**Z-graded Lie Superalgebras of Infinite Depth and Finite Growth**

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**Abstract.** In 1998 Victor Kac classified infinite-dimensional $\mathbb{Z}$-graded Lie superalgebras of finite depth. We construct new examples of infinite-dimensional Lie superalgebras with a $\mathbb{Z}$-gradation of infinite depth and finite growth and classify $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth under suitable hypotheses.

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**Introduction**

Simple finite-dimensional Lie superalgebras were classified by V. G. Kac in [K2]. In the same paper Kac classified the finite-dimensional, $\mathbb{Z}$-graded Lie superalgebras under the hypotheses of irreducibility and transitivity.

The classification of infinite-dimensional, $\mathbb{Z}$-graded Lie superalgebras of finite depth is also due to V. G. Kac [K3] and is deeply related to the classification of linearly compact Lie superalgebras. We recall that finite depth implies finite growth.

This naturally leads to investigate infinite-dimensional, $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth. The hypothesis of finite growth is central to the problem; indeed, it is well known that it is not possible to classify $\mathbb{Z}$-graded Lie algebras (and thus Lie superalgebras) of any growth (see [K1], [M]). The only known examples of infinite-dimensional, $\mathbb{Z}$-graded Lie superalgebras of finite growth and infinite depth are given by contragredient Lie superalgebras which were classified by V. G. Kac in [K2] in the case of finite dimension and by J.W. van de Leur in the general case [vdL]. Contragredient Lie superalgebras, as well as Kac-Moody Lie algebras, have a $\mathbb{Z}$-gradation of infinite depth and growth equal to 1, due to their periodic structure.

We construct three new examples of infinite-dimensional Lie superalgebras with a consistent $\mathbb{Z}$-gradation of infinite depth and finite growth, and we realize...
them as covering superalgebras of finite-dimensional Lie superalgebras. It turns out that if $G$ is an irreducible, simple Lie superalgebra generated by its local part, with a consistent $\mathbb{Z}$-gradation, and if we assume that $G_0$ is simple and that $G_1$ is an irreducible $G_0$-module which is not contragredient to $G_{-1}$, then $G$ is isomorphic to one of these three algebras (Theorem 3.1) and its growth is therefore equal to 1.

So far, any known example of a $\mathbb{Z}$-graded Lie superalgebra of infinite depth and finite growth is, up to isomorphism, either a contragredient Lie superalgebra or the covering superalgebra of a finite-dimensional Lie superalgebra. Since the aim of this paper is analyzing $\mathbb{Z}$-graded Lie superalgebras of infinite depth, we shall not describe the cases of finite depth which can be found in [K2], [K3].

Let $G$ be a $\mathbb{Z}$-graded Lie superalgebra. Suppose that $G_0$ is a simple Lie algebra and that $G_{-1}$ and $G_1$ are irreducible $G_0$-modules and are not contragredient. Let $F_\Lambda$ be a highest weight vector of $G_{-1}$ of weight $\Lambda$ and let $E_M$ be a lowest weight vector of $G_1$ of weight $M$. Since $G_{-1}$ and $G_1$ are not contragredient, the sum $\Lambda + M$ is a root of $G_0$, and, without loss of generality, we may assume that it is a negative root, i.e. $\Lambda + M = -\alpha$ for some positive root $\alpha$. The paper is based on the analysis of the relations between the $G_0$-modules $G_{-1}$ and $G_1$. It is organized in three sections: Section 1 contains some basic definitions and fundamental results in the general theory of Lie superalgebras. In Section 2 the main hypotheses on the Lie superalgebra $G$ are introduced. Section 2.1 is devoted to the case $(\Lambda, \alpha) = 0$. Since $\Lambda$ is a dominant weight, in this section the rank of $G_0$ is assumed to be greater than 1. The hypothesis $(\Lambda, \alpha) = 0$ always holds for $\mathbb{Z}$-graded Lie superalgebras of finite depth (see [K2], Lemma 4.1.4 and [K3], Lemma 5.3) but if the Lie superalgebra $G$ has infinite depth weaker restrictions on the weight $\Lambda$ are obtained (compare, for example, Lemma 4.1.3 in [K2] with Lemma 1.14 in this paper).

In Section 2.2 we examine the case $(\Lambda, \alpha) \neq 0$. In the finite-depth case this hypothesis may not occur (cf. [K3], Lemma 5.3). It turns out that, under this hypothesis, $G_0$ has necessarily rank one (cf. Theorem 2.17) namely it is isomorphic to $sl(2)$. Besides, a strong restriction on the possible values of $(\Lambda, \alpha)$ is obtained (cf. Corollary 2.12) so that $G_{-1}$ is necessarily isomorphic either to the adjoint module of $sl(2)$ or to the irreducible $sl(2)$-module of dimension 2.

Finally, Section 3 is devoted to the construction of the examples and to the classification theorem.

Throughout the paper the base field is assumed to be algebraically closed and of characteristic zero.

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1. – Basic definitions and main results

1.1. – Lie superalgebras

**Definition 1.1.** A superalgebra is a \( \mathbb{Z}_2 \)-graded algebra \( A = A_0 \oplus A_1 \); \( A_0 \) is called the even part of \( A \) and \( A_1 \) is called the odd part of \( A \).

**Definition 1.2.** A Lie superalgebra is a superalgebra \( G = G_0 \oplus G_1 \) whose product \([ \cdot, \cdot ]\) satisfies the following axioms:

(i) \( [a, b] = -(-1)^{\deg(a) \deg(b)} [b, a] \);

(ii) \( [a, [b, c]] = [[a, b], c] + (-1)^{\deg(a) \deg(b)} [b, [a, c]] \).

**Definition 1.3.** A \( \mathbb{Z} \)-grading of a Lie superalgebra \( G \) is a decomposition of \( G \) into a direct sum of finite-dimensional \( \mathbb{Z}_2 \)-graded subspaces \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) for which \([G_i, G_j] \subset G_i + j \). A \( \mathbb{Z} \)-grading is said to be consistent if \( G_0 = \bigoplus G_{2i} \) and \( G_1 = \bigoplus G_{2i+1} \).

**Remark 1.4.** By definition, if \( G \) is a \( \mathbb{Z} \)-graded Lie superalgebra, then \( G_0 \) is a subalgebra of \( G \) and \([G_0, G_i] \subset G_i \); therefore the restriction of the adjoint representation to \( G_0 \) induces linear representations of it on the subspaces \( G_i \).

**Definition 1.5.** A \( \mathbb{Z} \)-graded Lie superalgebra \( G \) is called irreducible if \( G_{-1} \) is an irreducible \( G_0 \)-module.

**Definition 1.6.** A \( \mathbb{Z} \)-graded Lie superalgebra \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) is called transitive if for \( a \in G_i, i \geq 0, [a, G_{-1}] = 0 \) implies \( a = 0 \), and bitransitive if, in addition, for \( a \in G_i, i \leq 0, [a, G_1] = 0 \) implies \( a = 0 \).

Let \( \hat{G} \) be a \( \mathbb{Z}_2 \)-graded space, decomposed into the direct sum of \( \mathbb{Z}_2 \)-graded subspaces, \( \hat{G} = G_{-1} \oplus G_0 \oplus G_1 \). Suppose that whenever \(|i + j| \leq 1 \) a bilinear operation is defined: \( G_i \times G_j \rightarrow G_{i+j}, (x, y) \mapsto [x, y] \), satisfying the axiom of anticommutativity and the Jacobi identity for Lie superalgebras, provided that all the commutators in this identity are defined. Then \( \hat{G} \) is called a local Lie superalgebra.

If \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) is a \( \mathbb{Z} \)-graded Lie superalgebra then \( G_{-1} \oplus G_0 \oplus G_1 \) is a local Lie superalgebra which is called the local part of \( G \). The following proposition holds:

**Proposition 1.7** [K2]. Two bitransitive \( \mathbb{Z} \)-graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.

**Definition 1.8.** A Lie superalgebra is called simple if it contains no nontrivial ideals.

**Proposition 1.9** [K2]. If in a simple \( \mathbb{Z} \)-graded Lie superalgebra \( G = \bigoplus_{i \in \mathbb{Z}} G_i \) the subspace \( G_{-1} \oplus G_0 \oplus G_1 \) generates \( G \) then \( G \) is bitransitive.
1.2. – On the growth of $G$

**Definition 1.10.** Let $G = \oplus_{i \in \mathbb{Z}} G_i$ be a $\mathbb{Z}$-graded Lie superalgebra. The limit

$$r(G) = \lim_{n \to \infty} \ln \left( \sum_{i=-n}^{n} \dim G_i \right) / \ln(n)$$

is called the growth of $G$. If $r(G)$ is finite then we say that $G$ has finite growth.

Let us fix some notation. Given a semisimple Lie algebra $L$, by $V(\omega)$ we shall denote its finite-dimensional highest weight module of highest weight $\omega$. $\omega_i$ will be the fundamental weights. It is well known that if $\lambda$ is a weight of a finite-dimensional representation of $L$ and $\beta$ is a root of $L$, then the set of weights of the form $\lambda + s\beta$ forms a continuous string: $\lambda - p\beta, \lambda - (p - 1)\beta, \ldots, \lambda - \beta, \lambda, \lambda + \beta, \ldots, \lambda + q\beta$, where $p$ and $q$ are nonnegative integers and $p - q = 2(\lambda, \beta)/(\beta, \beta)$. Let us put $2(\lambda, \beta)/(\beta, \beta) = \lambda(h_{\alpha_i})$. The numbers $\lambda(h_{\alpha_i})$, for a fixed basis of simple roots $\alpha_i$, are called the numerical marks of the weight $\lambda$.

For any positive root $\beta$ of $L$ we shall denote by $e_\beta$ a root vector of $L$ corresponding to $\beta$.

**Lemma 1.11 [K1].** Let $L$ be a Lie algebra containing elements $H \neq 0, E_i, F_i, i = 1, 2$, connected by the equations

$$[E_i, F_j] = \delta_{ij} H,$$

$$[H, E_1] = a E_1,$$  
$$[H, E_2] = b E_2,$$

$$[H, F_1] = -a F_1,$$  
$$[H, F_2] = -b F_2,$$

where $a \neq -b, b \neq -2a$, and $a \neq -2b$, then the growth of $L$ is infinite.

**Lemma 1.12 [K1].** Let $L = \oplus L_i$ be a graded Lie algebra, where $L_0$ is semisimple. Assume that there exist weight vectors $x_\lambda$ and $x_\mu$ corresponding to the weights $\lambda$ and $\mu$ of the adjoint representation of $L_0$ on $L$, and a root vector $e_\gamma$, corresponding to the root $\gamma$ of $L_0$, which satisfy the following relations:

$$[x_\mu, x_\lambda] = e_\gamma,$$

$$[x_\lambda, e_{-\gamma}] = 0 = [x_\mu, e_\gamma],$$

$$\lambda(h_\gamma) \neq -1, (\lambda, \gamma) \neq 0.$$  

Then the growth of $L$ is infinite.

**Lemma 1.13.** Let $G$ be a consistent, $\mathbb{Z}$-graded Lie superalgebra and suppose that $G_0$ is a semisimple Lie algebra. Let $E_i, F_i (i = 1, 2)$ be odd elements and $H$ a non zero element in $G_0$ such that:

(1) $$[E_i, F_j] = \delta_{ij} H, \quad [H, E_i] = a_i E_i, \quad [H, F_i] = -a_i F_i,$$

where $a_1 \neq -a_2, a_1 \neq -2a_2$ and $a_2 \neq -2a_1$. Then the growth of $G$ is infinite.
Proof. Suppose first that \( a_1 \neq 0 \neq a_2 \). Then the elements \( \tilde{E}_1 = a_1^{-1/2}[E_1, E_1] \), \( \tilde{E}_2 = a_2^{-1/2}[E_2, E_2] \), \( \tilde{F}_1 = a_1^{-1/2}[F_1, F_1] \), \( \tilde{F}_2 = a_2^{-1/2}[F_2, F_2] \), \( K = -4H \) satisfy the hypotheses of Lemma 1.11 in the Lie algebra \( G_0 \). Thus, the growth of \( G_0 \) is infinite and we get the thesis.

If, let us say, \( a_1 \neq 0 \), \( a_2 = 0 \) then the elements \( E_1' = [E_1, E_1] \), \( E_2' = [E_1, E_2] \), \( F_1' = -(4a_1)^{-1}[F_1, F_1] \), \( F_2' = a_1^{-1}[F_1, F_2] \), \( H \) satisfy the hypotheses of Lemma 1.11 in \( G_0 \), thus we conclude. \( \square \)

Lemma 1.14. Let \( G = \bigoplus G_i \) be a \( \mathbb{Z} \)-graded, consistent Lie superalgebra and suppose that \( G_0 \) is a semisimple Lie algebra. Assume that there exist odd elements \( x_\lambda \) and \( x_\mu \) that are weight vectors of the adjoint representation of \( G_0 \) on \( G \) of weight \( \lambda \) and \( \mu \) respectively, and a root vector \( e_{-\delta} \) of \( G_0 \), connected by the relations:

\[
\begin{align*}
[x_\lambda, x_\mu] &= e_{-\delta} \\
[x_\lambda, e_{\delta}] &= [x_\mu, e_{-\delta}] = 0
\end{align*}
\]

with \( 2(\lambda, \delta) \neq (\delta, \delta) \), \( (\lambda, \delta) \neq 0 \) and \( (\lambda, \delta) \neq (\delta, \delta) \). Then the growth of \( G \) is infinite.

Proof. We choose a root vector \( e_\delta \) in \( G_0 \) such that \( [e_\delta, e_{-\delta}] = h_\delta \) and consider the following elements:

\[
\begin{align*}
E_1 &= [e_\delta, x_\mu] \\
E_2 &= [[[x_\mu, e_\delta], e_\delta], e_\delta] \\
F_1 &= x_\lambda \\
F_2 &= -1/6\lambda(h_\delta)^{-1}(\lambda(h_\delta) - 1)^{-1}[[x_\lambda, e_{-\delta}], e_{-\delta}] \\
H &= h_\delta.
\end{align*}
\]

By a direct computation it is easy to check that \( E_i, F_i, H \) satisfy the hypotheses of Lemma 1.13 with \( a_1 = (\mu + \delta)(h_\delta) = -\lambda(h_\delta) \), \( a_2 = (\mu + 3\delta)(h_\delta) = -\lambda(h_\delta) + 4 \). By Lemma 1.13 the growth of \( G \) is therefore infinite. \( \square \)

We can reformulate Lemma 1.14 as follows:

Corollary 1.15. Suppose that \( G \) is a Lie superalgebra of finite growth. Let \( x_\lambda \), \( x_\mu, e_{-\delta} \) be as in Lemma 1.14. Then one of the following holds:

(i) \( (\lambda, \delta) = 0 \),
(ii) \( (\lambda, \delta) = (\delta, \delta) \),
(iii) \( (\lambda, \delta) = 1/2(\delta, \delta) \).

Theorem 1.16 [K1]. Let \( L = \bigoplus L_i \) be a \( \mathbb{Z} \)-graded Lie algebra with the following properties:

a) the Lie algebra \( L_0 \) has no center;
b) the representations \( \phi_{-1} \) and \( \phi_1 \) of \( L_0 \) on \( L_{-1} \) and \( L_1 \) are irreducible;
c) \( [L_{-1}, L_1] \neq 0 \);
d) $\Lambda + M = -\alpha$ where $\Lambda$ is the highest weight of $\phi_{-1}$, $M$ is the lowest weight of $\phi_1$ and $\alpha$ is a positive root of $L_0$;

e) the representations $\phi_{-1}$ and $\phi_1$ are faithful;

f) the growth of $L$ is finite.

Then $L_0$ is isomorphic to one of the Lie algebras $A_n$ or $C_n$, $\phi_{-1}$ is the corresponding standard representation and $\alpha$ is the highest root of $L_0$.

In the following $sl_n$, $sp_n$ and $so_n$ will denote the standard representations of the corresponding Lie algebras.

**Corollary 1.17.** Let $G = \oplus G_i$ be a Lie superalgebra with a consistent $\mathbb{Z}$-gradation. Suppose that $G_0$ is simple. Suppose that there exist a highest weight vector $x$ in $G_{-2}$ of weight $\lambda \neq 0$ and a lowest weight vector $y$ in $G_2$ of weight $\mu$ such that $[x, y] \neq 0$ and $\lambda + \mu = -\rho$ for a positive root $\rho$ of $G_0$. Then, if the growth of $G$ is finite, $G_0$ is isomorphic to one of the Lie algebras $A_n$ or $C_n$, $\rho$ is the highest root of $G_0$ and $G_{-2}$ is the standard $G_0$-module.

**Proof.** It follows from Theorem 1.16. \square

### 2. Main results

In this section we will consider an irreducible, consistent, simple $\mathbb{Z}$-graded Lie superalgebra $G$ generated by its local part, and we will always suppose that $G$ has finite growth. Besides, we will assume that $G_0$ is a simple Lie algebra and that $G_1$ is an irreducible $G_0$-module which is not contragredient to $G_{-1}$. Let us fix a Cartan subalgebra $H$ of $G_0$ and the following notation: let $F_\Lambda$ be a highest weight vector of $G_{-1}$ of weight $\Lambda$ (dominant weight) and let $E_M$ be a lowest weight vector of $G_1$ of weight $M$. As shown in [K2], Proposition 1.2.10, it turns out that $[F_\Lambda, E_M] = e_{-\alpha}$, where $\alpha = -(\Lambda + M)$ is a root of $G_0$ and $e_{-\alpha}$ is a root vector in $G_0$ corresponding to $-\alpha$. Interchanging, if necessary, $G_k$ with $G_{-k}$ we can assume that $\alpha$ is a positive root. Indeed, by transitivity, $[F_\Lambda, E_M] \neq 0$ and for any $t \in H$ we have:

$$[t, [F_\Lambda, E_M]] = (\Lambda + M)(t)[F_\Lambda, E_M].$$

Notice that $\Lambda + M \neq 0$ since the representations of $G_0$ on $G_{-1}$ and $G_1$ are not contragredient.

**Remark 2.1.** Under the above assumptions, $-M = \Lambda + \alpha$ is a dominant weight. Therefore $(\Lambda + \alpha, \beta) \geq 0$ for every positive root $\beta$ of $G_0$.

**Lemma 2.2.** Under the above hypotheses, $[E_M, E_M] = 0$ and $[E_M, [e_\rho, E_M]] = 0$ for every positive root $\rho$. 
Proof. We have $[F_{\Lambda}, [E_M, E_M]] = 2[e_{-\alpha}, E_M] = 0$ since $E_M$ is a lowest weight vector. Transitivity and irreducibility imply $[E_M, E_M] = 0$. Now, since $E_M$ is odd, for every positive root $\rho$ we have:

$$[E_M, [e_{\rho}, E_M]] = [[E_M, e_{\rho}], E_M] = -[E_M, [e_{\rho}, E_M]]$$

therefore $[E_M, [e_{\rho}, E_M]] = 0$.

\[ \qed \]

2.1. – Case $(\Lambda, \alpha) = 0$

In this paragraph we suppose $(\Lambda, \alpha) = 0$. If $\Lambda$ is zero then the depth of $G$ is finite. Therefore we suppose that $\Lambda$ is not zero. This implies that the rank of $G_0$ is greater than one.

Remark 2.3. Let $G$ be a bitransitive, irreducible $\mathbb{Z}$-graded Lie superalgebra. If $(\Lambda, \alpha) = 0$ then the vectors $[F_{\Lambda}, F_{\Lambda}]$ and $[[F_{\Lambda}, e_{-\rho}], F_{\Lambda}]$ are zero for every positive root $\rho$.

Proof. Once we have shown that $[F_{\Lambda}, F_{\Lambda}] = 0$, we proceed as in Lemma 2.2 and conclude that $[[F_{\Lambda}, e_{-\rho}], F_{\Lambda}] = 0$ for every positive root $\rho$. Since $[[F_{\Lambda}, F_{\Lambda}], E_M] = 2[F_{\Lambda}, e_{-\alpha}] = 0$, we conclude by bitransitivity.

Lemma 2.4. $\alpha$ is the highest root of one of the parts of the Dynkin diagram of $G_0$ into which it is divided by the numerical marks of $\Lambda$.

Proof. Suppose by contradiction that $\alpha$ is not the highest root of one of the parts of the Dynkin diagram of $G_0$ into which it is divided by the numerical marks of $\Lambda$. Then there exists a simple root $\beta$ such that $(\Lambda, \beta) = 0$ and $\alpha + \beta$ is a root. This gives a contradiction because: $0 = [[e_{-\beta}, F_{\Lambda}], E_M] = [e_{-\beta}, [F_{\Lambda}, E_M]] = e_{-\beta - \alpha} \neq 0$.

Lemma 2.5. If $\Lambda$ has at least two numerical marks then, for every numerical mark $\gamma$, we have:

$$(\Lambda + \alpha, \gamma) = 0.$$  

Proof. From Lemma 2.4 we know that $\alpha$ is the highest root of one of the parts of the Dynkin diagram of $G_0$ into which it is divided by the numerical marks of $\Lambda$. Therefore we can choose a numerical mark $\beta$ such that $\alpha + \beta$ is a root. Now suppose that $\gamma$ is a numerical mark, $\gamma \neq \beta$, such that $(\Lambda + \alpha, \gamma) \neq 0$.

Notice that $\gamma$ and $\beta$ are not subroots of $\alpha$, since $(\Lambda, \gamma) \neq 0$ and $(\Lambda, \beta) \neq 0$, therefore $\gamma(h_\alpha) \leq 0$, $\beta(h_\alpha) < 0$.

Consider the following vectors:

$$x := [[[F_{\Lambda}, e_{-\beta}], e_{-\gamma}], F_{\Lambda}]$$

$$y := [[[E_M, e_{\alpha}], e_{\gamma}], E_M].$$

First of all we want to show that $x$ is a highest weight vector in $G_{-2}$. By Remark 2.3, since $\beta$ and $\gamma$ are simple roots, it is sufficient to show that $x \neq 0$. In fact, $[e_{\gamma}, [x, E_M]] = (\Lambda + \alpha)(h_\gamma)[F_{\Lambda}, [e_{-\alpha}, e_{-\beta}]] \neq 0.$
Now let us prove that \( y \) is a lowest weight vector in \( G_2 \). First \( y \neq 0 \), indeed:

\[
[y, F_\Lambda] = (2 - \gamma(h_\alpha))[E_M, e_\gamma]
\]

which is different from 0 since \( \gamma(h_\alpha) \leq 0 \) and by the assumption \( (\Lambda + \alpha, \gamma) \neq 0 \).

We now compute the commutators \([y, e_{-\alpha_k}]\) for any simple root \( \alpha_k \). If \( \alpha_k = \gamma \) then, by Lemma 2.2, \([y, e_{-\alpha_k}] = 0\), since \( \alpha - \gamma \) is not a root. If \( \alpha_k \neq \gamma \), \([y, e_{-\alpha_k}] = [[F_M, e_{\alpha - \alpha_k}], e_\gamma], E_M]\), and this can be shown to be zero using the transitivity of \( G \).

Notice that \([x, y] = (2 - \gamma(h_\alpha))(\Lambda + \alpha)(h_\gamma)e_{-\alpha - \beta}\). By Theorem 1.16 we get a contradiction since \( \alpha + \beta \) cannot be the highest root of \( G_0 \). As a consequence, \( (\Lambda + \alpha, \gamma) = 0 \). In particular, \( \alpha + \gamma \) is a root and we can repeat the same argument interchanging \( \beta \) and \( \gamma \) in order to get \( (\Lambda + \alpha, \beta) = 0 \). \( \square \)

**Corollary 2.6.** If \( G_0 \) is of type \( A_n, B_n, C_n, F_4, G_2 \) then \( \Lambda \) has at most two numerical marks; if \( G_0 \) is of type \( D_n, E_6, E_7, E_8 \) then \( \Lambda \) has at most three numerical marks.

**Proof.** Immediate from Lemma 2.5. \( \square \)

**Lemma 2.7.** If \( \Lambda \) has only one numerical mark \( \beta \) then either \( (\Lambda + \alpha, \beta) = 0 \) or \( \Lambda(h_\beta) = 1 \).

**Proof.** Suppose both \( (\Lambda + \alpha, \beta) \neq 0 \) and \( \Lambda(h_\beta) > 1 \), and define

\[
x := [[[F_\Lambda, e_{-\beta}], e_{-\beta}], F_\Lambda]
\]

\[
y := [[[E_M, e_\alpha], e_\beta], E_M]\].

Then \( x \) is a highest weight vector in \( G_{-2} \) and \( y \) is a lowest weight vector in \( G_2 \). Besides, \([x, y] = 2(2 - \beta(h_\alpha))(\Lambda + \alpha)(h_\beta)e_{-\alpha - \beta}\). By Theorem 1.16, \( G_0 \) is either of type \( A_n \) or of type \( C_n \); \( \alpha + \beta \) is the highest root of \( G_0 \) and \( G_{-2} \) is its elementary representation. It is easy to show that these conditions cannot hold. \( \square \)

**Proposition 2.8.** Let \( \beta \) be a positive root such that:

- \( \alpha + \beta \) is a root;
- \( \alpha - \beta \) is not a root;
- \( 2\alpha + \beta \) is not a root.

Then either \( (\Lambda + \alpha, \beta) = 0 \) or \( \Lambda(h_\beta) = 1 \).

**Proof.** Let us first make some remarks:

(a) Since \( \beta + \alpha \) is a root but \( \beta + 2\alpha \) and \( \beta - \alpha \) are not, we have \( \beta(h_\alpha) = -1 \).

It follows that \( \alpha + \beta \) and \( \beta \) are roots of the same length.

(b) Since \( \beta - (\alpha + \beta) \) is a root and \( \beta - 2(\alpha + \beta) \) is not, then \( \beta(h_{\alpha + \beta}) \leq 1 \).

Now suppose that \( \Lambda(h_\beta) > 1 \), which implies \([F_\Lambda, e_{-\beta}] \neq 0\). Let \( x_\mu = E_M \) and \( x_\lambda = [F_\Lambda, e_{-\beta}] \). We have:

\[
[x_\lambda, x_\mu] = e_{-\alpha - \beta}
\]

\[
[e_{-\alpha - \beta}, x_\mu] = 0 = [x_\lambda, e_{\alpha + \beta}]\].
Therefore, by Lemma 1.14, we deduce that the difference $\Lambda(h_\beta) - \beta(h_{\alpha+\beta})$ is equal to 0, 1, or 2. In particular, $2 \leq \Lambda(h_\beta) \leq 3$ and $0 \leq \beta(h_{\alpha+\beta}) \leq 1$. We therefore distinguish the following two cases:

**Case A:** $\beta(h_{\alpha+\beta}) = 0$, i.e. $\alpha + 2\beta$ is a root, $2\alpha + 3\beta$ is not, and $\Lambda(h_\beta) = 2$.
In this case $(\beta, \beta) = - (\beta, \alpha)$ and $(\Lambda, \beta) = (\beta, \beta)$ therefore $(\Lambda + \alpha, \beta) = 0$ which concludes the proof in this case.

**Case B:** $\beta(h_{\alpha+\beta}) = 1$, i.e. $\alpha + 2\beta$ is not a root, and $\Lambda(h_\beta)$ is either 2 or 3.
In this case $\beta(h_\alpha) = -1 = \alpha(h_\beta)$, therefore $(\Lambda + \alpha, \beta) \neq 0$. The two cases $\Lambda(h_\beta) = 2$ and $\Lambda(h_\beta) = 3$ need to be analyzed separately.

(i) $\Lambda(h_\beta) = 2$

Let us define the following elements:

$$x_\lambda = [[F_\Lambda, e_{-\beta}], e_{-\beta}], F_\Lambda]$$

$$x_\mu = [[E_M, e_{\alpha+\beta}], e_{\beta}], E_M].$$

Then $[x_\lambda, x_\mu] = 6e_{-\alpha}$, $[x_\lambda, e_\alpha] = 0$ since $\alpha - \beta$ is not a root, and $[x_\mu, e_{-\alpha}] = 0$ since $(\Lambda + \alpha)(h_\beta) = 1$, thus $[[E_M, e_\beta], e_\beta] = 0$. Then we find a contradiction to Lemma 1.12 applied to the Lie algebra $\mathcal{G}_0$, since $\mathcal{G}$ was assumed to have finite growth. Indeed, using the same notation as in Lemma 1.12, we have: $\lambda(h_\gamma) = -\lambda(h_\alpha) = -(2\Lambda - 2\beta)(h_\alpha) = 2\beta(h_\alpha) = -2$.

(ii) $\Lambda(h_\beta) = 3$

Let us define the following elements:

$$E_1 = 1/8[[E_M, e_{\alpha+\beta}], [E_M, e_{\alpha+\beta}]]$$

$$F_1 = [[F_\Lambda, e_{-\beta}], [F_\Lambda, e_{-\beta}]]$$

$$E_2 = 1/64[[[E_M, e_{\alpha+\beta}], e_\beta], [[E_M, e_{\alpha+\beta}], e_\beta]]$$

$$F_2 = [[[F_\Lambda, e_{-\beta}], e_{-\beta}], [[F_\Lambda, e_{-\beta}], e_{-\beta}]]$$

$$H = h_{\alpha+\beta} = h_\alpha + h_\beta$$

Then the hypotheses of Lemma 1.11 are satisfied with $a_1 = -4$ and $a_2 = -2$, and this leads to a contradiction.

In the following, for what concerns simple Lie algebras, we will use the same notation as in [H, §11, §12]. In particular we shall adopt the same enumeration of the vertices in the Dynkin diagrams and refer to the bases of simple roots described by Humphreys [H].
Lemma 2.9. Let \( M \) be the lowest weight of the \( G_0 \)-module \( G_1 \).

(i) Let \( z := [[E_M, e_{\alpha + \beta}], [e_\gamma, E_M]], \) where \( \beta \) and \( \gamma \) are positive roots of \( G_0 \) such that \( [E_M, e_\beta] = 0, \alpha + \beta + \gamma \) is not a root, \( \beta + \gamma \) is not a root and \( \gamma - \alpha \) is a negative root. Then \( \langle z, F_\Lambda \rangle = 0. \)

(ii) Let \( \beta \) and \( \rho \) be positive roots such that \( \alpha + \beta \) and \( \beta + \rho \) are positive roots, \( \alpha + \beta + \rho \) is not a root, \( \beta + \rho \) is not a root, \( \rho - \alpha \) is a negative root. If \((M, \beta) = 0 \) and \((M, \rho) \neq 0\), then the vector \( [[E_M, e_{\alpha + \beta}], [e_\rho, E_M]] \) is non-zero.

(iii) Let \( \beta \) and \( \rho \) be as in (ii) and let \( \alpha_k \) be a simple root of \( G_0 \). Suppose, in addition, that either \( \rho + \beta - \alpha_k \) is not a root or \((M, \rho + \beta - \alpha_k) = 0\). Then \( [[E_M, e_{\alpha + \beta - \alpha_k}], [e_\rho, E_M]], F_\Lambda \rangle = 0. \)

(iv) If \( \rho \) is a positive root such that \( \alpha + \rho \) is not a root, \( \rho - \alpha \) is a negative root, \((M, \rho) \neq 0\) and \( \rho(h_\alpha) = 1 \), then \( [[E_M, e_\alpha], [e_\rho, E_M]], F_\Lambda \rangle = 0. \)

Proof. The proof consists of simple direct computations. \( \square \)

Theorem 2.10. Let \( G \) be an irreducible, simple, \( \mathbb{Z} \)-graded Lie superalgebra of finite growth, generated by its local part. Suppose that \( G_0 \) is simple, that the \( \mathbb{Z} \)-gradation of \( G \) is consistent and that \( (\Lambda, \alpha) = 0 \). If \( G \) has infinite depth then one of the following holds:

- \( G_0 \) is of type \( A_3 \), \( G_{-1} \) is its adjoint module, \( G_1 = V(2\omega_2) \);
- \( G_0 \) is of type \( B_n \) \((n \geq 2)\), \( G_{-1} \) is its adjoint module, \( G_1 = V(2\omega_1) \);
- \( G_0 \) is of type \( C_n \) \((n \geq 3)\), \( G_{-1} \cong \Lambda_{10}^{2} sp_{2n}, G_1 \) is its adjoint module;
- \( G_0 \) is of type \( D_n \) \((n \geq 4)\), \( G_{-1} \) is its adjoint module, \( G_1 = V(2\omega_1) \).

Proof. Let us analyze all the possible cases. Corollary 2.6 states that if \( G_0 \) is of type \( A_n, B_n, C_n, F_4 \) or \( G_2 \) then \( \Lambda \) might have one or two numerical marks while if \( G_0 \) is of type \( D_n, E_6, E_7 \) or \( E_8 \) then \( \Lambda \) might also have three numerical marks. Using Lemma 2.5 one can easily see that if \( G_0 \) is not of type \( A_n \) then the hypothesis that \( \Lambda \) has at least two numerical marks contradicts Proposition 2.8. It follows that if \( G_0 \) is not of type \( A_n \) then \( \Lambda \) has exactly one numerical mark and this numerical mark satisfies Lemma 2.7.

Using Remark 2.1 we immediately exclude the following possibilities, for which the weight \( M \) is not antidominant:

- \( G_0 \) of type \( B_n \) \((n \geq 2)\), \( G_{-1} = V(\omega_n) \);
- \( G_0 \) of type \( C_n \) \((n \geq 3)\), \( G_{-1} = V(\omega_1) \);
- \( G_0 \) of type \( C_n \) \((n \geq 3)\), \( G_{-1} = V(\omega_i) \) with \( 2 \leq i \leq n-1, \alpha = 2\alpha_{i+1} + \cdots + 2\alpha_{n-1} + \alpha_n \);
- \( G_0 \) of type \( F_4 \), \( G_{-1} = V(\omega_3) \), \( \alpha = \alpha_1 + \alpha_2 \);
- \( G_0 \) of type \( F_4 \), \( G_{-1} = V(\omega_4) \);
- \( G_0 \) of type \( G_2 \), \( G_{-1} = V(2\omega_1) \);
- \( G_0 \) of type \( G_2 \), \( G_{-1} = V(\omega_1) \) (simplest representation).
Proposition 2.8 allows us to rule out the cases summarized in Table 1, where we describe the irreducible modules \( G_{-1} \) and \( G_1 \) through their highest weights and indicate the positive root \( \beta \) used in Proposition 2.8.

On the other hand, Corollary 1.17 allows us to rule out the cases summarized in Table 2, where the vectors \( x \) and \( y \) used in Corollary 1.17 are indicated, and where the columns denoted by \( G_{-1} \) and \( G_1 \) contain the highest weights of these \( G_0 \)-modules. In order to show that the vectors \( x \) and \( y \) in Table 2 are highest and lowest weight vectors in the \( G_0 \)-modules \( G_{-2} \) and \( G_2 \) respectively, one can use the bitransitivity of \( G \) and, where needed, Lemma 2.9.

For the remaining cases let us point out what follows: suppose that \( G_{-2} \) contains a highest weight vector \( x \) of weight \( \lambda \) and that \( G_2 \) contains a lowest weight vector \( y \) of weight \(-\lambda \) such that \([x, y] \neq 0\). Then the irreducible submodules \( G_{-2} \) and \( G_2 \) generated respectively by \( x \) and \( y \) are dual \( G_0 \)-modules and the Lie subalgebra of \( G_0 \) with local part \( G_{-2} \oplus G_0 \oplus G_2 \) is an affine Kac-Moody algebra which will be denoted by \( A \).

Using the classification of affine Kac-Moody algebras we therefore exclude the cases in Table 3, where we indicate the highest weight vector \( x \) of \( G_{-2} \), the lowest weight vector \( y \) of \( G_2 \), and the highest weights of the \( G_0 \)-modules \( G_{-1} \) and \( G_1 \).

In the same way the classification of affine Kac-Moody algebras shows that the following cases are allowed:

1) \( G_0 \) of type \( A_3 \), \( G_{-1} = V(\omega_2 + \omega_3) \), \( G_1 = V(2\omega_2) \), \( \alpha = \alpha_2 \): under these hypotheses \( G_{-2} \) contains the highest weight vector \( x = [[F_\Lambda, e_{-\alpha_1}], [e_{-\alpha_3}, F_\Lambda]] \) and \( G_2 \) contains the lowest weight vector \( y = [[E_M, e_{\alpha_1 + \alpha_2}], [e_{\alpha_2 + \alpha_3}, E_M]] \).

The algebra \( A \) is an affine Kac-Moody algebra of type \( A_5^{(2)} \).

2) \( G_0 \) of type \( B_n \) \((n \geq 3)\), \( G_{-1} = V(\omega_2) \), \( G_1 = V(2\omega_1) \), \( \alpha = \alpha_1 \): \( G_{-2} \) contains the highest weight vector \( x = [[F_\Lambda, e_{-\alpha_2}], [e_{-\alpha_2 - 2\alpha_3 - \cdots - 2\alpha_n}, F_\Lambda]] \) and \( G_2 \) contains the lowest weight vector \( y = [[E_M, e_{\alpha_1 + \alpha_2}], [e_{\alpha_1 + \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n}, E_M]] \).

The algebra \( A \) is an affine Kac-Moody algebra of type \( A_{2n}^{(2)} \).

3) \( G_0 \) of type \( B_2 \), \( G_{-1} = V(2\omega_2) \), \( G_1 = V(2\omega_1) \), \( \alpha = \alpha_1 \): \( G_{-2} \) contains the highest weight vector \( x = [[F_\Lambda, e_{-\alpha_2}], [e_{-\alpha_2, F_\Lambda}]] \) and \( G_2 \) contains the lowest weight vector \( y = [[E_M, e_{\alpha_1 + 2\alpha_2}], [e_{\alpha_1}, E_M]] \).

The algebra \( A \) is an affine Kac-Moody algebra of type \( A_4^{(2)} \).

4) \( G_0 \) of type \( C_n \) \((n \geq 3)\), \( G_{-1} = V(\omega_2) \), \( G_1 = V(2\omega_1) \), \( \alpha = \alpha_1 \): \( G_{-2} \) contains the highest weight vector \( x = [[F_\Lambda, e_{-\alpha_2 - \cdots - \alpha_n}], [e_{-\alpha_1 - \cdots - \alpha_n - 1, F_\Lambda}]] \) and \( G_2 \) contains the lowest weight vector \( y = [[E_M, e_{\alpha_1}], [e_{2\alpha_1 + \cdots + 2\alpha_n - 1 + \alpha_n}, E_M]] \).

The algebra \( A \) is an affine Kac-Moody algebra of type \( A_{2n-1}^{(2)} \).

5) \( G_0 \) of type \( D_n \) \((n \geq 4)\), \( G_{-1} = V(\omega_2) \), \( G_1 = V(2\omega_1) \), \( \alpha = \alpha_1 \): in this case \( x = [[F_\Lambda, e_{-\alpha_2}], [e_{-\alpha_2 - 2\alpha_3 - \cdots - 2\alpha_n - 1 - \alpha_n}, F_\Lambda]] \) and \( y = [[E_M, e_{\alpha_1 + \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n - 1 + \alpha_n}, E_M]] \).

The algebra \( A \) is an affine Kac-Moody algebra of type \( A_{2n-1}^{(2)} \).
Table 1

<table>
<thead>
<tr>
<th>$G_0$</th>
<th>$G_{-1}$</th>
<th>$G_1$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>$\omega_1$</td>
<td></td>
<td>$\alpha_1 + \cdots + \alpha_{i-1}$</td>
<td>$\alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_n$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$2\omega_n$</td>
<td></td>
<td>$\alpha_1 + \cdots + \alpha_{n-1}$</td>
<td>$\alpha_{n-1} + 2\alpha_n$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\omega_1$</td>
<td></td>
<td>$\alpha_1 + \cdots + \alpha_{i-1}$</td>
<td>$\alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\omega_1$</td>
<td></td>
<td>$\alpha_1 + \cdots + \alpha_{i-1}$</td>
<td>$\alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$</td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_j$</td>
<td></td>
<td>$\alpha_1 + \alpha_2 + \cdots + \alpha_j$</td>
<td>$3 \leq i \leq n-1$</td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_2$</td>
<td></td>
<td>$\alpha_1 + \alpha_2 + \cdots + \alpha_j$</td>
<td>$n &gt; 2$</td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_3$</td>
<td></td>
<td>$\alpha_2 + \alpha_3 + \cdots + \alpha_j$</td>
<td>$3 \leq i \leq n-1$</td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_4$</td>
<td></td>
<td>$\alpha_3 + \alpha_4 + \cdots + \alpha_j$</td>
<td>$3 \leq i \leq n-2$</td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_5$</td>
<td></td>
<td>$\alpha_4 + \alpha_5 + \cdots + \alpha_j$</td>
<td></td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_6$</td>
<td></td>
<td>$\alpha_5 + \alpha_6 + \cdots + \alpha_j$</td>
<td></td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_7$</td>
<td></td>
<td>$\alpha_6 + \alpha_7 + \cdots + \alpha_j$</td>
<td></td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_8$</td>
<td></td>
<td>$\alpha_7 + \cdots + \alpha_j$</td>
<td></td>
</tr>
<tr>
<td>$E_n$</td>
<td>$\omega_{k+1}$</td>
<td></td>
<td>$\alpha_k + \cdots + \alpha_{j-1}$</td>
<td>$4 \leq k \leq 6$</td>
</tr>
</tbody>
</table>

Row $G_2$ is not shown in the table.
<table>
<thead>
<tr>
<th>G0</th>
<th>G−1</th>
<th>G1</th>
<th>x</th>
<th>y</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_n</td>
<td>ω_n</td>
<td>ω_{n+1} + ω_n</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-1}-ω_n-ω_{n+1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n-1}, E_M])</td>
<td>(n \geq 5, s \neq 1, 2, n)</td>
</tr>
<tr>
<td>A_n</td>
<td>ω_n</td>
<td>1 + ω_{n-1}</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-1}-ω_n-ω_{n-1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n-1}, E_M])</td>
<td>(n \geq 5, s \neq 1, n-1, n)</td>
</tr>
<tr>
<td>A_n</td>
<td>ω_n</td>
<td>0</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(s \geq 1, s + 2 &lt; i \leq n)</td>
</tr>
<tr>
<td>B_n</td>
<td>ω_n</td>
<td>ω_{n+2}</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-1}-ω_n-ω_{n+2}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(2 \leq s \leq n - 3)</td>
</tr>
<tr>
<td>B_n</td>
<td>ω_{n-2}</td>
<td>2ω_n</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-3}-ω_n-2-ω_{n-2}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(n \geq 3)</td>
</tr>
<tr>
<td>B_n</td>
<td>ω_{n-1}</td>
<td>2ω_n</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-2}-ω_n-1-α_n, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(1 \leq i \leq n - 2)</td>
</tr>
<tr>
<td>C_n</td>
<td>2α_n</td>
<td>2α_{n+1}</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(2 \leq i \leq n - 4)</td>
</tr>
<tr>
<td>C_n</td>
<td>ω_n</td>
<td>ω_1 + ω_{n-1}</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(n &gt; 4)</td>
</tr>
<tr>
<td>D_n</td>
<td>ω_{n-3}</td>
<td>ω_{n-1} + ω_n</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-4}-ω_n-3-ω_{n-2}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(n &gt; 4)</td>
</tr>
<tr>
<td>D_n</td>
<td>ω_n</td>
<td>ω_1 + ω_{n-1}</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-3}-2ω_n-2-ω_{n-1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(n &gt; 4)</td>
</tr>
<tr>
<td>D_n</td>
<td>ω_{n-1}</td>
<td>ω_1 + ω_{n-1}</td>
<td>([F_A, e_{-ω_n}], [e_{-ω_n-3}-ω_n-2-2ω_{n-1}, F_A])</td>
<td>([E_M, e_{α_n}], [e_{α_n}, E_M])</td>
<td>(n &gt; 4)</td>
</tr>
<tr>
<td>E_0</td>
<td>α_0</td>
<td>α_0</td>
<td>([F_A, e_{-α_0}], [e_{-α_0}, F_A])</td>
<td>([E_M, e_{α_0}], [e_{α_0}, E_M])</td>
<td>(γ = α_0 + α_1 + 2α_2 + 3α_3 + 3α_4 + 2α_5)</td>
</tr>
<tr>
<td>E_0</td>
<td>α_0</td>
<td>0</td>
<td>([F_A, e_{-α_0}], [e_{-α_0}, F_A])</td>
<td>([E_M, e_{α_0}], [e_{α_0}, E_M])</td>
<td>(γ = α_0 + α_1 + 2α_2 + 3α_3 + 3α_4 + 2α_5)</td>
</tr>
<tr>
<td>E_0</td>
<td>2α_0</td>
<td>2α_0</td>
<td>([F_A, e_{-α_0}], [e_{-α_0}, F_A])</td>
<td>([E_M, e_{α_0}], [e_{α_0}, E_M])</td>
<td>(γ = α_0 + α_1 + 2α_2 + 3α_3 + 3α_4 + 2α_5)</td>
</tr>
<tr>
<td>E_1</td>
<td>α_1</td>
<td>α_1</td>
<td>([F_A, e_{-α_1}], [e_{-α_1}, F_A])</td>
<td>([E_M, e_{α_1}], [e_{α_1}, E_M])</td>
<td>(γ = α_0 + α_1 + 2α_2 + 3α_3 + 3α_4 + 2α_5)</td>
</tr>
<tr>
<td>E_1</td>
<td>α_1</td>
<td>2α_1</td>
<td>([F_A, e_{-α_1}], [e_{-α_1}, F_A])</td>
<td>([E_M, e_{α_1}], [e_{α_1}, E_M])</td>
<td>(γ = α_0 + α_1 + 2α_2 + 3α_3 + 3α_4 + 2α_5)</td>
</tr>
<tr>
<td>G_2</td>
<td>Ad</td>
<td>2α_1</td>
<td>([F_A, e_{-α_1}], [e_{-α_1}, F_A])</td>
<td>([E_M, e_{2α_1}], [e_{2α_1}, E_M])</td>
<td>(γ = α_0 + α_1 + 2α_2 + 3α_3 + 3α_4 + 2α_5)</td>
</tr>
</tbody>
</table>
Table 3

<table>
<thead>
<tr>
<th>$G_0$</th>
<th>$G_{-1}$</th>
<th>$G_1$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\omega_s + \omega_{s+2}$</td>
<td>$2\omega_{n-s}$</td>
<td>$[F, e^{-\alpha_s}, [e^{-\alpha_{s+2}}, F]]$</td>
<td>$[E_M, e^{\alpha_1}, [e^{\alpha_{s+1}} + \alpha_{s+2}, E_M]]$</td>
</tr>
<tr>
<td>($n \geq 4$)</td>
<td>($1 \leq s \leq n - 2$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_n$</td>
<td>$2\omega_{n-1}$</td>
<td>$2\omega_n$</td>
<td>$[F, e^{-\alpha_{n-1}}, [e^{-\alpha_n}, F]]$</td>
<td>$[E_M, e^{\alpha_{n-1} + \alpha_n}, [e^{\alpha_n}, E_M]]$</td>
</tr>
<tr>
<td>($n \geq 3$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\omega_{n-2}$</td>
<td>$2\omega_n$</td>
<td>$[F, e^{-\alpha_{n-3} - \alpha_{n-2} - \alpha_n}, [e^{-\alpha_n}, F]]$</td>
<td>$[E_M, e^{\alpha_{n-3} + \alpha_{n-2} + \alpha_n - 1}, [e^{\alpha_n}, E_M]]$</td>
</tr>
<tr>
<td>($n &gt; 4$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\omega_5$</td>
<td>$2\omega_6$</td>
<td>$[F, e^{-\alpha_2}, [e^{-\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5}, F]]$</td>
<td>$[E_M, e_{\alpha_1}, [e_{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5}, E_M]]$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$\omega_5$</td>
<td>$2\omega_7$</td>
<td>$[F, e^{-\alpha_5}, [e^{-\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5}, F]]$</td>
<td>$[E_M, e_{\alpha_6}, [e_{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6}, E_M]]$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\omega_6$</td>
<td>$2\omega_7$</td>
<td>$[F, e^{-\alpha_6}, [e^{-\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6}, F]]$</td>
<td>$[E_M, e_{\alpha_7}, [e_{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7}, E_M]]$</td>
</tr>
</tbody>
</table>
We finally analyze and rule out the remaining cases:

• $G_0$ of type $A_n$, $G_{-1} = V(\omega_3)$:
  
  (i) if $s = 1$ (or, equivalently, $s = n$) then $G_1 = V(\omega_1 + \omega_{n-1})$ and $G_{-2} \subset S^2 G_{-1} = S^2 V(\omega_1) = 0$ since $S^2 V(\omega_1) = V(2\omega_1)$ and $[F_\Lambda, F_\Lambda] = 0$, therefore $G$ has finite depth;

  (ii) if $s = 2$, $G_1 = V(2\omega_n)$, $\alpha = \alpha_1$, (or, equivalently, $s = n - 1$, $G_1 = V(2\omega_1)$, $\alpha = \alpha_n$) then $G$ is isomorphic to the finite-dimensional Lie superalgebra $p(n)$ (for the definition of $p(n)$ see [K2]);

  (iii) if $n = 4$ and $s = 3$, i.e. $G_{-1} \cong \Lambda^2 sl_5^*$, $G_1 = V(\omega_3 + \omega_4)$, $\alpha = \alpha_1 + \alpha_2$ (or, equivalently, $G_{-1} = V(\omega_2)$, $G_1 = V(\omega_1 + \omega_2)$, $\alpha = \alpha_3 + \alpha_4$), then $G$ is isomorphic to the infinite-dimensional Lie superalgebra $E(5, 10)$ (for the definition of $E(5, 10)$ see [K3]).

• $G_0$ of type $B_n$ ($n \geq 2$), $G_{-1} = V(\omega_1)$ and:
  
  (i) $G_1 = V(\omega_2)$ if $n > 3$ ($\alpha = \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n$),

  (ii) $G_1 = V(2\omega_3)$ if $n = 3$ ($\alpha = \alpha_2 + 2\alpha_3$),

  (iii) $G_1 = V(\omega_2)$ if $n = 2$ ($\alpha = \alpha_2$).

For all these cases $G_{-2} \subset S^2 G_{-1} = S^2 V(\omega_1) = V(2\omega_1) + 1 = 1$ since $[F_\Lambda, F_\Lambda] = 0$. Thus $G$ has finite depth.

• $G_0$ of type $D_n$ ($n \geq 4$):
  
  (i) $G_{-1} = V(\omega_1)$, $G_1 = V(\omega_1 + \omega_3)$, then $G_{-2} \subset S^2 G_{-1} = S^2 V(\omega_1) = V(2\omega_1) + 1 = 1$ hence $G$ has finite depth.

  (ii) $n = 4$, $G_{-1} = V(\omega_4)$, $G_1 = V(\omega_1 + \omega_4)$ ($\alpha = \alpha_1 + \alpha_2 + \alpha_3$) (or, equivalently, $G_{-1} = V(\omega_5)$, $G_1 = V(\omega_1 + \omega_5)$), then we can use the same argument as in (i) and conclude. \(\square\)

2.2. – Case $(\Lambda, \alpha) \neq 0$

In the following we assume $(\Lambda, \alpha) \neq 0$.

Remark 2.11. Under the hypothesis $(\Lambda, \alpha) \neq 0$ the vector $[F_\Lambda, F_\Lambda]$ is different from 0: $[E_M, [F_\Lambda, F_\Lambda]] = 2[e_{-\alpha}, F_\Lambda] \neq 0$.

Nevertheless, $[E_M, E_M] = 0$ and therefore $[[E_M, e_\beta], E_M] = 0$ for every positive root $\beta$ (see Lemma 2.2).

Corollary 2.12. If $(\Lambda, \alpha) \neq 0$ then either $(\Lambda, \alpha) = (\alpha, \alpha)$ or $(\Lambda, \alpha) = (\alpha, \alpha)/2$.

Proof. It is enough to apply Lemma 1.14 to the following vectors:

$$x_\Lambda = F_\Lambda, \quad x_\mu = E_M.$$ \(\square\)
Lemma 2.13. Suppose that $\alpha$ is not simple. Then there exists $j$ such that: $\alpha = \alpha_j$ is a root and $\alpha + \alpha_j$, $2\alpha - \alpha_j$, $\alpha - 2\alpha_j$ are not roots, in all cases except those in the following list:

- $G_0$ of type $B_n$ and $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$, $\alpha = \alpha_{n-1} + 2\alpha_n$;
- $G_0$ of type $C_n$ and $\alpha = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$, $\alpha = \alpha_{n-1} + \alpha_n$;
- $G_0$ of type $F_4$ and $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha = \alpha_2 + \alpha_3$, $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\alpha = \alpha_2 + 2\alpha_3$, $\alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$;
- $G_0$ of type $G_2$ and $\alpha = 2\alpha_1 + \alpha_2$, $\alpha = \alpha_1 + \alpha_2$, $\alpha = 3\alpha_1 + \alpha_2$.

Proof. Case by case check. \qed

Lemma 2.14. Let $\alpha$ be a positive root of $G_0$ and suppose that it is not simple. If $\alpha_j$ is a simple root of $\alpha$ such that $\alpha_j \alpha_j$ is a root and $\alpha + \alpha_j$, $2\alpha - \alpha_j$, $\alpha - 2\alpha_j$ are not roots, then either

$$\tilde{x} := [[[E_M, e_{\alpha}], e_{\alpha - \alpha_j}], E_M]$$

is a lowest weight vector in $G_{-2}$ or $\tilde{x} = 0$ and

$$x := [[[E_M, e_{\alpha}], e_{\alpha}], E_M]$$

is a lowest weight vector in $G_{-2}$.

Proof. If $\tilde{x} \neq 0$ then, using the transitivity of $G$, one can show that it is a lowest weight vector in $G_{-2}$. If $\tilde{x} = 0$ then $[x, e_{-k}]=0$ for every $k=1, \ldots, n$. \qed

Proposition 2.15. If $\alpha$ is not a simple root and the growth of $G$ is finite then either $(G_0, \alpha)$ belongs to the list in Lemma 2.13 or $(G_0, \alpha) = (A_n, longest root)$ and $\tilde{x} := [[[E_M, e_{\alpha}], e_{\alpha - \alpha_j}], E_M] \neq 0$.

Proof. Suppose that $(G_0, \alpha)$ is not in the list in Lemma 2.13. Since $\alpha$ is not a simple root we can apply Lemma 2.14: in the case $\tilde{x} = 0$ we take $y = [F_\Lambda, F_\Lambda]$. Then $[x, y] = 2\Lambda(h_\alpha)e_{-\alpha + \alpha_j} \neq 0$, and, by Theorem 1.16, we get infinite growth.

If $\tilde{x} := [[[E_M, e_{\alpha}], e_{\alpha - \alpha_j}], E_M] \neq 0$ then, by bitransitivity, $[\tilde{x}, F_\Lambda] = (\Lambda(h_\alpha) - 2\Lambda(h_j))E_M \neq 0$, thus $[\tilde{x}, y] = (\Lambda(h_\alpha) - 2\Lambda(h_j))e_{-\alpha}$ is different from zero. Then the thesis follows from Theorem 1.16. (Notice that the case $G_0$ of type $C_n$, $\alpha$ its longest root, is in the list of Lemma 2.13 and is therefore excluded by the hypotheses.) \qed

Lemma 2.16. If the growth of $G$ is finite and $\beta$ is a positive root such that $\alpha + \beta$ and $\alpha - \beta$ are not roots, then $(\Lambda, \beta) = 0$.

Proof. Suppose $(\Lambda, \beta) \neq 0$. We define:

$$E_1 = [e_\alpha, E_M], \quad E_2 = [[[E_M, e_\alpha], e_\beta], E_M],$$

$$F_1 = F_\Lambda, \quad F_2 = \Lambda(h_\beta)^{-1}[F_\Lambda, e_{-\beta}],$$

$$H = h_\alpha.$$

It is easy to verify that the conditions of Lemma 1.13 are satisfied with $a_1 = a_2 = -\Lambda(h_\alpha)$, thus $r(G) = \infty$. \qed
Theorem 2.17. Let $G = \bigoplus_{i \in \mathbb{Z}} G_i$ be a $\mathbb{Z}$-graded, consistent, simple, irreducible Lie superalgebra of finite growth. Assume that $G_0$ is a simple Lie algebra, that $G_1$ is an irreducible $G_0$-module which is not contragredient to $G_{-1}$ and that the local part generates $G$. Let $F_{\Lambda}$ be a highest weight vector in $G_{-1}$ and $E_M$ a lowest weight vector in $G_1$ so that $\Lambda + M = -\alpha$ for a positive root $\alpha$. If $(\Lambda, \alpha) \neq 0$ then $G_0$ has rank 1.

Proof. By Proposition 2.15 and its proof only the following cases may occur:

- $\alpha$ is a simple root;
- $(G_0, \alpha) = (A_n, \text{longest root});$
- $(G_0, \alpha)$ is in the list of Lemma 2.13.

Let us analyze these possibilities case by case:

1) $G_0$ of type $A_n$, $\alpha = \alpha_1 + \cdots + \alpha_n$. If $n = 1$ we get the thesis. Now suppose $n \geq 2$. The proof of Proposition 2.15 shows that this possibility holds if

$$\tilde{x} = [[[E_M, e_j], e_{\alpha_i}], E_M]$$

is a nonzero vector, thus either $j = 1$ or $j = n$. If we apply Lemma 2.16 to $\alpha = \alpha_1 + \cdots + \alpha_n$ and $\beta = \alpha_2 + \cdots + \alpha_{n-1}$ we deduce that $(\Lambda, \alpha_i) = 0$ for every $i = 2, \ldots, n - 1$, therefore $(\Lambda, \alpha) = (\Lambda, \alpha_1) + (\Lambda, \alpha_n)$.

As we already noticed in the proof of Proposition 2.15, for every $k = 1, \ldots, n$, $[\tilde{x}, e_{-k}] = 0$ thus, since we assume $\tilde{x} \neq 0$, transitivity implies $[\tilde{x}, F_{\Lambda}] \neq 0$. Since $[\tilde{x}, F_{\Lambda}] = (\Lambda(h_\alpha) - 2\Lambda(h_j))E_M$, it turns out that $\Lambda(h_1) \neq \Lambda(h_n)$. Corollary 2.12 now implies that either $(\Lambda, \alpha_1) = 0$ or $(\Lambda, \alpha_n) = 0$. But this hypothesis contradicts Theorem 1.16, since if we take the highest weight vector $y = [F_{\Lambda}, F_{\Lambda}]$ in $G_{-2}$, then $[\tilde{x}, y] \neq 0$ but the irreducible submodule of $G_{-2}$ generated by $[F_{\Lambda}, F_{\Lambda}]$ is not the standard $A_n$-module.

2) $G_0$ of type $A_n$, $\alpha$ simple, $n \geq 2$.

2a) $n \geq 3$, $\alpha = \alpha_j$ with $j \neq 1, n$.

If we apply Lemma 2.16 with $\alpha = \alpha_j$ and $\beta = \alpha_{j-1} + \alpha_j + \alpha_{j+1}$ we find a contradiction.

2b) $\alpha = \alpha_1$ (or, equivalently, $\alpha = \alpha_n$).

Again, by applying Lemma 2.16 with $\beta = \alpha_3 + \cdots + \alpha_n$, we find $(\Lambda, \alpha_i) = 0$ for every $i \geq 3$. On the other hand, $(\Lambda, \alpha_2) \neq 0$ since $[E_M, [F_{\Lambda}, e_{-\alpha_2}]] = e_{-\alpha_1 - \alpha_2} \neq 0$. We distinguish two cases:

Case 1: $(\Lambda, \alpha_2) \neq 1$

Under this hypothesis let us consider the following vectors:

$$x_{\mu} = [[[E_M, e_1], [E_M, e_2]], [E_M, e_1]]$$

$$x_{\lambda} = \Lambda(h_1)^{-1}(1 - \Lambda(h_2))^{-1}(3 + \Lambda(h_1))^{-1}[F_{\Lambda}, [F_{\Lambda}, [F_{\Lambda}, e_{-2}]]].$$
Then \(x_\lambda\) and \(x_\mu\) satisfy the hypotheses of Lemma 1.14 with \(\delta = \alpha_1\). Since \((3\Lambda - \alpha_2, \alpha_1) = 3(\Lambda, \alpha_1) + 1 \geq 4\) we find a contradiction.

**Case 2:** \((\Lambda, \alpha_2) = 1\)

By Corollary 2.12, either \(\Lambda(h_1) = 1\) or \(\Lambda(h_1) = 2\). Notice that \(x := [F_\Lambda, F_\Lambda]\) is a highest weight vector in \(G_{-2}\) and \(y := [[[E_M, e_1], [E_M, e_1]]\) is a lowest weight vector in \(G_2\). Since \([x, y] = -4\Lambda(h_1)h_1\), \(G_0\) contains a \(\mathbb{Z}\)-graded Lie subalgebra with local part \(s_{-2} \oplus G_0 \oplus s_2\), where \(s_{-2}\) is the irreducible submodule of \(G_{-2}\) generated by \(x\) and \(s_2\) is the irreducible submodule of \(G_2\) generated by \(y\). The classification of Kac-Moody Lie algebras immediately allows us to rule out the case \(\Lambda(h_1) = 2\) and the case \(\Lambda(h_1) = 1, n > 2\).

Now suppose \(n = 2\), \(\Lambda(h_1) = 1 = \Lambda(h_2)\). Under these hypotheses \(G_{-2}\) contains the highest weight vector

\[
z := -4[[F_\Lambda, e_{-\alpha_1-\alpha_2}], F_\Lambda] + 5[[[F_\Lambda, e_{-\alpha_1}], e_{-\alpha_2}], F_\Lambda] - 3[[F_\Lambda, e_{-\alpha_2}], F_\Lambda, e_{-\alpha_1}]
\]

of weight \(\Lambda\). Besides, \([z, y] = -24e_{-\alpha_1-\alpha_2}\) and this contradicts Theorem 1.16 since the irreducible \(G_0\)-submodule of \(G_{-2}\) containing \(z\) is the adjoint module and not the standard one.

3) \(G_0\) of type \(B_n (n \geq 2), \alpha = \alpha_i + \cdots + \alpha_n (1 \leq i \leq n - 1)\).

3a) If \(i > 1\) take \(\beta = \alpha_{i-1} + \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_n\), then \(\alpha + \beta\) and \(\alpha - \beta\) are not roots and, by Lemma 2.16, \((\Lambda, \beta) = 0\), i.e. \((\Lambda, \alpha_j) = 0\) for every \(j \geq i - 1\) which contradicts the hypothesis \((\Lambda, \alpha) \neq 0\).

3b) If \(i = 1\) and \(n \geq 3\) take \(\beta = \alpha_2 + \cdots + 2\alpha_n\). Then, by Lemma 2.16, \((\Lambda, \alpha_i) = 0\) for every \(i \neq 1\). This implies the following contradiction:

\[
0 = [E_M, [F_\Lambda, e_{-\alpha_n}]] = [e_\alpha, e_{-\alpha_n}] \neq 0.
\]

3c) Let \(i = 1\) and \(n = 2\), i.e. \(\alpha = \alpha_1 + \alpha_2\).

If \((\Lambda, \alpha_2) = 0\), as above we have:

\[
0 = [E_M, [F_\Lambda, e_{-\alpha_2}]] = [e_\alpha, e_{-\alpha_2}] \neq 0.
\]

Thus suppose \((\Lambda, \alpha_2) \neq 0\). Since \(\alpha\) and \(\alpha_2\) have both length 1, Corollary 2.12 implies \((\Lambda, \alpha_1) = 0\) and either \(\Lambda(h_2) = 1\) or \(\Lambda(h_2) = 2\). Notice that \(G_{-2}\) contains the highest weight vector \(x := [F_\Lambda, F_\Lambda]\).

Now, if \(\Lambda(h_2) = 1\) then \(G_2\) contains the lowest weight vector \(y := [[[E_M, e_1], [E_M, e_2]]\) and \([x, y] = 2e_\alpha\) thus \(G_0\) has infinite growth according to Theorem 1.16.

If \(\Lambda(h_2) = 2\), by bitransitivity, then \(y = 0\) and the vector

\[
z := [[[E_M, e_{\alpha_1+\alpha_2}], [E_M, e_{\alpha_2}]]\]
is a lowest weight vector in $G_2$. Again, since $[x, z] = -8e_{-\alpha_1}$, this
contradicts Theorem 1.16.

4) $G_0$ of type $B_n$, $\alpha$ simple.

4a) If $\alpha = \alpha_i$ with $i \neq 1, n$, we proceed as for $A_n$.

4b) If $\alpha = \alpha_1$ we take $\beta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$ and apply Lemma 2.16.

4c) If $\alpha = \alpha_n$ and $n \geq 3$ we take $\beta = \alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n$. Then Lemma
2.16 holds and we get a contradiction.

4d) $n = 2$, $\alpha = \alpha_2$. In this case relation $[E_M, [F_\Lambda, e_{-\alpha_1}]] = e_{-\alpha_2-\alpha_1}$
implies $(\Lambda, \alpha_1) \neq 0$. This possibility is therefore ruled out by the
classification of Kac-Moody Lie algebras once we have noticed that
since $G_{-2}$ contains the highest weight vector $x := [F_\Lambda, F_\Lambda]$ and $G_2$
contains the lowest weight vector $y := [[E_M, e_{\alpha_2}], [E_M, e_{\alpha_2}]]$, with
$x, y \neq 0$, $G_0$ contains an affine Kac-Moody, $\mathbb{Z}$-graded Lie subalgebra
with local part $s_{-2} + G_0 \oplus s_2$, where $s_{-2}$ is the $G_0$-irreducible module
with highest weight $2\Lambda$ and $s_2$ is the $G_0$-module contragredient to $s_{-2}$.

5) $G_0$ of type $B_n$, $\alpha = \alpha_{n-1} + 2\alpha_n$.

5a) If $n \geq 3$ take $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ and use Lemma 2.16.

5b) Let $n = 2$, $\alpha = \alpha_1 + 2\alpha_2$. If we take $\beta = \alpha_1$ then Lemma 2.16 implies
$(\Lambda, \alpha_1) = 0$ thus $\Lambda(h_\alpha) = \Lambda(h_2)$ is either 1 or 2. One can easily verify,
using the bitransitivity of $G$, that the vector $z := [[E_M, e_{\alpha_1+\alpha_2}], [E_M, e_{\alpha_2}]]$
is equal to 0, the vector $y := [[E_M, e_{\alpha_1+2\alpha_2}], [E_M, e_{\alpha_2}]]$ is a lowest
weight vector in $G_2$ and, as in the previous cases, $x := [F_\Lambda, F_\Lambda]$ is
a highest weight vector in $G_{-2}$. Since $[x, y] = 24(\Lambda(h_2) + 1)e_{-\alpha_1-\alpha_2}$
this contradicts Theorem 1.16.

6) $G_0$ of type $C_n$ ($n \geq 3$), $\alpha = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$ ($1 \leq i \leq n - 1$).
If $i \neq 1$ we apply Lemma 2.16 to $\beta = \alpha_{i-1} + \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-1} + \alpha_n$
and get a contradiction.
If $i = 1$ take $\beta = 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. Then Lemma 2.16 implies
$(\Lambda, \alpha_i) = 0$ for every $i \geq 2$. Thus $(\Lambda, \alpha) = 2(\Lambda, \alpha_1)$.
Consider the following vectors:

$$x = [F_\Lambda, F_\Lambda]$$
$$y = [[E_M, e_1], e_\alpha], E_M].$$

Then $x$ is a highest weight vector in $G_{-2}$ and $y$ is a lowest weight vector
in $G_2$. Besides, $[x, y] = 2\Lambda(h_\alpha)e_{\alpha_1-\alpha}$. This contradicts Theorem 1.16 since
$\alpha - \alpha_1$ is not the highest root of $G_0$.

7) $G_0$ of type $C_n$ ($n \geq 3$), $\alpha = \alpha_{n-1} + \alpha_n$.
If we take $\beta = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n$, by Lemma 2.16, we get a contradiction.
8) \( G_0 \) of type \( C_n \) (\( n \geq 3 \)), \( \alpha \) simple.

8a) If \( \alpha = \alpha_i \) with \( i \neq 1, n - 1, n \) then we proceed as for \( A_n \), case 2a).

8b) If \( \alpha = \alpha_{n-1} \), take \( \beta = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n \) and apply Lemma 2.16.

8c) If \( \alpha = \alpha_1 \), \( \left[ E_M, [F_{\Lambda}, e_{-\alpha_2}] \right] = e_{-\alpha_1 - \alpha_2} \) implies \( (\Lambda, \alpha_2) \neq 0 \). Thus we apply the same argument as in case 4d) with \( x = [F_{\Lambda}, F_\Lambda] \) and \( y = [[E_M, e_{\alpha_1}].[E_M, e_{\alpha_1}]]. \)

8d) If \( \alpha = \alpha_n \) we take \( \beta = 2\alpha_{n-1} + \alpha_n \). By Lemma 2.16 we find a contradiction.

9) \( G_0 \) of type \( D_n \) (\( n \geq 4 \)), \( \alpha \) simple.

9a) If \( \alpha = \alpha_i \), \( i \neq 1, n - 1, n \) we proceed as for \( A_n \), case 2a).

9b) If \( \alpha = \alpha_1 \) we apply Lemma 2.16 to \( \beta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \) and find a contradiction.

9c) If \( \alpha = \alpha_n \) (or, equivalently, \( \alpha = \alpha_{n-1} \)) we apply Lemma 2.16 to \( \beta = \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \).

10) \( G_0 \) of type \( E_6 \), \( \alpha \) simple, \( \alpha = \alpha_1 \).

If \( i \neq 1, 2, 6 \) we proceed as for \( A_n \), case 2a).

Otherwise we apply Lemma 2.16 as follows:

- \( i = 1 \) we take \( \beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \);
- \( i = 6 \) we take \( \beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \);
- \( i = 2 \) we take \( \beta = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \).

11) \( G_0 \) of type \( E_7 \) or \( E_8 \).

The situation is analogous to case 10).

12) \( G_0 \) of type \( F_4 \) and \( \alpha \) in the list.

We apply Lemma 2.16 with the following roots \( \alpha \) and \( \beta \):

- \( \alpha = \alpha_1 + \alpha_2 + \alpha_3 \), \( \beta = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \);
- \( \alpha = \alpha_2 + \alpha_3 \), \( \beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \);
- \( \alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \), \( \beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \);
- \( \alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \), \( \beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \);
- \( \alpha = \alpha_2 + 2\alpha_3 \), \( \beta = \alpha_2 + \alpha_3 + \alpha_4 \);
- \( \alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4 \), \( \beta = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \).

13) \( G_0 \) of type \( F_4 \), \( \alpha \) simple.

We apply Lemma 2.16 with the following roots \( \alpha \) and \( \beta \):

- \( \alpha = \alpha_1 \), \( \beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 \);
- \( \alpha = \alpha_2 \), \( \beta = \alpha_1 + \alpha_2 + \alpha_3 \);
- \( \alpha = \alpha_3 \), \( \beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \);
- \( \alpha = \alpha_4 \), \( \beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \).
14) $G_0$ of type $G_2$, $\alpha$ in the list.

14a) $\alpha = 2\alpha_1 + \alpha_2$

If we apply Lemma 2.16 with $\beta = \alpha_2$ we find $(\Lambda, \alpha_2) = 0$ thus $(\Lambda, \alpha) = 2(\Lambda, \alpha_1)$. Besides, Corollary 2.12 implies $\Lambda(h_{a_1}) = 2$, i.e. $\Lambda(h_{a_1}) = 1$.

Consider the vector $x := [[E_M, e_{a}], [E_M, e_{a_1}]]$. Then one can verify that $x$ is a lowest weight vector. Now, if we take $y := [F_{a}, F_{a}]$ in $G_{-2}$, then $[x, y] \neq 0$ and this contradicts Theorem 1.16.

14b) $\alpha = \alpha_1 + \alpha_2$

In this case we apply Lemma 2.16 with $\beta = 3\alpha_1 + \alpha_2$ and find a contradiction.

14c) $\alpha = 3\alpha_1 + \alpha_2$

We proceed as in 14b) with $\beta = \alpha_1 + \alpha_2$.

15) $G_0$ of type $G_2$, $\alpha$ simple.

If $\alpha = \alpha_1$ apply Lemma 2.16 with $\beta = 3\alpha_1 + 2\alpha_2$.

If $\alpha = \alpha_2$ apply Lemma 2.16 with $\beta = 2\alpha_1 + \alpha_2$.

3. – The classification theorem

Let $L$ be a finite-dimensional Lie superalgebra and let $\sigma$ be an automorphism of $L$ of finite order $k$. Then

$$L = \bigoplus_{i=0}^{k-1} L_i$$

where $L_i = \{x \in L | \sigma(x) = \epsilon^i x\}$, $\epsilon = e^{2\pi i/k}$. Notice that (3) is a mod-$k$ gradation of $L$.

Consider the Lie superalgebra $C[x, x^{-1}] \otimes L = \bigoplus_{i=-\infty}^{+\infty} x^i \otimes L$ and its subalgebra

$$G^k(L, \sigma) := \bigoplus_{i=-\infty}^{+\infty} x^i \otimes L_i$$

called the covering superalgebra of $L$. Then $G^k(L, \sigma)$ is a $\mathbb{Z}$-graded Lie superalgebra of infinite depth and growth 1.

EXAMPLE 1 (The Lie superalgebra $S_1(n)$). We recall that $sl(m, n)$ is the Lie superalgebra of $(m + n) \times (m + n)$ matrices with supertrace equal to 0, i.e., in suitable coordinates, the set of matrices $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \text{tr}(a) = \text{tr}(d) \right\}$.

Let $\tilde{Q}(n)$ be the subalgebra of $sl(n+1, n+1)$ consisting of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $\text{tr}(b) = 0$. Then $\tilde{Q}(n)$ has a one-dimensional centre $C = \langle I_{2n+2} \rangle$ and we define $Q(n) = \tilde{Q}(n)/C$. Notice that $Q(n)$ has even part isomorphic to the Lie algebra of type $A_n$ and odd part isomorphic to
$ad\,sl_{n+1}$ and has therefore dimension $2(n^2 + 2n)$. We consider the following automorphism $\sigma$ of $Q(n)$:

$$\sigma \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} -a' & ib' \\ ib' & -a' \end{pmatrix}.$$ 

Then $\sigma$ has order 4 and $Q(n) = \oplus_{i=0}^{3} Q(n)_i$ where $Q(n)_0 \cong so_{n+1}$, $Q(n)_1 = \{b \in sl_{n+1} | b = b^t\}$, $Q(n)_2 = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a = a^t \} / C$, $Q(n)_3 = \{b \in sl_{n+1} | b = -b^t\}$.

Let us suppose $n \neq 3$ and denote by $S_1(n)$ the covering superalgebra $G^4(Q(n), \sigma)$. Notice that $Q(n)_3$ is isomorphic to the adjoint module of $so_{n+1}$ and if $n > 2$ then $Q(n)_1$ and $Q(n)_2$ are isomorphic, as $so_{n+1}$-modules, to the highest weight module $V(2\omega_1)$, while if $n = 2$ $Q(n)_1$ and $Q(n)_2$ are $sl(2)$-irreducible modules of dimension 5.

**Example 2** (The Lie superalgebra $S_2(m)$). Suppose $m = 2n - 1$ and consider the following automorphism $\tau$ of $Q(m)$:

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \begin{pmatrix} r & s \\ v & w \end{pmatrix} = \begin{pmatrix} -d^t & b^t \\ c^t & -a^t \end{pmatrix} | \begin{pmatrix} -i w^t & is^t \\ iv^t & -ir^t \end{pmatrix}.$$

where $a, b, c, d, r, s, v, w$ are $n \times n$-blocks and $\text{tr}(r) + \text{tr}(w) = 0$. Then $\tau^4 = 1$ and $Q(m) = \oplus_{i=0}^{3} Q(m)_i$ where $Q(m)_0 \cong sp(2n)$, $Q(m)_1 = \{\begin{pmatrix} r & s \\ v & w \end{pmatrix} | r = -w^t, s = s^t, v = v^t\}$, $Q(m)_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | b^t = -b, c^t = -c, a^t = d\} / C$, $Q(m)_3 = \{\begin{pmatrix} r & s \\ v & w \end{pmatrix} | w^t = r, s^t = -s, v^t = -v, \text{tr}(r) = 0\}$.

Let us denote by $S_2(m)$ the covering superalgebra $G^4(Q(m), \tau)$. Notice that $Q(m)_1$ is isomorphic to the adjoint module of the Lie algebra $sp(2n)$ and $Q(m)_2, Q(m)_3$ are isomorphic to the $sp(2n)$-module $\Lambda_0^2 sp_{2n}$.

**Example 3** (The Lie superalgebra $S_3$). Let $D(2, 1; \alpha)$ be the one-parameter family of 17-dimensional Lie superalgebras with even part isomorphic to $A_1 \oplus A_1 \oplus A_1$ and odd part isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. We recall that two members $D(2, 1; \alpha)$ and $D(2, 1; \beta)$ of this family are isomorphic if and only if $\alpha$ and
\( \beta \) lie in the same orbit of the group \( V \) of order 6 generated by \( \alpha \mapsto -1 - \alpha, \alpha \mapsto 1/\alpha \).

\( D(2, 1; \alpha) \) is the contragredient Lie superalgebra associated to the matrix

\[
\begin{pmatrix}
0 & 1 & -1 - \alpha \\
1/\alpha & 0 & 1 \\
1 & -\alpha/(1 + \alpha) & 0
\end{pmatrix}.
\]

Suppose that \( \alpha^2 + \alpha + 1 = 0 \) and consider the following automorphism \( \varphi \) of \( D(2, 1; \alpha) \):

\[
\varphi(e_1) = -e_2, \quad \varphi(f_1) = -f_2, \quad \varphi(h_1) = h_2,
\]

\[
\varphi(e_2) = -e_3, \quad \varphi(f_2) = -f_3, \quad \varphi(h_2) = h_3,
\]

\[
\varphi(e_3) = -e_1, \quad \varphi(f_3) = -f_1, \quad \varphi(h_3) = h_1.
\]

Then \( \varphi \) has order 6 and \( D(2, 1; \alpha) = \oplus_{i=0}^5 V_i \) where

- \( V_0 \) is isomorphic to the Lie algebra of type \( A_1 \);
- \( V_1 \) is isomorphic, as a \( V_0 \)-module, to the \( sl(2) \)-irreducible module of dimension 4;
- \( V_2 \) is isomorphic, as a \( V_0 \)-module, to the adjoint module of \( sl(2) \);
- \( V_3 \) is isomorphic to the \( sl(2) \)-irreducible module of dimension 2;
- \( V_4 \) is isomorphic to the adjoint module of \( sl(2) \);
- \( V_5 \) is isomorphic to the \( sl(2) \)-irreducible module of dimension 2.

We denote by \( S_3 \) the covering superalgebra \( G^0(D(2, 1; \alpha), \varphi) \).

**Theorem 3.1.** Let \( G = \oplus_{i \in \mathbb{Z}} G_i \) be an infinite-dimensional \( \mathbb{Z} \)-graded Lie superalgebra. Suppose that:

- \( G \) is simple and generated by its local part,
- the \( \mathbb{Z} \)-gradation is consistent and has infinite depth,
- \( G_0 \) is simple,
- \( G_{-1} \) and \( G_1 \) are irreducible \( G_0 \)-modules which are not contragredient.

Then \( G \) has finite growth if and only if it is isomorphic to one of the Lie superalgebras \( S_i \) for some \( 1 \leq i \leq 3 \).

**Proof.** Theorems 2.10 and 2.17 show that under our hypotheses either \( G_0 \) has rank 1 or one of the following possibilities occur:

a) \( G_0 \) is of type \( A_3, G_{-1} \) is its adjoint module and \( G_1 = V(2\omega_2) \);

b) \( G_0 \) is of type \( B_n, G_{-1} \) is its adjoint module and \( G_1 = V(2\omega_1) \);

c) \( G_0 \) is of type \( C_n \) \((n \geq 3), G_1 \) is its adjoint module and \( G_{-1} \cong \Lambda^2_0 sp_{2n} \);

d) \( G_0 \) is of type \( D_n \) \((n \geq 4), G_{-1} \) is its adjoint module and \( G_1 = V(2\omega_1) \).

Besides, if \( G_0 \) has rank 1, by Corollary 2.12, either

- \( G_{-1} \cong V(\omega) \) and \( G_1 \cong V(3\omega) \) or
- \( G_{-1} \) is isomorphic to the adjoint module of \( A_1 \) and \( G_1 \cong V(4\omega) \).

By Propositions 1.7 and 1.9 we conclude that \( G \) is isomorphic to the Lie superalgebra \( S_1(m) = G^4(Q(m), \sigma) \) with \( m = 5 \) in case a), \( m = 2n \) in case b), \( m = 2 \) in case f) and \( m = 2n - 1 \) in case d); in case c) \( G \) is isomorphic to the Lie superalgebra \( S_2(m) = G^4(Q(m), \tau) \) with \( m = 2n - 1 \). Finally, in case e) \( G \) is isomorphic to the Lie superalgebra \( S_3 \).
REFERENCES


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