \wedge \sigma_3;\;\; d\sigma_3 = \epsilon_3\sigma_1 \wedge \sigma_2$$

for a triple $(\epsilon_1, \epsilon_2, \epsilon_3)$, $n_i \in \{-1, 0, 1\}$, depending on the chosen group, according to the following table:

class	ϵ_1	ϵ_2	ϵ_3	$\widetilde{G}$
I	0	0	0	$\mathbb{R}^3$
II	0	0	1	Nil^3
VI_0	1	-1	0	Sol^3
VII_0	1	1	0	$\mathrm{Isom}(\mathbb{R}^2)$
VIII	1	1	-1	SU(1, 1)
IX	1	1	1	SU(2)

Except for Class VI_0, when $A = B$, all these metrics admit a further (local) symmetry: it comes from the right action of the vector field dual to σ_3 for the metrics in the classes VII_0, VIII and IX, or from the rotation in the $\{\sigma_1, \sigma_2\}$-plane for the metrics in the classes I and II. We thus get the so-called *biaxial* Bianchi metrics, i.e. diagonal Bianchi metrics of Type A, admitting a local isometric action of a four-dimensional Lee group G, where G is U(2), $\mathrm{U}(1, 1)$, $\mathbb{R} \times \mathrm{Isom}(\mathbb{R}^2)$ (corresponding to biaxial Class I and VI_0 metrics), or the non-trivial central extension of $\mathrm{Isom}(\mathbb{R}^2)$ (corresponding to biaxial Class II metrics), see e.g. [17]. Clearly, any such metric admits a positive *and* a negative invariant Hermitian structure, J and I, whose Kähler forms are given by

$$F = \sqrt{C}dt \wedge \sigma_3 + A\sigma_1 \wedge \sigma_2\,,$$

and

$$\bar{F} = \sqrt{C}dt \wedge \sigma_3 - A\sigma_1 \wedge \sigma_2\,,$$

respectively. When imposing the Einstein and the selfduality conditions, we obtain an ODE system for the unknown functions A and C, which can be explicitly solved, cf. e.g. [5], [17], [20], [36], [40], [46].

In the sequel, we shall simply refer to these (selfdual, Einstein, Hermitian) metrics as *diagonal Bianchi* metrics.

Note that the 4-dimensional locally symmetric metrics, the real and the complex space forms, can also be written (in several ways) as diagonal Bianchi metrics. For example, all selfdual Einstein Hermitian metrics in Class VII_0 are real space forms, while those in Class I are flat, cf. [46].

Our next result shows that, apart from locally symmetric spaces, diagonal Bianchi metrics in the above sense are actually *all* (non locally symmetric) cohomogeneity-one selfdual Einstein Hermitian metrics, and, in fact, can be characterized by the property $m_0 \equiv 0$ in the notation of the preceding section. More precisely, we have:

THEOREM 1. *Let (M, g) be a selfdual Einstein 4-manifold. Suppose that (M, g) is not locally symmetric. Then the following three conditions are equivalent:*

(i) *(M, g) is locally of cohomogeneity one and the spectrum of W^+ is degenerate.*

(ii) *(M, g) admits a local isometric action of a Lie group of dimension at least four, with three-dimensional orbits, and is locally isometric to a diagonal Bianchi selfdual Einstein Hermitian metric belonging to one of the classes* II, VIII *or* IX.

(iii) *(M, g) admits a positive, non-Kähler Hermitian structure, J, and a negative Hermitian structure, I, such that I is equal to J on $\mathcal{D} = \mathrm{span}\{\theta, J\theta\}$ and to $-J$ on the orthogonal complement $\mathcal{D}^\perp$; equivalently, the 1-form m_0 of (g, J) vanishes identically.*

PROOF. (i) $\Rightarrow$ (iii). By Propositions 1 and 2, W^+ has two distinct, non-constant eigenvalues at any point and there exists a positive, non-Kähler Hermitian structure J whose Kähler form F generates the eigenspace of W^+ corresponding to the simple eigenvalue. It follows that the Hermitian structure is preserved by the action of G, and therefore both functions $|D^g F|^2 = 2|\theta|^2$ and $|W^+|^2 = \frac{\kappa^2}{24}$ are constant along the orbits of G; in particular, $\mathrm{d}\ln|\theta|$ is colinear to $\theta = \frac{\mathrm{d}\kappa}{3\kappa}$, at any point; this means that $m_0 = 0$; by (31) and (26), the vanishing of m_0 is equivalent to the integrability of the negative almost Hermitian structure I.

(iii) $\Rightarrow$ (ii). If $m_0 \equiv 0$ or, equivalently, if the negative almost Hermitian structure I is integrable, then, by (31), the Lie form θ_I of (g, I) reads:

$$\theta_I = \left(2p + \frac{(\kappa - s)}{12|\theta|^2}\right)\theta\,. \tag{36}$$

According to (26) we also have $m_1 = \mathrm{d}\ln|\theta| = (p - \frac{(\kappa-s)}{24|\theta|^2} + \frac{1}{2})\theta$ and $\theta = \frac{1}{3}\mathrm{d}\ln|\kappa|$; it follows that (locally) $\theta_I = \mathrm{d}f$ for a positive K-invariant function f, i.e. g is conformal to a Kähler metric $g' = f^2 g$. Since $W^- = 0$, the Kähler metric g' is of zero scalar curvature. Clearly, the Killing field K preserves both J and g, hence, also, the Kähler structure (g', I). Two cases occur, according as g' is homothetic or not to g.

(a) Suppose g' is *not* homothetic to g; equivalently, the scalar curvature s of g does not vanish; then, by [22], $K' = I\,\mathrm{grad}_g(f^{-1})$ is a Killing vector field for g and g' and is holomorphic with respect I. By the very definition of I we have that $J|_{\mathcal{D}} = I|_{\mathcal{D}}$; the Killing vector fields K' and K are thus colinear everywhere (see (36)); it follows that K' is a constant multiple of K. By considering $z = f^2$ as a local coordinate on M and, by introducing a holomorphic coordinate $x + iy$ on the (locally defined) orbit-space for the holomorphic action of $K + \sqrt{-1}IK$ on (M, I), the metric g can be written in the following form:

$$g = \frac{1}{z^2}\left(e^u w(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) + w\mathrm{d}z \otimes \mathrm{d}z + w^{-1}\omega \otimes \omega\right), \tag{37}$$

where $u(x, y, z)$ is a smooth function satisfying the $\mathrm{SU}(\infty)$ Toda field equation:

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0,$$

w is a positive function given by

$$w = \frac{6(zu_z - 2)}{s},$$

and ω is a connection 1-form of the $\mathbb{R}$-bundle $M \mapsto N = \{(x, y, z)\} \subset \mathbb{R}^3$, whose curvature is given by

$$\mathrm{d}\omega = -w_x \mathrm{d}y \wedge \mathrm{d}z - w_y \mathrm{d}z \wedge \mathrm{d}x - (we^u)_z \mathrm{d}x \wedge \mathrm{d}y, \tag{38}$$

(see e.g. [48]). Moreover, the Killing field K is dual to $\frac{1}{wz^2}\omega$, and the (anti-selfdual) Kähler form of the negative Hermitian structure I is given by

$$\bar{F} = \frac{1}{z^2}\left(we^u \mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}z \wedge \omega\right). \tag{39}$$

By (36) we have that $\mathcal{D} = \mathrm{span}\{\theta, J\theta\} = \mathrm{span}\{\theta_I, I\theta_I\} = \mathrm{span}\{K^{\sharp_g}, IK^{\sharp_g}\}$, so that the Kähler form F of the positive Hermitian structure J is given by

$$F = \frac{1}{z^2}\left(we^u \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge \omega\right). \tag{40}$$

It is now easily seen that (39) and (40) simultaneously define integrable almost complex structures if and only if $w_x = w_y = 0$, or equivalently if and only if $u(x, y, z) = u_1(x, y) + u_2(z)$. This means that u is a *separable* solution to the $\mathrm{SU}(\infty)$ Toda field equation. Up to a change of the holomorphic coordinate $x + iy$, it is explicitly given by [48]

$$e^u = \frac{4(c + bz + az^2)}{(1 + a(x^2 + y^2))^2},$$

for properly chosen constants a, b, c. Any such solution gives rise to a diagonal Bianchi selfdual Einstein Hermitian metric pertaining to one of the classes I, II, VII_0, VIII and IX, depending on the choice of the constants a, b, c (see e.g. [17, Sec. 8]). The case when $b = 0$ corresponds to Class I or VII_0 metrics; we then have that g' is the product metric of two Riemann surfaces, one of constant curvature a and another one of constant curvature $-a$; correspondingly, the Einstein metric g is conformally-flat, i.e. g is a real space form, a contradiction. We thus conclude that g is in one of the classes II, VIII or IX.

(b) If g' is homothetic to g, i.e. (g, I) is itself a Kähler structure of zero scalar curvature, then g is locally hyperkähler and K is a Killing vector field preserving the Kähler structure I. Then, one of the two following situations occurs:

(b1) *K is triholomorphic*, i.e. K preserves each Kähler structure in the hyperkähler family: Then the quotient space, N, for the (real) action of K is flat and is endowed with a field of parallel straight lines. This situation is described by the Gibbons-Hawking Ansatz [32], and the metric g has the form:

$$g = w(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z) + \frac{1}{w}\omega \otimes \omega\,,$$

for a positive harmonic function $w(x, y, z)$ on N and a 1-form ω on M satisfying

$$\mathrm{d}\omega = -w_x \mathrm{d}y \wedge \mathrm{d}z - w_y \mathrm{d}z \wedge \mathrm{d}x - w_z \mathrm{d}x \wedge \mathrm{d}y\,.$$

The Killing field K is dual to $\frac{1}{w}\omega$ and one may consider that the positive and negative Hermitian structures, J and I, correspond to the 2-forms

$$F = w\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge \omega; \quad \bar{F} = w\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}z \wedge \omega\,,$$

respectively. We again conclude that $w_x = 0$, $w_y = 0$, and therefore $w = az+b$. The case $a = 0$ corresponds to flat metrics in Class I, whereas, when $a \neq 0$, by putting $at = az + b$, $\sigma_1 = \mathrm{d}x$, $\sigma_2 = \mathrm{d}y$, $\sigma_3 = \omega$, the metric becomes a diagonal Bianchi metric of Class II.

(b2) *K is not triholomorphic*: Since, nevertheless, K preserves (g, I), the metric g takes the form [8]

$$g = e^u w(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) + w\mathrm{d}z \otimes \mathrm{d}z + w^{-1}\omega \otimes \omega\,,$$

where $u(x, y, z)$ is a solution to the $\mathrm{SU}(\infty)$ Toda field equation, $w = au_z$, ω satisfies (38) and a is a constant. Moreover, K is dual to $\frac{1}{w}\omega$, and I is defined by the anti-selfdual form

$$\bar{F} = we^u \mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}z \wedge \omega\,.$$

Similar arguments as above show that $w_x = w_y = 0$, i.e. u is a separable solution to the $\mathrm{SU}(\infty)$ Toda field equation, and therefore our metric is again a diagonal Bianchi metric in one of the classes II, VIII or IX.

The implication (ii) $\Rightarrow$ (i) is clear. $\square$

REMARK 2. A weaker version of Theorem 1 was announced in [21] (see [21, Rem. 1.3] and Lemma 2 above).

2.2. – The generic case

We now consider the generic case, when m_0 a *non-vanishing* section of $\mathcal{D}^{\perp}$, hence determines a gauge ϕ such that $r \equiv 0, q \neq 0$ in (27). According to (26), the 1-form α is then given by

$$m_1 = \mathrm{d} \ln |\theta| = q\alpha + \left(p - \frac{(\kappa - s)}{24|\theta|^2} + \frac{1}{2} \right) \theta \,; \tag{41}$$

moreover, by (32)-(34), we have that

$$\beta = \frac{1}{q} \left(p(2p + \frac{(\kappa - s)}{12|\theta|^2} - 1) - \frac{\kappa}{24|\theta|^2} + 2q^2 \right) J\alpha \frac{(\kappa - s)}{12|\theta|^2} J\theta \,, \tag{42}$$

$$\mathrm{d}p = \left(2q^2 - p(2p - \frac{(\kappa - s)}{12|\theta|^2} - 1) - \frac{\kappa}{24|\theta|^2} \right) \theta - q \left(4p + \frac{(\kappa - s)}{12|\theta|^2} - 1 \right) \alpha \,, \tag{43}$$

$$\mathrm{d}q = -q \left(4p - \frac{(\kappa - s)}{12|\theta|^2} - 1 \right) \theta - \left(2q^2 - p(2p + \frac{(\kappa - s)}{12|\theta|^2} - 1) + \frac{\kappa}{24|\theta|^2} \right) \alpha \,. \tag{44}$$

By differentiating (41) and by making use of (43)-(44), we get

$$\mathrm{d}\alpha = \frac{(\kappa - s)}{12|\theta|^2} \alpha \wedge \theta = \alpha \wedge J\beta \,; \tag{45}$$

this is nothing but the first relation in (22), when β is given by (42); by substituting the expression (42) for β into the second relation of (22), we obtain

$$\mathrm{d}(J\alpha) = J\alpha \wedge J\beta \,. \tag{46}$$

In view of (41) and (43)-(44), it is not hard to check that the 1-form $J\beta$ is equivalently given by

$$J\beta = \mathrm{d} \ln \left(\frac{|\kappa|}{|q||\theta|^4} \right) \,, \tag{47}$$

so that (46) becomes

$$\mathrm{d} \left(\frac{\kappa}{q|\theta|^4} J\alpha \right) = 0 \,; \tag{48}$$

from (25) we get

$$\mathrm{d}(J\theta) = J\theta \wedge \left(\frac{1}{3} \mathrm{d} \ln |\kappa| - 2 \mathrm{d} \ln |\theta| \right) + J\alpha \wedge \eta \,, \tag{49}$$

or, equivalently,

$$\mathrm{d}\left(\frac{\kappa^{\frac{1}{3}}}{|\theta|^2}J\theta\right) = \frac{\kappa^{\frac{1}{3}}}{|\theta|^2}J\alpha\wedge\eta\,, \tag{50}$$

where

$$\eta = -2q\theta + \left(2p + \frac{(\kappa - s)}{12|\theta|^2} - 1\right)\alpha\,.$$

We are now ready to prove the existence of selfdual Einstein Hermitian metrics with $m_0 \neq 0$. More precisely, we exhibit a 1–1-correspondence between these metrics and the set of solutions of the integrable Frobenius system (43)-(44). We start with the data $(s, \kappa, |\theta|)$ consisting of a constant s (the scalar curvature), a nowhere vanishing smooth function κ (the conformal scalar curvature), and a positive smooth function $|\theta|$ (the norm of the Lie form $\theta = \frac{\mathrm{d}\kappa}{3\kappa}$), defined on an open subset U of M, such that $\theta\wedge\mathrm{d}|\theta|^2$ has no zero on U (equivalently, m_0 does not vanish on U). We then introduce local coordinates $x = \kappa^{\frac{1}{3}} \neq 0$ and $y = |\theta|^2 > 0$. Observe that x is a *momentum map* for the Killing field K with respect to the selfdual Kähler metric $\bar{g} = \kappa^{\frac{2}{3}}g$ while $y = |K|^2_{\bar{g}}$ is the square-norm of K with respect to $\bar{g}$ (see Proposition 2). The Lee form θ is then given by

$$\theta = \frac{\mathrm{d}x}{x}\,, \tag{51}$$

and the 1-form α is given by (41) for some smooth functions $p(x, y)$ and $q(x, y) \neq 0$ of x, y, i.e.

$$\alpha = \frac{1}{q}\left(\frac{\mathrm{d}y}{2y} - \frac{1}{x}\left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2}\right)\mathrm{d}x\right)\,. \tag{52}$$

Then, (43)-(44) can be made into the following Frobenius system for the (unknown) functions p and q^2:

$$\begin{aligned}\mathrm{d}p = \frac{1}{x}&\left(2q^2 + 2\left(p + \frac{(x^3 - s)}{24y}\right)\left(p - \frac{(x^3 - s)}{24y} + 1\right) - \frac{1}{2} - \frac{x^3}{24y}\right)\mathrm{d}x\\ &- \frac{1}{y}\left(2p + \frac{(x^3 - s)}{24y} - \frac{1}{2}\right)\mathrm{d}y\end{aligned} \tag{53}$$

$$\begin{aligned}\mathrm{d}(q^2) = &-\frac{1}{y}\left(2q^2 - 2p\left(p + \frac{(x^3 - s)}{24y} - \frac{1}{2}\right) + \frac{x^3}{24y}\right)\mathrm{d}y\\ &- \frac{2}{x}\Bigg(\left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2}\right)\left(2p\left(p + \frac{(x^3 - s)}{24y} - \frac{1}{2}\right) - \frac{x^3}{24y}\right)\\ &- 2q^2(1 - p)\Bigg)\mathrm{d}x\,.\end{aligned} \tag{54}$$

A straightforward computation shows that the integrability condition $\mathrm{d}(\mathrm{d}p) = \mathrm{d}(\mathrm{d}q^2) = 0$ is satisfied (as a matter of fact, explicit solutions are given in Lemma 3 below). The above mentioned correspondence between solutions to (53)-(54) and selfdual Einstein Hermitian metrics with $m_0 \neq 0$ now goes as follows. Since (53)-(54) is integrable, each value of (p, q) at a given point (x_0, y_0) can be extended to a solution of (53)-(54) in some neighborhood V of (x_0, y_0); moreover, by choosing $q(x_0, y_0) \neq 0$, we may assume that q has no zero on V; by (52) and (53)-(54), one immediately obtains (45) for the corresponding 1-form α. We then introduce a third local coordinate, z, such that

$$J\alpha = \frac{qy^2}{x^3}\mathrm{d}z\,, \tag{55}$$

see (48). Finally, since the 1-form $J\theta$ satisfies (49) or, equivalently, (50), the integrability condition reads as follows:

$$\mathrm{d}\left(\frac{qy}{x^2}\eta\right) = 0\,,$$

see (48) and (49); by using (43)-(46), one easily checks that the integrability condition is actually satisfied, so that

$$J\theta = \frac{y}{x}(\mathrm{d}t + h\mathrm{d}z)\,, \tag{56}$$

where t is a suitable transversal coordinate to (x, y, z), and $h(x, y)$ is a smooth function on V, defined by

$$\mathrm{d}h = -\frac{qy}{x^2}\eta\,.$$

It is an easy consequence of (53) that the above equation is solved by

$$h = \frac{yp}{x^2} + \frac{x}{24}\,. \tag{57}$$

The metric g and the orthogonal almost-complex structure J are then given by

$$g = \frac{1}{|\theta|^2}(\theta \otimes \theta + J\theta \otimes J\theta + \alpha \otimes \alpha + J\alpha \otimes J\alpha)\,.$$

According to (51), (52), (55) and (56), and by using the coordinates (x, y, z, t) introduced above, the metric g takes the form

$$\begin{aligned} g = \frac{1}{y}\Bigg(&\frac{1}{q^2}\left(\frac{\mathrm{d}y}{2y} - \frac{1}{x}\left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2}\right)\mathrm{d}x\right) \otimes \left(\frac{\mathrm{d}y}{2y} - \frac{1}{x}\left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2}\right)\mathrm{d}x\right) \\ &+ \frac{1}{x^2}\mathrm{d}x \otimes \mathrm{d}x + \frac{q^2y^4}{x^6}\mathrm{d}z \otimes \mathrm{d}z + \frac{y^2}{x^2}(\mathrm{d}t + h\mathrm{d}z) \otimes (\mathrm{d}t + h\mathrm{d}z)\Bigg)\,, \end{aligned} \tag{58}$$

where, we recall, s is a constant (equal to the scalar curvature of the metric), (p, q) is a solution of (53)-(54), and h is given by (57). This shows that any selfdual Einstein Hermitian metric with $m_0 \neq 0$ is locally isometric to a metric of the above form for some solution (p, q) to (53)-(54).

Conversely, for any solution to (53)-(54), the corresponding almost-hermitian metric (g, J) is selfdual Einstein Hermitian metric with $m_0 \neq 0$. Indeed, by (45), (46) and (50), J is integrable and it is easily checked that $\theta = \frac{dx}{x}$ is the Lee form for (g, J), i.e.

$$dF = -2\theta \wedge F\,;$$

moreover, the 1-form α corresponds to the gauge

$$\phi = -\frac{1}{y}\big(\alpha \wedge J\theta + J\alpha \wedge \theta\big)\,,$$

meaning that $\alpha = \phi(J\theta)$; one directly computes

$$d\phi = (\theta + J\beta) \wedge \phi\,,$$

where the 1-form β is given by (42); it follows that β is precisely the 1-form defined by (21) and that (45)-(46) are nothing but the Ricci identities (22); this allows us to recognize the curvature: By (22), the Ricci tensor of (g, J) is J-invariant, and, since $\theta = \frac{dx}{x}$, the dual vector field K of $\kappa^{-\frac{1}{3}} J\theta = \frac{1}{x} J\theta$ is Killing, cf. e.g. [2]; by (50) and (18), the covariant derivative of θ is given by (24) for p and q constructed as above, and $r \equiv 0$; hence, (42) and (43)-(44) (equivalently, (53)-(54)) are the same as (32)-(34); these, in turn, are a way of re-writing (30); it follows that the projection of the curvature to $\Lambda^- M$ reduces to $\frac{s}{12}\mathrm{Id}|_{\Lambda^- M}$, i.e. the Hermitian metric g is Einstein and selfdual, with scalar curvature equal to s, see (3); turning back to (45), we conclude that the conformal scalar curvature is $\kappa = x^3$, see (22); the metric constructed in this way is not of cohomogeneity one, as $m_0 \neq 0$, see Theorem 1. Different solutions (p, q) of (53)-(54) give rise to non-isometric metrics, as p and q are determined by $|W^+|, d|W^+|$ and $d|D^g W^+|$.

We finally observe that the metric (58) admits two commuting vector fields, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ and summarize the results obtained so far as follows:

THEOREM 2. *Let (M, g, J) be a selfdual Einstein Hermitian* 4*-manifold. Suppose that (M, g, J) is neither locally symmetric nor of cohomogeneity one. Then, on an open dense subset of M, g is locally given by* (58). *In particular, (M, g) admits a local isometric action of $\mathbb{R}^2$ almost-everywhere.*

REMARK 3. (i) It is easily seen that the metrics (58) have only 2-dimensional continuous symmetries. Moreover, as we already observed, the coordinate $x_1 = x = \kappa^{\frac{1}{3}}$ is a momentum map of the Killing vector field $K_1 = \frac{\partial}{\partial t}$ with respect to the Kähler metric $\bar{g} = x^2 g$ while, by (53) and (57), a momentum map x_2 of the second Killing field, $K_2 = \frac{\partial}{\partial z}$, is given by

$$x_2 = \frac{y}{2x} + \frac{x^3 + s}{24x}\,.$$

The momentum map x_1 is also equal to the scalar curvature of the Kähler metric $\bar{g}$. A straighforward computation shows that x_2 is related to the Pfaffian of the *normalized Ricci form* $\bar{\sigma}$ of the Kähler metric $\bar{g}$ by $x_2 = 6\,(\mathrm{Pf}(\bar{\sigma}) + 2b)$, where b is the constant appearing in (61) below. This fits with an observation of R. Bryant in [12]. Recall that for any 2-form ψ, the Pfaffian of ψ with respect to $\bar{g}$ is defined by $\psi \wedge \psi = \mathrm{Pf}(\psi)\, v_{\bar{g}}$, where $v_{\bar{g}}$ is the volume form of $\bar{g}$; the normalized Ricci form $\bar{\sigma}$ is the $(1,1)$-form associated to the normalized Ricci tensor, $\bar{S}$, appearing in the usual decomposition $\bar{R} = \bar{S} \wedge \bar{g} + W$ of the curvature operator of $\bar{g}$; it is related to the usual Ricci form $\bar{\rho}$ by $\bar{\sigma} = \frac{1}{2}\,(\bar{\rho}_0 + \frac{x}{12}\,\bar{\omega})$, where $\bar{\rho}_0$ is the trace-free part of $\bar{\rho}$; since $g = x^{-2}\bar{g}$ is Einstein and $d^c x$ is the dual of a Killing vector field, we have that $\bar{\rho}_0 = -\frac{1}{x}\,(dd^c x)_0$ and the claim follows easily.

(ii) It follows from Theorems 1 and 2 that every selfdual Einstein Hermitian 4-manifold admits a (local) isometric $\mathbb{R}^2$-action compatible with a *product structure* in the sense of [35]; the general considerations in [35, Sec. 2] therefore apply to the present situation; a detailed analysis of selfdual Einstein 4-manifolds admitting $\mathbb{R}^2$-continuous symmetry has been recently carried out by D. Calderbank and H. Pedersen [18] (see also [14]).

We close this section by providing an explicit form for the metric (58).

LEMMA 3. *The solutions $p(x, y)$ and $q(x, y)$ of the system* (53)-(54) *are explicitly given by*

$$p = \frac{f}{y^2} - \frac{(x^3 - s)}{24y} + \frac{1}{4}\,; \tag{59}$$

$$q^2 = \frac{1}{y^2}\left(\frac{x}{2}f' - f + \left(\frac{x^3 - s}{24}\right)^2\right) - \frac{x^3}{24y} - p^2\,, \tag{60}$$

where

$$f(x) = ax^2 + bx^4 - \frac{(x^6 - s^2)}{576}\,, \tag{61}$$

s, a and b are constants defined by positivity in (60)*, and f' stands for the first derivative of f.*

PROOF. We first observe that (53) can be equivalently written as

$$d\left(y^2\left(p + \frac{(x^3 - s)}{24y} - \frac{1}{4}\right)\right) = \frac{y^2}{x}\left(2q^2 + 2\left(p + \frac{(x^3 - s)}{24y}\right)\left(p - \frac{(x^3 - s)}{24y}\right)\right.$$
$$\left. + 2\left(p + \frac{(x^3 - s)}{24y} - \frac{1}{4}\right) + \frac{x^3}{12y}\right) dx\,;$$

this shows that $y^2(p + \frac{(x^3-s)}{24y} - \frac{1}{4})$ is function of x, say f; from the above equality, we get (59) and (60), where f is a (still unknown) smooth function; in order to determine f, we differentiate (60) by using (59) and substitute into (54); then, cancellations occur and (54) eventually reduces to

$$x^2 f'' - 5xf' + 8f + \frac{(x^6 - s^2)}{72} = 0\,; \tag{62}$$

the solutions of (62) are given by (61). □

3. – Selfdual Einstein Hermitian metrics with hyperhermitian structures

In this section, we consider selfdual, Einstein, Hermitian metrics which in addition admit a *non-closed* hyperhermitian structure compatible with the negative orientation. It is well-known that LeBrun-Pedersen metrics, which are of cohomogeneity one under the action of the unitary group U(2), carry such hyperhermitian structures; in LeBrun's coordinates [36] these metrics read as follows:

$$\begin{aligned} g = \frac{1}{(bt^2+4c)^2}\Bigg(&\left(1+\frac{8b}{t^2}+\frac{16c}{t^4}\right)^{-1} \mathrm{d}t\otimes\mathrm{d}t \\ &+\frac{t^2}{4}\left(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2+\left(1+\frac{8b}{t^2}+\frac{16c}{t^4}\right)\sigma_3\otimes\sigma_3\right)\Bigg)\,, \end{aligned} \tag{63}$$

where b and c are properly chosen constants [38]; more precisely, we have the following

Proposition 4 ([38]). *Let (M, g) be an oriented selfdual Einstein 4-manifold. Assume that (M, g) admits a* U(2) *isometric action with generically three-dimensional* SU(2)*-orbits. If g admits a non-closed,* U(2)*-invariant negative hyperhermitian structure, then g is isometric to* (63) *with $c > b^2$, and actually admits exactly two distinct invariant hyperhermitian structures.*

We here prove the following more general result:

Theorem 3. *A selfdual Einstein Hermitian 4-manifold (M, g, J) locally admits a non-closed, negative hyperhermitian structure if and only if g is locally isometric to one of the* U(2)*-invariant metrics* (63) *with $c > b^2$; then, (M, g) actually carries exactly two distinct hyperhermitian structures, each of them* U(2)*-invariant.*

We first establish general facts concerning selfdual Einstein 4-manifolds which carry a *non-closed* hyperhermitian structure compatible with the negative orientation. As already observed in Section 2, a (negative) hyperhermitian structure (g, I_1, I_2, I_3) is determined by a real 1-form θ —the common Lee form of (g, I_i), also the Lee form of the Obata connection— satisfying conditions (10) and (15), and such that $\Phi := \mathrm{d}\theta$ is selfdual; in particular, the 2-form Φ is harmonic. The next lemma shows that the selfdual Weyl tensor of g is completely determined by θ, Φ and the first covariant derivative $D^g\Phi$ of Φ.

Lemma 4. *Let (M, g) be an oriented selfdual Einstein 4-manifold and assume that (M, g) carries a negative hyperhermitian structure. Then, as a symmetric operator acting on $\Lambda^+ M$, the selfdual Weyl tensor W^+ is given by*

$$W^+(\psi) = \frac{1}{2}[\psi, \Phi] + \frac{1}{|\theta|^2} D^g_{\psi(T)}\Phi\,, \tag{64}$$

where ψ is any selfdual 2-form, T is the Riemannian dual vector field of θ, and $[\cdot, \cdot]$ denotes the commutator of 2-forms, viewed as skew-symmetric endomorphisms of the tangent bundle. Moreover, θ and Φ are related by

$$D^g_T\Phi = 2|\theta|^2\Phi\,. \tag{65}$$

$$\mathrm{d}|\theta|^2 - \left(\frac{s}{12} + |\theta|^2\right)\theta + \Phi(\theta) = 0\,, \tag{66}$$

Proof. By using (10), the right-hand side of

$$R_{X,Y}\theta = (D^g)^2_{Y,X}\theta - (D^g)^2_{X,Y}\theta$$

is easily computed; we thus obtain:

$$\begin{aligned} R(\theta \wedge Z) = &-\frac{1}{2}\mathrm{d}|\theta|^2 \wedge Z - \frac{1}{2}\left(\frac{s}{12} - |\theta|^2\right)\theta \wedge Z \\ &- \frac{1}{2}\Phi(Z) \wedge \theta - \frac{1}{2}D^g_Z\Phi + \theta(Z)\Phi\,. \end{aligned} \tag{67}$$

Since g is selfdual and Einstein, $R = \frac{s}{12}\mathrm{Id}|_{\Lambda^2 M} + W^+$, see (3). Then, by projecting (67) to $\Lambda^- M$, we get (66), whereas the projection of (67) to $\Lambda^+ M$ gives (64) and (65). □

Corollary 1 ([13], [23]). *Every hyperhermitian structure on a conformally-flat 4-manifold is closed.*

Proof. If we assume that $\Phi \neq 0$ somewhere on M and that the anti-selfdual Weyl tensor is identically zero, then, after contracting (64) and (65) with Φ, we obtain $\theta = \frac{1}{4}\mathrm{d}\ln|\Phi|^2$, which contradicts $\Phi = \mathrm{d}\theta \neq 0$. □

We compute the covariant derivative $D^g_T W^+$ of W^+ along T by using (64) together with (65) and (66) (the latter are used for evaluating the term $(D^g)^2_{T,\psi(T)}\Phi$ which appears in the calculation); we thus get

LEMMA 5. *Let (M, g) be an oriented selfdual Einstein 4-manifold, admitting a negative hyperhermitian structure; then, the covariant derivative $D^g_T W^+$ of the selfdual Weyl tensor W^+ along the dual vector field T of the Lee form θ is given by*

$$\begin{aligned} \big((D^g_T W^+)(\psi), \phi\big) = {} & \big([W^+(\phi), \psi] + [W^+(\psi), \phi], \Phi\big) \\ & + \left(4|\theta|^2 - \frac{s}{6}\right)\big(W^+(\psi), \phi\big) \\ & + |\Phi|^2\big(\psi, \phi\big) - 3\big(\Phi, \psi\big)\big(\Phi, \phi\big)\,, \end{aligned} \tag{68}$$

for any sections, ϕ and ψ, of $\Lambda^+ M$.

From Lemma 5 and Propositions 1 and 2, we infer

PROPOSITION 5. *Let (M, g) be an oriented selfdual Einstein 4-manifold, admitting a non-closed hyperhermitian structure compatible with the negative orientation. Then the following three conditions are equivalent:*

(i) *the spectrum of W^+ is everywhere degenerate;*
(ii) *W^+ has two distinct eigenvalues at any point;*
(iii) *the selfdual 2-form Φ is a nowhere vanishing eigenform for W^+ with respect to the simple eigenvalue, and is proportional to a positive Hermitian structure J.*

PROOF. (i) $\Rightarrow$ (ii). According to Proposition 1, if the spectrum of W^+ is everywhere degenerate, then either W^+ vanishes identically (and therefore the hyperhermitian structure is closed by Corollary 1) or W^+ has two distinct eigenvalues λ and $-\frac{\lambda}{2}$ at any point.

(ii) $\Rightarrow$ (iii). By Proposition 1, we know that a normalized generator F of the λ-eigenspace of W^+ is the Kähler form of a positive Hermitian structure J. Let ϕ be any selfdual 2-form orthogonal to F, with $|\phi|^2 = 2$; then, ϕ and $\psi = J\phi$ are orthogonal, $(-\frac{\lambda}{2})$-eigenforms of W^+; by substituting into (68), we get

$$\begin{aligned} 0 &= \big((D^g_T W^+)(\phi), \psi\big) = -3\big(\Phi, \psi\big)\big(\Phi, \phi\big)\,, \\ -\mathrm{d}\lambda(T) &= \big((D^g_T W^+)(\phi), \phi\big) = -\left(4|\theta|^2 - \frac{s}{6}\right)\lambda + 2|\Phi|^2 - 3\big(\Phi, \phi\big)^2\,, \\ -\mathrm{d}\lambda(T) &= \big((D^g_T W^+)(\psi), \psi\big) = -\left(4|\theta|^2 - \frac{s}{6}\right)\lambda + 2|\Phi|^2 - 3\big(\Phi, \psi\big)^2\,. \end{aligned}$$

From the last two equalities, we get $\big(\Phi, \psi\big) = \pm\big(\Phi, \phi\big)$, and by the first one we conclude that $\big(\Phi, \psi\big) = \big(\Phi, \phi\big) = 0$. This shows that Φ is a multiple of F. It remains to prove that Φ does not vanish on M; by taking a twofold cover of M if necessary, we may assume that the Hermitian structure J is globally defined on M; by Proposition 2, (g, J) is conformally Kähler and $\lambda^{\frac{2}{3}} F$ is the corresponding closed Kähler form; but Φ is also a closed, selfdual 2-form, and a multiple of F, hence a constant (non-zero) multiple of $\lambda^{\frac{2}{3}} F$.

(iii) $\Rightarrow$ (i). This is an immediate consequence of Proposition 1. □

CONVENTION. Up to the end of this section, we assume that (M, g) is an oriented selfdual Einstein 4-manifold whose selfdual Weyl W^+ has degenerate spectrum, and which admits a *non-closed* hyperhermitian structure compatible with the negative orientation of M. According to Proposition 5, W^+ has two distinct eigenvalues which we denote by λ and $-\frac{\lambda}{2}$, and the harmonic selfdual 2-form Φ defines a positive Hermitian structure J on (M, g) whose Kähler form, F, is an λ-eigenform for W^+. Moreover, as a consequence of Proposition 2, after rescaling the metric if necessary we may assume:

$$\Phi = \frac{\lambda^{\frac{2}{3}}}{2} F \,. \tag{69}$$

The conformal scalar curvature κ of (g, J) is then equal to 6λ (notations of Sec. 2.1); the Lee form θ_J and the Killing vector field K (suitably rescaled by a positive constant) are therefore given by

$$\theta_J = \frac{\mathrm{d}\lambda}{3\lambda}; \quad K = J \operatorname{grad}_{\mathrm{g}}(\lambda^{-\frac{1}{3}}) \,, \tag{70}$$

see Proposition 2.

At this point, our main technical result reads as follows:

PROPOSITION 6. *A selfdual Einstein Hermitian 4-manifold (M, g, J) admits a non-closed, hyperhermitian structure compatible with the negative orientation if and only if the Lee form θ_J satisfies*

$$\begin{aligned} D^g\theta_J = & \frac{(1+\lambda^{\frac{2}{3}})(s+3\lambda^{\frac{1}{3}})}{12} g \\ & + \frac{(1+2\lambda^{\frac{2}{3}})}{(1+\lambda^{\frac{2}{3}})} \theta_J \otimes \theta_J + \frac{\lambda^{\frac{2}{3}}}{(1+\lambda^{\frac{2}{3}})} J\theta_J \otimes J\theta_J \,. \end{aligned} \tag{71}$$

If this holds, (M, g) actually admits two, and only two, non-closed hyperhermitian structures $\{I_1', I_2', I_3'\}$ and $\{I_1'', I_2'', I_3''\}$, whose Lee forms, θ' and θ'', are given by

$$\theta' = \frac{1}{(1+\lambda^{\frac{2}{3}})}\big(\theta_J - \lambda^{\frac{1}{3}} J\theta_J\big), \quad \theta'' = \frac{1}{(1+\lambda^{\frac{2}{3}})}\big(\theta_J + \lambda^{\frac{1}{3}} J\theta_J\big) \,. \tag{72}$$

Moreover, the Killing vector field K is triholomorphic for both hyperhermitian structures, i.e. K preserves all complex structures I_i' and I_i'', $i = 1, 2, 3$.

PROOF. We first show that if (M, g, J) admits a non-closed hyperhermitian structure compatible with the negative orientation, then the corresponding Lee form θ must be one of the forms θ' and θ'' given by (72).

From (65) and the fact that Φ is an λ-eigenform of W^+, we infer

$$\mathrm{d}|\Phi|^2 = 4|\Phi|^2\theta + 4\lambda\Phi(\theta) \,. \tag{73}$$

By differentiating (73) and by using (66) to compute $\mathrm{d}(\Phi(\theta))$, we obtain

$$(\mathrm{d}\lambda - 3\lambda\theta) \wedge \Phi(\theta) + \left(|\Phi|^2 + \lambda\left(\frac{s}{12} + |\theta|^2\right)\right)\Phi = 0\,;$$

we infer

$$|\Phi|^2 = -\lambda\left(\frac{s}{12} + |\theta|^2\right). \tag{74}$$

By substituting the above expression of $|\Phi|^2$ in (73), and by using (66) again, we get

$$\mathrm{d}\lambda - 3\lambda\theta = \frac{3\lambda^2}{|\Phi|^2}\Phi(\theta)\,. \tag{75}$$

By using (70) and (69) (see the above convention) we finally obtain

$$\theta = \frac{1}{(1+\lambda^{\frac{2}{3}})}(\theta_J - \lambda^{\frac{1}{3}} J\theta_J)\,. \tag{76}$$

This shows that every non-closed hyperhermitian structure is completely determined by the selfdual harmonic 2-form Φ. It remains to prove that Φ itself is determined, up to sign, by the metric g; then, the possible two values of θ appearing in (72) will only differ by conjugating J or, equivalently, by substituting $-\Phi$ to Φ. Notice that, according to our convention, at this stage we have the freedom to rescal Φ by a non-zero constant. In other words, by fixing a non-closed hyperhermitian structure and by following our convention, we know that any other non-closed hyperhermitian structure corresponds to a harmonic 2-form of the form $a\Phi = \frac{a}{2}\lambda^{\frac{2}{3}}F$, where a is a non-zero constant. Our claim is that $a = \pm 1$; to prove that we calculate

$$|D^g\Phi|^2 = 2|\theta|^2(3|\Phi|^2 + |W^+|^2)\,,$$

by using (64) and (65); in the present situation, when W^+ has degenerate spectrum, the norm of W^+ is given by $|W^+|^2 = \frac{3}{2}\lambda^2$; then, by (74), the above equality reduces itself to

$$|D^g\Phi|^2 = -\left(\frac{|\Phi|^2}{\lambda} + \frac{s}{12}\right)(6|\Phi|^2 + 3\lambda^2)\,; \tag{77}$$

it is readily checked that if Φ and $a\Phi$ simultaneously satisfy (77), then $a = \pm 1$.

We now check that conditions (10)-(15) for either θ' or θ'' are equivalent to (71). Keeping (69) in mind, we see that (75) can be equivalently re-written as

$$\theta_J = \theta + \lambda^{\frac{1}{3}} J\theta\,; \tag{78}$$

then, the equivalence (71) $\Leftrightarrow$ (10)-(15) follows by a straightforward computation involving the expressions (76) and (78) and using formula (12); 1-forms θ' and θ'' thus correspond to two distinct, non-closed, hyperhermitian structures $\{I_1', I_2', I_3'\}$ and $\{I_1'', I_2'', I_3''\}$ provided that (71) holds, see Section 1.2.

As a final step, we prove that K is triholomorphic with respect to both hyperhermitian structures. For a general hyperhermitian structure $I_i, i = 1, 2, 3$, with Lee form θ, and for any Killing field K, we have

$$\mathcal{L}_K I_i = D_K I_i - [DK, I_i]\,,$$

where D is the Weyl derivative given by (9); we thus only need to check that in our specific situation DK commutes with I_i; by using (9), (70), (12) and (71), we get

$$DK = \theta(K)\mathrm{Id}|_{TM} + \frac{(1+\lambda^{\frac{2}{3}})}{4}J\,;$$

the claim follows immediately. □

Corollary 2 ([23]). *A locally symmetric selfdual Einstein* 4*-manifold does not admit non-closed hyperhermitian structures.*

Proof. Any such manifold is either a space of constant curvature, hence conformally-flat, or a Kähler manifold of constant holomorphic sectional curvature (see Propositions 1 and 2). In the former case, the claim follows by Corollary 1, whereas in the latter case $\theta = 0$; we then conclude by using Proposition 6. □

Remark 4. In [15], D. Calderbank proved that any conformal selfdual 4-manifold admitting two distinct Einstein-Weyl structures is equipped with a canonical conformal submersion to an Einstein-Weyl 3-manifold. Under the hypothesis of Proposition 6, this conformal submersion can be described as follows: the hyperhermitian structures $\{I_1', I_2', I_3'\}$ and $\{I_1'', I_2'', I_3''\}$ determine an SO(3)-valued function, p, on M defined by:

$$I_i'' = \sum_{j=1}^{3} a_{ij} I_j';\quad A = (a_{ij}) \in \mathrm{SO}(3)\,;$$

we claim that p is a conformal submersion of (M, g) to SO(3)=$\mathbb{R}P^3$: The differential of p is easily computed by using the fact that I_i'' and I_j' are both integrable; we thus obtain:

$$\mathrm{d}(a_{ij}) + \frac{\lambda^{\frac{2}{3}}}{2(1+\lambda^{\frac{2}{3}})}\Sigma_{k=1}^{3} a_{ik}\big([I_k', I_j']K\big)^{\sharp_g} = 0\,; \tag{79}$$

here, $[\cdot,\cdot]$ denotes the commutator of endomorphisms of TM and $\sharp_g$ stands for the Riemannian duality; from (79), we infer:

$$\mathcal{L}_K a_{ij} = 0\,,$$
$$\sum_{i,j} \big(da_{ij}(X)\big)^2 = \frac{\lambda^{\frac{4}{3}}}{2(1+\lambda^{\frac{2}{3}})^2} g(X,X), \ \ \forall X \in K^\perp\,.$$

The first equality shows that p coincides with the projection of M to the space, N, of orbits of K, whereas the second equality means that the K-invariant metric $\bar{g} = \frac{\lambda^{\frac{2}{3}}}{(1+\lambda^{\frac{2}{3}})} g$ descends to the round metric of $\mathrm{SO}(3) = \mathbb{R}P^3$; in other words, K defines a Riemannian submersion from $(M,\bar{g})$ to SO(3).

Proof of Theorem 3. We first notice that the Killing vector field K is trivial if and only if λ is constant (see (70)), or, equivalently, $\theta_J = 0$. Thus, according to Propositions 5 and 6, if (M,g,J) is a selfdual Einstein Hermitian 4-manifold admitting a *non-closed* hyperhermitian structure, the Killing vector field K does not vanish on an open, dense subset of M. It then follows from [31], [16], [17] that selfdual Einstein 4-manifolds admitting two distinct hyperhermitian structures and anon-trivial triholomorphic Killing vector field are locally given by Proposition 4.

For completeness, however, we give here a different and more direct argument adapted to our "Hermitian" situation.

By Proposition 4 it is sufficient to show that our metric can be written in the diagonal form (35). Since the eigenvalues of W^+ are not constant, i.e. $\theta_J \neq 0$ (Proposition 6), we introduce the variable $t = \lambda^{\frac{1}{3}}$; the Lee form θ_J is then equal to $\frac{\mathrm{d}t}{t}$, whereas the dual 1-form of the Killing vector field is given by $-\frac{1}{t^2} J\mathrm{d}t$. We set $\sigma_3 = f(t) J\mathrm{d}t$ for some smooth function f of t, and we look for 1-forms σ_1 and σ_2 such that

$$\mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2\,, \tag{80}$$

where σ_1 and $\sigma_2 = J\sigma_1$ are both orthogonal to $\mathrm{d}t$ and satisfy

$$\mathrm{d}\sigma_1 = \sigma_2 \wedge \sigma_3; \quad \mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1\,. \tag{81}$$

We can derive f from (80): by differentiating (76) and by making use of (69), we obtain

$$\mathrm{d}(J\mathrm{d}t) = -\frac{(1+t^2)t^2}{2} F + \frac{2t}{(1+t^2)} \mathrm{d}t \wedge J\mathrm{d}t\,. \tag{82}$$

By (78), (74) and (69), we also get

$$|\mathrm{d}t|^2 = -\left(\frac{t}{2} + \frac{s}{12}\right)\left(t^4 + t^2\right);$$

it follows that $\big(d\sigma_3, dt \wedge J dt\big) = 0$ if and only if $(\ln f)' = -\frac{2t}{(1+t^2)} - \frac{1}{(t+\frac{s}{6})}$, where the prime stands for $\frac{d}{dt}$; we then have $f = \frac{a}{(1+t^2)(t+\frac{s}{6})}$, hence

$$\sigma_3 = \frac{a}{(1+t^2)\left(t+\frac{s}{6}\right)} J dt \tag{83}$$

for a positive constant a.

In order to determine the 1-forms σ_1 and σ_2, we choose a gauge ϕ or, equivalently, a 1-form $\alpha = \phi(J\theta) \in \mathcal{D}^{\perp}$; since σ_1 and $\sigma_2 = J\sigma_1$ are orthogonal to dt, there certainly exist a smooth function h of t and a smooth function φ on M, such that

$$\sigma_1 = h(\cos\varphi\alpha + \sin\varphi J\alpha); \sigma_2 = h(-\sin\varphi\alpha + \cos\varphi J\alpha)\,;$$

by (83) and (80), we obtain the following expression for h:

$$h^2 = \frac{at^2}{\left(t+\frac{s}{6}\right)^2 (1+t^2)}\,; \tag{84}$$

by using (83) and (22), we now see that the conditions (81) are equivalent to

$$d\varphi + \beta + \frac{\left(\frac{s}{6} - t^3 + at\right)}{t(1+t^2)\left(\frac{s}{6}+t\right)} J dt = 0\,; \tag{85}$$

therefore, the existence of a smooth function φ satisfying (85) is equivalent to the following condition:

$$d\left(\beta + \frac{\left(\frac{s}{6} - t^3 + at\right)}{t(1+t^2)\left(\frac{s}{6}+t\right)} J dt\right) = 0\,;$$

a straightforward computation involving (23) and (82) shows that the above equality holds whenever the constant a is chosen equal to $1 + \frac{s^2}{36}$.

4. – Hermitian structures on quaternionic quotients

Let (N, g) be a quaternionic-Kähler manifold of real dimension $4n$, endowed with a non-trivial Killing field K which preserves the quaternionic structure. According to [24], [25], [26], under a certain non-degeneracy condition for K one can define a $4(n-1)$-dimensional quaternionic-Kähler orbifold (M, g^*)

via the so-called *quaternionic reduction construction*. This can be described as follows. We first consider the orthogonal splitting of the bundle of 2-forms

$$\Lambda^2 N = \Lambda^+ N \oplus \Lambda^{1,1} N \oplus \Lambda^{\perp} N \,, \tag{86}$$

where

- $\Lambda^+ N$ is the three-dimensional sub-bundle of "selfdual" 2-forms, which determines the quaternionic structure (also identified to a sub-bundle A^+N of skew-symmetric endomorphism of TN). Both A^+N and Λ^+N are preserved by the Levi-Civita connection D^g and at each point x of N there is an orthonormal basis $\{I_1, I_2, I_3\}$ of $A^+N \subset \mathrm{End}(T_x N)$ with the property $I_i \circ I_j = -\delta_{ij} \mathrm{Id}|_{TN} + \epsilon_{ijk} I_k$. Then $\Lambda^+ N = \mathrm{span}(\omega_1, \omega_2, \omega_3)$, where ω_i are the orthogonal Kähler forms of the almost-hermitian structures (g, I_l). In the sequel, we will refer to any such choice of I_l's (resp. ω_l's) as a *trivialization* of A^+N (resp. Λ^+N);
- $\Lambda^{1,1}N$ is the sub-bundle of 2-forms which are I_ℓ-invariant for any section I_ℓ of A^+N;
- $\Lambda^{\perp}N$ denotes the orthogonal complement of $\Lambda^+ N \oplus \Lambda^{1,1}N$ in $\Lambda^2 N$.

We denote by Π^+ the projection of $\Lambda^2 N$ onto Λ^+N; for any trivialization $\{\omega_1, \omega_2, \omega_3\}$ of Λ^+N we have

$$\Pi^+ = \frac{1}{2n} \sum_l \omega_l \otimes \omega_l \,,$$

and $\Pi^+_K := \frac{1}{2n} \sum_l (i_K \omega_l \otimes \omega_l)$ is a section of $T^*N \otimes \Lambda^+ N$.

In [26, Th. 2.4], K. Galicki and H. B. Lawson show the existence of a section f_K of Λ^+N, such that

$$\mathrm{d}^{D^g} f_K = D^g f_K = \Pi^+_K \,;$$

the section f_K is called *the momentum map* associated to (N, g, K) and it is easily seen that the level set

$$L_K := \{x \in N : f_K(x) = 0\}$$

is K-invariant.

If we assume that $K_x \neq 0$ at each point x of L_K, then the level set L_K is regular, i.e. L_K is a smooth submanifold of N (see [26]). When the quotient space $M := L_K/K$ is a $(4n-4)$-dimensional manifold (or an orbifold) it then becomes a quaternionic-Kähler manifold (resp. orbifold) with respect to the "projected" quaternionic structure, g^*, from N. The case of interest here is when N is 8-dimensional; then, the quaternonic reduction gives rise to a four dimensional *anti-selfdual* Einstein manifold (resp. orbifold) with respect to the canonical orientation induced by N. Note that if K is the generator of an S^1-quaternionic action on N then, under the non-degeneracy condition as above, M always inherits an orbifold structure, cf. [26, Th. 3.1 & Cor. 3.2].

The above construction applies in particular to the quaternionic projective space $N = \mathbb{H}P^2$ endowed with certain *weighted* S^1-actions; one thus obtains a wealth of examples of *compact* anti-selfdual Einstein orbifolds; as shown by Galicki-Lawson, the corresponding orbifolds are all isomorphic to weighted projective planes $\mathbb{C}P^{[p_1,p_2,p_3]}$ for some integers $0 < p_1 \leq p_2 \leq p_3$ satisfying $p_3 < p_1 + p_2$, [26, Sec. 4]. Notice that, with respect to the orientation induced by the canonical complex structure, the metric becomes *selfdual*. (In the case when $p_1 = p_2 = p_3$ one obtains the Fubini-Study metric on $\mathbb{C}P^2$). However, it is not clear from the consideration of [26] whether or not the selfdual Einstein metrics are compatible with the complex orbifold structure, i.e. whether or not these metrics are Hermitian with respect to the *non-standard* orientation. On the other hand, R. Bryant showed that each weighted projective plane admits a selfdual Kähler metric which, under the above assumption for the weights, has everywhere positive scalar curvature [12, Sec. 4.2]. Therefore, according to Lemma 2, Bryant's metric is conformal to a selfdual *Einstein* Hermitian metric on $\mathbb{C}P^{[p_1,p_2,p_3]}$, $p_3 < p_1 + p_2$. When considering both results together, a natural question arises:

Question ([37]). Are Galicki-Lawson's metrics on $\mathbb{C}P^{[p_1,p_2,p_3]}$ Hermitian with respect to some anti-selfdual complex structure?

In this section we show that this is indeed the case. Specifically, we prove the following

Theorem 4. *Let (N, g) be either the quaternionic projective space $\mathbb{H}P^2$ or the quaternionic hyperbolic space $\mathbb{H}H^2$. Then, any anti-selfdual, non-conformally-flat Einstein 4-orbifold (M, g^*) obtained as quaternionic reduction of (N, g) admits a (negatively oriented) Hermitian structure J. In particular, on the smooth part of M, the metric g^* is locally given by the explicit constructions in Section 2.*

The proof is based on the following simple observation.

Lemma 6. *Let (N, g) be a quaternionic-Kähler manifold of non-zero scalar curvature and let K be a Killing field on N. Denote by $\Psi(X, Y) = (D^g_X K, Y)$ the 2-form corresponding to $D^g K$ and let $\Psi^+ = \Pi^+(\Psi)$ be the projection of Ψ in $\Lambda^+ N$. Then, up to multiplication by a constant, the momentum map f_K of K is given by Ψ^+.*

Proof. Since K is Killing, the Kostant identity (29) holds. But for a quaternionic-Kähler manifold the curvature operator R acts on $\Lambda^+ N$ by $\lambda \mathrm{Id}|_{\Lambda^+ N}$, where λ is a positive multiple of the scalar curvature, cf. e.g. [45]. Thus, projecting (29) to $\Lambda^+ N$ we get $D^g_X \Psi^+ = \lambda \Pi^+_K$. □

By Lemma 6 the level set L_K of K is the same as the set of points $x \in N$ where $\Psi^+_x = 0$. Thus, at any point $x \in L_K$ the tangent space $T_x L_K$ is given by $T_x L_K = \{T_x N \ni X : D^g_X \Psi^+ = 0\}$. Since by assumption K does not vanish on L_K, by (29) and by the fact that $R|_{\Lambda^+ N} = \lambda \mathrm{Id}|_{\Lambda^+ N}$ we obtain

$$T_x L_K = \mathrm{span}(I_1 K, I_2 K, I_3 K)^{\perp},$$

where $\{I_1, I_2, I_3\}$ is any trivialization of $A^+ N$.

We also observe that the 2-form Ψ is a section of $\Lambda^+N \oplus \Lambda^{1,1}N$, provided that K preserves the quaternionic structure. Indeed,

$$[D^g K, I_l] = D^g_K I_l - \mathcal{L}_K I_l \,,$$

where $[\cdot,\cdot]$ stands for the commutator of $\mathrm{End}(TN)$. Since K is quaternionic, the left-hand-side of the above equality is a section of Λ^+N. By summing over l in the above relation we get

$$\Psi - \Pi^{1,1}(\Psi) \in \Lambda^+N \,, \tag{87}$$

where $\Pi^{1,1}$ denotes the projection to $\Lambda^{1,1}N$:

$$\Pi^{1,1}\psi(\cdot,\cdot) = \frac{1}{4}\left[\psi(\cdot,\cdot) + \sum_l \psi(I_l\cdot, I_l\cdot)\right], \quad \forall\, \psi \in \Lambda^2 N \,. \tag{88}$$

Thus, Ψ is a section of $\Lambda^+N \oplus \Lambda^{1,1}N$, and at $x \in L_K$, Ψ_x actually belongs to $\Lambda^{1,1}_x N$.

Since $\Psi = \frac{1}{2}\mathrm{d}K^\sharp$, where $K^\sharp$ is the g-dual 1-form of K, we conclude that

$$\mathcal{L}_K\Psi = \mathrm{d}(i_K(\Psi)) = -\frac{1}{2}\mathrm{d}(\mathrm{d}|K|^2) = 0 \,,$$

i.e. Ψ is a closed K-invariant 2-form. This shows that Ψ projects to $M = L_K/K$ to define an *anti-selfdual* form on (M, g^*), then denoted by Ψ^*. Considering the Riemannian submersion

$$\pi : L_K \longmapsto M = L_K/K \,,$$

the *horizontal* space, H, of TL_K is given by

$$H = \mathrm{span}(K, I_1K, I_2K, I_3K)^\perp \,.$$

Note that H is I_l-invariant for any section I_l of A^+N. Using the above remarks we calculate:

$$(D^{g^*}_{U^*}\Psi^*)(V^*, T^*) = (D^g_U\Psi)(V, T) - \frac{4}{|K|^2_g}\Pi^{1,1}(i_U\Psi \wedge i_K\Psi)(V, T) \,, \tag{89}$$

where D^{g^*} is the Levi-Civita connection of g^*, U^*, V^*, T^* are any vectors on M, and U, V, T are the corresponding horizontal lifts.

By assumption, K has no zero on L_K; it then follows from (89) and (29) that Ψ^* does not vanish identically on M. Thus, on the open subset of (M, g^*) where $\Psi^* \neq 0$ the normalised ASD form $\sqrt{2}\frac{\Psi^*}{|\Psi^*|_{g^*}}$ determines a *negative* almost-hermitian structure J.

Lemma 7. *When N is $\mathbb{H}P^2$ or $\mathbb{H}H^2$, the almost-complex structure J is integrable.*

PROOF. We denote by Z_i^* any complex (1,0)-vector field of (M, J) and by Z_i the corresponding horizontal lift (considered as complex vector in $T_x^{\mathbb{C}}N$); then, J is integrable if and only if the following identity holds:

$$D^{g^*}_{Z_i^*}\left(\frac{\sqrt{2}\Psi^*}{|\Psi^*|_{g^*}}\right)(Z_j^*, Z_k^*) = \frac{\sqrt{2}}{|\Psi^*|_{g^*}}(D^{g^*}_{Z_i^*}\Psi^*)(Z_j^*, Z_k^*) = 0 \quad \forall i, j, k\,; \tag{90}$$

by the very definition of J we have $\Psi(Z_i, Z_j) = 0$; moreover, since Ψ belongs to $\Lambda^{1,1}N$ on L_K, the almost-complex structure J (defined on H) commutes with I_l's for any trivialization $\{I_1, I_2, I_3\}$ of A^+N. Then, by (89) and (29) it is easily seen that the integrability condition (90) for J is the same as

$$(D^{g^*}_{Z_i^*}\Psi^*)(Z_j^*, Z_k^*) = (D^g{}_{Z_i}\Psi)(Z_j, Z_k) = (R(K \wedge Z_i), Z_j \wedge Z_k) = 0\,. \tag{91}$$

We now derive (91) from the structure of the Riemannian curvature tensor of the symmetric space $\mathbb{H}P^2$ (or its non-compact dual $\mathbb{H}H^2$); we refer to [45], [29] for a general description of the curvature operator of a Riemannian symmetric space.

When $N = \mathbb{H}P^2 = Sp(3)/(Sp(1)Sp(2))$, the eigenspaces of the curvature operator R are the simple factors $\mathbf{sp}(1)$ and $\mathbf{sp}(2)$ of the isotropy Lie subalgebra $\mathbf{h} = \mathbf{sp}(1) \oplus \mathbf{sp}(2)$, and the orthogonal complement $\mathbf{h}^\perp$ of $\mathbf{h}$ in the space $\mathrm{Skew}(\mathbf{m})$ of the skew-symmetric endomorphisms of $\mathbf{m} = \mathbf{sp}(3)/\mathbf{h}$ (note that R acts trivially on $\mathbf{h}^\perp$); the decomposition $\mathrm{Skew}(\mathbf{m}) = \mathbf{sp}(1) \oplus \mathbf{sp}(2) \oplus \mathbf{h}^\perp$ into eigenspaces of R fits in with the splitting (86) as follows: Λ^+N is identified to $\mathbf{sp}(1)$, $\Lambda^{1,1}N$ to $\mathbf{sp}(2)$, whereas $\Lambda^\perp N$ corresponds to the kernel of R, the space $\mathbf{h}^\perp$. This shows that the curvature operator acts on the first two factors in (86) by multiplication with a non-zero constant (a certain multiple of the scalar curvature), and R acts trivially on the third factor (i.e. R has thus three distinct eigenvalues, λ, μ and 0). This observation also shows that any Killing field on $\mathbb{H}P^2$ is necessarily quaternionic.

As already observed, the almost-complex structure J (defined on H) commutes with the I_l's, so that $I_l(Z_k)$ is again a (1,0)-vector of (H, J); we thus get

$$\Pi^+(Z_j \wedge Z_k) = \sum_l (Z_j, I_l(Z_k))\omega_l = 0\,,$$

which means that $Z_j \wedge Z_k$ is an element of $\Lambda_x^{1,1}N \oplus \Lambda_x^\perp N$. It then follows that

$$\begin{aligned}(R(K \wedge Z_i), Z_j \wedge Z_k) &= (R(Z_j \wedge Z_k), K \wedge Z_i)\\ &= \mu(\Pi^{1,1}(Z_j \wedge Z_k), K \wedge Z_i)\,.\end{aligned}$$

But $\Pi^{1,1}(Z_j \wedge Z_k)$ is again a (2,0)-vector of (M, J) (see formula (88)), so that $(\Pi^{1,1}(Z_j \wedge Z_k), K \wedge Z_i) = 0$; this implies (91).

The same argument applies to the non-compact dual space $\mathbb{H}H^2$. □

PROOF OF THEOREM 4. By (89) and Lemma 7, we see that $1/|K|^2\Psi^*$ is a harmonic selfdual 2-form, i.e. it is the Kähler form of a selfdual Kähler metric in the conformal class of g^*, defined on the open subset where $\Psi^* \neq 0$ (see also Proposition 2). Since by hypothesis W^+ is not identically zero, we conclude by Proposition 1 that W^+ has no zero on M and that Ψ^* is an eigenform of W^+, corresponding to (the unique) simple eigenvalue; according to Proposition 2, the norm of $1/|K|^2\Psi^*$ must be a constant multiple of $|W^+|^{4/3}$, showing that Ψ^* has no zero on M; thus, the complex structure J is globally defined on M. The last part of Theorem 4 is a consequence of Theorems 1 and 2.

REMARK 5. (i) We cannot expect that a similar result would hold for quaternionic quotients of the remaining 8-dimensional quaternionic-Kähler Wolf spaces; we are grateful to D. Calderbank and K. Galicki for pointing this out to us.

(ii) The harmonic 2-form $2/|K|^2\Psi^*$ can be thought of as the curvature of the Riemannian submersion $\pi : L_K \to M$. Thus, L_K is a Sasakian manifold fibered over a Kähler selfdual —equivalently, a Bochner-flat— orbifold. It is well known that the corresponding CR-structure of L_K has vanishing fourth-order Chern-Moser curvature; therefore L_K is uniformized over S^5 with respect to $\mathrm{Aut}_{CR}(S^5) = PU(3,1)$, cf. [52].

(iii) As observed in [26, p. 20], the quaternionic reduction procedure can be applied to the quaternionic hyperbolic space to obtain *smooth*, *complete* (non locally symmetric) Einstein selfdual metrics of negative scalar curvature, which are necessarily Hermitian by Lemma 7; see also [12] for another construction of complete Einstein selfdual Hermitian metrics.

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