Abstract. Small data scattering for nonlinear Schrödinger equations (NLS), nonlinear wave equations (NLW), nonlinear Klein-Gordon equations (NLKG) with power type nonlinearities is studied in the scheme of Sobolev spaces on the whole space $\mathbb{R}^n$ with order $s < n/2$. The assumptions on the nonlinearities are described in terms of power behavior $p_1$ at zero and $p_2$ at infinity such as $1 + 4/n \leq p_1 \leq p_2 \leq 1 + 4/(n - 2s)$ for NLS and NLKG, and $1 + 4/(n - 1) \leq p_1 \leq p_2 \leq 1 + 4/(n - 2s)$ for NLW.

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1. Introduction

We consider the small data scattering for nonlinear Schrödinger equations (NLS), nonlinear wave equations (NLW), nonlinear Klein-Gordon equations (NLKG) in the Sobolev space $H^s$ for NLS and NLKG, and in the homogeneous Sobolev space $\dot{H}^s$ for NLW. The equations in this paper take the form

\begin{align*}
\text{NLS} &\quad i \partial_t u - \Delta u = f(u), \quad n \geq 1, \quad 0 \leq s < n/2, \\
\text{NLW} &\quad \partial^2_t u - \Delta u = f(u), \quad n \geq 2, \quad 1/2 \leq s < n/2, \\
\text{NLKG} &\quad \partial^2_t u - \Delta u + u = f(u), \quad n \geq 2, \quad 1/2 \leq s < n/2,
\end{align*}

where $u$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\Delta$ is the Laplacian in $\mathbb{R}^n$, and $f$ is a nonlinear interaction. A typical form of $f(u)$ is given by

\begin{align*}
f(u) = \begin{cases}
c_1 |u|^{p(0)} + c_2 |u|^{p(s)} & \text{for NLS, NLKG} \\
c_1 |u|^{p(1/2)} + c_2 |u|^{p(s)} & \text{for NLW},
\end{cases}
\end{align*}

where \( c_1, c_2 \in \mathbb{C} \), \( p(s) \equiv 1 + 4/(n - 2s) \). A scaling argument shows that \( p(s) \) is the critical power at the level of \( H^s \).

There is a large literature on the small data scattering for \textbf{NLS}, \textbf{NLW} and \textbf{NLKG}. See for instance [4], [7], [10], [13], [15], [16], [18], [19], [21] and [22], and references therein.

In [7], Kato has shown the small data scattering theory for \textbf{NLS} with \( f \) satisfying \( f \in C^1(\mathbb{C}; \mathbb{C}), f^{(j)}(0) = 0 \),

\[
 f^{(j)}(z) = \begin{cases} 
 O(|z|^{p-j}) & \text{as } |z| \to \infty, \\
 O(|z|^{p(0)-j}) & \text{as } |z| \to 0,
\end{cases}
\]

for \( 0 \leq j \leq \{s\} \) and \( \max(|s|, p(0)) \leq p \leq p(s) \), where \( \{s\} = [s] + 1 \) for \( s \notin \mathbb{Z}_+ \), \( \{s\} = s \) for \( s \in \mathbb{Z}_+ \), and \([s]\) denotes the largest integer less than or equal to \( s \).

It is also shown that the auxiliary spaces of solutions are removable when \( p \) satisfies

\[
p < 1 + \frac{\min(2s + 2, 4)}{n - 2s} \quad \text{if } n \geq 2, \quad p \leq \frac{2}{1 - 2s} \quad \text{if } n = 1.
\]

In [13], we have shown the small data scattering theory for \textbf{NLW} with \( f \) satisfying \( f \in C^{\max(1,\{s-1/2\})}(\mathbb{C}; \mathbb{C}) \),

\[
 f^{(j)}(z) = O(|z|^{p(s)-j}) \quad \text{for } \quad z \in \mathbb{C},
\]

\[
 (1.2) \quad |f^{(l-1/2)}(z) - f^{(l-1/2)}(w)| \leq C|z - w|^{p(s) - \{s-1/2\}} 
\]

if \( s - 1/2 < p(s) < [s + 1/2] \)

for \( 0 \leq j \leq \max(1,\{s - 1/2\}) \).

In [22], Wang has shown the small data scattering theory for \textbf{NLKG} with \( f \) satisfying one of two conditions

\[(a) \quad f \in C^1(\mathbb{C}; \mathbb{C}), |f'(z)| \leq C|z|^{p-1}, p(0) \leq p < p(1/2) \quad \text{for } \frac{1}{2} \leq s \leq \frac{3}{2},
\]

\[(b) \quad f \in C^\kappa(\mathbb{C}; \mathbb{C}), |f^{(j)}(z)| \leq C(|z|^{p1-j} + |z|^{p2-j}), 0 \leq j \leq \kappa .
\]

\[\text{for } p(1/2) \leq p_1 \leq p_2 \leq p(s) \text{ and } 1/2 \leq s < n/2,\]

where \( \kappa \) is a constant greater than \( \{s\} \). Thus the sum of two powers \(|u|^{1+4/n} + |u|^{1+4/(n-2s)}\) is excluded.

In this paper, we extend all those results above to the full range of powers of the nonlinearities such as (1.1) with minimal regularity assumption on \( f \) using the property (1.2). For that purpose we exploit refined Strichartz estimates, Corollary 2.8 below, which is needed to show the uniqueness of solutions without auxiliary spaces, as well as to deal with the sum of two powers \(|u|^{1+4/n}\) and \(|u|^{1+4/(n-2s)}\) for \textbf{NLKG}. The proof of such refined Strichartz estimates is inspired by [5], [9], [11]. In order to state our results precisely, we introduce the following notation.
For any $r$ with $1 \leq r \leq \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any $r$ with $1 < r < \infty$, $H^s_r = (1 - \Delta)^{-s/2}L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbb{R}$ and any $r$ with $1 < r < \infty$, $\dot{H}^s_r = (\Delta)^{-s/2}L^r$ denotes the homogeneous Sobolev space defined in terms of Riesz potentials. To introduce the Besov space and the homogeneous Besov space, let $\phi_0$ be a nonnegative function on $\mathbb{R}^n$ with

$$\text{supp } \phi_0 \subset \{ \xi \in \mathbb{R}^n ; \ 1/2 \leq |\xi| \leq 2 \}$$

such that $\{ \phi_0(2^{-j} \cdot) \}_{j = -\infty}^{\infty}$ forms the Littlewood-Paley dyadic decomposition on $\mathbb{R}^n \setminus \{0\}$. Let $\{ \psi_j \}_{j = -\infty}^{\infty}$ and $\tilde{\psi}$ be functions defined by

$$\mathcal{F} \psi_j(\xi) \equiv \phi_0(2^{-j} \xi), \quad \mathcal{F} \tilde{\psi}(\xi) \equiv 1 - \sum_{j=1}^{\infty} \phi_0(2^{-j} \xi),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and its inverse, respectively. For any $s \in \mathbb{R}$ and any $r, m$ with $1 \leq r, m \leq \infty$, the Besov space $B^s_{r, m}$ and the homogeneous Besov space $\dot{B}^s_{r, m}$ are defined by

$$B^s_{r, m} \equiv \left\{ u \in \mathcal{S}'(\mathbb{R}^n) ; \| u; B^s_{r, m} \| \equiv \left\| \tilde{\psi} * u; L^r \right\|^m + \sum_{j=1}^{\infty} (2^{sj} \| \psi_j * u; L^r \|^m)^{1/m} < \infty \right\}$$

$$\dot{B}^s_{r, m} \equiv \left\{ u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) ; \| u; \dot{B}^s_{r, m} \| \equiv \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \| \psi_j * u; L^r \|^m)^{1/m} < \infty \right\}$$

where $\ast$ denotes the convolution in $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ denote the sets of tempered distributions and of polynomials on $\mathbb{R}^n$, respectively. We refer to [1], [6], [20] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity we make abbreviations such as $H^s = H^s_2$, $\dot{H}^s = \dot{H}^s_2$, $B^s_r = B^s_{r, 2}$, $\dot{B}^s_r = \dot{B}^s_{r, 2}$. For any Banach spaces $X$ and $Y$ having a common dense subspace, we put

$$\| a; X \cap Y \| \equiv \max(\| a; X \|, \| a; Y \|)$$

for any $a \in X \cap Y$. For any interval $I \subset \mathbb{R}$ and any Banach space $X$ we denote by $C(I; X)$ the space of strongly continuous functions from $I$ to $X$, by $L^q(I; X)$ (or by $L^q X$, for simplicity) the space of measurable functions $u$ from $I$ to $X$ such that $\| u(\cdot); X \| \in L^q(I)$. For any $r$ with $1 \leq r \leq \infty$, $r'$ is the exponent dual to $r$ defined by $1/r + 1/r' = 1$. For $a, b \in \mathbb{R}$ we denote by $a \vee b$ and $a \wedge b$ the maximum and minimum of $a$ and $b$, respectively, and we denote by $a \lesssim b$ the inequality $a \leq Cb$ with some constant $C > 0$ which is independent on other constants and variables in question.
The behavior of nonlinearity $f$ is described by the following assumptions $N(s, p)$ and $N(s, p_1, p_2)$ with $s \geq 0$, $1 \leq p, p_1, p_2 < \infty$.

$N(s, p)$: $f \in C^s(\mathbb{C}; \mathbb{C})$, $f^{(j)}(z) = O(|z|^{p-j})$ for $0 \leq j \leq |s| \wedge p$.

Moreover $f^{(j)}(z) = 0$ for $p < j \leq |s|$ if $p < |s|$.

\[
|f^{(j)}(z)| \leq \begin{cases} 
\frac{1}{1} |z|^{p-s} & \text{if } s + 1 \leq p, \\
|z|^{p-s} & \text{if } s < p < |s| + 1, \\
0 & \text{if } p \leq s.
\end{cases}
\]

$N(s, p_1, p_2)$: $f$ is written as a finite sum $\sum_{j=1}^{\ell} f_j$ with $f_j$ satisfying $N(s, p_j^s)$ for some $p_j^s$ with $p_1 \leq p_j^s \leq p_2$.

In this paper, $\tilde{H}^s$, $\tilde{\mathcal{H}}^s$, and $\tilde{\mathcal{H}}^t$ denote the class of solutions, the class of data, and the class in which the data is needed to be small, respectively, which are defined by the following correspondence.

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{H}^s$</th>
<th>$\tilde{\mathcal{H}}^s$</th>
<th>$\tilde{\mathcal{H}}^t$</th>
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<tbody>
<tr>
<td>NLS</td>
<td>$H^s$</td>
<td>$H^s$</td>
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<tr>
<td>NLW</td>
<td>$H^{1/2} \cap \tilde{H}^s$</td>
<td>$H^{1/2} \cap \tilde{H}^s$</td>
<td>$\tilde{\mathcal{H}}^s$</td>
</tr>
<tr>
<td>NLKG</td>
<td>$H^s$</td>
<td>$\tilde{\mathcal{H}}^s$</td>
<td>$\tilde{\mathcal{H}}^t$</td>
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Here $\tilde{\mathcal{H}}^s = H^s \oplus H^{s-1}$ and $\tilde{\mathcal{H}}^s = H^s \oplus H^{s-1}$. We denote by $\tilde{\phi}$ the elements of $\tilde{\mathcal{H}}^s$ with $\tilde{\phi} = \tilde{\phi} = u(0)$ for NLS, $\tilde{\phi} = (\phi, \psi) = (u(0), \partial_t u(0))$ for NLW and NLKG. Accordingly, we define $\tilde{u}(t) = u(t)$ for NLS and $\tilde{u}(t) = (u(t), \partial_t u(t))$ for NLW and NLKG. For $s, s_1, s_2 \in \mathbb{R}$ and $\gamma > 0$, we put

\[
A^\gamma(s_1, s_2, \gamma) = \{ \tilde{\phi} \in \tilde{\mathcal{H}}^s \mid \|\tilde{\phi}; \tilde{\mathcal{H}}^{s_1} \cap \tilde{\mathcal{H}}^{s_2}\| \leq \gamma \},
\]

so that the first index $s$ denotes the required regularity of the data and the next two indices $s_1$ and $s_2$ are related to the smallness of the data, which is realized by the norm of $\tilde{\mathcal{H}}^{s_1} \cap \tilde{\mathcal{H}}^{s_2}$ of size less than or equal to $\gamma$. Let $\omega \equiv -\Delta$ for NLS, $\omega \equiv \sqrt{-\Delta}$ for NLW, $\omega \equiv \sqrt{1-\Delta}$ for NLKG. Let $U(t) \equiv \exp(it\omega)$. For $\tilde{\phi} \in \tilde{\mathcal{H}}^s$, let

\[
\Gamma(t)\tilde{\phi} = \begin{cases} 
U(t)\phi & \text{for NLS}, \\
(\cos t\omega)\phi + \frac{\sin t\omega}{\omega}\psi & \text{for NLW and NLKG}.
\end{cases}
\]

For $t_0, t \in \mathbb{R}$, let

\[
\widetilde{G}_{t_0} h(t) = \begin{cases} 
-i \int_{t_0}^t U(t - \tau) h(\tau) d\tau & \text{for NLS}, \\
\int_{t_0}^t \frac{\sin(t - \tau)\omega}{\omega} h(\tau) d\tau & \text{for NLW and NLKG}.
\end{cases}
\]

Then NLS, NLW and NLKG are rewritten by the following integral equation

\[
\text{INT} \quad u(t) = \Phi(u)(t) \equiv \Gamma(t)\tilde{\phi} + \widetilde{G}_{t_0}(f(u))(t).
\]
The above integral equation is considered in the following auxiliary spaces. For \( s \in \mathbb{R} \), we put

\[
X^s = L^\infty \tilde{H}^s \cap \left\{ \begin{aligned}
L^{q_0} B^{s}_{q_0} & , \\
L^{q_1} (\dot{B}^{s-1/2}_{q_1} \cap \dot{B}^{0}_{q_1} ) & , \\
\cap_{q_0 \leq q \leq q_1} L^{q} B^{s-1/2}_{q} &
\end{aligned} \right\}
\]

endowed with the metric \( d \) given by

\[
d(u, v) = \left\{ \begin{aligned}
\| u - v ; X^0 \| & for \text{ NLS} , \\
\| u - v ; X^{1/2} \| & for \text{ NLW} \text{ and NLKG} .
\end{aligned} \right.
\]

For \( 1 \leq p < \infty \), let \( s(p) \) be the critical regularity associated with the power \( p \), given by

\[
s(p) = \left\{ \begin{aligned}
\frac{n}{2} - 2/(p-1) & for \text{ NLS and NLW} , \\
1/2 \lor (n/2 - 2/(p-1)) & for \text{ NLKG} .
\end{aligned} \right.
\]

With the notation above, our main results read as:

**Theorem 1.1.** Let \( n, s, p_1, p_2 \) satisfy

\[
\begin{align*}
n & \geq 1, \quad 0 \leq s < n/2, \quad s_0 = 0, \quad p(0) \leq p_1 \leq p_2 \leq p(s) \quad for \text{ NLS} , \\
n & \geq 2, \quad 1/2 \leq s < n/2, \quad s_0 = 1/2, \quad p(1/2) \leq p_1 \leq p_2 \leq p(s) \quad for \text{ NLW} , \\
n & \geq 2, \quad 1/2 \leq s < n/2, \quad s_0 = 1/2, \quad p(0) \leq p_1 \leq p_2 \leq p(s) \quad for \text{ NLKG} .
\end{align*}
\]

Let \( f \) satisfy \( N(s-s_0, p_1, p_2) \). Then there exists \( \gamma > 0 \) with the following property.

1. For any data \( \tilde{\phi} \in A^s(s(p_1), s(p_2), \gamma) \), INT has a unique global solution \( u \) in \( C(\mathbb{R}; \tilde{H}^s) \cap X^s \).
2. There exists unique two states \( \tilde{\phi}_+, \tilde{\phi}_- \) in \( \tilde{H}^s \) such that

\[
\| \tilde{u}(t) - \Gamma(t) \tilde{\phi}_\pm ; \tilde{H}^s \| \to 0 
\]

as \( t \to \pm \infty \). More precisely,

\[
\left\| u(t) - U(t) \phi_\pm ; H^s \right\| \to 0 \quad for \text{ NLS} ,
\]

\[
\left\| \left( u(t) - (\cos t \omega) \phi_\pm - \frac{\sin t \omega}{\omega} \psi_\pm \right) , \partial_t \left( u(t) - (\cos t \omega) \phi_\pm - \frac{\sin t \omega}{\omega} \psi_\pm \right) ; \tilde{H}^{1/2} \cap \tilde{H}^s \right\| \to 0 \quad for \text{ NLW} ,
\]

\[
\left\| \left( u(t) - (\cos t \omega) \phi_\pm - \frac{\sin t \omega}{\omega} \psi_\pm \right) , \partial_t \left( u(t) - (\cos t \omega) \phi_\pm - \frac{\sin t \omega}{\omega} \psi_\pm \right) ; H^s \right\| \to 0 \quad for \text{ NLKG}
\]

with \( \tilde{\phi}_\pm = (\phi_\pm, \psi_\pm) \) for \text{ NLW and NLKG}.
(3) For any data $\phi_-$ in $A^s(s(p_1), s(p_2), \gamma)$, there exists a unique global solution $u$ of \textbf{INT} in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$ and a unique state $\phi_+ \in \tilde{H}^s$ such that (1.5) holds. Moreover the scattering operator $S : \phi_- \mapsto \phi_+$ is well-defined on $A^s(s(p_1), s(p_2), \gamma)$ to $\tilde{H}^s$ and is continuous in the following sense. If $\{\tilde{\phi}_j^j\}_{j=1}^{\infty} \subset A^s(s(p_1), s(p_2), \gamma)$ satisfies $\|\tilde{\phi}_- - \tilde{\phi}_+ \| \tilde{H}^{s_0} \rightarrow 0$ as $j \rightarrow \infty$, then

\begin{equation}
\|\tilde{\phi}_+ - \tilde{\phi}_+ \| \rightarrow 0
\end{equation}

as $j \rightarrow \infty$ for any $\mu$ with $s_0 \leq \mu < s(p_2)$ where $\tilde{\phi}_+^j = S(\tilde{\phi}_j^j)$.

(4) In parts (1) and (3), the auxiliary space $X^s$ is removable if $p_2$ satisfies

\begin{equation}
\begin{cases}
p_2 \leq 1 + (1 + 2s)/(1 - 2s) & \text{for } n = 1, \\
p_2 < 1 + (1 + s)/(1 - s) & \text{for } n = 2, \\
p_2 \leq 1 + (2n + 4(n - 1)s)/n(n - 2s) & \text{for } n \geq 3 \text{ and } 0 \leq s < n/(2(n - 1)), \\
p_2 < 1 + 4(n - 1)/n(n - 2) & \text{for } n \geq 3 \text{ and } s = n/2(n - 1), \\
p_2 \leq 1 + 4/(n - 2s) & \text{for } n \geq 3 \text{ and } n/2(n - 1) < s < n/2
\end{cases}
\end{equation}

for \textbf{NLS},

\begin{equation}
p_2 \leq 1 + 4n/(n + 1)(n - 2s)
\end{equation}

for \textbf{NLW} and \textbf{NLKG}. In particular, uniqueness of solutions holds in $C(I; \tilde{H}^s)$ for any interval $I$ in $\mathbb{R}$ without smallness condition on the data if $p_2 \neq p(s)$.

**Proposition 1.1.** Let us consider \textbf{NLW} and \textbf{NLKG}. Let $n \geq 3$. Let

$$1 \leq s < 3/2, \text{ for } n = 3, \quad 2/3 \leq s < 2 \text{ for } n = 4, \quad (n - 2)/(n + 1) \leq s < n/2 \text{ for } n \geq 5.$$  

Then the state (4) in Theorem 1.1 holds with (1.8) replaced by $p_2 < p(s)$. In particular, uniqueness of solutions holds in $C(I; \tilde{H}^s)$ for any interval $I$ in $\mathbb{R}$ without smallness condition on the data.

**Remark 1.** The assumptions of the theorem above cover for instance nonlinearities of the form

- $cu^p$ with $p \in \mathbb{Z}_+$,
- $c|u|^{p-1}u$ with $p = 2m + 1$ and $m \in \mathbb{Z}_+$,
- $c|u|^p$ and $c|u|^{p-1}u$ with $p > s - s_0$,

where $c \in \mathbb{C}$, $p(0) \leq p \leq p(s)$ for \textbf{NLS} and \textbf{NLKG}, $p(1/2) \leq p \leq p(s)$ for \textbf{NLW}, and

- $c|u|^p \log |u|$, $c|u|^{p-1}u \log |u|$ with $p > [s - s_0] + 1$. 
where $p(0) < p < p(s)$ for NLS and NLKG, $p(1/2) < p < p(s)$ for NLW, and any finite sum of the above nonlinearities. In particular, for NLS, the quintic correction of cubic interaction

$$f(u) = c_1|u|^2u + c_2|u|^4u, \quad c_1, c_2 \in \mathbb{C}$$

falls within the scope with $n \geq 2$, $(n - 1)/2 \leq s < n/2$. The logarithmic correction of cubic interaction (see Section 3.5 in [17])

$$f(u) = c_1|u|^2u + c_2|u|^2u \log |u|^2,$$

also, falls within the scope with $3 \leq n \leq 5$, $(n - 2)/2 < s < 2 \wedge (n/2)$. On the other hand, the nonlinearity $f(u) = cu \log |u|^2$ requires a special treatment (see [2], [3]).

Remark 2. The regularity assumptions $C^{[s]}$ in [7] and $C^x$ in [22] for $f$, described above, are refined in the theorem as $C^{[s-s_0]}$ with the property (1.3). For NLKG, $p_1$ and $p_2$ can be taken in the full range $p(0) \leq p_1 \leq p_2 \leq p(s)$ as compared to $p(1/2) \leq p_1 \leq p_2 \leq p(s)$ in [22].

Remark 3. As for NLKG for $n = 1$, we have shown in [12] the scattering theory for small data at the level of $H^{1/2}$ for nonlinearities with exponential growth at infinity such as $f(u) = c|u|^{p(0)} \exp(v|u|^2)$, $c \in \mathbb{C}$, $v > 0$.

Acknowledgments. We are grateful to S. Machihara and K. Nakanishi for useful comments on the Strichartz estimates in Section 2.

2. – Strichartz estimates

Let $\lambda_j, \sigma_j, j = 1, 2$, satisfy

$$\lambda_1 = \lambda_2 = 0, \quad \sigma_1 = \sigma_2 = n$$

for NLS,

$$(2.10) \lambda_1 = \lambda_2 = (n + 1)/2, \quad \sigma_1 = \sigma_2 = n - 1$$

for NLW,

$$(n + 1)/2 \leq \lambda_j \leq (n + 2)/2, \quad 2n - 2\lambda_j \leq \sigma_j \leq 2\lambda_j - 2, \quad \sigma_j > 0, j = 1, 2,$$

for NLKG.

Our purpose in this section is to show the estimates

$$\|Gh; L^q(\mathbb{R}; \vec{B}_{\tau}^{-\lambda_1\alpha(\tau)})\| \lesssim \|h; L^{\tilde{q}}(\mathbb{R}; \vec{B}_{\tau}^{-\lambda_2\alpha(\tilde{\tau})})\|,$$

(2.11)

$$Gh(t) \equiv \int J U(t - \tau)h(\tau)d\tau$$
for various $r, q, \tilde{r}$ and $\tilde{q}$ in $[1, \infty)$, where $\alpha(r) \equiv 1/2 - 1/r$ and $J$ denotes an interval in $\mathbb{R}$ of the form $[0, t]$, $[t, \infty)$ and $(-\infty, t]$, and $\tilde{B}_r^s$ denotes $\tilde{B}_r^s$ for NLS and NLW, $\tilde{B}_r^s$ for NLKG. For that purpose we first recall a basic reasoning of the problem. We start from the following well-known decay estimate

(2.12) \[ \| U(t)\phi; \tilde{B}_\infty^{-\lambda_1/2} \| \lesssim |t|^{-\sigma_1/2} \| \phi; \tilde{B}_1^{\lambda_1/2} \|. \]

Interpolating the energy estimate and (2.12) and applying the Hardy-Littlewood-Sobolev inequality in the time variable, we have

(2.13) \[ \left\| \int f U(\cdot - \tau)h(\tau)d\tau; L^q \tilde{B}_r^{-\lambda_1\alpha(r)} \right\| \lesssim \| h; L^q \tilde{B}_r^{-\lambda_1\alpha(r')} \|. \]

Applying the bilinear estimate associated with (2.13), we also have

(2.14) \[ \left\| \int f U(\cdot - \tau)h(\tau)d\tau; L^\infty \tilde{B}_2^0 \right\| \lesssim \| h; L^q \tilde{B}_r^{-\lambda_1\alpha(r')} \| \]

for $1 \leq r, q \leq \infty$ with $1/r + 2/\sigma_1 q = 1/2$ and $1/2 - 1/\sigma_1 < 1/r \leq 1/2$. Considering the dual operator $U'$ of $U$ in $\mathbb{R}^{1+n}$ of the form $U'h \equiv \int_\mathbb{R} U(-\tau)h(\tau)d\tau$, (2.14) yields

(2.15) \[ \| U(\cdot)\phi; L^q \tilde{B}_r^{-\lambda_1\alpha(r)} \| \lesssim \| \phi; \tilde{B}_2^0 \|. \]

Interpolating (2.13) and (2.14) and applying the duality argument, we obtain the following well-known lemma (see [5], [8], [22] and references therein).

**Lemma 2.1.** Let $1 \leq r, q, \tilde{r}, \tilde{q} \leq \infty$ satisfy

\[
\frac{1}{r} + \frac{2}{\sigma_1 q} = \frac{1}{2}, \quad \frac{1}{\tilde{r}} + \frac{2}{\sigma_1 \tilde{q}} = \frac{1}{2} + \frac{2}{\sigma_1},
\]

\[
\frac{1}{2} - \frac{1}{\sigma_1} < \frac{1}{r} \leq \frac{1}{2} < \frac{1}{\tilde{r}} = \frac{1}{2} + \frac{1}{\sigma_1}.
\]

Then (2.11) and (2.15) hold with $\lambda_1 = \lambda_2$.

To use the bilinear estimate, we consider the sequence

\[
P(h, g) \equiv \left\{ \left\langle \int_{f \cap I_{t,j}} U(t - \tau)h(\tau)d\tau, g(t) \right\rangle_{t,x} \right\}_{j = -\infty}^{j = \infty}, \quad I_{t,j} \equiv [t + 2^j, t + 2^{j + 1}],
\]

in the following lemma. For $r, q, \tilde{r}$ and $\tilde{q}$, we put

\[
\beta(r, q, \tilde{r}, \tilde{q}) \equiv \frac{\sigma_1}{2} \left( \frac{1}{r} + \frac{2}{\sigma_1 q} - \frac{1}{2} \right) - \frac{\sigma_2}{2} \left( \frac{1}{\tilde{r}} + \frac{2}{\sigma_2 \tilde{q}} - \frac{1}{2} - \frac{2}{\sigma_2} \right).
\]

For $x \in \mathbb{R}$, $x \sim 0$ means that $x$ is sufficiently close to 0.
Lemma 2.2. Let \( r, q, \tilde{r}, \tilde{q} \) satisfy \( q \geq \tilde{q} \),
\[
1/r + 2/(\sigma_1 + \sigma_2)q < 1/2, \quad \beta(r, q, \tilde{r}, \tilde{q}) \sim 0,
\]
(2.16)
\[
1/r' \leq 1/\tilde{r}\begin{cases}
< 1 - (\sigma_2 - 2)/\sigma_2 r & \text{if } \sigma_2 \geq 2, \\
\leq 1 & \text{if } \sigma_2 < 2.
\end{cases}
\]
Then the following estimate holds.

(2.17)
\[
\left\| \int_{J \cap I_{t,j}} U(t - \tau)h(\tau)d\tau, g \right\|_{t,x} \lesssim 2^{\beta(r,q,\tilde{r},\tilde{q})} \|h; L^q B^{-\lambda_2 \alpha(\tilde{r})}_{\tilde{r}}\| g; L^{q'} B^{-\lambda_1 \alpha(r')}_{r'}.
\]

Proof of Lemma 2.2. By the decay estimate (2.12) with \( \sigma_1 \mapsto (\sigma_1 + \sigma_2)/2 \) and \( \lambda_1 \mapsto (\lambda_1 + \lambda_2)/2 \), we have

(2.18)
\[
\left\| \int_{J \cap I_{t,j}} U(t - \tau)h(\tau)d\tau; B^{-\lambda_1/2}_{\tilde{r}} \right\| \lesssim 2^{-(\sigma_1 + \sigma_2)/4} \|h; L^1(I_{t,j}; B^{-\lambda_2/2}_{\tilde{r}})\|.
\]

Let \( 1/\tilde{r}_0 = 1 - r/2\tilde{r}' \). Then by (2.14) we have

(2.19)
\[
\left\| \int_{J \cap I_{t,j}} U(t - \tau)h(\tau)d\tau; B^0_{\tilde{r}_0} \right\| \lesssim \|h; L^{\tilde{q}_0}(I_{t,j}; B^{-\lambda_2(\tilde{r}_0)})\|,
\]
where \( \tilde{q}_0 \) is given by \( 1/\tilde{r}_0 + 2/\sigma_2 \tilde{q}_0 = 1/2 + 2/\sigma_2 \). Interpolating (2.18) and (2.19), we have

(2.20)
\[
\left\| \int_{J \cap I_{t,j}} U(t - \tau)h(\tau)d\tau; B^{-\lambda_1 \alpha(r)}_{\tilde{r}} \right\| \lesssim 2^{-(\sigma_1 + \sigma_2)\alpha(r)/2} \|h; L^{\ell}(I_{t,j}; B^{-\lambda_2 \alpha(\tilde{r})}_{\tilde{r}})\|,
\]
where \( 1/\ell \equiv 1 - \sigma_2(1/r + 1/\tilde{r} - 1)/2 \). Therefore by the Hölder inequality, we have

\[
\left\| \int_{J \cap I_{t,j}} U(t - \tau)h(\tau)d\tau, g \right\|_{t,x} \lesssim 2^{-(\sigma_1 + \sigma_2)\alpha(r)/2} \|h; L^{\ell}(I_{t,j}; B^{-\lambda_2 \alpha(\tilde{r})}_{\tilde{r}})\| g; L^{q'} B^{-\lambda_1 \alpha(r')}_{r'}.
\]

Applying the Hölder and Minkowski inequalities to the norm of \( h \) by \( q \geq \tilde{q} \), we have

\[
\|h; L^{\ell}(I_{t,j}; B^{-\lambda_2 \alpha(\tilde{r})}_{\tilde{r}})\| L^q_t \| \lesssim 2^{(1/\ell - 1/\tilde{q} + 1/q)/2} \|h; L^{\tilde{q}} B^{-\lambda_2 \alpha(\tilde{r})}_{\tilde{r}}\|
\]
if \( \tilde{q} \geq \ell \). Then we have the required results. The condition \( \tilde{q} \geq \ell \) is satisfied by (2.16). \( \square \)
Lemma 2.2 is rewritten as the following corollary by the duality argument.

**Corollary 2.1.** Let $r, q, \tilde{r}, \tilde{q}$ satisfy $q \geq \tilde{q}$,
\[
1/\tilde{r} + 2/(\sigma_1 + \sigma_2)\tilde{q} > 1/2 + 2/(\sigma_1 + \sigma_2), \quad \beta(r, q, \tilde{r}, \tilde{q}) \sim 0,
\]
\[
1/\tilde{r} \leq 1/r' \begin{cases} < 1 - (\sigma_1 - 2)/\sigma_1 \tilde{r}' & \text{if } \sigma_1 \geq 2, \\ \leq 1 & \text{if } \sigma_1 < 2. \end{cases}
\]

Then the estimate (2.17) holds.

To show Proposition 2.1, below, we recall the following bilinear interpolation due to O’Neil.

**Lemma 2.3** [1, Section 3.13.5 (b)], [9, Lemma 6.1]. Let $A_j, B_j, C_j, j = 0, 1,$ be Banach spaces. Let $T$ be a bilinear operator with boundedness properties as
\[
T : \begin{cases} A_0 \times B_0 \to C_0, \\ A_0 \times B_1 \to C_1, \\ A_1 \times B_0 \to C_1. 
\end{cases}
\]

Let $\theta_0, \theta_1, r_0, r_1, r$ satisfy $0 < \theta_0, \theta_1, \theta_0 + \theta_1 < 1, 1 \leq r_0, r_1, r \leq \infty, 1 \leq 1/r_0 + 1/r_1$. Then $T$ is a bounded bilinear operator as
\[
T : (A_0, A_1)_{\theta_0, r_0} \times (B_0, B_1)_{\theta_1, r_1} \to (C_0, C_1)_{\theta_0 + \theta_1, r},
\]
where $(A_0, A_1)_{\theta, r}$ denotes the real interpolation space.

**Proposition 2.1.** Let $r, q, \tilde{r}, \tilde{q}$ satisfy $q > 1, \tilde{q} \leq q, \beta(r, q, \tilde{r}, \tilde{q}) = 0$,
\[
1/r + 2/(\sigma_1 + \sigma_2)q < 1/2 \quad \text{if} \quad 1/r' \leq 1/\tilde{r} \leq (1 - (\sigma_2 - 2)/\sigma_2 r) \land 1,
\]
\[
1/\tilde{r} + 2/(\sigma_1 + \sigma_2)\tilde{q} > 1/2 + 2/(\sigma_1 + \sigma_2)
\]
\[
\text{if} \quad 1/\tilde{r} \leq 1/r' \leq (1 - (\sigma_1 - 2)/\sigma_1 \tilde{r}') \land 1,
\]
where both conditions in (2.21) are assumed when $r' = \tilde{r}$. Then (2.11) holds.

**Proof of Proposition 2.1.** Let $r, q, \tilde{r}, \tilde{q}$ satisfy the assumptions in the proposition, and let $r_*$ and $\tilde{r}_*$ be sufficiently close to $r$ and $\tilde{r}$, respectively. Then by Lemma 2.2 and Corollary 2.1, we have
\[
P : L^\tilde{q} B_{\tilde{r}_*}^{-\lambda_2 / (\tilde{r}_*)} \times L^q B_{r_*}^{-\lambda_1 / (r_*)} \to L^* B_{\tilde{r}}^{-\beta(r_*, q, \tilde{r}_*, \tilde{q})},
\]
where $\ell^* = L^q(\mathbb{Z}, 2^sj \, dj)$ for $s \in \mathbb{R}, 1 \leq q \leq \infty$, where $d_j$ denotes the pointmass 1 at $j$. Applying Lemma 2.3 with $r_0 = r_1 = 2, r = 1$ to (2.22), and using the facts that
- $(L^p, L^p)_{\theta, r} = L^p$ for $1 \leq p, r \leq \infty$,
- $(B^0_{p_0}, B^1_{p_1})_{\theta, 2} \leftrightarrow (B^0_{p_0}, B^1_{p_1})_{\theta, p} = B^s_p$
  for $s_0, s_1 \in \mathbb{R}, 1 \leq p_0, p_1 \leq 2, s = (1 - \theta)s_0 + \theta s_1, 1/p = (1 - \theta)/p_0 + \theta/p_1$,
- $(\ell^s_{\infty}, \ell^s_1)_{\theta, 1} = \ell^s_1$ for $s = (1 - \theta)s_0 + \theta s_1, s_0 \neq s_1$, $0 < \theta < 1$ (see [1, Theorems 3.4.1, 6.45. 5.6.1]), we have
\[
P : L^\tilde{q} B_{\tilde{r}}^{-\lambda_2 / (\tilde{r})} \times L^q B_{r'}^{-\lambda_1 / (r')} \to \ell^0_1.
\]
Since $q', r' \neq \infty$, we obtain (2.11). □
The next corollary follows from the above proposition and Lemma 2.1 immediately.

**Corollary 2.2.** Let \( r, q, \tilde{r}, \tilde{q} \) satisfy
\[
1/r + 2/\sigma_1 q = 1/2, \quad \sigma_1/2(\sigma_1 + 2) \leq 1/r \leq 1/2, \quad 1/\tilde{r} = 1/\tilde{q} = (\sigma_2 + 4)/2(\sigma_2 + 2).
\]
Then (2.11) and (2.15) hold.

We put \( a, b \) as follows.
\[
a = (\sigma_1 + \sigma_2 - 4)/2(\sigma_1 + \sigma_2 - 2), \quad b = (\sigma_1 + \sigma_2 - 4)/2(\sigma_1 + \sigma_2).
\]

**Corollary 2.3.** Let \( \sigma_1, \sigma_2, r, q, \tilde{r}, \tilde{q} \) satisfy
\[
\begin{align*}
\sigma_1, \sigma_2 & \geq 2, \quad 1/r' \leq 1/\tilde{r}, \quad b \leq 1/r < 1/2, \\
1/r + 2/\sigma_1 q & = 1/2, \quad 1/\tilde{r} + 2/\sigma_2 \tilde{q} = 1/2 + 2/\sigma_2, \\
1/\tilde{r} & \begin{cases} < 1 - (\sigma_2 - 2)/\sigma_2 r \quad \text{if } a \leq 1/r, \\ \leq 1/2 + 2/\sigma_2 - \sigma_1 \alpha(r)/\sigma_2 \quad \text{if } 1/r < a. \end{cases}
\end{align*}
\]
Then (2.11) holds. Moreover \( q = \tilde{q} \) if \( 1/\tilde{r} \) equals to the RHS of the last inequality, and \( q > \tilde{q} \) otherwise.

**Proof of Corollary 2.3.** The condition \( b \leq 1/r \) is needed for the existence of \( \tilde{r} \) in (2.25) when \( 1/r < a \). The conditions \( \beta(r, q, \tilde{r}, \tilde{q}) = 0 \) and (2.21) follow from (2.24). The index \( r \) which satisfies (2.24) with \( q = \tilde{q} \) and \( 1/\tilde{r} = 1 - (\sigma_2 - 2)/\sigma_2 r \) is given by \( 1/r = a \). If \( a \leq 1/r \) and \( 1/\tilde{r} < 1 - (\sigma_2 - 2)/\sigma_2 r \), then \( q > \tilde{q} \). If \( 1/r < a \) and \( 1/\tilde{r} \leq 1/2 + 2/\sigma_2 - \sigma_1 \alpha(r)/\sigma_2 \), then \( q \geq \tilde{q} \). □

**Corollary 2.4.** Let \( 2 < \sigma_2 \leq \sigma_1 \). Let \( r, q, \tilde{r}, \tilde{q} \) be given by
\[
(1/r, 1/q) = (1/2 - 1/\sigma_1, 1/2), \quad (1/\tilde{r}, 1/\tilde{q}) = (1/2 + 1/\sigma_2, 1/2).
\]
Then (2.11) holds.

**Proof of Corollary 2.4.** Since \( r \) satisfies \( b \leq 1/r < a \) by \( 2 < \sigma_2 \leq \sigma_1 \) and \( \tilde{r} \) satisfies \( 1/\tilde{r} = 1/2 + 2/\sigma_2 - \sigma_1 \alpha(r)/\sigma_2 \), (2.11) follows from Corollary 2.3. □

Applying the duality argument described in the proof of Lemma 2.1 to Corollary 2.4 with \( \sigma_1 = \sigma_2 \), we have the following endpoint estimates for Lemma 2.1.

**Corollary 2.5.** Let \( \sigma_1 > 2 \). Let \( r, q, \tilde{r} \) and \( \tilde{q} \) satisfy
\[
\begin{align*}
1/r + 2/\sigma_1 q & = 1/2, \quad 1/\tilde{r} + 2/\sigma_1 \tilde{q} = 1/2 + 2/\sigma_1, \\
1/2 - 1/\sigma_1 & \leq 1/r \leq 1/2 \leq 1/\tilde{r} \leq 1/2 + 1/\sigma_1.
\end{align*}
\]
Then (2.11) and (2.15) hold with \( \lambda_1 = \lambda_2 \).
Remark 4. Corollary 2.5 has been shown in [9] for Schrödinger and wave equations, and is applied to show the local well-posedness of NLW with

\begin{equation}
\tag{2.26}
|f^{(j)}(u)| = O(|u|^{p-j}) \quad j = 0, 1, \quad p = 1 + 4(n-1)/(n^2 - 2n + 5),
\end{equation}

where \( r, q, \tilde{r}, \tilde{q} \) are taken as

\begin{equation}
1/r = 1/2 + 1/(n-1), \quad 1/\tilde{q} = 1/2, \quad 1/r = 1/\tilde{r} - 2/(n+1), \quad 1/q = (n-1)\alpha(r)/2.
\end{equation}

By Corollary 2.5, we can show the local well-posedness of NLKG with (2.26) analogously.

**Corollary 2.6.** The conclusion of Corollary 2.3 holds with (2.23) replaced by the following conditions

\( \sigma_1 > 2, \sigma_2 \geq 2, 1/2 \leq 1/\tilde{r}, b \vee (1/2 - 1/\sigma_1) \leq 1/r < 1/2. \)

**Proof of Corollary 2.6.** By Corollary 2.5, we have

\[ \| Gh; L^q B^{-\lambda_1 \alpha(r)}_r \| \lesssim \| h; L^1 \tilde{B}^0_2 \|. \]

Interpolating this estimate and the conclusion of Corollary 2.3, we may replace

\[ 1/r' \leq 1/\tilde{r} \] in (2.23) with \( 1/2 \leq 1/\tilde{r}. \)

In the following lemma we show (2.11) for \( 1/\tilde{r} = 1 - (\sigma_2 - 2)/\sigma_2 r \) when \( a < 1/r \) in Corollary 2.3. However, we were not able to obtain the estimate when \( a = 1/r. \)

**Lemma 2.4.** Let \( \sigma_1 > 2. \) Let \( r, q, \tilde{r}, \tilde{q} \) satisfy \( \beta(r, q, \tilde{r}, \tilde{q}) = 0, \)

\begin{equation}
\tag{2.27}
(\sigma_1 - 2)/\sigma_1 \tilde{r}' \leq 1/r \leq 1/\tilde{r}',
\end{equation}

\[ 1/\tilde{r} + 2/(\sigma_1 + \sigma_2)\tilde{q} > 1/2 + 2/(\sigma_1 + \sigma_2), \quad 0 \leq 1/q < 1/\tilde{q} < 1. \]

Then (2.11) holds.

**Proof of Lemma 2.4.** Let \( r_*, q_* \) be given by

\[ 1/r_* = \tilde{r}'/2r, \quad 1/r_* + 2/\sigma_1 q_* = 1/2. \]

By (2.12) and (2.15), we have

\begin{equation}
\tag{2.28}
\| t^{(\sigma_1 + \sigma_2)/4} U(t) \phi; L^\infty (\tilde{B}^{-\lambda_1/2}_0, dw) \| \lesssim \| \phi; \tilde{B}^{\lambda_2/2}_1 \|,
\end{equation}

\begin{equation}
\tag{2.29}
\| t^{(\sigma_1 + \sigma_2)/4} U(t) \phi; L^{q_*} (\tilde{B}^{-\lambda_1 \alpha(r_*)}_0, dw) \| \lesssim \| \phi; \tilde{B}_2^0 \|,
\end{equation}

where \( dw \) is the measure given by

\[ dw(t) = |t|^{-(\sigma_1 + \sigma_2)q_*/4} dt. \]
and \( L^p(X, dw) \) denotes the weighted Lebesgue space given by

\[
\|u; L^p(X, dw)\| = \left\{ \int \|u(t); X\|^p dw(t) \right\}^{1/p}.
\]

Interpolating (2.28) and (2.29), we have

\[
\| |t|^{(\sigma_1 + \sigma_2)/4} U(t) \phi; L^\ell (\bar{B}_r^{-\lambda_1 \alpha(r)}, dw) \| \lesssim \| \phi; \bar{B}_r^{-\lambda_2 \alpha(\bar{r})} \|,
\]

which is rewritten by

\[
(2.30) \quad \| |t|^{\gamma} U(t) \phi; L^\ell \bar{B}_r^{-\lambda_1 \alpha(r)} \| \lesssim \| \phi; \bar{B}_r^{-\lambda_2 \alpha(\bar{r})} \|,
\]

where \( 1/\ell \equiv \sigma_1(1 - 1/\bar{r} - 1/r)/2, \gamma \equiv -\sigma_1 + \sigma_2)\alpha(\bar{r})/2 \). Applying (2.30), we have

\[
\langle Gh, g \rangle = \int ds \int d\tau \langle U(\tau) h(s), g(\tau + s) \rangle
\]

\[
\lesssim \| h; L^{\tilde{q}} \bar{B}_r^{-\lambda_2 \alpha(\bar{r})} \| \cdot \| \tau^{\gamma} g(\tau + s); L^\ell \bar{B}_r^{-\lambda_1 \alpha(r')} \|; L^{\tilde{q}'}\| .
\]

If \( \bar{q}, \ell \) satisfy

\[
(2.31) \quad 0 < \ell'/\bar{q}' < \ell'/q' < 1 ,
\]

then applying the Hardy-Littlewood-Sobolev inequality with \( \beta(r, q, \bar{r}, \bar{q}) = 0, \) we have

\[
\langle Gh, g \rangle \lesssim \| h; L^{\tilde{q}} \bar{B}_r^{-\lambda_2 \alpha(\bar{r})} \| \| g; L^{q'} \bar{B}_r^{-\lambda_1 \alpha(r')} \|.
\]

Therefore we obtain (2.11). The condition (2.31) is satisfied by (2.27).

Applying the duality argument to Lemma 2.4, we have the following Corollary.

**Corollary 2.7.** Let \( \sigma_2 > 2 \). Let \( r, q, \bar{r}, \bar{q} \) satisfy \( \beta(r, q, \bar{r}, \bar{q}) = 0, \bar{q} < q < \infty, \)

\[
1/r' \leq 1/\bar{r} \leq 1 - (\sigma_2 - 2)/\sigma_2 r, \quad 1/r + 2/(\sigma_1 + \sigma_2) q < 1/2 .
\]

Then (2.11) holds.

Combining Corollary 2.6 with Corollary 2.7, we obtain the following corollary.

**Corollary 2.8.** Let \( \sigma_1 > 2, \sigma_2 > 2 \). Let \( r, q, \bar{r}, \bar{q} \) satisfy

\[
1/r + 2/\sigma_1 q = 1/2, \quad 1/\bar{r} + 2/\sigma_2 \bar{q} = 1/2 + 2/\sigma_2, \quad b \vee ((\sigma_1 - 2)/2\sigma_1) \leq 1/r < 1/2 ,
\]

\[
\begin{align*}
1/2 \leq 1/\bar{r} \quad \left\{ \begin{array}{ll}
\leq 1 - (\sigma_2 - 2)/\sigma_2 r & \text{if } a < 1/r, \\
< 1 - (\sigma_2 - 2)/\sigma_2 r & \text{if } 1/r = a, \\
\leq 1/2 + 2/\sigma_2 - \sigma_1 \alpha(r)/\sigma_2 & \text{if } 1/r < a .
\end{array} \right.
\end{align*}
\]

Then (2.11) holds. Here \( q = \bar{q} \) if \( 1/\bar{r} \) equals to the RHS of the last inequality, and \( q > \bar{q} \) otherwise.
3. – Proof of Theorem 1.1

(1) Since \( f \) is written as a finite sum such that \( f = \sum_{j=1}^{\ell} f_j \) with \( f_j \) satisfying \( N(s - s_0, p_j) \) with \( p_1 \leq p_j \leq p_2 \), we define \( \sigma_j \) and \( \lambda_j \), \( 1 \leq j \leq \ell \), as \( \sigma_j = \sigma_1 \), \( \lambda_j = \lambda_1 \) for \( \text{NLS} \) and \( \text{NLW} \),

\[
\sigma_j = 4/(p_j - 1) \text{ if } p(0) \leq p_j \leq p(1/2), \quad \sigma_j = n - 1 \text{ if } p(1/2) < p_j \leq p(s)
\]

and \( \lambda_j = (\sigma_j + 2)/2 \) for \( \text{NLKG} \). Then \( \sigma_j, \lambda_j, 1 \leq j \leq \ell \), satisfy (2.10). For \( \mu \geq 0 \), we define

\[
Y^\mu \equiv \begin{cases} 
L^\infty \dot{H}^\mu \cap L^{q_1} \dot{B}^{\mu}_{q_1} & \text{for NLS}, \\
L^\infty \dot{H}^\mu \cap L^{q_1} \dot{B}^{\mu-1/2}_{q_1} & \text{for NLW}, \\
L^\infty H^\mu \cap \bigcap_{j=1}^{\ell} L^{q_j} B^{\mu-1/2}_{q_j} & \text{for NLKG},
\end{cases}
\]

where \( q_j \equiv 2(\sigma_j + 2)/\sigma_j \), \( 1 \leq j \leq \ell \). By Corollary 2.2, in particular for \( \text{NLKG} \), we have

\[
\max_{1 \leq j \leq \ell} \|G h; L^{q_j} B^{\mu-1/2}_{q_j}\| \lesssim \min_{1 \leq j \leq \ell} \|h; L^{q_j} B^{\mu-1/2}_{q_j}\|.
\]

Therefore we have

\[
(3.32) \|\Phi(u); Y^\mu\| \lesssim \|\tilde{\Phi}; \tilde{H}^\mu\| + \sum_{j=1}^{\ell} \|f_j(u); L^{q_j} \dot{B}^{\mu}_{q_j}\| \quad \text{for NLS},
\]

\[
\quad \|f_j(u); L^{q_j} \dot{B}^{\mu-1/2}_{q_j}\| \quad \text{for NLW, NLKG},
\]

where \( \tilde{H}^\mu \) denotes \( \dot{H}^\mu \) for \( \text{NLS} \), \( \dot{H}^\mu \) for \( \text{NLW} \), \( \dot{H}^\mu \) for \( \text{NLKG} \), and we have used the property that \((-\Delta)^{\mu/2} \text{ for NLS}, \dot{H}^\mu \text{ for NLW, } \dot{H}^\mu \text{ for NLKG} \) is an isomorphism on \( \dot{B}^0_r \) [resp. \( B^0_r \)] to \( \dot{B}^{-\mu}_r \) [resp. \( B^{-\mu}_r \)], \( 1 \leq r \leq \infty \).

**Lemma 3.1** [14, Lemma 2.2]. Let \( s > 0 \), \( 1 \leq p < \infty \). Let \( f \) satisfy \( N(s, p) \). Let \( r, m, r \) satisfy \( 1 \leq r < \infty \), \( 2 \leq m, r < \infty \),

\[
1/r = (p - 1)/m + 1/r.
\]

Then

\[
\|f(u); \dot{B}^\mu_r\| \lesssim \|u; \dot{B}^0_r\|^{p-1} \|u; \dot{B}^\mu_r\|,
\]

where \( \dot{B} \) may be replaced by \( B \).

There exist \((r^*_j, q^*_j)\) such that

\[
1/r^*_j + 2/\sigma_j q^*_j = 1/2, \quad 0 < 1/q^*_j \leq 1/q_j, \quad 1/q_j' = (p_j - 1)/q_j + 1/q_j, \quad 1/q_j^* \equiv \begin{cases} 
1/r^*_j - s(p_j)/n & \text{for NLS}, \\
1/r^*_j - (s(p_j) - \lambda_j \alpha(r^*_j))/n & \text{for NLW, NLKG}.
\end{cases}
\]
Indeed, the above $q_j^*$ is given by $(p_j - 1)(\sigma_j + 2)/2$. Therefore by Lemma 3.1
with the embedding $\tilde{B}_m^\mu \hookrightarrow \tilde{B}_m^0$ for $1/r - \mu/n = 1/m$, $\mu \geq 0$, and
the Hölder inequality in time variable, the second terms on the RHS of (3.32) are estimated by

$$
\sum_{j=1}^\ell \|u; L^q_j \tilde{B}_j^{s(p_j)}\|_{p_j-1} \|u; L^q_1 \tilde{B}_1^\mu\|
$$
\text{for NLS},

$$
\sum_{j=1}^\ell \|u; L^q_j \tilde{B}_j^{s(p_j)-\lambda_j}\|_{p_j-1} \|u; L^q_1 \tilde{B}_1^{\mu-1/2}\|
$$
\text{for NLW, NLKG}.

For $\delta > 0$, $M > 0$, $R > 0$, we put

$$
A^s(s(p_1), s(p_2), \gamma, \delta) \equiv \left\{ \tilde{\phi} \in A^s(s(p_1), s(p_2), \gamma) \ ; \ \|\tilde{\phi}; \tilde{H}^s\| \leq \delta \right\},

X^s(R, M) \equiv \left\{ u \in X^s \ ; \ \max_{1 \leq j \leq \ell} \|u; Y^s(p_j)\| \leq R, \ \|u; Y^s\| \leq M \right\}.
$$

Then by (3.32), (3.33) and applying an analogous argument to $d$, we have

$$
\left\{ \begin{array}{l}
\|\Phi(u); Y^s\| \lesssim \delta + \sum_{j=1}^\ell R^{p_j-1} M,

\max_{1 \leq j \leq \ell} \|\Phi(u); Y^s(p_j)\| \lesssim \gamma + \sum_{j=1}^\ell R^{p_j-1} R,

d(u, v) \lesssim \sum_{j=1}^\ell R^{p_j-1} d(u, v)
\end{array} \right.
$$

(3.34)

for $\tilde{\phi} \in A^s(s(p_1), s(p_2), \gamma, \delta)$ and $u, v \in X^s(R, M)$. $\Phi$ is a contraction map
on $X^s(R, M)$ if the RHS of (3.34) are dominated by $M$, $R$, and $d(u, v)/2$, respectively. Indeed, this is realized by putting
$M = 2\delta$ and $R = 2\gamma$ with $\gamma$ satisfying

$$
C \sum_{j=1}^\ell (2\gamma)^{p_j-1} \leq 1/2.
$$

(3.35)

Since $X^s(R, M)$ endowed with metric $d$ is a complete metric space, we obtain
the global solution $u$ of INT if $\gamma$ is sufficiently small. The continuity of $u$
in time variable follows from the unitarity of $U$ and Lebesgue’s convergence
theorem.

To show the uniqueness of solutions in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$, let $u$ and $v$ be two
solutions with the same data. Then we have

$$
d(u, v_{\chi T}) \leq C \sum_{j=1}^\ell \left( \|u; Y^s(p_j)\| \vee \|v_{\chi T}; Y^s(p_j)\| \right)^{p_j-1} d(u, v_{\chi T}).
$$
where $\chi_T(t) = 1$ for $|t| \leq T$, $\chi_T(t) = 0$ for $|t| \geq T$. Since the sum on the RHS is smaller than $1/2$ for sufficiently small $T$, we have $u(t) = v(t)$ for $|t| \leq T$. Repeating this procedure, we obtain $u(t) = v(t)$ for all $t$.

(2) The asymptotic states $\bar{\phi}_\pm = \phi_\pm$ for NLS, $\bar{\phi}_\pm = (\phi_\pm, \psi_\pm)$ for NLW and NLKG are given by

$$
\phi_\pm = \phi + \int_0^{\pm \infty} U(-\tau) f(u(\tau)) d\tau \quad \text{for NLS,}
$$

$$
\phi_\pm = \phi - \int_0^{\pm \infty} \frac{\sin \tau \omega}{\omega} f(u(\tau)) d\tau,
$$

$$
\psi_\pm = \psi + \int_0^{\pm \infty} \cos \tau \omega f(u(\tau)) d\tau \quad \text{for NLW, NLKG.}
$$

Let us consider NLKG. The required regularity $\phi_\pm \in H^s$ follows from

$$
\|\phi_\pm; H^s\| \leq \|\phi; H^s\| + \left\| \int_0^{\pm \infty} \frac{\sin(t-\tau)\omega}{\omega} f(u(\tau)) d\tau; L^\infty H^s \right\|
$$

since the second term on RHS is finite by (3.32) and (3.33). Similarly we have $\psi_\pm \in H^{s-1}$. Since $\bar{\phi}_\pm$ satisfies

$$
\Gamma(t)\bar{\phi}_\pm = \Gamma(t)\bar{\phi} + \int_0^{\pm \infty} \frac{\sin(t-\tau)\omega}{\omega} f(u(\tau)) d\tau,
$$

we have

$$
\|u(t) - \Gamma(t)\bar{\phi}_\pm; H^s\| \leq \left\| \int_v^{\infty} \frac{\sin(v-\tau)\omega}{\omega} f(u(\tau)) d\tau; L^\infty_v([t, \infty); H^s) \right\|
$$

$$
\lesssim \sum_{j=1}^\ell \left\| f_j(u); L^q_j([t, \infty); B^s_{qj}^{s-1/2}) \right\|.
$$

Therefore we have $\|u(t) - \Gamma(t)\bar{\phi}_\pm; H^s\| \to 0$ as $t \to \infty$. We obtain (1.5) by an analogous argument for NLS and NLW. The uniqueness of states $\bar{\phi}_\pm$ in $\tilde{H}^s$ with (1.5) follows from the unitarity of $U(t)$.

(3) The proof of (1) shows that there exists $\gamma > 0$ such that for any $\delta > 0$ and any data $\bar{\phi}_0 \in A^s(s(p_1), s(p_2), \gamma, \delta)$, the operator $\Phi_{t_0}, t_0 \in [-\infty, \infty]$, given by

$$
\Phi_{t_0}(u)(t) \equiv \Gamma(t - t_0)\bar{\phi}_0 + \widetilde{G}_{t_0}(f(u))(t) \quad \text{for } t_0 \in \mathbb{R},
$$

$$
\Phi_{\pm \infty}(u)(t) \equiv \Gamma(t)\bar{\phi}_0 + \widetilde{G}_{\pm \infty}(f(u))(t),
$$

has a unique fixed point in $C(\mathbb{R}; \tilde{H}^s) \cap X^s(2\gamma, 2\delta)$, and if $t_0 \neq \pm \infty$, then the fixed point is also unique in $C(\mathbb{R}; \tilde{H}^s) \cap X^s$. Let $\bar{\phi}_- \in A^s(s(p_1), s(p_2), \gamma/2)$. 

In particular, we have \( \vec{\phi}^- \in A^s(s(p_1), s(p_2), \gamma, 2\|\vec{\phi}^-; \tilde{H}^s\|) \). Then there exists a unique fixed point \( u \) of \( \Phi_{-\infty} \) with \( \vec{\phi}_0 \) replaced by \( \vec{\phi}^- \) in
\[
W \equiv C(\mathbb{R}; \tilde{H}^s) \cap X^s(2\gamma, 4\|\vec{\phi}^-; \tilde{H}^s\|) .
\]

Let \( \vec{\phi}, \vec{\phi}^+ \) be defined by (3.36). Then it follows that \( \vec{\phi}, \vec{\phi}^+ \in \tilde{H}^s \), and \( u \) is a global solution of INT which satisfies (1.5). The above \( u \) is also unique in \( C(\mathbb{R}; \tilde{H}^s) \cap X^s \) by the following argument. Let \( v \) be another solution in \( C(\mathbb{R}; \tilde{H}^s) \cap X^s \) such that (1.5) holds for \( \vec{\phi}^- \). Let
\[
(3.37) \quad \vec{\phi}_0 \equiv v(t_0) \quad \text{for NLS}, \quad \vec{\phi}_0 \equiv (v(t_0), \partial_t v(t_0)) \quad \text{for NLW, NLKG} .
\]

Then we have \( \vec{\phi}_0 \in A^s(s(p_1), s(p_2), \gamma, 2\|\vec{\phi}^-; \tilde{H}^s\|) \) for sufficiently small \( t_0 \) by the convergence of the norms \( \|\vec{\phi}_0; \tilde{H}^\mu\| \to \|\vec{\phi}^-; \tilde{H}^\mu\| \) as \( t_0 \to -\infty \) for \( \mu = s(p_1), s(p_2), s \). Therefore \( \Phi_{t_0} \) with \( \vec{\phi}_0 \) replaced by \( \vec{\phi}_0 \) has a unique fixed point in \( W \), and the fixed point is also unique in \( C(\mathbb{R}; \tilde{H}^s) \cap X^s \). Since \( v \) satisfies \( v = \Phi_{t_0}(v) \) with \( \vec{\phi}_0 \) replaced by \( \vec{\phi}_0 \), we have \( v \in W \). Moreover since \( v \) satisfies \( v = \Phi_{-\infty}(v) \) with \( \vec{\phi}_0 \) replaced by \( \vec{\phi}^- \), we obtain \( u = v \) by the uniqueness of the fixed point of \( \Phi_{-\infty} \) in \( W \).

The correspondence between \( \vec{\phi}^- \) and \( \vec{\phi}^+ \) are given by
\[
(3.38) \quad \phi_- = \int_{-\infty}^{\infty} U(-\tau) f(u(\tau)) d\tau \quad \text{for NLS}, \quad \phi_+ = \int_{-\infty}^{\infty} \frac{\sin \tau \omega}{\omega} f(u(\tau)) d\tau , \quad \psi_+ = \int_{-\infty}^{\infty} \cos \tau \omega f(u(\tau)) d\tau \quad \text{for NLW and NLKG} ,
\]
via global solutions with (1.5). The continuity of the scattering operator \( S : \vec{\phi}^- \mapsto \vec{\phi}^+ \) is shown as follows. Let \( u^k, k \geq 1 \), be global solutions with asymptotic states \( \vec{\phi}^k_\pm \). In particular \( u^k \) is the fixed point of \( \Phi_{-\infty} \) with \( \vec{\phi}_0 \) replaced by \( \vec{\phi}^-_k \). Let us consider NLKG. Then we have
\[
d(u, u^k) \leq C \|\vec{\phi}^- - \vec{\phi}_0; \mathcal{H}^{1/2}\| + C \sum_{j=1}^{\ell} \| f_j(u) - f_j(u^k); L^q_{B_0} \| .
\]

Since the last term is dominated by \( d(u, u^k)/2 \), we have \( d(u, u^k) \to 0 \) as \( k \to \infty \). By (3.38), we have
\[
\|\phi_+ - \phi^+_k; \mathcal{H}^{1/2}\| \leq \|\phi_- - \phi^-_k; \mathcal{H}^{1/2}\| + \left\| \int_{-\infty}^{\infty} \frac{\sin(t - \tau) \omega}{\omega} (f(u(\tau)) - f(u^k(\tau))) d\tau ; L^\infty \mathcal{H}^{1/2}\right\| .
\]
Since the last term is dominated by \( d(u, u^k)/2 \), we obtain \( \phi^+_k \to \phi_+ \) in \( \mathcal{H}^{1/2} \) as \( k \to \infty \). Similarly we also have \( \psi^+_k \to \psi_+ \) in \( \mathcal{H}^{-1/2} \) as \( k \to \infty \), and (1.6)
follows from the complex interpolation $H^\mu = [H^{1/2}, H^s(p_2)]$. The results for NLS and NLW are obtained by an analogous argument.

(4) We use the following argument to show the uniqueness of solutions. Let $u, v \in C([-T, T]; \tilde{H}^s)$, $T > 0$, be two solutions with the same data. Then $u$ and $v$ satisfy

$$u - v = \sum_{j=1}^{\ell} \tilde{G}_0(\tilde{f}_j(u) - \tilde{f}_j(v)).$$

If there exist $r, q, \tilde{r}_j, \tilde{q}_j, r^*_j, 1 \leq j \leq \ell$, such that

$$(3.39) \quad \|\tilde{G}_0h; L^q L^r\| \lesssim \|h; L^{\tilde{q}_j} L^{\tilde{r}_j}\|,$$

$$(3.40) \quad 1/\tilde{r}_j = (p_j - 1)/r^*_j + 1/r, \quad q \geq \tilde{q}_j, \quad \tilde{H}^s \hookrightarrow L^r \cap L^{r^*_j},$$

then by the Hölder inequality we have

$$\|u - v; L^q L^r\| \lesssim \sum_{j=1}^{\ell} T^{1/\tilde{q}_j-1/q} (\|u; L^\infty \tilde{H}^s\| \lor \|v; L^\infty \tilde{H}^s\|)^{p_j-1} \|u - v; L^q L^r\|.$$ 

Therefore taking $T$ sufficiently small if $q > \tilde{q}_j$ for $1 \leq j \leq \ell$, and moreover taking $\|u; L^\infty \tilde{H}^s\|$ sufficiently small if there exists $j$ with $q = \tilde{q}_j$, we have $u(t) = v(t), t \in [-T, T]$, in $\tilde{H}^s$. Repeating this procedure, we obtain the uniqueness in $C(\mathbb{R}; \tilde{H}^s)$.

**Lemma 3.2.** Let us consider NLS. Let $s$, $p_1$ and $p_2$ satisfy $0 \leq s < n/2$, $1 + \min(2s/n, 2/n) \leq p_1 \leq p_2$,

- $p_2 \leq 1 + (1 + 2s)/(1 - 2s)$ if $n = 1$, $p_2 \leq 1 + (1 + s)/(1 - s)$ if $n = 2$,
- $p_2 \leq 1 + (2n + 4(n - 1)s)/n(n - 2s)$ if $n \geq 3$ and $0 \leq s < n/2(n - 1)$,
- $p_2 \leq 1 + 4(n - 1)/n(n - 2)$ if $n \geq 3$ and $s = n/2(n - 1)$,
- $p_2 \leq 1 + 4/(n - 2s)$ if $n \geq 3$ and $n/2(n - 1) < s < n/2$.

Then there exist $r, q, \tilde{r}_j, \tilde{q}_j, r^*_j, 1 \leq j \leq \ell$, with (3.39) and (3.40). Moreover $q = \tilde{q}_j$ if and only if $p_j = 1 + 4/(n - 2s)$.

**Proof of Lemma 3.2.** Let $r, q$ satisfy $1/r + 2/nq = 1/2$ and

$$1/2 = (n - 2s)/2n \lor (n - 2)/2n.$$

Let $\tilde{r}_j, \tilde{q}_j, 1 \leq j \leq \ell$, satisfy $1/\tilde{r}_j + 2/n\tilde{q}_j = 1/2 + 2/n, 1 \leq \tilde{r}_j \leq 2$ if $n = 1, 1 < \tilde{r}_j \leq 2$ if $n = 2$, $1 \leq 1/\tilde{r}_j \leq 1 - (n - 2)/nr$ if $n \geq 3, (n - 2)/2(n - 1) < 1/r \leq 1/2$,

$$1/2 \leq 1/\tilde{r}_j \leq (n^2 + 2n - 4)/2(n - 1) \text{ if } n \geq 3, 1/r = (n - 2)/2(n - 1),$$

$$1/r + 2/n \leq (n - 2)/2n \leq 1/r < (n - 2)/2(n - 1).$$

Then (3.39) follows from Lemma 2.1 for $n = 1, 2$ and Corollary 2.8 for $n \geq 3$. Let $r^*_j$ satisfy

$$(n - 2s)/2n \leq 1/r^*_j \leq 1/2.$$

We now take $\tilde{r}_j$ and $r^*_j$ such that (3.40) holds. It happens $q = q_j$ if and only if $r = 2n/(n - 2), \tilde{r}_j = r^*, r^*_j = 2n/(n - 2s), n \geq 3$. Namely $p_j = 1 + 4/(n - 2s)$.  \(\square\)
Lemma 3.3. Let us consider NLW and NLKG. Let \( s, p_1 \) satisfy \( 1/2 \leq s < n/2 \), \( p_1 \geq 1 + 4n/(n^2 - 1) \) for NLW, \( p_1 \geq 1 + 4/(n + 1) \) for NLKG, and let \( p_2 \) satisfy
\[
p_1 \leq p_2 \leq 1 + 4n/(n + 1)(n - 2s).
\]
Then there exist \( r, q, \tilde{r}_j, \tilde{q}_j, r^*_j \), \( 1 \leq j \leq \ell \) with (3.39) and (3.40). Moreover \( q \) and \( \tilde{q}_j \) satisfy \( q > \tilde{q}_j \) for \( 1 \leq j \leq \ell \).

Proof of Lemma 3.3. Let \( r, q \) satisfy \( 1/r + 2/(n - 1)q = 1/2 \) and
\[
1/r = \left( (n - 2s)/2n \right) \vee \left( (n - 1)/2(n + 1) \right).
\]
Let \( \tilde{r}_j, \tilde{q}_j \) satisfy \( 1/\tilde{r}_j + 2/(n - 1)\tilde{q}_j = 1/2 + 2/(n - 1) \) and
\[
(3.41) \quad 1/\tilde{r}_j = 1/r + 2/(n + 1)
\]
for \( 1 \leq j \leq \ell \). Then (3.39) follows from Lemma 2.1 for \( n \leq 3 \) and Corollary 2.8 for \( n \geq 4 \). Let \( r^*_j \) satisfy
\[
(n - 2s)/2n \leq 1/r^*_j \leq \begin{cases} (n - 1)/2n & \text{for NLW,} \\ 1/2 & \text{for NLKG.} \end{cases}
\]
We now take \( r^*_j \) such that (3.40) holds. Note that \( q \) and \( q_j \) always satisfy \( q > q_j \) by (3.41).

If \( p_1 \) and \( p_2 \) satisfy the assumption in Theorem 1.1 with (1.7) and (1.8), then by Lemmas 3.2 and 3.3 solutions of INT with the same data are unique in \( C(\mathbb{R}; \dot{H}^s) \). Therefore the space \( X^s \) in (1) in the theorem is redundant.

To show that \( X^s \) in (3) in the theorem is removable, we use a similar argument in (3) in this section. Let \( v \) be another solution in \( C(\mathbb{R}; \dot{H}^s) \) such that (1.5) holds for \( \phi_- \). Then for sufficiently small \( t_0 \), \( v \) is a fixed point of \( \Phi_{t_0} \) with \( \tilde{\phi}_0 \) replaced by \( \tilde{\phi}_{t_0} \) which is defined by (3.37). Since the fixed point of \( \Phi_{t_0} \) is unique in \( C(\mathbb{R}; \dot{H}^s) \) by Lemma 3.2 and 3.3, we have \( v \in X^s \). Therefore we obtain \( u = v \) by (3) in the theorem. So that we may remove \( X^s \) in (3) in the theorem.

4. – Proof of Proposition 1.1

In Lemma 4.6, below, we show the uniqueness of solutions of NLW and NLKG with the same data in \( C(I; \dot{H}^s) \) for any interval \( I \) in \( \mathbb{R} \). We follow the method of the proof of Proposition 3.1 in [6]. Throughout this section, we consider NLW and NLKG with \( n \geq 3 \) and \( 1/2 \leq s < n/2 \). We use Lemma 2.1 for \( n = 3 \), Corollary 2.8 for \( n \geq 4 \) with \( \sigma \equiv \sigma_1 = \sigma_2 = n - 1 \), \( \lambda \equiv \lambda_1 = \lambda_2 = (n + 1)/2 \). We put \( \delta(r) \equiv n\alpha(r) \).
Lemma 4.1. Let $r$ satisfy
\[
1/2(n - 2) \leq \alpha(r) \leq 1/(n - 1) \text{ for } n \geq 4, \quad 0 \leq \alpha(r) < 1/2 \text{ for } n = 3.
\]

Let $p$ satisfy
\[
\frac{3n + 1}{n^2 - 1} \leq p - 1 \left\{ \begin{array}{ll}
\leq \frac{2}{n - 2s} \min \left( \frac{2n}{n + 1}, 2 - \frac{n - 1}{2} \alpha(r) \right) & \text{for } n \geq 4, \\
< \frac{3}{3 - 2s} & \text{for } n = 3.
\end{array} \right.
\]

Let $f$ satisfy $N(s - 1/2, p)$. Then
\[
\left\| \int_{t_0}^{t} \frac{U(t - \tau)}{\omega} f(u(\tau)) d\tau : L^q(I; \tilde{B}^{s-\lambda \alpha(r)}_r) \right\| \lesssim |I|^\nu \|u; L^\infty(I; \tilde{H}^s)\|_p
\]
for any $u \in L^\infty(I; \tilde{H}^s)$, where $q \equiv 2/(n - 1) \alpha(r)$ and $\nu > 0$ is some constant, and $t_0$ is any point with $t_0 \in I \subset \mathbb{R}$.

Proof of Lemma 4.1. By the conditions on $p$, there exists $s_1$ with $1/2 \leq s_1 \leq s$,
\[
\frac{n + \lambda}{\lambda} \leq (p - 1)(n - 2s_1) \left\{ \begin{array}{ll}
\leq \min \left( \frac{2n}{\lambda}, 4 - (n - 1) \alpha(r) \right) & \text{for } n \geq 4, \\
< \frac{3}{\lambda} & \text{for } n = 3.
\end{array} \right.
\]

Let $\tilde{r}$ be given by $\alpha(\tilde{r}) = (2 - (p - 1)(n - 2s_1))/(n - 1)$. Then $\tilde{r}$ satisfies the condition $\alpha(r) - 2/(n - 1) < \alpha(\tilde{r}) \leq 0$ for $n \geq 4$ since $2 \leq (p - 1)(n - 2s_1) \leq 4 - (n - 1) \alpha(r)$, the condition $-1/2 < \alpha(\tilde{r}) \leq 0$ for $n = 3$ since $2 \leq (p - 1)(n - 2s_1) < 3$. Therefore by Lemma 2.1 and Corollary 2.8, we have
\[
\left\| \int_{t_0}^{t} \frac{U(t - \tau)}{\omega} f(u(\tau)) d\tau : L^q(I; \tilde{B}^{s-\lambda \alpha(r)}_{\tilde{r}}) \right\| \lesssim \|f(u); L^q(I; \tilde{B}^{\rho \alpha}_{\tilde{r}})\|,
\]
where $q, \tilde{q}, \rho_*$ are given by
\[
q = 2/(n - 1) \alpha(r), \quad 1/\tilde{q} = (n - 1) \alpha(\tilde{r})/2 + 1, \quad \rho_* = s - \lambda \alpha(\tilde{r}) - 1.
\]

Here $\rho_*$ satisfies $\rho_* \geq 0$ since $s \geq 1/2$ and $(n + \lambda)/\lambda \leq (p - 1)(n - 2s_1)$. Let $r_*, r_{**}$ be numbers with $\delta(r_*) = s_1, 1/\tilde{r} = (p - 1)/r_* + 1/r_{**}$. Then $r_{**}$ satisfies $\delta(r_{**}) = \lambda \alpha(\tilde{r}) + 1, 0 \leq \alpha(r_{**}) < 1/2$ since the last inequality is rewritten by $n(5 - n)/(n + 1) < (p - 1)(n - 2s_1) \leq 4n/(n + 1)$, which is satisfied by the condition on $s_1$. By Lemma 3.1 and the embeddings $\tilde{H}^{s_1} \hookrightarrow \tilde{B}^0_{r_*}, \tilde{H}^s \hookrightarrow \tilde{B}^{\rho \alpha}_{r_{**}}$, we have
\[
\|f(u); \tilde{B}^{\rho \alpha}_{\tilde{r}}\| \lesssim \|u; \tilde{B}^0_{r_*}\|^{p-1}\|u; \tilde{B}^{\rho \alpha}_{r_{**}}\| \lesssim \|u; \tilde{H}^{s_1}\|^{p-1}\|u; \tilde{H}^s\|,
\]
so that we obtain the required result as $\nu = 1/\tilde{q} > 0$. \qed
Lemma 4.2. Let \( r, s, \mu \) satisfy \( 2 \leq r < \infty, 0 \leq s \leq \mu < s + \delta(r) \). Let \( r_* \) satisfy
\[
s \leq \delta(r_*) \leq \min\left( \frac{n}{2}, \mu, \frac{s \delta(r)}{s + \delta(r) - \mu} \right).
\]
Then
\[
\| \phi; B^0_{r_*} \| \lesssim \| \phi; H^s \|^{1-\theta} \| \phi; B^\mu_{r-\delta(r)} \| relevant \theta \]
for any \( \phi \in H^s \cap B^\mu_{r-\delta(r)} \), where \( \theta = (\delta(r_*) - s)/(\mu - s) \), \( B \) and \( H \) may be replaced with \( \tilde{B} \) and \( \tilde{H} \), respectively.

Proof of Lemma 4.2. By \( s \leq \delta(r_*) \leq \mu \), \( \theta \) satisfies \( 0 \leq \theta \leq 1 \). Let \( \ell \) be a number given by \( \alpha(\ell) = \theta \alpha(r) \). Then \( \ell \) satisfies \( \delta(m) - \delta(\ell) = (1 - \theta)s + \theta(\mu - \delta(r)) \), and \( 1 \leq \ell \leq m \) by \( s \leq \delta(r_*) \leq \delta(r)/(s + \delta(r) - \mu) \). Therefore by the embedding and interpolation, \( B^0_{r_*} \leftrightarrow B^{\delta(r_*)-\delta(\ell)}_{\ell} = (H^s, B^\mu_{r-\delta(r)}|_{\ell}] \), we obtain the conclusion.

Lemma 4.3. Let \( s \) satisfy the condition in Proposition 1.1. Let \( r, q \) satisfy
\[
\max \left( \frac{1}{2(n-2)}, \frac{4}{n^2-1}, \frac{n-2s}{n^2-1} \right) \leq \alpha(r) = \min \left( \frac{1}{n-1}, \frac{2 + (n-1)s}{n^2-1} \right) \quad \text{for} \quad n \geq 4,
\]
\[
\alpha(r) = \frac{1 - \varepsilon_0}{2} \quad \text{for} \quad n = 3,
\]
where \( \varepsilon_0 > 0 \) is a sufficiently small number. Let \( \varepsilon \) be a number with \( 0 < \varepsilon \leq (n - \lambda) \alpha(r) \) for \( n \geq 4 \), \( \varepsilon_0 \leq \varepsilon \leq \alpha(r) \) for \( n = 3 \). Let \( p \) satisfy \( p = 1 + 4(1 - \varepsilon)/(n - 2s) \). Let \( f \) satisfy \( N(s - 1/2, p) \). Then the following estimate holds
\[
\left\| \int_{t_0}^t \frac{U(t - \tau)}{\omega} f(u(\tau)) d\tau; L^q(I; \tilde{B}^\mu_{r+\varepsilon-\delta(r)}(\ell)) \right\| \lesssim |I|^\varepsilon \| u; L^\infty(I; \tilde{H}^s) \|^{p-1} \| u; L^q(I; \tilde{B}^\mu_{r-\delta(r)}(\ell)) \|
\]
for any \( u \in L^\infty(I; \tilde{H}^s) \cap L^q(I; \tilde{B}^\mu_{r-\delta(r)}(\ell)) \), and any \( \mu \) with \( s \leq \mu \leq s + (n-\lambda) \alpha(r) - \varepsilon \), where \( t_0 \) is any point with \( t_0 \in I \subset \mathbb{R} \).

Proof of Lemma 4.3. First we note that there exists \( r \) with (4.42) for \( n \geq 4 \) by the conditions on \( s \). Let \( r \) be any number with (4.42), (4.43), where \( \varepsilon_0 > 0 \) is a sufficiently small number determined later. Let \( \tilde{r}, \tilde{q} \) be given by
\[
\alpha(\tilde{r}) = \alpha(r) - \frac{1 - \varepsilon}{n - \lambda}, \quad \frac{1}{\tilde{q}} = \frac{\sigma}{2} \alpha(\tilde{r}) + 1.
\]
Then by Lemma 2.1 for \( n = 3 \), Corollary 2.8 for \( n \geq 4 \), we have
\[
\left\| \int \frac{U(t - \tau)}{\omega} f(u(\tau)) d\tau; L^q(I; \tilde{B}^\mu_{r+\varepsilon-\delta(r)}(t)) \right\| \lesssim \| f(u); L^\tilde{q}(I; \tilde{B}^\mu_{r+\varepsilon-\delta(r)}(t)) \|.
\]
where \( \rho_\ast = \mu + \varepsilon - (n - \lambda)\alpha(r) - \lambda\alpha(\tilde{r}) - 1 \). Here \( \rho_\ast \) satisfies \( \rho_\ast \geq 0 \) by \( \mu \geq s \), \( \varepsilon \leq (n - \lambda)\alpha(r) \) and
\[
\alpha(r) \leq (2 + (n - 1)s)/(n^2 - 1).
\]
Let \( \eta = (s - \rho_\ast)/(s + \delta(r) - \mu) \). Then \( \eta \) satisfies \( 0 \leq \eta \leq 1 \) by \( \mu \leq s + (n - \lambda)\alpha(r) - \varepsilon \) and
\[
4(1 - \varepsilon)/(n^2 - 1) \leq \alpha(r)
\]
Let \( r_\ast, r_{\ast\ast} \) satisfy
\[
\alpha(r_{\ast\ast}) = \eta\alpha(r), \quad 1/\tilde{r} = (p - 1)/r_\ast + 1/r_{\ast\ast}.
\]
Then \( r_\ast \) satisfies \( r_\ast < \infty \) by \( 2 < r \), and
\[
s \leq \delta(r_\ast) \leq \mu_\ast \equiv \min\left(\frac{n}{2}, \mu, \frac{s\delta(r)}{s + \delta(r) - \mu}\right).
\]
Indeed, the above inequalities are rewritten as
\[
(s + (n - \lambda)\alpha(r) - \mu + 1 - \varepsilon)\alpha(r) - (s + \delta(r) - \mu)(p - 1)(n - 2s)/2n
\]
\[
\leq (s + (n - \lambda)\alpha(r) - \mu)\alpha(\tilde{r})
\]
\[
\leq (s + (n - \lambda)\alpha(r) - \mu + 1 - \varepsilon)\alpha(r) - (s + \delta(r) - \mu)(p - 1)(n - 2\mu_\ast)/2n.
\]
The first inequality is easily shown, while the second is rewritten as \( s + (n - 1)\alpha(r) - \mu \geq a_\ast \), where \( a_\ast \) is given by
\[
a_\ast = 0 \quad \text{for} \quad \mu \geq n/2, \quad a_\ast = \frac{(s + \delta(r) - \mu)(n - 1)(n - 2\mu)}{n(n - 2s)} \quad \text{for} \quad \delta(r) \leq \mu < n/2,
\]
\[
a_\ast = \frac{(n - 1)(n(s - \mu) + (n - 2s)\delta(r))}{n(n - 2s)} \quad \text{for} \quad s \leq \mu < \delta(r).
\]
The cases for \( \mu \geq n/2 \) and \( s \leq \mu < \delta(r) \) follow from \( \mu \leq s + (n - \lambda)\alpha(r) - \varepsilon \) and \( s \geq 1/2 \), respectively. The case for \( \delta(r) \leq \mu < n/2 \) is rewritten as
\[
g(\mu) \equiv (n - 1)(s + \delta(r) - \mu)(n - 2\mu) - n(n - 2s)(s + (n - 1)\alpha(r) - \mu) \leq 0.
\]
Since \( g(s) = 0 \) and \( g''(\mu) > 0 \), \( g(\mu) \leq 0 \) holds for \( s \leq \mu \leq s + (n - \lambda)\alpha(r) - \varepsilon \) if \( g(s + (n - \lambda)\alpha(r) - \varepsilon) \leq 0 \), which is rewritten as
\[
g_1(\varepsilon) \equiv \left(\varepsilon - \frac{n - 1}{2}\alpha(r)\right)\left(2\varepsilon + 2\lambda\alpha(r) - \frac{n - 2s}{n - 1}\right) \leq 0.
\]
Since \( g_1((n - 1)\alpha(r)/2) = 0 \) and \( g''(\varepsilon) > 0 \), \( g_1(\varepsilon) \leq 0 \) holds for \( 0 < \varepsilon \leq (n - \lambda)\alpha(r) \) if \( g_1(0) \leq 0 \), which is satisfied by the condition
\[
(n - 2s)/(n^2 - 1) \leq \alpha(r).
\]
By Lemma 3.1, Lemma 4.2 and the interpolation \( \bar{B}_{\ast\ast}^{\rho_\ast} = (\bar{H}^s, \bar{B}^{\mu - \delta(r)}_{\ast\ast})[\eta] \), we have
\[
\|f(u); \bar{B}^{\rho_\ast}_{\ast}\| \lesssim \|u; \bar{H}^s\|^{(1 - \theta)(p - 1) + 1 - \eta}\|u; \bar{B}^{\mu - \delta(r)}_{\ast\ast}\|^{\theta(p - 1) + \eta},
\]
where \( \theta = (\delta(r_\ast) - s)/(\mu - s) \). Therefore we have
\[
\|f(u); L^q(I; \bar{B}^{\rho_\ast}_{\ast})\| \lesssim \|I\|^{\theta} \|u; L^\infty(I; \bar{H}^s)\|^{p - 1}\|u; L^q(I; \bar{B}^{\mu - \delta(r)}_{\ast\ast})\|,
\]
where we have used the properties \( \theta(p - 1) + \eta = 1 \) and \( \varepsilon = 1/\bar{q} - 1/q \). So that we obtain the required result for \( n \geq 4 \). For \( n = 3 \), the conditions (4.44), (4.45), (4.46) follow from \( s \geq 1, \varepsilon \geq \varepsilon_0, \varepsilon_0 \leq 2/(n + 1) \), respectively. \qed
Lemma 4.4. Let $s$ satisfy the conditions in Proposition 1.1. Let $r$ satisfy (4.42), (4.43), where $\varepsilon_0 > 0$ be sufficiently small, and let $q \equiv 2/(n-1\alpha(r))$. Let $p_1, p_2$ satisfy

$$\frac{3n+1}{n^2-1} \leq p_1 - 1 \leq p_2 - 1 \begin{cases} \leq \frac{4(1-\varepsilon_0)}{3-2s} & \text{for } n = 3, \\ \leq \frac{4}{n-2s} & \text{for } n \geq 4. \end{cases}$$

Let $f$ satisfy $N(s-1/2, p_1, p_2)$. Then any solution $u$ of NLW, or NLKG, in $C(I; \tilde{H}^s)$ satisfies $u \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)})$, where $I$ is any bounded interval in $\mathbb{R}$.

Proof of Lemma 4.4. Let $t_0$ be any point with $t_0 \in \tilde{I}$. The solution $u$ is rewritten as

$$u(t) = (\cos(t-t_0)\omega)u(t_0) + \frac{\sin(t-t_0)\omega}{\omega} \partial_t u(t_0) + \sum_{j=1}^{k} \tilde{G}_{t_0} f_j(u) + \sum_{j=k+1}^{\ell} \tilde{G}_{t_0} f_j(u),$$

where $f_j$ satisfies $N(s-1/2, p_j^*)$ with

$$p_1 \leq p_1^* \leq \cdots \leq p_k^* < 1 + \frac{4(1-(n-\lambda)\alpha(r))}{n-2s} \leq p_{k+1}^* \leq \cdots \leq p_{\ell}^* \leq p_2.$$

Let $\varepsilon_j, k+1 \leq j \leq \ell$, be numbers given by

$$p_j - 1 = \frac{4(1-\varepsilon_j)}{(n-2s)}.$$

We note that $\varepsilon_{k+1} \geq \cdots \geq \varepsilon_{\ell}$. By Lemma 2.1 and Lemma 4.1, we have

$$u(t) - \sum_{j=k+1}^{\ell} \tilde{G}_{t_0} f_j(u) \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{s-\lambda s(r)}).$$

By Lemma 4.3, we have

$$\tilde{G}_{t_0} f_j(u) \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{s+\varepsilon_j-\lambda s(r)}) \hookrightarrow L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{s+\varepsilon_j-\lambda s(r)}),$$

for $k+1 \leq j \leq \ell$, so that we have $u \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{s+\varepsilon_j-\lambda s(r)})$. Again by Lemma 4.3, we have

$$\tilde{G}_{t_0} f_j(u) \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{\rho_1}) \hookrightarrow L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{\rho_2}),$$

where $\rho_1 = (s+\varepsilon_{\ell}+\varepsilon_j-\lambda s(r)) \wedge (s-\lambda s(r))$, $\rho_2 = (s+2\varepsilon_{\ell}-\lambda s(r)) \wedge (s-\lambda s(r))$, so that we have $u \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)} \cap \bar{B}_{r}^{\rho_2})$. Repeating this procedure, we obtain $u \in L^q(I; \bar{B}_{r}^{s-\lambda s(r)})$. □
Lemma 4.5. Let $s$ satisfy $1 \leq s < 3/2$ for $n = 3$, $5/8 \leq s < 2$ for $n = 4$, $1/2 \leq s < n/2$ for $n \geq 5$. Let $p$ satisfy $p - 1 \leq 4/(n - 2s)$, and $p - 1 \geq 4n/(n^2 - 1)$ for NLW, $p - 1 \geq 4/(n + 1)$ for NLKG. Let $f$ satisfy $N(s - 1/2, p)$. Let $r$ satisfy

$$
\max(1/2(n-2), (n-2s)/(n^2-1)) \leq \alpha(r) \leq \min(1/(n-1), s/\lambda) \quad \text{for } n \geq 4,
$$

$$
\alpha(r) = (1 - \varepsilon_0)/2 \quad \text{for } n = 3,
$$

where $\varepsilon_0 > 0$ is some sufficiently small number. Then

$$
\left\| \int_{t_0}^{t} \frac{U(t - \tau)}{\omega} (f(u(\tau)) - f(v(\tau))) d\tau; L^q(I; \tilde{B}_r^0) \right\| \leq \begin{cases} 
\left| I \right|^{2/(n+1)} \max_{w = u, v} \|w; L^\infty(I; \tilde{H}^s)\|^{p-1} \|u - v; L^q(I; \tilde{B}_r^0)\| 
& \text{for } p - 1 < 4n/(n + 1)(n - 2s), \\
\left| I \right|^{2-(p-1)(n-2s)/2} \max_{w = u, v} \|w; L^\infty(I; \tilde{H}^s) \cap L^q(I; \tilde{B}_r^{s-\lambda\alpha(r)})\|^{p-1} \|u - v; L^q(I; \tilde{B}_r^0)\| 
& \text{for } p - 1 \geq 4n/(n + 1)(n - 2s), 
\end{cases}
$$

where $q \equiv 2/(n - 1)\alpha(r)$, $t_0$ is any number with $t_0 \in \tilde{I} \subset \mathbb{R}$.

Proof of Lemma 4.5. First we note that there exists $r$ with (4.47) by the conditions on $s$. Let $\tilde{r}$ satisfy $1/\tilde{r} = 1/r + 2/(n + 1)$. Then $\tilde{r}$ satisfies

$$
\alpha(r) - 2/(n - 1) < \alpha(\tilde{r}) \leq 0 \quad \text{for } n \geq 4, \quad -1/2 < \alpha(\tilde{r}) \leq 0 \quad \text{for } n = 3.
$$

By Lemma 2.1 and Corollary 2.8, we have

$$
\left\| \int_{t_0}^{t} \frac{U(t - \tau)}{\omega} (f(u(\tau)) - f(v(\tau))) d\tau; L^q(I; \tilde{B}_r^0) \right\| \leq \left\| f(u) - f(v); L^\tilde{q}(I; \tilde{B}_r^0) \right\| ,
$$

where $1/\tilde{q} = (n - 1)\alpha(\tilde{r})/2 + 1$. Let $r_*$ satisfy $r_* = (p - 1)(n + 1)/2$. If $p - 1 < 4n/(n + 1)(n - 2s)$, then $r_*$ satisfies $1/2 \leq \delta(r_*) < s$ for NLW, $0 \leq \delta(r_*) < s$ for NLKG. By the Hölder inequality and the embedding $\tilde{H}^s \hookrightarrow \tilde{H}^{\delta(r_*)} \hookrightarrow \tilde{B}_r^{0}$, we have

$$
\| f(u) - f(v); \tilde{B}_r^0 \| \leq \max_{w = u, v} \|w; \tilde{H}^s\|^{p-1} \|u - v; \tilde{B}_r^0\|.
$$

So that we obtain the required result. If $p - 1 \geq 4n/(n + 1)(n - 2s)$, then $r_*$ satisfies

$$
s \leq \delta(r_*) \leq \min(n/2, s + (n - \lambda)\alpha(r), ns/\lambda)
$$

since the above inequalities are satisfied if $p$ and $s$ satisfy

$$
\frac{4n}{(n + 1)(n - 2s)} \leq p - 1 \leq \frac{4}{n - 2s},
$$

$$
\frac{n}{2} - \min\left(\frac{n}{2}, s + (n - \lambda)\alpha(r), \frac{ns}{\lambda}\right) \leq \frac{n(n - 2s)}{2(n + 1)},
$$
where the last inequality is satisfied by $1/2 \leq s < n/2$ and $(n - 2s)/(n^2 - 1) \leq \alpha(r)$. By Lemma 4.2, we have

$$\| f(u) - f(v); \tilde{B}_r^0 \| \leq \max_{\nu = u, v} \| w; \tilde{H}^s \|^{(1-\theta)(p-1)} \| w; \tilde{B}_r^{s-\lambda \alpha(r)} \|^{\theta(p-1)} \| u - v; \tilde{B}_r^0 \|,$$

where $\theta = (\delta(\tau_s) - s) / (\mu - s)$. By the Hölder inequality in time variable, we obtain the required result, where we use the property

$$1/\tilde{q} - \theta(p-1)/q - 1/q = 2 - (p-1)(n-2s)/2.$$  

\[ \square \]

**Lemma 4.6.** Let $n, s$ satisfy the conditions in Proposition 1.1. Let $p_1, p_2$ satisfy $p_1 \leq p_2 < p(s)$, and $p_1 \geq p(1/2)$ for NLW, $p_1 \geq p(0)$ for NLKG. Let $f$ satisfy $N(s - 1/2, p_1, p_2)$. Let $u, v$ be solutions of NLW, or NLKG, in $C(I; \tilde{H}^s)$ with $(u(t_0), \partial_t u(t_0)) = (v(t_0), \partial_t v(t_0))$ for some $t_0 \in I$. Then $u = v$ in $C(I; \tilde{H}^s)$.

**Proof of Lemma 4.6.** By the condition on $s$, there exists $r$ which satisfies (4.42), (4.43) with $\alpha(r) \leq s/\lambda$, where $\varepsilon_0 > 0$ is taken as $p_2 - 1 \leq 4(1 - \varepsilon_0)/(n - 2s)$ for $n = 3$. By Lemma 4.4, we have $u, v \in L^q(I; \tilde{B}_r^{s-\lambda \alpha(r)})$ with $q = 2/(n-1)\alpha(r)$. Since $u - v$ is rewritten by

$$u - v = \sum_{j=1}^k \int_{t_0}^t \frac{U(t - \tau)}{\omega} (f_j(u(\tau)) - f_j(v(\tau))) d\tau,$$

where $f_j, 1 \leq j \leq k$, satisfies $N(s-1/2, p_j^*)$ with $p_1 \leq p_j^* \leq p_2$, by Lemma 4.5 we have

$$\| u - v; L^q(I_1; \tilde{B}_r^0) \| \leq \sum_{j=1}^k |I_j|^{\nu_j} \max_{\nu=u,v} \| w; L^\infty(I_1; \tilde{H}^s) \cap L^q(I_1; \tilde{B}_r^{s-\lambda \alpha(r)}) \|^{p_j^*-1} \| u - v; L^q(I_1; \tilde{B}_r^0) \|$$

for any $I_1 \subset \mathbb{R}$ with $t_0 \in \tilde{I}_1$, $|I_1| < 1$, where $\nu_j = \min(2/(n+1), 2 - (p_j^* - 1)(n-2s)/2)$, we note that $u, v \in L^q(I_1; \tilde{B}_r^0)$ by the embeddings $B_r^{s-\lambda \alpha(r)} \hookrightarrow B_r^0$ for NLKG, and $\dot{H}^{1/2} \cap B_r^{s-\lambda \alpha(r)} \hookrightarrow \tilde{B}_r^0$ for NLW. Taking $I_1$ sufficiently small, we have $u = v$ in $C(I_1; \tilde{H}^s)$. Repeating this procedure, we obtain the required results.  

\[ \square \]

**REFERENCES**


