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Curvature Flows on Surfaces

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Abstract. Prompted by recent work of Xiuxiong Chen, a unified approach to the Hamilton-Ricci and Calabi flows on a closed, compact surface is presented, recovering global existence and exponentially fast asymptotic convergence from concentration-compactness results for conformal metrics.

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1. – Introduction

Let (M, g_0) be a compact Riemann surface without boundary. Consider the normalized Hamilton-Ricci flow

$$\frac{\partial g}{\partial t} = rg - \text{Ric} = (r - R)g, \tag{1}$$

where R is the scalar curvature of g with average r and where $\text{Ric} = Rg$ is the Ricci curvature of g. Since Ric is proportional to g, the flow (1) generates a flow of conformal metrics $g(t)$ of fixed volume. Hamilton [14] and Chow [11] established global existence and exponential convergence for this flow. The most difficult case is the case when M is the sphere S^2. For this case a simpler proof of the above result was later given by Bartz-Struwe-Ye [4] along the lines of Ye's [22] proof of the corresponding result for the Yamabe flow in higher dimensions.

Also consider the Calabi flow

$$\frac{\partial g}{\partial t} = \Delta_g K \cdot g, \tag{2}$$

where Δ_g is the Laplace-Beltrami operator on (M, g) – with the analysts' sign! – and where $K = R/2$ is the Gauss curvature. Again, (2) generates a flow of

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conformal metrics $g(t)$ of fixed volume. Surprisingly, equation (2) also has a physical interpretation which is apparently unrelated to its differential geometric origins (Calabi [7]). Solutions to equation (2) describe the time evolution of the compact factor in a Robinson-Trautman [18] solution to the Einstein equations. Using the a-priori bounds on solutions that derive from this interpretation, in particular, Singleton's [20] Bondi mass estimate, Chrusciel [12] deduced global existence and exponential convergence of solutions to (2) to a limit metric of constant scalar curvature.

In a recent preprint, Xiuxiong Chen [9] suggested a completely different approach to Chrusciel's result based on ideas from geometric analysis and using, in particular, his analysis of the compactness properties of conformal metrics with bounded Calabi energy and area and their possible concentration behavior. Inspired by his work, here we propose a much simplified and essentially self-contained proof where we also clarify some points left open by Chen. The central ideas of our argument may be carried over to the Ricci flow, for which we achieve a proof of global existence and exponential convergence based only on elementary integral estimates and avoiding the use of the maximum principle completely. Prompted by this work, Schwetlick [19] has shown that the same ideas also may be applied to the class of curvature flows studied by Polden [17].

A core ingredient is the characterization of possible singularities of the flow as concentration points for the integrated curvature. In Section 3 below we show that this is a direct consequence of the L^1-estimates of Brezis-Merle [6] for the Gauss equation

$$-\Delta u = Ke^{2u} \text{ in } \Omega \subset \mathbb{R}^2, \tag{3}$$

relating the Gauss curvature of a metric $e^{2u}g_{\text{eucl}}$ to the conformal factor.

Moreover, in the analytically most subtle case when M is the sphere we adapt Singleton's [20] idea to study a flow of metrics that are normalized with respect to the Möbius group action, which in turn is a variant of De Turck's [13] trick; finally, we apply the Kazdan-Warner [15] identity to deal with the problem of a non-trivial kernel of the linearized Gauss equation.

We start with an analysis of the Calabi flow and return to the Ricci flow in Section 6. We may suppose that M is orientable; otherwise we consider the oriented double cover. Moreover, for simplicity, throughout the following we assume that a background metric g_0 of constant scalar curvature $R_0 = 2K_0$ has been fixed in the conformal class of the flow $g(t)$. Although the Calabi and Ricci flows may be used to reprove the uniformization theorem, here we will not pursue this. In the following, a subscript will usually indicate the metric used to compute a norm or scalar product, etc.; otherwise, all norms by default will refer to the metric g_0; the letter C denotes a generic constant, sometimes numbered for clarity.

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2. – The Calabi flow

Representing $g(t) = e^{2u(t)} g_0$, and letting $u_t = \frac{\partial u}{\partial t}$, etc., equation (2) translates into the evolution equation

$$u_t = \frac{1}{2} \Delta_g K \tag{4}$$

for u, where $\Delta_g = e^{-2u} \Delta_{g_0}$ and with

$$K = e^{-2u}(-\Delta_{g_0} u + K_0) = -\Delta_g u + K_0 e^{-2u}. \tag{5}$$

In the following we simply write $\Delta_0 = \Delta_{g_0}$, etc., for brevity. By (4) the area element $d\mu = d\mu_g = e^{2u} d\mu_0$ evolves according to

$$\frac{d}{dt}(d\mu) = 2u_t d\mu = \Delta_g K \cdot d\mu. \tag{6}$$

From this conservation of volume is immediate; indeed,

$$\frac{d}{dt}\left(\int_M d\mu\right) = \int_M \Delta_g K \, d\mu = 0.$$

By adding a suitable constant to a solution u of (4), with no loss of generality we may therefore assume that the corresponding metrics have unit volume.

Our aim is to show the following:

Theorem 2.1. *For any $u_0 \in H^2(M, g_0)$ there exists a unique, global solution u of* (4) *with $u(0) = u_0$ and a smooth limit function u_∞ corresponding to a smooth metric $g_\infty = e^{2u_\infty} g_0$ of constant Gauss curvature such that*

$$||u(t) - u_\infty||_{H^2} \leq C e^{-\alpha t}$$

for some constant $\alpha > 0$ and all $t \geq 0$.

Higher regularity of u for $t > 0$ and exponential convergence in stronger norms may easily be derived from this result. Our proof is based on the following a priori estimates.

As was pointed out by Chen [9], the Calabi flow decreases a number of curvature functionals, in particular, the Calabi energy

$$Ca(g) = \int_M |K - K_0|^2 \, d\mu = \int_M K^2 \, d\mu - C_0,$$

where $C_0 \geq 0$ by the Gauss-Bonnet theorem only depends on the genus and the volume of (M, g_0), and the Liouville energy

$$E(u) = \frac{1}{2} \int_M (|\nabla u|_0^2 + 2K_0 u) \, d\mu_0,$$

where the norm $|\cdot|_0$ refers to the metric g_0.

In fact, using (5), we find that K evolves under (4) according to

$$K_t = -2u_t K - \Delta_g u_t = -K\Delta_g K - \frac{1}{2}\Delta_g^2 K. \tag{7}$$

Thus, from (6) we obtain

$$\begin{aligned}\frac{d}{dt}Ca(g(t)) &= 2\int_M (K_t K + K^2 u_t)\,d\mu \\ &= -\int_M (K\Delta_g^2 K + K^2\Delta_g K)\,d\mu \\ &= -\int_M ((\Delta_g K)^2 - 2K|\nabla K|_g^2)\,d\mu,\end{aligned} \tag{8}$$

But for any function f on M, upon integrating by parts and commuting derivatives, as in Aubin [1], Theorem 4.19, formula (25), we have

$$\int_M (\Delta_g f)^2\,d\mu = \int_M (|\nabla^2 f|_g^2 + K|\nabla f|_g^2)\,d\mu; \tag{9}$$

therefore

$$\frac{d}{dt}\,Ca(g(t)) = -2\int_M |\nabla^2 K - \frac{1}{2}\Delta_g K\cdot g|_g^2\,d\mu \le 0. \tag{10}$$

Here, in local coordinates we denote

$$|\nabla^2 f|_g^2 = g^{ik}g^{jl}\nabla_i\nabla_j f\nabla_k\nabla_l f,$$

thereby tacitly summing over repeated indices.

Similarly, we compute

$$\begin{aligned}\frac{d}{dt}\,E(u(t)) &= \int_M (\langle\nabla u, \nabla u_t\rangle_0 + K_0 u_t)\,d\mu_0 \\ &= \int_M u_t(-\Delta_{g_0}u + K_0)\,d\mu_0 = \int_M u_t K\,d\mu \\ &= \frac{1}{2}\int_M K\Delta_g K\,d\mu = -\frac{1}{2}\int_M |\nabla K|_g^2\,d\mu \le 0,\end{aligned} \tag{11}$$

where $\langle\cdot,\cdot\rangle_0$ denotes the inner product defined by g_0.

In particular, for any $T > 0$ we have

$$E(u(T)) + \frac{1}{2}\int_0^T\int_M |\nabla K|_g^2\,d\mu\,dt = E(u(0)), \tag{12}$$

thus providing an important space-time a priori integral estimate for the flow.

Indeed, by classical results the Liouville energy is bounded from below in any class of conformal metrics with prescribed volume. For $K_0 \le 0$ and unit volume, this easily follows from Jensen's inequality

$$2\int_M u\,d\mu_0 \le \log\left(\int_M e^{2u}\,d\mu_0\right) = 0. \tag{13}$$

For $K_0 > 0$ the lower bound on E is a consequence of Moser's [16] sharp version of Trudinger's [21] inequality; see Aubin [1], p. 63, 65, and 231.

Finally, Chen observes that the Calabi flow also decreases the Mabuchi energy, introduced in [3]. However, our aim being to discuss curvature flows on surfaces in a unified manner, here we will not make use of his Mabuchi energy estimate or Singleton's Bondi mass bounds and only rely on estimates (8), (11) directly related to curvature.

3. – Concentration-compactness

We recall the following L^1-estimate of Brezis-Merle [6], Theorem 1.

THEOREM 3.1. *Let B be the unit ball in $\mathbb{R}^2$, u a distribution solution to the equation*

$$-\Delta u = f \text{ on } B,\, u = 0 \text{ on } \partial B,$$

where $f \in L^1(B)$. Then for any $p < 4\pi/||f||_{L^1}$ there holds $e^{|u|} \in L^p(B)$ with

$$\int_B e^{p|u|}\,dx \le C(4\pi - p\,||f||_{L^1})^{-1}.$$

This result may be applied to obtain concentration-compactness results for families of solutions to equation (5), as follows. Let $B_R(x_0) \subset M$ be the ball of radius R around x_0 in the metric defined by g_0.

THEOREM 3.2. *Let $g_n = e^{2u_n} g_0$ be a family of smooth conformal metrics on M with unit volume and bounded Calabi energy. Then, either i) the sequence (u_n) is bounded in $H^2(M, g_0)$, or ii) there exist points $x_1, \dots, x_L \in M$ and a subsequence (u_n) such that for any $R > 0$ and any $l \in \{1, \dots, L\}$ there holds*

$$\liminf_{n\to\infty} \int_{B_R(x_l)} |K_n|\,d\mu_n \ge 2\pi, \tag{14}$$

where $d\mu_n = d\mu_{g_n}$ and where K_n is the Gauss curvature of g_n. Moreover, there holds

$$2\pi L \le \limsup_{n\to\infty} (Ca(g_n) + C_0)^{1/2} < \infty,$$

and either $u_n \to -\infty$ as $n \to \infty$ locally uniformly on $M \setminus \{x_1, \dots, x_L\}$, or (u_n) is locally bounded in H^2 on (M, g_0) away from $x_1, \dots, x_L$.

PROOF. Choose $R_0 > 0$ such that for $R \le R_0$ any metric ball $B_R(x_0)$ on (M, g_0) is diffeomorphic to the flat unit disc B in $\mathbb{R}^2$. By the Riemann mapping theorem we may choose these diffeomorphisms φ to be conformal. Identifying B with $\varphi(B)$, then we may regard $g_0 = e^{2u_0} g_{\mathrm{eucl}}$, $g_n = e^{2(u_0+u_n)} g_{\mathrm{eucl}}$, $n \in \mathbb{N}$, as conformal metrics on B. Letting $v_n = u_n + u_0$, from (5) then we have

$$-\Delta v_n = K_n e^{2v_n} \text{ on } B,$$

where Δ is the standard Laplacian.

Suppose that for a point $x_0 \in M$ and some $R > 0$ there holds

$$\sup_n \int_{B_R(x_0)} |K_n|\, d\mu_n \le a < 2\pi. \tag{15}$$

We may assume $R \le R_0$. Then, letting u_0, $v_n = u_n + u_0$ as above, we find that

$$-\Delta v_n = K_n e^{2v_n} =: f_n$$

is bounded in $L^1(B)$ with

$$||f_n||_{L^1(B)} = \int_{B_R(x_0)} |K_n|\, d\mu_n \le a < 2\pi.$$

Splitting $v_n = v_n^{(0)} + w_n$, where $v_n^{(0)}$ is harmonic in B and where $w_n = 0$ on ∂B, from Theorem 3.1 we obtain a uniform bound

$$\int_B e^{p|w_n|}\, dx \le C(p, a) \tag{16}$$

for any $p < 4\pi/a$. Since $a < 2\pi$, we may fix some such number $p > 2$.

On the other hand, by the mean value property of harmonic functions, for any $y \in B_{1/2}(0) \subset \mathbb{R}^2$ we may estimate

$$v_n^{(0)}(y) = \fint_{B_{1/2}(y)} v_n^{(0)}\, dx \le \fint_{B_{1/2}(y)} v_n\, dx + \fint_{B_{1/2}(y)} |w_n|\, dx,$$

where $\fint$ denotes mean value.

By Jensen's inequality, and recalling that all g_n have unit volume, moreover, we can bound

$$2\fint_{B_{1/2}(y)} v_n\, dx \le \log\left(\fint_{B_{1/2}(y)} e^{2v_n}\, dx\right) \le C + \log\left(\int_B e^{2v_n}\, dx\right) \le C.$$

Similarly, using (16) we can bound

$$\fint_{B_{1/2}(y)} |w_n|\, dx \le C + \log\left(\int_B e^{|w_n|}\, dx\right) \le C,$$

uniformly for $y \in B_{1/2}(0)$.

Thus, $e^{v_n} \le Ce^{w_n}$ is bounded in $L^p(B_{1/2}(0))$. By Hölder's inequality, and recalling that

$$||K_n e^{v_n}||^2_{L^2(B)} = \int_{B_R(x_0)} |K_n|^2 \, d\mu_n \le \int_M |K_n|^2 \, d\mu_n \le C,$$

therefore we find that

$$-\Delta v_n = K_n e^{v_n} \cdot e^{v_n}$$

is bounded in $L^q(B_{1/2}(0))$ for some $q > 1$.

Now suppose that (15) is satisfied for every $x_0 \in M$ for some number $a < 2\pi$ and some $R = R(x_0) > 0$. Then, upon covering M with finitely many balls $B_{R_i}(x_i)$ corresponding to the ball $B_{1/2}(0)$ under the conformal maps in the above construction, we obtain that $(\Delta_0 u_n)$ is bounded in $L^q(M, g_0)$ for some $q > 1$. Hence, by the Calderón-Zygmund inequality, $u_n \in W^{2,q}(M, g_0)$ with

$$||\nabla^2 u_n||_{L^q} \le C||\Delta_0 u_n||_{L^q} \le C.$$

In particular, since $W^{2,q} \hookrightarrow C^0$, this implies that

$$\max u_n - \min u_n \le C,$$

and hence, in view of the volume constraint, that

$$||u_n||_{L^\infty} \le C.$$

Going back to (5), then (u_n) is bounded in $H^2(M, g_0)$ with

$$||u_n||^2_{H^2} \le C(||\Delta_0 u_n||^2_{L^2} + ||u_n||^2_{L^\infty}) \le C||K_n||^2_{L^2} + C \le C, \tag{17}$$

proving i).

If (15) is not satisfied at a point $x_1 \in M$ for any $a < 2\pi$ and any $R > 0$, we choose a subsequence (u_n) (relabelled) such that

$$\liminf_{n\to\infty} \int_{B_{1/n}(x_1)} |K_n| \, d\mu_n \ge 2\pi. \tag{17}$$

Iterating, we choose points x_l, $l \in \mathbb{N}$, and further subsequences such that (17) holds for every x_l. This iteration terminates after finitely many steps. Indeed, having determined $x_1, \ldots, x_L$ as above, choose $R < \frac{1}{2} \min_{1 \le k \ne l \le L} \operatorname{dist}(x_k, x_l)$. The balls $B_R(x_l)$ thus being disjoint, for any n we can bound

$$\begin{aligned}\sum_{l=1}^{L} \int_{B_R(x_l)} |K_n| \, d\mu_n &= \int_{\cup_{l=1}^{L} B_R(x_l)} |K_n| \, d\mu_n \\ &\le \int_M |K_n| \, d\mu_n \le \left(\int_M |K_n|^2 \, d\mu_n\right)^{1/2} \left(\int_M d\mu_n\right)^{1/2} \\ &= (Ca(g_n) + C_0)^{1/2}.\end{aligned}$$

Therefore, upon letting $n \to \infty$, we find

$$2\pi L \le \limsup_{n\to\infty} (Ca(g_n) + C_0)^{1/2}$$

and the latter is at most equal to the limes superior for the original sequence.

Given any compact, connected domain $D \subset M \setminus \{x_1, \dots, x_L\}$, a covering argument as in the proof of i) then shows that $(\Delta_0 u_n)$ is bounded in $L^q(U, g_0)$ on a neighborhood $U \subset D$ for some $q > 1$, and hence

$$(\max_D u_n - \min_D u_n) \le C||\nabla^2 u_n||_{L^q} \le C \tag{18}$$

for some $C = C(D)$. The volume constraint now implies a uniform upper bound $u_n \le C$ on D, and (5) yields the L^2-bound $||\Delta_0 u_n||^2_{L^2(D)} \le C \cdot (Ca(g_n) + C_0) \le C$. Finally, the Calderón-Zygmund inequality implies that $(u_n - \min_D u_n)$ is bounded in $H^2(D, g_0)$.

It remains to observe that in view of (18) there holds $\min_D u_n \to -\infty$ as $n \to \infty$ on *one* such domain D_1 if and only if this happens on *any* such domain D_2, as is seen by applying (18) to a connected domain D containing $D_1 \cup D_2$. □

REMARK. By Hölder's inequality, we have

$$\left(\int_{B_R(x_0)} |K_n| \, d\mu_n \right)^2 \le \int_{B_R(x_0)} |K_n|^2 \, d\mu_n \cdot \int_{B_R(x_0)} d\mu_n \le C \int_{B_R(x_0)} d\mu_n. \tag{19}$$

Thus, (14) implies Chen's criterion

$$\limsup_{n\to\infty} \left(\int_{B_R(x_0)} |K_n|^2 \, d\mu_n \cdot \int_{B_R(x_0)} d\mu_n \right) \ge 4\pi^2$$

for any $R > 0$, characterizing concentration points $x_0 \in M$; see [10], Lemma 2, p. 201. Chen remarks that the latter constant may be improved to $16\pi^2$, which is optimal if we measure concentration in the L^2-norm. However, for the L^1-norm the constant 2π is the best possible, as is illustrated by long thin cylinders with spherical caps.

4. – Local existence

Combining (4), (5) we deduce the equation

$$u_t + \frac{1}{2}\Delta_g^2 u = \frac{1}{2} K_0 \Delta_g e^{-2u} \tag{20}$$

for the evolution of the conformal factor. We show the existence of a solution $u \in C^0([0,T]; H^2(M,g_0))$ with $u_t, \nabla^4 u \in L^2([0,T] \times M)$ of equation (20) for data $u(0) = u_0 \in H^2(M, g_0)$ for sufficiently small $T > 0$.

Fix a constant $C_1 > ||u_0||_{H^2}$. Given a family of metrics $g = g(t) = e^{2u(t)} g_0$ satisfying the uniform bound

$$C^{-1}||u||_{L^\infty} \le ||u||_{H^2} \le C_1, \tag{21}$$

let v be the solution to the linear biharmonic evolution equation

$$v_t + \frac{1}{2}\Delta_g^2 v = \frac{1}{2} K_0 \Delta_g e^{-2u} \tag{22}$$

with Cauchy data $v = u_0$ at $t = 0$.

Upon multiplying (22) with the testing function $\Delta_g^2 v$ and integrating by parts, we then see that the a-priori bound (21) also holds for v on a sufficiently short time interval $[0,T]$ and $v_t, \nabla^4 v$ are bounded in $L^2([0,T] \times M)$. Similarly, subtracting equations (22) for the solutions v_i corresponding to the choice of metrics $g_i = e^{2u_i(t)} g_0$, $i = 1, 2$, upon multiplying by $\Delta_g^2(v_1 - v_2)$ we find that the operator taking u to the solution v of the initial value problem (22) defines a contraction mapping in the space $\{u \in L^\infty([0,T]; H^2(M,g_0)); ||u||_{H^2} \le C_1\}$ for sufficiently small $T > 0$. Existence of a solution with the desired properties then follows from Banach's fixed point theorem.

Moreover, the solution u is smooth for $t > 0$, as is most easily seen by applying standard L^2-estimates to the equation (7) for the evolution of curvature. For instance, a local uniform bound for $||\nabla K||_{L^2}$ – and hence for $||u||_{H^3}$ – may be obtained upon multiplying (7) by $\Delta_g K$ and integrating by parts with respect to $d\mu_g$, etc. These estimates are tedious but standard and need not be discussed here.

5. – Long time existence and asymptotic convergence

We now use the geometric a-priori bounds on the Calabi and Liouville energies guaranteed by (10) and (11) to show that any local solution $g = g(t)$ to the Calabi flow (2) may be extended for all time and, as $t \to \infty$, converges exponentially fast to a limit metric of constant Gauss curvature. To illustrate the argument, we first discuss the case when M has positive genus. Note that in this case the background metric g_0 is the only metric of constant scalar curvature in its conformal class having unit volume; moreover, $K_0 \le 0$.

5.1. – Positive genus

Since $K_0 \le 0$, from (11) and (13) we have the uniform bound

$$E(u(0)) \ge E(u) = \frac{1}{2}\int_M (|\nabla u|_0^2 + 2K_0 u)\, d\mu_0 \ge \frac{1}{2}\int_M |\nabla u|_0^2\, d\mu_0 \ge 0 \tag{23}$$

for $u = u(t)$ and all $t \geq 0$. Thus, by the Moser-Trudinger inequality we can uniformly bound

$$\int_M e^{4|u|}\, d\mu_0 \leq C$$

and therefore, by (19) and Hölder's inequality,

$$\left(\int_{B_R(x_0)} |K|\, d\mu\right)^2 \leq (Ca(g(t)) + C_0) \cdot \int_{B_R(x_0)} e^{2u}\, d\mu_0 \leq C \left(\int_{B_R(x_0)} d\mu_0\right)^{1/2},$$

uniformly in t, ruling out concentration. From Theorem 3.2 it then follows that $u(t)$ is bounded in $H^2(M, g_0) \hookrightarrow L^\infty$, uniformly in t; in particular, the local solution constructed in Section 4 may be extended for all $t > 0$ and all metrics $g(t), 0 < t < \infty$, are uniformly equivalent to the background metric g_0.

In the fixed background metric g_0 we may use the Sobolev embedding $H^1 \hookrightarrow L^4$ with the estimate

$$||f||^2_{L^4} \leq C||f||_{L^2}||f||_{H^1} \leq C||f||^2_{H^1} \tag{24}$$

for any function f on M. Moreover, letting $\bar{f}$ be the average of f, upon integrating by parts we have

$$\begin{aligned}\lambda_1 \int_M |f - \bar{f}|^2\, d\mu_0 &\leq \int_M |\nabla f|_0^2\, d\mu_0 = \int_M (f - \bar{f})(-\Delta_0 f)\, d\mu_0 \\ &\leq \left(\int_M |f - \bar{f}|^2\, d\mu_0\right)^{1/2} \left(\int_M |\Delta_0 f|^2\, d\mu_0\right)^{1/2},\end{aligned}$$

where λ_1 is the first positive eigenvalue of the operator $-\Delta_0$ on M. Hence we obtain

$$\lambda_1 \int_M |f - \bar{f}|^2\, d\mu_0 \leq \int_M |\nabla f|_0^2\, d\mu_0 \leq \lambda_1^{-1} \int_M |\Delta_0 f|^2\, d\mu_0, \tag{25}$$

which in combination with (9) also yields the Calderón-Zygmund type estimate

$$\int_M |\nabla^2 f|_0^2\, d\mu_0 \leq C \int_M |\Delta_0 f|^2\, d\mu_0. \tag{26}$$

THEOREM 5.1. *Suppose $K_0 \leq 0$. Then for any $\alpha < \lambda_1^2$ there holds $||u(t)||^2_{H^2} \leq Ce^{-\alpha t}$ for some constant C and all $t \geq 0$.*

PROOF. From (12) and (23) we have the bound

$$\int_0^\infty \int_M |\nabla K|_0^2 \, d\mu_0 \, dt = \int_0^\infty \int_M |\nabla K|_g^2 \, d\mu_g \, dt \le 2E(u(0)) < \infty.$$

Hence, and using (25), for a suitable sequence $t_i \to \infty \quad (i \to \infty)$ we obtain

$$\lambda_1 \int_M |K(t_i) - \overline{K}(t_i)|^2 \, d\mu_0 \le \int_M |\nabla K(t_i)|_0^2 \, d\mu_0 \to 0 \quad (i \to \infty).$$

Observing that for any t the constant $a = K_0$ minimizes the integral $\int_M |K - a|^2 \, d\mu$, from equivalence of g_0 and $g(t_i)$ then we conclude that

$$Ca(g(t_i)) \le \int_M |K(t_i) - \overline{K}(t_i)|^2 \, d\mu \le C \int_M |K(t_i) - \overline{K}(t_i)|^2 \, d\mu_0 \to 0 \tag{27}$$

as $i \to \infty$, which in view of (10) implies that

$$Ca(g(t)) \to 0 \text{ as } t \to \infty. \tag{28}$$

Writing (5) as

$$-\Delta_0 u - K_0(e^{2u} - 1) = (K - K_0)e^{2u},$$

upon multiplying by $-\Delta_0 u$ and integrating by parts, we deduce that

$$\begin{aligned} &||\Delta_0 u||_{L^2}^2 + 2|K_0| \int_M e^{2u} |\nabla u|_0^2 \, d\mu_0 \\ &\qquad \le \int_M |\Delta_0 u||K - K_0| e^{2u} \, d\mu_0 \le C||\Delta_0 u||_{L^2} (Ca(g))^{1/2}. \end{aligned}$$

Thus from (28) we conclude that

$$||\Delta_0 u(t)||_{L^2}^2 \le C \cdot Ca(g(t)) \to 0, \tag{29}$$

which together with (25), (26) and our volume constraint implies that

$$||u(t)||_{H^2} \to 0 \text{ as } t \to \infty. \tag{30}$$

Also using (25), we can then improve (27) to read

$$Ca(g) \le (1 + o(1))||K - \overline{K}||_{L^2}^2 \le (1 + o(1))\lambda_1^{-2} ||\Delta_0 K||_{L^2}^2 \tag{31}$$

with error $o(1) \to 0$ as $t \to \infty$.

Thus, and observing that by (30) we have uniform convergence $g_t \to g_0$ as $t \to \infty$, we deduce from (8) and (24)-(26) that with error $o(1) \to 0$ as $t \to \infty$ there holds

$$\begin{aligned}
\frac{d}{dt} Ca(g(t)) &= -\int_M (|\Delta_g K|^2 - 2K|\nabla K|_g^2)\, d\mu \\
&= -\int_M (|\Delta_g K|^2 + 2|K_0||\nabla K|_g^2)\, d\mu + 2\int_M (K - K_0)|\nabla K|_g^2\, d\mu \\
&\le -(1 - o(1)) \int_M |\Delta_0 K|^2\, d\mu_0 + C||K - K_0||_{L^2}||\nabla K||_{L^4}^2 \\
&\le -(1 - o(1))||\Delta_0 K||_{L^2}^2 + C \cdot Ca(g(t))^{1/2}||\nabla K||_{H^1}^2 \\
&\le -(1 - o(1))||\Delta_0 K||_{L^2}^2 \le -\alpha Ca(g(t)),
\end{aligned}$$

for any $\alpha < \lambda_1^2$, if $t \ge t_0$ is sufficiently large. It follows that

$$Ca(g(t)) \le Ce^{-\alpha t}$$

and hence, in view of (29), (25), and (26) that

$$||u(t) - \overline{u}(t)||_{H^2}^2 \le Ce^{-\alpha t}$$

for all $t \ge 0$. But from conservation of volume we also have

$$\begin{aligned}
(2 + o(1))\overline{u}(t) &= \int_M (e^{2\overline{u}(t)} - 1)\, d\mu_0 \\
&= \int_M (e^{2\overline{u}(t)} - e^{2u(t)})\, d\mu_0 \le C||u(t) - \overline{u}(t)||_{L^2}^2 \le Ce^{-\alpha t}.
\end{aligned}$$

The claim follows. □

5.2. – The sphere

Any metric g on S^2 being conformal to the standard one, we may take this to be our background metric g_0, now normalized to have volume 4π, and we may regard (S^2, g_0) as the submanifold $\{x \in \mathbb{R}^3; |x| = 1\}$ of Euclidean space.

The special difficulties arising in the case $M = S^2$ may be attributed to the action of the Möbius group $G = \text{Möb}^+(2)$ of oriented conformal diffeomorphisms on S^2, which is a non-compact, six-dimensional Lie group containing the group $SO(3)$ of rigid rotations as a subgroup; see for instance Berger [5], Theorem 18.10.4.

The Möbius group acts on metrics $g = e^{2u} g_0$ and their curvatures, as follows. Given $\varphi \in G$, let $\varphi^* g$ be the pull-back of g under φ. Observing that φ is conformal with

$$\varphi^* g_0 = \det d\varphi \cdot g_0,$$

we have

$$\varphi^* g = e^{2u\circ\varphi}\varphi^* g_0 = e^{2v} g_0$$

with

$$v = u \circ \varphi + \frac{1}{2} \log \det d\varphi;$$

moreover,

$$\int_M d\mu_{\varphi^* g_0} = \int_M d\mu_0 = 4\pi.$$

Thus, and in view of

$$K_{\varphi^* g} = K_g \circ \varphi, \;\; K_{\varphi^* g_0} = K_0 \circ \varphi \equiv 1, \tag{32}$$

we have

$$\begin{aligned} Ca(\varphi^* g) &= \int_M |K_{\varphi^* g} - K_0|^2 \, d\mu_{\varphi^* g} \\ &= \int_M |K_g - K_0|^2 \circ \varphi \, e^{2u\circ\varphi} \det \, d\varphi \, d\mu_0 = Ca(g). \end{aligned} \tag{33}$$

From the identity

$$\Delta_{\varphi^* g}(f \circ \varphi) = (\Delta_g f) \circ \varphi, \tag{34}$$

and writing $h = \varphi^* g$ for brevity, we also deduce that

$$\int_M (|\Delta_h K_h|^2 + K_h^2 \Delta_h K_h) \, d\mu_h = \int_M (|\Delta_g K_g|^2 + K_g^2 \Delta_g K_g) \, d\mu_g, \tag{35}$$

and similarly for any other geometric expression that is naturally defined independent of the coordinate representation. Finally, there holds

$$E(v) = E(u); \tag{36}$$

see for instance Chang [8], Lemma 1, p. 85.

Now let $g = g(t)$ be a solution to (4). Following Singleton's [20] adaptation of De Turck's [13] trick to the Calabi flow we define a flow of gauge-equivalent metrics $h = \varphi^* g$. However, instead of Singleton's condition we require the identity

$$\int_M x d\mu_h = 0 \tag{37}$$

to be satisfied for all t, where x denotes the position vector in $\mathbb{R}^3$.

Condition (37) being invariant under rotations, in addition we require that each φ fixes a prescribed point $p_0 \in S^2$ and maps some given $e_0 \in T_{p_0} S^2$, $e_0 \neq 0$, to a positive multiple of itself.

By [8], Lemma 2, p. 85, we can find a conformal transformation φ_0 so that (37) is satisfied at the initial time $t = 0$. The above conditions then determine a unique gauge flow $\varphi = \varphi(t)$, generated by a vector field

$$\xi = \xi(t) = (d\varphi)^{-1}\varphi_t \in T_{id}G.$$

Interpreting φ_t as a vector field on S^2 with $\varphi_t(x) \in T_{\varphi(x)}S^2$ for all $x \in S^2$, we may also regard ξ as the pull-back vector field $\xi = \varphi^*\varphi_t$. Similarly, we may identify any member of $T_{id}G$ with a vector field on S^2. An equation for ξ now may be obtained, as follows.

Letting $g(t) = e^{2u(t)}g_0$, $h(t) = \varphi(t)^*g(t) = e^{2v(t)}g_0$ with

$$v(t) = u(t) \circ \varphi(t) + \frac{1}{2}\log\det(d\varphi(t))$$

as above, upon differentiating we have

$$v_t = u_t \circ \varphi + \xi \cdot dv + \frac{1}{2}\mathrm{div}_0\xi. \tag{38}$$

Observing that (4) and (32), (34) imply

$$u_t \circ \varphi = \frac{1}{2}\Delta_g K_g \circ \varphi = \frac{1}{2}\Delta_h K_h = \frac{1}{2}e^{-2v}\Delta_0 K_h,$$

from the normalization condition (37) we obtain the relation

$$\begin{aligned} 0 = \frac{d}{dt}\left(\int_M x\, d\mu_h\right) &= 2\int_M x v_t\, d\mu_h \\ &= \int_M x\Delta_0 K_h\, d\mu_0 + \int_M x\mathrm{div}_0(\xi e^{2v})\, d\mu_0. \end{aligned}$$

Also using the equation for the coordinate function $F(x) = x$ on S^2,

$$-\Delta_0 F = 2F,$$

upon integrating by parts then we arrive at the identity

$$2\int_M x K_h\, d\mu_0 + \int_M \xi\, d\mu_h = 0. \tag{39}$$

Now the linear map Ξ taking a vector field $\xi \in T_{id}G$ (with the identification made above) to the vector

$$X = \int_M \xi\, d\mu_h \in \mathbb{R}^3$$

by (37) vanishes on the space $T_{id}SO(3)$; moreover, Ξ is surjective. Indeed, arguing indirectly, suppose that there is $p \in S^2$ such that $\langle X, p\rangle_{\mathbb{R}^3} = 0$ for all X in the range of Ξ. For $\lambda > 0$ then consider the map

$$\varphi_{p,\lambda} = \pi_p^{-1} \circ \delta_\lambda \circ \pi_p \in G,$$

where $\pi_p \colon S^2 \setminus \{p\} \to \mathbb{R}^2$ denotes stereographic projection from p and where δ_λ denotes dilation $x \mapsto \lambda x$ in $\mathbb{R}^2$. Observe that $\varphi_{p,1} = id$; moreover, letting $\eta = \frac{\partial \varphi_{p,\lambda}}{\partial \lambda}\big|_{\lambda=1}$, there holds $\langle \eta, p\rangle_{\mathbb{R}^3} > 0$ away from p and $-p$. Hence the vector field η integrates to a vector $Y = \Xi(\eta)$ such that $\langle Y, p\rangle_{\mathbb{R}^3} > 0$, contrary to assumption. In addition, this argument shows that for metrics $h = e^{2v}g_0$ with a uniform bound for e^{2v} in $L^2(M, g_0)$ the quantity $\langle Y, p\rangle_{\mathbb{R}^3}$ is uniformly bounded away from 0, and the map Ξ is boundedly invertible on a 3-dimensional subspace of $T_{id}G$ complementing the space $T_{id}SO(3)$, uniformly for all such h.

Taking account of the normalization with respect to the action of $SO(3)$, we thus may choose a unique vector $\xi \in T_{id}G$ satisfying (39) and preserving the point p_0 and the direction $e_0 \in T_{p_0}S^2$ infinitesimally. Moreover, ξ is bounded in terms of $X = 2\int_M xK_h\, d\mu_0$ as long as e^{2v} is bounded in $L^2(M, g_0)$. The vector field $\xi = \xi(t)$ uniquely determines a normalized gauge flow $\varphi = \varphi(t)$ satisfying (37) by solving the initial value problem

$$\varphi_t = d\varphi \cdot \xi$$

with initial data $\varphi(0) = \varphi_0$.

The associated flow $v = v(t)$ is bounded in $H^2(M, g_0)$. Indeed, by a result of Aubin [2], in view of our volume normalization and condition (37), for any $\epsilon > 0$ with a constant $C(\epsilon)$ we have

$$1 = \fint_{S^2} e^{2v}\, d\mu_0 \le C(\epsilon)\, exp\left((1/2+\epsilon)\fint_{S^2} |\nabla v|_0^2\, d\mu_0 + \fint_{S^2} 2K_0 v\, d\mu_0\right).$$

Choosing $\epsilon = 1/6$, from (36) and (11) then we obtain the uniform bound

$$2/3\fint_{S^2} |\nabla v|_0^2\, d\mu_0 + \fint_{S^2} 2K_0 v\, d\mu_0 \ge C,$$

and hence

$$||\nabla v(t)||_{L^2}^2 \le 6E(v(t)) + C \le C;$$

confer also [8], Lemma 3, p. 86. Thus, the claim follows as in Section 5.1. In particular, all metrics $h(t)$ are equivalent to g_0, uniformly in $t \ge 0$, and the vector field $\xi(t)$ is uniformly bounded in terms of $X(t) = \Xi(\xi(t))$

Similar to the proof of Theorem 5.1 we now deduce exponential decay of the function v.

LEMMA 5.2. *There exist constants $\alpha > 0$, C such that $Ca(g(t)) + ||v(t)||_{H^2}^2 \le Ce^{-\alpha t}$ for all $t \ge 0$.*

PROOF. i) We first show exponential decay of the Calabi energy. The same reasoning as used in the proof of Theorem 5.1 yields a sequence $t_i \to \infty$ such that

$$Ca(g(t_i)) = Ca(h(t_i)) \le C \int_M |\nabla K_h(t_i)|_h^2 \, d\mu_h = C \int_M |\nabla K_g(t_i)|_g^2 \, d\mu_g \to 0$$

as $i \to \infty$, and by (8) it follows that

$$Ca(h(t)) = Ca(g(t)) \to 0 \text{ as } t \to \infty. \tag{40}$$

But then also

$$||v(t)||_{H^2} \to 0 \text{ as } t \to \infty. \tag{41}$$

Indeed, being bounded in $H^2(M, g_0)$, any sequence $v_i = v(t_i)$, where $t_i \to \infty$ as $i \to \infty$, admits a subsequence (v_i) (relabelled) converging weakly in $H^2(M, g_0)$ and uniformly to a limit v^*. Writing (5) in the form

$$-\Delta_0 v_i = K_h(t_i) e^{2v_i} - K_0 = (K_h(t_i) - K_0) e^{2v_i} + K_0(e^{2v_i} - 1),$$

from (40) we then conclude that $v_i \to v^*$ strongly in $H^2(M, g_0)$ as $i \to \infty$, where v^* solves the equation

$$-\Delta_0 v^* + K_0 = K_0 e^{2v^*}, \ \ K_0 = 1.$$

All solutions $v = v^*$ to this equation with volume $\int_{S^2} e^{2v} \, d\mu_0 = 4\pi$ are of the form $v = \frac{1}{2} \log|\det d\varphi|$ for some Möbius transformation $\varphi\colon S^2 \to S^2$. Taking into account (37), it follows that $v^* = 0$; confer [8], p. 87. Since (t_i) was arbitrary, this gives the desired conclusion.

Hence, from (8) and (35) with error $o(1) \to 0$ as $t \to \infty$ we obtain

$$\begin{aligned} \frac{d}{dt} Ca(h(t)) &= -\int_M (|\Delta_h K_h|^2 - 2K_h |\nabla K_h|_h^2) \, d\mu_h \\ &= -\int_M (|\Delta_0 K_h|^2 - 2K_0 |\nabla K_h|_0^2) \, d\mu_0 + o(1) \int_M |\Delta_0 K_h|^2 \, d\mu_0 + I, \end{aligned} \tag{42}$$

where the "error term" I in view of (25), (26) may be bounded

$$\begin{aligned} I &= 2 \int_M (K_h - K_0) |\nabla K_h|_h^2 \, d\mu_h \le 2Ca(h(t))^{1/2} \left(\int_M |\nabla K_h|_h^4 \, d\mu_h \right)^{1/2} \\ &\le o(1) ||\nabla K_h||_{L^4}^2 \le o(1) ||\nabla K_h||_{H^1}^2 \le o(1) ||\Delta_0 K_h||_{L^2}^2. \end{aligned} \tag{43}$$

Obtaining the desired bound for the principal term is slightly more subtle than in the case where $K_0 \le 0$ since the number $2K_0 = 2$ belongs to the spectrum of $-\Delta_0$ on (M, g_0).

Let (φ_i) be an L^2-orthonormal basis of eigenfunctions for $-\Delta_0$ on (S^2, g_0) with eigenvalues $\lambda_i, i \in \mathbb{N}_0$. Recall that $\lambda_0 = 0, \lambda_1 = \lambda_2 = \lambda_3 = 2 < \lambda_4 \leq \dots$ Expand $K_h - K_0 = \sum_{i=0}^{\infty} K_h^i \varphi_i$. Writing

$$-\Delta_0 K_h = \sum_{i=0}^{\infty} \lambda_i K_h^i \varphi_i,$$

with the constant $\vartheta = \frac{\lambda_4 - 2}{\lambda_4} > 0$ we then obtain

$$\begin{aligned}
&\int_M (|\Delta_0 K_h|^2 - 2K_0 |\nabla K_h|_0^2)\, d\mu_0 \\
&= \int_M (|\Delta_0 K_h|^2 + 2K_0(K_h - K_0)\Delta_0 K_h)\, d\mu_0 = \sum_{i=4}^{\infty} (\lambda_i - 2)\lambda_i (K_h^i)^2 \qquad (43)\\
&\geq \vartheta \sum_{i=4}^{\infty} \lambda_i^2 (K_h^i)^2 \geq \vartheta \left(\int_M |\Delta_0 K_h^i|^2\, d\mu_0 - 4 \sum_{i=1}^{3} (K_h^i)^2 \right).
\end{aligned}$$

We can estimate K_h^1, K_h^2, and K_h^3 via the Kazdan-Warner [15] identity, stating that

$$\int_M \langle \nabla K_h, \nabla F \rangle_0\, d\mu_h = 0$$

for any first order spherical harmonic F on (S^2, g_0), that is, for the restriction of any linear function in $\mathbb{R}^3$ to S^2 or – equivalently – for any $F \in \text{span}\{\varphi_1, \varphi_2, \varphi_3\}$. Indeed, using the relation

$$-\Delta_0 F = 2F,$$

upon integrating by parts in view of the Kazdan-Warner identity we find that

$$\int_M (K_h - K_0) F\, d\mu_h = \int_M (K_h - K_0) \langle \nabla F, \nabla v \rangle_0\, d\mu_h.$$

In particular, letting $F = \varphi_i, i = 1, 2, 3$, we then obtain the estimate

$$\begin{aligned}
|K_h^i| &= \left| \int_M (K_h - K_0)\varphi_i\, d\mu_0 \right| \\
&= \left| \int_M (K_h - K_0)\langle \nabla \varphi_i, \nabla v \rangle_0\, d\mu_h + \int_M (K_h - K_0)\varphi_i (e^{-2v} - 1)\, d\mu_h \right| \qquad (44)\\
&\leq C \cdot Ca(h(t))^{1/2} ||v(t)||_{H^2} = o(1) Ca(h(t))^{1/2}.
\end{aligned}$$

Combining this estimate with (43), from (42) we deduce that with error $o(1) \to 0$ as $t \to \infty$ there holds

$$\frac{d}{dt} Ca(h(t)) \leq -(\vartheta + o(1)) \int_M |\Delta_0 K_h|^2\, d\mu_0 + o(1) Ca(h(t)). \qquad (45)$$

Thus, from (31) we conclude that for any $\alpha < \vartheta\lambda_1^2$ and sufficiently large t there holds

$$\frac{d}{dt}Ca(h(t)) < -\alpha Ca(h(t))$$

and it follows that

$$Ca(g(t)) = Ca(h(t)) \le Ce^{-\alpha t}$$

for all $t \ge 0$ with some constant C.

ii) We can now show exponential decay of v. Observing that

$$\int_M (e^{2v} - 1)\, d\mu_0 = \int_M d\mu_h - \int_M d\mu_0 = 0$$

and recalling the normalization condition

$$\int_M x d\mu_h = \int_M (e^{2v} - 1) x d\mu_0 = 0,$$

we have an expansion

$$e^{2v} - 1 = \sum_{i=0}^{\infty} V^i \varphi_i$$

in terms of an L^2-orthonormal basis of eigenfunctions φ_i as in the proof of Lemma 5.2, where $V^0 = \ldots = V^3 = 0$.

Also let $v = \sum_{i=0}^{\infty} v^i \varphi_i$. Observe that

$$2v^i = 2\int_M v\varphi_i\, d\mu_0 = \int_M (e^{2v} - 1)\varphi_i\, d\mu_0 + O(||v||^2_{L^\infty}) = V^i + o(1)||v||_{H^2},$$

where $o(1) \to 0$ as $t \to \infty$ on account of (41).

In particular, we have

$$\sum_{i=0}^{3} (v^i)^2 \le o(1)||v||^2_{H^2}. \tag{46}$$

Again writing (5) as

$$-\Delta_0 v = (K_h - K_0)e^{2v} + K_0(e^{2v} - 1),$$

from Young's inequality and uniform boundedness of v for any $\varepsilon > 0$ we obtain

$$\begin{aligned}\int_M |\Delta_0 v|^2\, d\mu_0 &\le C(\varepsilon)Ca(h(t)) + (1+\varepsilon)\int_M (e^{2v} - 1)^2\, d\mu_0 \\ &\le C(\varepsilon)e^{-\alpha t} + (4(1+\varepsilon) + o(1))\int_M |v|^2\, d\mu_0,\end{aligned}$$

where $o(1) \to 0$ as $t \to \infty$.

In terms of the coefficients v^i, this implies that

$$\sum_{i=1}^{\infty} \lambda_i^2 (v^i)^2 \leq C(\varepsilon) e^{-\alpha t} + (4(1+\varepsilon) + o(1)) \sum_{i=0}^{\infty} (v^i)^2.$$

Hence, if we choose $\varepsilon > 0$ so that

$$4(1+\varepsilon) < \lambda_4^2$$

and take (46) into account, we find

$$||v(t)||_{H^2}^2 \leq C \sum_{i=0}^{\infty} (1+\lambda_i^2)(v^i)^2 \leq C e^{-\alpha t} + o(1)||v(t)||_{H^2}^2$$

where $o(1) \to 0$ as $t \to \infty$, proving our claim. □

We can now complete the proof of Theorem 2.1 for the sphere.

Proof of Theorem 2.1 (completed). In view of Lemma 5.2, from (39) we deduce the estimate

$$||\xi||_{L^\infty}^2 \leq C \left| \int_M x K_h \, d\mu_0 \right|^2 = C \left| \int_M x (K_h - K_0) \, d\mu_0 \right|^2 \leq C \cdot Ca(h(t)) \leq C e^{-\alpha t}$$

for all $t \geq 0$. Thus, and recalling that $T_{id}G$ is a finite-dimensional space of smooth vector fields, we have smooth exponential convergence $\varphi(t) \to \varphi_\infty$ as $t \to \infty$. By Lemma 5.2 again, therefore also $g(t) = (\varphi(t)^{-1})^* h(t) \to g_\infty = (\varphi_\infty^{-1})^* g_0$ and hence $u(t) \to u_\infty$ exponentially fast in $H^2(M, g_0)$ as $t \to \infty$, where $g_\infty = e^{2u_\infty} g_0$ satisfies $Ca(g_\infty) = 0$. □

6. – The Ricci flow

Similar ideas may be applied in the case of the Ricci flow (1) on a closed, compact surface M. Denoting $g(t)$ as $g(t) = e^{2u(t)} g_0$ for a background metric g_0 of constant curvature K_0 conformal to $g(0)$ and of equal volume, equation (1) takes the simple form

$$u_t = K_0 - K. \tag{47}$$

Inserting (5), this gives

$$\frac{1}{2} \frac{d}{dt}(e^{2u}) - \Delta_0 u + K_0(1 - e^{2u}) = 0, \tag{48}$$

which immediately implies conservation of volume. For convenience, we again normalize the volume to unity if $K_0 \le 0$ and to the value 4π in the case of the sphere. Moreover, from (11) we see that (1) may be regarded as the gradient flow for the Liouville energy, satisfying

$$\frac{d}{dt}E(u(t)) = \int_M u_t K\, d\mu = -\int_M (K_0 - K)^2\, d\mu = -\int_M |u_t|^2\, d\mu \le 0. \tag{49}$$

In [14], [11] Hamilton and Chow established global existence and exponential convergence $g(t) \to g_\infty$ as $t \to \infty$ for (1), where $K_{g_\infty} \equiv$ const. Here we reobtain their result by a simpler method, based essentially on the concentration-compactness result in Theorem 3.2. In terms of a solution u to (47) the result may be phrased as follows.

Theorem 6.1. *For any* $u_0 \in H^2(M, g_0)$ *there exists a unique, global solution* u *of* (47) *with* $u(0) = u_0$ *and a smooth limit* u_∞ *corresponding to a smooth metric* $g_\infty = e^{2u_\infty} g_0$ *of constant curvature such that*

$$||u(t) - u_\infty||_{H^2} \le Ce^{-\alpha t}$$

for some constant $\alpha > 0$ *and all* $t \ge 0$.

Again, smoothness for $t > 0$ and exponential convergence in stronger norms can be easily deduced from Theorem 6.1.

We first establish long-time existence, the delicate case being, of course, the case when $M = S^2$.

6.1. – Existence

Local existence for data $u(0) = u_0 \in H^2(M, g_0)$ can be obtained by a standard fixed point argument in the space $L^\infty([0, T]; H^2(M, g_0))$ for sufficiently small $T > 0$. The method is similar to the method we sketched in Section 4; however, we now use $\Delta_0 v_t$ as testing function in the equation

$$v_t - \Delta_g v = K_0(1 - e^{-2u})$$

replacing (22), etc. We show that the local solution remains bounded in $H^2(M, g_0)$ for any finite time.

Lemma 6.2. *For any* $T > 0$*, any solution* $u \in L^\infty_{\mathrm{loc}}([0, T[; H^2(M, g_0))$ *of* (47) *there holds*

$$\sup_{0 \le t < T} \int_M e^{4|u(t)|}\, d\mu_0 < \infty.$$

Proof. If $K_0 \le 0$, from (49) we obtain the uniform bound

$$\frac{1}{2}||\nabla u(t)||^2_{L^2} \le E(u(t)) \le E(u(0))$$

and the assertion follows from the Moser-Trudinger inequality and in view of the volume constraint.

If $M = S^2$, then as above we may choose the standard metric as our background metric with volume 4π and $K_0 = 1$. Given $u = u(t)$, we determine Möbius transformations $\varphi = \varphi(t)$ such that $h = h(t) = \varphi(t)^* g(t) = e^{2v(t)} g_0$ with

$$v(t) = u(t) \circ \varphi(t) + \frac{1}{2} \log \det(d\varphi(t))$$

satisfying (37), where $\varphi(t)$ is normalized with respect to the action of $SO(3)$ as in Section 5 above.

From (38), (47), (32), and (5) we have

$$v_t = u_t \circ \varphi + \frac{1}{2} e^{-2v} \mathrm{div}_0(\xi e^{2v}) = e^{-2v} \left(\Delta_0 v + K_0(e^{2v} - 1) + \frac{1}{2} \mathrm{div}_0(\xi e^{2v}) \right),$$

where, as in Section 5.2, we regard

$$\xi = \xi(t) = (d\varphi)^{-1} \varphi_t \in T_{id} G$$

as a vector field on S^2. Our normalization then implies that

$$\begin{aligned} 0 &= \frac{d}{dt} \left(\int_M x \, d\mu_h \right) = 2 \int_M (x v_t) \, d\mu_h \\ &= 2 \int_M x(\Delta_0 v + K_0(e^{2v} - 1)) \, d\mu_0 + \int_M x \mathrm{div}_0(\xi e^{2v}) \, d\mu_0 \qquad (50) \\ &= 2 \int_M (\Delta_0 x) v \, d\mu_0 - \int_M \xi \, d\mu_h = -4 \int_M x v \, d\mu_0 - \int_M \xi \, d\mu_h. \end{aligned}$$

In view of (36) and (49) for v we have the uniform a-priori bound

$$E(v(t)) = E(u(t)) \leq E(u(0)).$$

Therefore, from Aubin's result and the Moser-Trudinger inequality we obtain

$$||v(t)||^2_{H^1} \leq C, \ \int_M e^{4|v|} \, d\mu_0 \leq C \qquad (51)$$

in the same way as in Section 5.2 above. Hence by (50) we can bound

$$||\xi||_{L^\infty} \leq C \left| \int_M x v \, d\mu_0 \right| \leq C ||v||_{L^2} \leq C$$

uniformly in time. Upon integration, this yields that

$$\sup_{0 \leq t < T} (||\varphi(t) - \varphi(0)||_{C^1} + ||\varphi(t)^{-1} - \varphi(0)^{-1}||_{C^1}) \leq C(T)$$

and therefore that

$$\int_M e^{4|u|}\,d\mu_0 \le C(T)\int_M e^{4|v\circ\varphi(t)^{-1}|}\,d\mu_0 \le C(T)\int_M e^{4|v|}\,d\mu_0 \le C(T). \qquad \square$$

LEMMA 6.3. *Under the hypotheses of Lemma* 6.2 *for any* $T > 0$ *there holds*

$$\sup_{0\le t<T} ||u(t)||^2_{H^2} < \infty.$$

PROOF. Upon multiplying (48) by $-\Delta_0 u_t$ and integrating by parts with respect to $d\mu_0$, we obtain

$$\int_M e^{2u}|\nabla u_t|_0^2\,d\mu_0 + \frac{1}{2}\frac{d}{dt}\left(\int_M |\Delta_0 u|^2 d\mu_0\right) = -2\int_M e^{2u}(\nabla u_t,\nabla u)_0(u_t - K_0)d\mu_0$$
$$\le \frac{1}{2}\int_M e^{2u}|\nabla u_t|_0^2\,d\mu_0 + C\int_M e^{2u}|\nabla u|_0^2(|u_t|^2+1)d\mu_0.$$

Hence for any $0 \le t_0 < t_1 \le T$ we find that

$$\begin{aligned} I := \int_{t_0}^{t_1}\int_M e^{2u}|\nabla u_t|_0^2\,d\mu_0\,dt + \sup_{t_0\le t<t_1} ||u(t)||^2_{H^2} \\ \le C\int_{t_0}^{t_1}\int_M e^{2u}|\nabla u|_0^2(|u_t|^2+1)\,d\mu_0\,dt + C||u(t_0)||^2_{H^2} + C(T), \end{aligned} \tag{52}$$

thereby using that we can bound

$$||u||^2_{H^2} \le C(||\Delta_0 u||^2_{L^2} + ||u||^2_{L^2})$$

for any t, together with the estimate

$$||u||^2_{L^2} \le C\int_M e^{2|u|}\,d\mu_0 \le C(T).$$

From boundedness of e^{2u} in $L^2(M, g_0)$ and (24), for $t_0 \le t < t_1 \le T$ we deduce

$$\int_M e^{2u}|\nabla u|_0^2\,d\mu_0 \le C(T)||\nabla u||^2_{L^4} \le C(T)||u||^2_{H^2} \le C(T)\sup_{t_0\le t<t_1}||u(t)||^2_{H^2}. \tag{53}$$

Also observing that

$$||\nabla u||^2_{L^2} \le E(u) + C||u||_{L^2} \le C(T),$$

in the same fashion for $t_0 \le t < t_1$ we find

$$
\begin{aligned}
\int_M e^{2u} |\nabla u|_0^2 |u_t|^2 \, d\mu_0 &= \int_M |\nabla u|_0^2 |(e^u)_t|^2 \, d\mu_0 \le ||\nabla u||_{L^4}^2 ||(e^u)_t||_{L^4}^2 \\
&\le C ||\nabla u||_{L^2} ||(e^u)_t||_{L^2} ||(e^u)_t||_{H^1} ||\nabla u||_{H^1} \\
&\le C(T) ||(e^u)_t||_{L^2} ||(e^u)_t||_{H^1} \cdot \sup_{t_0 \le t < t_1} ||u(t)||_{H^2}.
\end{aligned} \tag{54}
$$

Noting that

$$||(e^u)_t||_{L^2}^2 = \int_M e^{2u} |u_t|^2 \, d\mu_0 = \int_M |u_t|^2 \, d\mu$$

and recalling (49), we have

$$\int_{t_0}^{t_1} ||(e^u)_t||_{L^2}^2 \, dt = E(u(t_0)) - E(u(t_1)) \le C,$$

where we define $E(u(T)) := \lim_{t \to T} E(u(t))$. Similarly, we estimate

$$\int_{t_0}^{t_1} ||(e^u)_t||_{H^1}^2 \, dt \le C \int_{t_0}^{t_1} \int_M e^{2u} (|\nabla u_t|_0^2 + |\nabla u|_0^2 |u_t|^2 + |u_t|^2) \, d\mu_0 \, dt.$$

Thus, upon integrating (54) and observing that

$$\sup_{t_0 \le t < t_1} ||u(t)||_{H^2}^2 \le I,$$

from Hölder's inequality we obtain

$$II := \int_{t_0}^{t_1} \int_M e^{2u} |\nabla u|_0^2 |u_t|^2 \, d\mu_0 \, dt \le C_1(T) (E(u(t_0)) - E(u(t_1)))^{1/2} (I + II + C).$$

Since $t \mapsto E(u(t))$ is continuous, non-increasing and uniformly bounded on $[0, T]$, given $\varepsilon \in]0, 1/2[$ there is $\tau > 0$ such that for any $0 \le t_0 < t_1 \le T$ satisfying $t_1 \le t_0 + \tau$ there holds

$$C_1(T)(E(u(t_0)) - E(u(t_1)))^{1/2} \le \varepsilon \le 1/2.$$

For such t_0, t_1 then we have

$$II \le 2\varepsilon I + C.$$

Hence from (52), (53) with a constant $C_2 = C_2(T)$ we deduce that

$$I \le C_2(\tau + \varepsilon) I + C ||u(t_0)||_{H^2}^2 + C(T).$$

Choosing $\varepsilon = \frac{1}{3C_2}$, clearly we may also assume that $\tau \le \frac{1}{3C_2}$. Upon covering $[0, T]$ by finitely many intervals of length τ, we obtain the desired conclusion. □

6.2. – Convergence

Finally, we establish exponential convergence. Our key estimate is (49). Moreover, using the evolution equation

$$K_t = -2u_t K - \Delta_g u_t = \Delta_g K + 2K(K - K_0)$$

analogous to (7) we obtain the relation

$$\begin{aligned} \frac{d}{dt}Ca(g(t)) &= 2\int_M ((K-K_0)K_t + (K-K_0)^2 u_t)\, d\mu_g \\ &= 2\int_M ((K-K_0)^2(K+K_0) - |\nabla K|_g^2)\, d\mu_g \\ &= -2\int_M (|\nabla K|_g^2 - 2K_0(K-K_0)^2)\, d\mu_g + 2\int_M (K-K_0)^3\, d\mu_g. \end{aligned} \tag{55}$$

First consider the case $K_0 \le 0$.

THEOREM 6.4. *Suppose $K_0 \le 0$. There exist constants C, $\alpha > 0$ such that $||u(t)||_{H^2}^2 \le Ce^{-\alpha t}$ for all $t \ge 0$.*

PROOF. Observe that (49) and (23) imply

$$\int_0^\infty Ca(g(t))\, dt \le E(u_0) =: E_0.$$

Hence

$$\liminf_{t_0\to\infty} Ca(g(t_0)) \le \liminf_{t_0\to\infty} \int_{t_0}^\infty Ca(g(t))\, dt \to 0. \tag{56}$$

and there exist arbitrarily large numbers $t_0 \ge 0$ such that

$$Ca(g(t_0)) \le E_0. \tag{57}$$

Choose such a time t_0 and let $t_1 \ge t_0$ be any number such that

$$\sup_{t_0\le t\le t_1} Ca(g(t)) \le 4E_0. \tag{58}$$

Since (49) and our assumption $K_0 \le 0$ as in Section 5.1 imply uniform bounds

$$||u(t)||_{H^1} \le C,\ ||e^{2|u(t)|}||_{L^2} \le C$$

independent of t, Theorem 3.2 then implies that

$$\sup_{t_0\le t\le t_1} ||u(t)||_{H^2} \le C;$$

in particular the metrics $g(t)$ and g_0 are uniformly equivalent for $t_0 \le t \le t_1$. Thus, by the Poincaré-Sobolev inequality (24) we may bound

$$\begin{aligned}\int_M |K-K_0|^3\,d\mu &\le C||K-K_0||^3_{L^3} \le C||K-K_0||_{L^2}||K-K_0||^2_{L^4} \\ &\le C||K-K_0||^2_{L^2}||K||_{H^1} \le \frac{1}{2}||\nabla K||^2_{L^2} + C(1+Ca(g(t)))Ca(g(t)).\end{aligned} \tag{59}$$

From (55) we then obtain

$$\frac{d}{dt}Ca(g(t)) \le -\int_M |\nabla K|^2_g\,d\mu_g + C(1+Ca(g(t)))Ca(g(t)).$$

Integrating in time, we deduce

$$Ca(g(t_1)) \le Ca(g(t_0)) + C_1(1+\sup_{t_0\le t\le t_1} Ca(g(t)))\int_{t_0}^{t_1} Ca(g(t))\,dt \tag{60}$$

with a constant $C_1 = C_1(E_0)$. Choose $T > 0$ such that

$$C_1\int_T^\infty Ca(g(t))\,dt \le \frac{1}{2}\min\{1, E_0\}.$$

Then for $t_0 \ge T$ satisfying (57) and any $t_1 \ge t_0$ such that

$$Ca(g(t_1)) = \sup_{t_0\le t\le t_1} Ca(g(t)) \le 4E_0$$

from (60) we obtain the bound

$$Ca(g(t_1)) \le 2Ca(g(t_0)) + E_0 \le 3E_0;$$

hence this bound, in fact, is valid for all $t \ge t_0$. But then (60) and (56) imply convergence

$$Ca(g(t)) \to 0 \text{ as } t \to \infty,$$

and (59) may be improved to read

$$\int_M |K-K_0|^3\,d\mu_g \le o(1)\left(||\nabla K||^2_{L^2} + Ca(g(t))\right) \tag{61}$$

with error $o(1) \to 0$ as $t \to \infty$. Moreover, as in deducing (30) from (28) in the proof of Theorem 5.1, in addition we obtain convergence

$$||u(t)||_{H^2} \to 0 \text{ as } t \to \infty.$$

From (25), (27), and (55) we then deduce the differential inequality

$$\frac{d}{dt}Ca(g(t)) \le -(2+o(1))\int_M |\nabla K|^2_g\,d\mu_g + o(1)Ca(g(t)) \le -\alpha Ca(g(t))$$

for any $\alpha < 2\lambda_1$ and sufficiently large t, proving exponential decay

$$Ca(g(t)) \le Ce^{-\alpha t}.$$

Exponential decay of u now follows exactly as in the proof of Theorem 5.1. □

If $M = S^2$ with the spherical metric g_0 of curvature $K_0 \equiv 1$ we argue similarly for the normalized flow $h = h(t) = e^{2v(t)} g_0$ introduced in Lemma 6.2 above.

LEMMA 6.5. *There exist constants $C, \alpha > 0$ such that there holds*

$$||v(t)||^2_{H^2} \le Ce^{-\alpha t}$$

for all $t \ge 0$.

PROOF. Again we start by showing exponential decay of $Ca(h(t))$. As demonstrated in Section 5.2, all terms in (55) transform invariantly under Möbius transformations and we obtain the identity

$$\text{(62)}\quad \frac{d}{dt}Ca(h(t)) = -2\int_M (|\nabla K_h|_h^2 - 2K_0|K_h - K_0|^2)\,d\mu_h + 2\int_M (K_h - K_0)^3\,d\mu_h.$$

Moreover, as in the case $K_0 \le 0$, for $t_0 \ge 0$ satisfying (57) with $E_0 = E(u_0) - \lim_{t\to\infty} E(t)$ and any $t_1 \ge t_0$ such that (58) and hence (59) are valid, we obtain that

$$Ca(g(t_1)) = Ca(h(t_1)) \le Ca(h(t_0)) + C(1 + \sup_{t_0 \le t \le t_1} Ca(h(t))) \int_{t_0}^{t_1} Ca(h(t))\,dt.$$

Since E is uniformly bounded from below, again (56) is valid and we deduce that

$$Ca(g(t)) = Ca(h(t)) \to 0 \text{ as } t \to \infty$$

exactly as before. But then, by the argument leading to (41) in the proof of Lemma 5.2 it follows that

$$||v(t)||_{H^2} \to 0 \text{ as } t \to \infty$$

and (61), (62) imply that

$$\text{(63)}\quad \begin{aligned} \frac{d}{dt}Ca(h(t)) \le &-2\int_M (|\nabla K_h|_0^2 - 2K_0|K_h - K_0|^2)\,d\mu_0 \\ &+ o(1)\left(\int_M |\nabla K_h|_0^2\,d\mu_0 + Ca(h(t))\right), \end{aligned}$$

where $o(1) \to 0$ as $t \to \infty$.

Expanding

$$K_h - K_0 = \sum_{i=0}^{\infty} K_h^i \varphi_i$$

in terms of an L^2-orthonormal basis of eigenfunctions for $-\Delta_0$ on (S^2, g_0) as in Section 5.2, for $i = 0$ by the Gauss-Bonnet theorem we have

$$\text{(64)}\quad \begin{aligned} 4\pi K_h^0 &= \int_M (K_h - K_0)\,d\mu_0 = \int_M (K_h - K_0)\,d\mu + o(1)Ca(h(t))^{1/2} \\ &= o(1)Ca(h(t))^{1/2}; \end{aligned}$$

moreover, as in (44), in view of the Kazdan-Warner identity we have

$$K_h^i = \int_M (K_h - K_0)\varphi_i \, d\mu_0 = o(1)Ca(h(t))^{1/2},$$

where $o(1) \to 0$ as $t \to \infty$, $i = 1, 2, 3$.

Thus, with $\vartheta = \frac{\lambda_4 - 2}{\lambda_4} > 0$ we obtain

$$\int_M (|\nabla K_h|_0^2 - 2K_0|K_h - K_0|^2)\, d\mu_0 = \sum_{i=4}^{\infty}(\lambda_i - 2)(K_h^i)^2 + o(1)Ca(h(t))$$

$$\geq \vartheta \sum_{i=4}^{\infty} \lambda_i (K_h^i)^2 + o(1)Ca(h(t)) \geq (\vartheta\lambda_1 + o(1))Ca(h(t)).$$

But then (63) implies that

$$\frac{d}{dt}Ca(h(t)) \leq -\alpha Ca(h(t))$$

for any $\alpha < 2\vartheta\lambda_1$ and sufficiently large t. It follows that

$$Ca(g(t)) \leq Ca(h(t)) \leq Ce^{-\alpha t},$$

as desired.

Exponential convergence

$$||v(t)||_{H^2}^2 \leq Ce^{-\alpha t}$$

then follows exactly as in Lemma 5.2. □

We can now complete the proof of Theorem 6.1 for the sphere.

PROOF OF THEOREM 6.1 (completed). In view of (50) and Lemma 6.5 we have

$$||\xi(t)||_{L^\infty}^2 \leq C\left|\int_M xv\, dx_h\right|^2 \leq C||v(t)||_{H^2}^2 \leq Ce^{-\alpha t}$$

and we conclude that $\varphi(t) \to \varphi_\infty$ smoothly and therefore also $g(t) = e^{2u(t)}g_0 = (\varphi(t)^{-1})^* h(t) \to g_\infty = e^{2u_\infty}g_0$ exponentially fast in $H^2(M, g_0)$ as $t \to \infty$, where

$$Ca(g_\infty) = \lim_{t\to\infty} Ca(g(t)) = 0.$$ □

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