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A Remark on Quiver Varieties and Weyl Groups

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Abstract. In this paper we define an action of the Weyl group on the quiver varieties $M_{m,\lambda}(v)$ with generic (m, λ).

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In [11], [12] Nakajima defined a particular class of quiver varieties and showed how to use them to give a geometric construction of integrable representations of Kac-Moody algebras. Luckily these varieties can be used also to give a geometric construction of representations of Weyl groups. In [6], Lusztig constructed an action of the Weyl group on the homology of Nakajima's quiver varieties. His construction is similar to the construction of Springer representations. In [12], Nakajima gave a construction of isomorphisms $\Phi_{\sigma,\zeta}(d, v)$: $\mathfrak{M}_\zeta(d, v) \longrightarrow \mathfrak{M}_{\sigma\zeta}(d, \sigma(v - d) + d)$ in the case of a quiver of finite type (Nakajima's conventions are different from the ones we adopt here, for us this case would be a particular case of quiver varieties of affine type). In the same paper he also suggested that it would have been possible to construct these isomorphisms using reflection functors in the general case. His construction is analytic and relies on a description of quiver varieties as moduli spaces of instantons on ALE spaces.

The main result of this paper is a direct and algebraic construction of these isomorphisms which works for a general quiver without simple loops. To do it we also describe a set of generators for the algebra of covariant functions.

The paper is organized as follows. In the first section we fix the notation and we give the definition of a quiver variety: $M_{m,\lambda}(v)$ where m and λ are two parameters, and v is a dimension vector. We are interested in quiver varieties as algebraic varieties but in order to explain one of the applications we need to give also the hyperKähler construction of a quiver variety. We use a result of Migliorini [9] to explain the connection between the two constructions.

Algebraic quiver varieties are defined as the **Proj** scheme of a ring of

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covariants. In the second section we describe a set of generators for this ring. In a special case which is not directly related with quiver varieties we are also able to give a more precise result and to describe a basis for the vector space of χ-covariants functions.

In the third section we use this description to generalize a construction of Lusztig [6]. Namely, if m and λ are generic, for any element σ of the Weyl group we construct an isomorphism Φ_σ between $M_{m,\lambda}(v)$ and $M_{\sigma m,\sigma\lambda}(\sigma v)$. More precisely we construct Φ_s for all simple reflections s and we verify the Coxeter relations.

In the fourth section we give a result which reduces the study of the geometric and algebraic properties of the quiver varieties $M_{m,\lambda}(v)$ to the case v dominant, for all m and λ.

In the fifth section, following Nakajima [11], we show how to use the action constructed in section 3 (and the connection between the hyperKähler construction and the algebraic construction) to describe an action of the Weyl group on the homology of a class of quiver varieties. This action is different from the one constructed by Lusztig in [6].

In the first version of this paper only Nakajima's quiver varieties were considered. In an attempt to make the paper clearer I found that the notation required to explain the general case was simpler. In the last section I restrict my attention to Nakajima's quiver varieties, which depend on two dimension vectors v and d. In particular in the case of a quiver of finite type and $d - v$ a regular weight, I prove the normality of the quiver variety $M_0(d, v)$ and the connectedness of $M(d, v)$.

After this paper was available Nakajima gave a construction of the isomorphisms described in Section 3 based on the hyperKähler construction of quiver varieties and Crawley-Boevey gave a complete proof of the connectedness of $M(d, v)$.

1. – Notation and definitions

In this section we associate to a quiver many different objects: a Cartan matrix, a Weyl group, a variety and an associative algebra.

1.1. – Double quivers

Let $Q = (I, H, h \mapsto h_0, h \mapsto h_1)$ be a finite oriented quiver: I is the set of vertices, H the set of arrows and the orientation is given by the two maps

$$h \longmapsto h_0 \text{ and } h \longmapsto h_1$$

from H to I. A double quiver is a quiver as above equipped with the following extra structure:

1. For all $h \in H$ we have $h_0 \neq h_1$;

2. An involution $h \mapsto \bar{h}$ of H without fixed points and satisfying $\bar{h}_0 = h_1$;
3. A map $\varepsilon : H \longrightarrow \{-1, 1\}$ such that $\varepsilon(\bar{h}) = -\varepsilon(h)$.

We define $\Omega = \{h \in H : \varepsilon(h) = 1\}$ and $\overline{\Omega} = \{h \in H : \varepsilon(h) = -1\}$.

Observe that given a symmetric graph without simple loops it is always possible to define ε and an involution $\bar{\ }$ as above.

1.2. – The Cartan matrix, the root lattice and the Weyl group

To a double quiver Q we associate a card $I \times$ card I matrix A with the following entries:

$$a_{ij} = \text{card}\{h \in H : h_0 = i \text{ and } h_1 = j\}.$$

We define a generalized symmetric Cartan matrix as follows: $C = 2\mathrm{I} - \mathrm{A} = (\mathrm{c_{ij}})$.

We define also a lattice $R = \mathbb{Z}[I]$, its dual $P = \text{Hom}(R, \mathbb{Z})$ and its positive part $R^+ = \mathbb{N}[I]$. To avoid confusion we indicate with α_i the element of R corresponding to $i \in I$. The set $\{\alpha_i : i \in I\}$ is a basis for R, we indicate with ω_i the dual basis of P.

We define a symmetric bilinear form $\langle\ ,\ \rangle$ on R by $\langle \alpha_i, \alpha_j \rangle = c_{ij}$. If $r \in R$ we let $\bar{r}$ be the element of P defined by this bilinear form, so $\overline{\alpha}_i = \sum_{j \in I} c_{ij} \omega_j$.

The Weyl group W attached to C is defined as the subgroup of $Aut(R)$ generated by the reflections

$$s_i : x \longmapsto x - \langle x, \alpha_i \rangle \alpha_i. \tag{1}$$

A presentation of the group W is given by the following Coxeter relations:

$$s_i^2 = 1 \text{ for all } i \in I, \tag{2a}$$

$$(s_i s_j)^2 = 1 \text{ for all } i, j \in I \text{ such that } c_{ij} = 0, \tag{2b}$$

$$(s_i s_j)^3 = 1 \text{ for all } i, j \in I \text{ such that } c_{ij} = -1. \tag{2c}$$

1.3. – Representations of a quiver

If $Q = (I, H)$ is any oriented quiver and $v = \sum_{i \in I} v_i \alpha_i \in R$, we fix the vector spaces $V_i = \mathbb{C}^{v_i}$ and we define the space of representations of Q of dimension v to be the vector space

$$S(Q, v) = \bigoplus_{h \in H} \text{Hom}(V_{h_0}, V_{h_1}). \tag{3}$$

If there is no ambiguity about Q and v we will write S instead of $S(Q, v)$.

If $s \in S$ we will write B_h or $B_h(s)$ for the component of s relative to the arrow h.

In the case of a double quiver Q when v is fixed and $s \in S$ we define:

$$T_i = \bigoplus_{h\,:\,h_1=i} V_{h_0}\,, \tag{4a}$$

$$a_i = a_i(s) = (B_{\bar{h}}(s))_{h\,:\,h_1=i} : V_i \longrightarrow T_i\,, \tag{4b}$$

$$b_i = b_i(s) = (\varepsilon(h) B_h(s))_{h\,:\,h_1=i} : T_i \longrightarrow V_i\,. \tag{4c}$$

A natural symplectic form is defined on the space $S(Q, v)$ by the formula:

$$\omega(s,t) = \sum_{h\in H} \varepsilon(h)\,\mathrm{Tr}(B_h(s)B_{\bar{h}}(t)) = \sum_{i\in I} \mathrm{Tr}(b_i(s)a_i(t)).$$

1.4. – Hermitian structure on S

We suppose now that a double quiver $Q = (I, H)$ is fixed and that the spaces V_i are endowed with hermitian metrics. So we can speak of the adjoint φ^* of a linear map between these spaces, and we have a positive definite hermitian structure h on S defined by

$$h(s,t) = \sum_{h\in H} \mathrm{Tr}(B_h(s)B^*_{\bar{h}}(t)) = \sum_{i\in I} \mathrm{Tr}(a_i(s)a_i^*(t) + b_i^*(t)b_i(s))$$

and an associated real symplectic form $\omega_I(s,t) = \mathrm{Re}\, h(\sqrt{-1}s, t)$.

1.5. – Group actions and moment maps

We can define an action of the group $GL(V) = \prod GL(V_i)$ on the set S in the following way:

$$g(B_h) = (g_{h_1} B_h g_{h_0}^{-1}) \text{ for } g = (g_i) \in GL(V).$$

Observe that ω is $GL(V)$ invariant. Observe also that the 1-dimensional subgroup $\mathbb{C}^* \cdot (\mathrm{Id}_{V_i})_{i\in I}$ acts trivially, so an action of the group $G_v = GL(V)/\mathbb{C}^*(\mathrm{Id}_{V_i})_{i\in I}$ is defined. Moreover if $U(V) = \prod U(V_i)$ is the group of unitary transformations in $GL(V)$ and $U_v = U(V)/S^1 \subset G_v$ then the real symplectic form ω_I is U_v invariant.

We want now to give explicit formulas for the relative moment maps.

Given two finite dimensional vector spaces E and F we will identify the dual of $\mathrm{Hom}_{\mathbb{C}}(E, F)$ with $\mathrm{Hom}_{\mathbb{C}}(F, E)$ through the pairing $\langle \varphi, \psi \rangle = \mathrm{Tr}(\varphi \circ \psi)$.

Using this identification we see that we can identify $\mathfrak{g}_v^*$ with the space $\{(x_i) \in \oplus_{i\in I}\mathfrak{gl}(V_i) : \sum_i \mathrm{Tr}(x_i) = 0\}$ and $\mathfrak{u}_v^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{u}, \mathbb{R})$ with the subspace of $\mathfrak{g}_v$ of skew-hermitian matrices.

We can now give the following explicit formulas for the moment map $\mu\colon S \longrightarrow \mathfrak{g}_v^*$ relative to the symplectic form ω and for the moment map $\mu_I\colon S \longrightarrow \mathfrak{u}_v^*$ relative to the symplectic form ω_I:

$$(\mu(s))_i = \sum_{h\in H\,:\,h_1=i} \varepsilon(h) B_h B_{\bar{h}} = b_i a_i,$$

$$(\mu_I(s))_i = \frac{\sqrt{-1}}{2}\left(\sum_{h\in H\,:\,h_1=i} B_h B_h^* - B_{\bar{h}}^* B_{\bar{h}}\right) = \frac{\sqrt{-1}}{2}(b_i b_i^* - a_i^* a_i).$$

It is common to group these moment maps together and to define an hyperKähler moment map

$$\widetilde{\mu} = (\mu_I, \mu) : S \longrightarrow \mathfrak{u}_v \oplus \mathfrak{g}_v = (\mathbb{R} \oplus \mathbb{C}) \otimes_{\mathbb{R}} \mathfrak{u}_v.$$

1.6. – Quiver varieties as hyperKähler quotients

Let $\mathfrak{Z} = (\mathbb{R}\oplus\mathbb{C}) \otimes_{\mathbb{Z}} P$ and for any $v \in R$ define $\mathfrak{Z}_v = \{\zeta \in \mathfrak{Z} : \langle \zeta, v\rangle = 0\}$. If $v \in R^+$ and $\zeta = \sum_{i\in I}(\xi_i, \lambda_i)\omega_i \in \mathfrak{Z}_v$ define:

$$\mathfrak{L}_\zeta(Q, v) = \{s \in S \,:\, \mu_i(s) - \lambda_i \operatorname{Id}_{V_i} = 0 \ \text{ and } \ \mu_{I,i}(s) - \sqrt{-1}\xi_i \operatorname{Id}_{V_i} = 0\}.$$

We observe that $\mathfrak{L}_\zeta(Q, v)$ is stable for the action of U_v, so, at least as a topological Hausdorff space we can define the *quiver variety of type* ζ as

$$\mathfrak{M}_\zeta(Q, v) = \mathfrak{L}_\zeta(Q, v)/U_v.$$

It will be convenient to define also $\mathfrak{M}_\zeta(Q, v) = \varnothing$ if $v \in R - R^+$.

REMARK 1. There is a surjective map from $\mathfrak{Z}_v$ to $Z(\mathfrak{u}_v) \oplus Z(\mathfrak{g}_v)$ given by:

$$\sum_{i\in I}(\xi_i, \lambda_i)\omega_i \longmapsto \left(\sum_i \sqrt{-1}\xi_i \operatorname{Id}_{V_i}, \sum_i \lambda_i \operatorname{Id}_{V_i}\right)$$

Observe that $\mathfrak{L}_\zeta$ is the fiber of $\widetilde{\mu}$ over the image of ζ in $Z(\mathfrak{u}_v) \oplus Z(\mathfrak{g}_v)$.

REMARK 2. If $v \geq 0$ define: $I^* = \{i \in I : v_i \neq 0\}$, $H^* = \{h \in H : h_0, h_1 \in I^*\}$, $\varepsilon^* = \varepsilon\big|_{H^*}$, $v^* = \sum_{i\in I^*} v_i\alpha_i$, and $\zeta^* = \sum_{i\in I^*} \zeta_i\omega_i$ then it is clear that

$$\mathfrak{L}_{\zeta^*}(v^*) \simeq \mathfrak{L}_\zeta(v) \ \text{ and } \ \mathfrak{M}_{\zeta^*}(v^*) \simeq \mathfrak{M}_\zeta(v).$$

1.7. – Geometric invariant theory and moment map

In this section we explain the relation between the moment map and the GIT quotient proved by Kempf, Ness [4], Kirwan [10] and others. To be more precise we need a generalization of their results in the case of an action on an affine variety proved by Migliorini [9].

Let X be an affine variety over $\mathbb{C}$ and G a reductive group acting on X. We can assume that X is a closed sub-variety of a vector space V where G acts linearly. Let h be an hermitian form on V invariant by the action of a maximal compact subgroup U of G and define a real U-invariant symplectic form on V by

$$\eta(x, y) = \operatorname{Re} h(\sqrt{-1}x, y).$$

Then we can define a moment map $\nu : V \longrightarrow \mathfrak{u}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{u}, \mathbb{R})$:

$$\langle \nu(x), u \rangle = \frac{1}{2}\eta(u \cdot x, x).$$

We observe that the real symplectic form η restricted to a complex submanifold is always non degenerate and that μ restricted to the non singular locus of X is a moment map for the action of U on X.

Now let χ be a multiplicative character of G. We observe that for all $g \in U$ we have $|\chi(g)| = 1$ so $\sqrt{-1}\, d\chi$ is a map with values in $\mathbb{R}$. In particular we can consider $\sqrt{-1}\, d\chi$ as an element of $\mathfrak{u}^*$. Moreover we observe that it is invariant by the dual adjoint action, hence it makes sense to consider the quotient:

$$\mathfrak{M} = \nu^{-1}(\sqrt{-1}\, d\chi)/U.$$

As we saw our varieties are a particular case of this construction.

On the other side we can consider the GIT quotient. Let us recall the definition. If φ is a character of G we consider the line bundle $L_\varphi = V \times \mathbb{C}$ on V with the following G-action:

$$g(x, z) = (g \cdot x, \varphi(g)z).$$

An invariant section of L_φ is determined by an algebraic function $f : V \longrightarrow \mathbb{C}$ such that $f(gx) = \varphi(g) f(x)$ for all $g \in G$ and $x \in V$. We use the same symbol L_φ also for the restriction of L_φ to X.

Given a rational action of G on a $\mathbb{C}$-vector space A we define

$$A_{\varphi,n} = \{a \in A \,:\, g \cdot a = \varphi^{-n}(g)a \text{ for all } g \in G\},$$

$$A_\varphi = \bigoplus_{n=0}^{\infty} A_{\varphi,n} \qquad \text{as a graded vector space.}$$

Hence we have that $H^0(X, L_\varphi)^G = \mathbb{C}[X]_{\varphi,1}$. We observe that if I is the ideal of algebraic function on V vanishing on X then

$$H^0(X, L_\varphi)^G = \frac{H^0(V, L_\varphi)^G}{I_{\varphi,1}}.$$

This last fact can be easily proved (for example) by averaging a φ-equivariant function f on X in the following way:

$$\tilde{f}(v) = \int_U \varphi^{-1}(u) f(u \cdot v)\, du.$$

Definition 3. A point x of X is said to be χ-*semi-stable* if there exist $n > 0$ and $f \in H^0(X, L_\chi^{\otimes n})^G$ such that $f(x) \neq 0$. We observe that by the remark above a point of X is χ-semi-stable if and only if it is χ-semi-stable as a point of V. We denote by X_χ^{ss} (resp. V_χ^{ss}) the open subset of χ-semi-stable points of X (resp. V).

Proposition 4 ([10], [13]). *There exists a good quotient of X_χ^{ss} by the action of G and we have that*

$$X_\chi^{ss}//G = \operatorname{Proj} \mathbb{C}[X]_\chi.$$

Moreover $\operatorname{Proj} \mathbb{C}[X]_\chi$ *is a finitely generated $\mathbb{C}$-algebra and a natural projective map*

$$\pi : X_\chi^{ss}//G \longrightarrow X//G = \operatorname{Spec} \mathbb{C}[X]^G$$

is defined.

In the case of $\chi \equiv 1$ the following fact is well known:

$$\operatorname{Proj} \mathbb{C}[X]_\chi = \operatorname{Spec} \mathbb{C}[X]^G \simeq \nu^{-1}(0)/U.$$

The following result is less well known, and its proof requires some adjustment of the classical proof for the case $\chi \equiv 1$ (see for example an appendix of [9] or par.I.2 in [8]).

Proposition 5 (Migliorini, [9]). *Let $x \in X$ then*

$$\exists g \in G \,:\, \nu(gx) = \sqrt{-1}d\chi \iff Gx \text{ is a closed orbit in } X_\chi^{ss}.$$

Proposition 6. *The inclusion $\nu^{-1}(\sqrt{-1}d\chi) \subset X_\chi^{ss}$ induces a homeomorphism*

$$\nu^{-1}(\sqrt{-1}d\chi)/U \simeq X_\chi^{ss}//G.$$

1.8. – Quiver varieties as algebraic varieties

If $v \in R$ and $m = \sum_i m_i\omega_i \in P$ is such that $\langle v, m\rangle = 0$ then we define a character χ_m of G_v by $\chi_m = \prod_{i\in I} \det_{GL(V_I)}^{m_i}$. Let $Z = \mathbb{C} \otimes_{\mathbb{Z}} P$ and if $v \in R$ set $Z_v = \{\lambda \in Z \,:\, \langle \lambda, v\rangle = 0\}$. If $\lambda = \sum_i \lambda_i\omega_i \in Z$, $m \in P$ and $v \in R^+$ then we define the varieties:

$$\Lambda_\lambda(Q, v) = \{s \in S \,:\, \mu_i(s) - \lambda_i \operatorname{Id}_{V_i} = 0 \text{ for all } i\},$$
$$\Lambda_{m,\lambda}(Q, v) = \{s \in \Lambda_\lambda(Q, v) \,:\, s \text{ is } \chi_m - \text{semi-stable}\}$$

and the associated *quiver varieties*

$$M_{m,\lambda}(Q,v) = \Lambda_{m,\lambda}(Q,v)//G_v.$$

We call $p^v_{m,\lambda}\colon \Lambda_{m,\lambda}(v) \longrightarrow M_{m,\lambda}(v)$ the quotient map. Observe that the inclusion $\Lambda_{m,\lambda}(v) \subset \Lambda_\lambda(v)$ induces a projective morphism

$$\pi^v_{m,\lambda} : M_{m,\lambda}(v) \longrightarrow M_{0,\lambda}(v).$$

Finally it will be convenient to define $M_{m,\lambda}(v) = \varnothing$ if $v \in R - R^+$.

REMARK 7. As in Remark 1 we have a surjective map from Z to $Z(\mathfrak{g}_v)$ and $\Lambda_\lambda(v)$ is the fiber of μ over the image of λ in $Z(\mathfrak{g}_v)$.

REMARK 8. Remark 2 holds without changes also in this case.

REMARK 9. Observe that $P \oplus Z \subset \mathfrak{Z}$. Observe also that the map $m \longrightarrow \chi_m$ defines a surjective morphism from P to $\mathrm{Hom}(G_v, \mathbb{C}^*)$ and that the following diagram commutes:

$$\begin{array}{ccccc} P & \longrightarrow & \mathrm{Hom}(G,\mathbb{C}^*) & \ni & \chi \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^n & \longrightarrow & Z(\mathfrak{u}) \simeq (\mathfrak{u}^*)^U & \ni & \sqrt{-1}d\chi \end{array}$$

In particular we can apply Proposition 6 to the action of G_v on $\Lambda_\lambda(v)$ and we obtain:

$$\mathfrak{M}_{(m,\lambda)}(Q,v) \simeq M_{m,\lambda}(Q,v).$$

1.9. – Path algebra

To describe functions on quiver varieties we need some notations about the path algebra.

DEFINITION 10. A *path* α in our graph is a sequence $h^{(m)} \dots h^{(1)}$ such that $h^{(i)} \in H$ and $h^{(i)}_1 = h^{(i+1)}_0$ for $i = 1, \dots, m-1$. We define also the source $\alpha_0 = h^{(1)}_0$, and the target $\alpha_1 = h^{(m)}_1$ and we say that the length of α is m. If $\alpha_0 = \alpha_1$ we say that α is a closed path. We consider also the empty paths $\varnothing_i$ for $i \in I$ and we define $(\varnothing_i)_0 = (\varnothing_i)_1 = i$. The product of paths is defined in the obvious way.

Given a path $\alpha = h^{(m)} \dots h^{(1)}$ we define an evaluation of α on S in the following way: if $s \in S$ then

$$\varnothing_i(s) = \mathrm{Id}_{V_i} \in \mathrm{Hom}(V_i, V_i) \quad\text{and}\quad \alpha(s) = B_{h^{(m)}} \circ \dots \circ B_{h^{(1)}} \in \mathrm{Hom}(V_{\alpha_0}, V_{\alpha_1}).$$

The path algebra $\mathcal{R}$ is the vector space spanned by all the paths, and with the product induced by the product of paths. If $i, j \in I$ we say that an element in $\mathcal{R}$ is of type (i, j) if it is in the linear span of the paths with source in i and target in j.

REMARK 11. We observe that the evaluation on S is a morphism of algebras from $\mathcal{R}$ to the algebra defined by the morphisms of the category of vector spaces. Moreover if f is of type (i, j) we observe that $f(s) \in \mathrm{Hom}(V_i, V_j)$.

2. – Generators of the projective ring of a quiver variety

In this section we want to describe a set of generators of the graded ring $\mathbb{C}[S]_\chi$ and by consequence of the projective ring of a quiver variety $\mathbb{C}[\Lambda_\lambda]_\chi$. More precisely we will give a set of generators as $\mathbb{C}[S]^G$-module of its l-homogeneous part: $\mathbb{C}[S]_{\chi,l}$. This result is a generalization of the one obtained by Le Bruyn and Procesi in [1] in the case of invariants: $\chi \equiv 1$. Other bases of the ring of covariants have been obtained by Derksen and Weyman for all characteristics (see [3]).

First of all recall the result of Le Bruyn and Procesi.

THEOREM 12 (Le Bruyn and Procesi [1]). *The ring $\mathbb{C}[S]^G$ is generated by the polynomials $s \longmapsto \mathrm{Tr}\,(\alpha(s))$ for α a closed path.*

2.0.1. – Determinants

To describe our result we make first some general remarks. Forget for a moment our quiver, and suppose to have a finite set of finite dimensional vector spaces $X_1, \dots, X_k$ of dimensions $u_1, \dots, u_k$, a pair of nonnegative integers (m_i^+, m_i^-) for each of them and assume that $N = \sum_{i=1}^k m_i^+ u_i = \sum_{i=1}^k m_i^- u_i$. Construct the vector spaces:

$$Y = \bigoplus_{i=1}^{k} \mathbb{C}^{m_i^-} \otimes X_i, \qquad Z = \bigoplus_{i=1}^{k} \mathbb{C}^{m_i^+} \otimes X_i$$

and observe that $\dim Y = \dim Z = N$. Define an action of the general linear group $GL(X_i)$ of X_i on Y by

$$g_i \cdot \left(\sum_{j=1}^{k} v_j \otimes x_j \right) = \sum_{j \neq i} v_j \otimes x_j + v_i \otimes g_i x_i,$$

and also a similar action on Z. Hence the vector space $\mathrm{Hom}(Y, Z)$ acquires a natural structure of $G_X = \prod_{i=1}^k GL(X_i)$ module. If we choose an isomorphism σ between $\mathrm{Hom}(\bigwedge^N Y, \bigwedge^N Z)$ and $\mathbb{C}$ we can define a function det on $\mathrm{Hom}(Y, Z)$ by

$$\det(A) = \sigma \left(\bigwedge^{n} A \right).$$

For simplicity we do not emphasize the role of σ in this definition, so strictly speaking, det is a function defined only up to a nontrivial constant factor. We observe also that $\bigwedge^n Y \simeq \left(\bigwedge^{u_i} X_1\right)^{\otimes m_1^-} \otimes \dots \otimes \left(\bigwedge^{u_k} X_k\right)^{\otimes m_k^-}$ (and similarly for Z) so an isomorphism σ is determined if we choose orientations, or bases, of the X_j's. Finally observe that for any $g = (g_j) \in G_X$ we have

$$\det(g \cdot A) = \prod_{i=1}^{k} (\det_{GL(X_i)}(g_i))^{m_i^+ - m_i^-} \det(A).$$

2.0.2. – Description of the generators

We go back now to our quiver and we describe a set of covariant polynomials on S. Any character χ of the group G_v is of the form $\chi = \chi_m = \prod_{i \in I} \det_{GL(V_i)}^{m_i}$. We fix such a character. We use now the construction explained in 2.0.1 in the case $X_i = V_i$ and $m_i^+ - m_i^- = m_i$. In particular we have

$$Y = \bigoplus_{i \in I} \bigoplus_{h=1}^{m_i^-} V_i^{(h)}, \qquad Z = \bigoplus_{i \in I} \bigoplus_{k=1}^{m_i^+} V_i^{[k]}$$

where $V_i^{(l)}, V_i^{[l]}$ are isomorphic copies of V_i. For any $i, j \in I$ and for any $1 \leq h \leq m_i^-$, $1 \leq k \leq m_j^+$ we choose an element $\alpha_{j,k}^{i,h}$ of the path algebra of type (i, j). We call the data $\Delta = (m_i^+, m_i^-, \alpha_{j,k}^{i,h})$ a χ-data and we attach to it a χ-covariant function f_Δ on S through the formula:

$$f_\Delta(s) = \det(\Psi_\Delta(s))$$

where $\Psi_\Delta(s)$ is a linear map from Y to Z defined by

$$[\Psi_\Delta]_{V_j^{[k]}}^{V_i^{(h)}}(s) = \alpha_{j,k}^{i,h}(s).$$

The functions f_Δ form a set of generators as $\mathbb{C}[S]^G$-module of $\mathbb{C}[S]_{\chi,1}$, but we will need to define a smaller set of generators. To define this set we give a notion of good Δ.

Definition 13. Data Δ as above is said to be χ-*good* if it satisfies $m_i^+ + m_i^- = |m_i|$ for all $i \in I$.

Theorem 14. *The set of polynomials f_Δ with Δ χ-good generates $\mathbb{C}[S]_{\chi,1}$ as a $\mathbb{C}[S]^{G_v}$-module.*

Remark 15. In a previous version of this paper [7] a slightly more general and precise result was proved. In that case we used a different definition of quiver varieties where extra vector spaces were considered.

2.1. – Some remark on the invariant theory of $GL(n)$

If V is a finite dimensional representation of a linearly reductive Lie group G and S is a simple representation of S we write $V[S]$ for the S-isotypic component of type S of V.

We now fix n and $V = \mathbb{C}^n$ and we make some remarks on the representations of $GL(n)$. To any partition of height less then or equal to n we associate an irreducible representation of $GL(n)$ in the usual way. If we multiply these representations by a power of the inverse of the determinant representation we obtain a complete list of irreducible representations of $GL(n)$. If λ is a partition we define $\lambda^{op} = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_1)$. We call δ the determinant representation of $GL(n)$ and $\varepsilon = 1^n$ the associated partition. Finally we call V the natural representation.

LEMMA 16.
1. ${L_\lambda}^* = \delta^{-\lambda_1} \otimes L_{\lambda^{op}}$,
2. $\operatorname{Hom}_{GL(n)}(\delta^m, L_\lambda \otimes L_\mu) = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu^{op} + (m - \mu_1)\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$
3. $\operatorname{Hom}_{GL(n)}(\delta^m, L_\lambda \otimes L_\mu^*) = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu + m\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$

We want now to describe $\operatorname{Hom}_{GL(n)}\big(\delta^m, V^{\otimes i} \otimes (V^*)^{\otimes j}\big)$. To do it we will use Schur-duality. Remind that the irreducible representations of the groups S_m are parameterized by the partitions of m and we call S_λ the irreducible representation associated with λ. Consider now the action of S_m on $V^{\otimes m}$ given by permuting the factors. This action commute with the $GL(n)$ action. Schur duality asserts that the action of the group $S_m \times GL(n)$ on $V^{\otimes m}$ decomposes in the following way:

$$V^{\otimes m} = \bigoplus_{\substack{\lambda \vdash m \\ ht(\lambda) \le n}} S_\lambda \otimes L_\lambda.$$

We describe a set of elements of $\operatorname{Hom}_{GL(n)}\big(\delta^m, V^{\otimes i} \otimes \big(V^*\big)^{\otimes j}\big)$. Let m be a nonnegative integer and choose a permutation σ of $\{1, \dots, i + mn\}$. To σ we associate maps:

$$\Phi_\sigma : \Big(V^{\otimes i} \otimes (V^*)^{\otimes i}\Big)^{GL(n)} \longrightarrow V^{\otimes i+mn} \otimes (V^*)^{\otimes i}[\delta^m],$$
$$\Psi_\sigma : \Big(V^{\otimes i} \otimes (V^*)^{\otimes i}\Big)^{GL(n)} \longrightarrow V^{\otimes i} \otimes (V^*)^{\otimes i+mn}[\delta^{-m}],$$

by

$$\Phi_\sigma(t \otimes s) = \sigma(o \otimes \cdots \otimes o \otimes t) \otimes s,$$
$$\Psi_\sigma(t \otimes s) = t \otimes \sigma(o^* \otimes \cdots \otimes o^* \otimes s)$$

where o is a nonzero vector in $\bigwedge^n V$, o^* is a non zero vector in $\bigwedge^n V^*$, $t \in V^{\otimes i}$ and $s \in (V^*)^{\otimes i}$.

LEMMA 17.
1. *If* $i \neq j + mn$ *then* $\operatorname{Hom}_{GL(n)}\Big(\delta^m, V^{\otimes i} \otimes (V^*)^{\otimes j}\Big) = 0$.
2. *If* $m > 0$ *then* $V^{\otimes i+mn} \otimes (V^*)^{\otimes i}[\delta^m] = \sum_\sigma \operatorname{Im} \Phi_\sigma$.
3. *If* $m > 0$ *then* $V^{\otimes i} \otimes (V^*)^{\otimes i+mn}[\delta^{-m}] = \sum_\sigma \operatorname{Im} \Psi_\sigma$.

We want now to give a slightly different formulation of the Lemma above. Let $M = V^{\otimes i} \otimes (V^*)^{\otimes j}$ we want to describe $M^*_{\delta^m} = \{\varphi \in M^* : g \cdot \varphi = \delta^{-m}(g)\varphi\}$. Of course this problem is completely equivalent to the previous one. We shall now reformulate the description of a set of generators of $M^*_{\delta^m}$ in a more convenient way for our purposes. Let $m \ge 0$ and choose a collection $\mathcal{I} = \{I_1, \dots, I_m\}$ of m disjoint subsets of $\{1, \dots, i + mn\}$ of cardinality n. Let

$I_j = \{i_{j1} < \cdots < i_{jn}\}$ and $\{1, \ldots, i+mn\} - \bigcup \mathcal{I} = \{j_1 < \cdots < j_i\}$. To $\mathcal{I}$ and to a permutation $\sigma \in S_i$ we associate elements

$$\phi_{\mathcal{I},\sigma} \in \left(V^{\otimes i+mn} \otimes (V^*)^{\otimes i}\right)^*_{\delta^m} \quad \text{and} \quad \psi_{\mathcal{I},\sigma} \in \left(V^{\otimes i} \otimes (V^*)^{\otimes i+mn}\right)^*_{\delta^{-m}}$$

defined by

$$\phi_{\mathcal{I},\sigma}(v_1 \otimes \ldots v_{i+mn} \otimes \varphi_1 \ldots \varphi_i) = \prod_{j=1}^{m} \langle o^*, v_{i_{j1}} \wedge \cdots \wedge v_{i_{jn}} \rangle \cdot \prod_{h=1}^{i} \langle v_{j_h}, \varphi_{\sigma_h} \rangle$$

$$\psi_{\mathcal{I},\sigma}(v_1 \otimes \ldots v_i \otimes \varphi_1 \ldots \varphi_{i+mn}) = \prod_{j=1}^{m} \langle o, \varphi_{i_{j1}} \wedge \cdots \wedge \varphi_{i_{jn}} \rangle \cdot \prod_{h=1}^{i} \langle v_{\sigma_h}, \varphi_{j_h} \rangle$$

where o is a nonzero vector in $\bigwedge^n V$ and o^* is a non zero vector in $\bigwedge^n V^*$.

LEMMA 18.
1. *If $i \neq j + mn$ then* $\left(V^{\otimes i} \otimes (V^*)^{\otimes j}\right)^*_{\delta^m} = 0$.
2. *If $m \geq 0$ then* $\left(V^{\otimes i+mn} \otimes (V^*)^{\otimes i}\right)^*_{\delta^m}$ *is generated by the functions* $\phi_{\mathcal{I},\sigma}$.
3. *If $m \geq 0$ then* $\left(V^{\otimes i} \otimes (V^*)^{\otimes i+mn}\right)^*_{\delta^{-m}}$ *is generated by the functions* $\psi_{\mathcal{I},\sigma}$.

The proofs of the three Lemmas are straightforward.

2.2. – A special case

To simplify the exposition in this subsection we prove two special cases of Theorem 14. In one of these cases we obtain a more precise result interesting in its own.

Let J^+, J^- be two sets of indexes and define $J = J^+ \times J^-$. For each $i \in J^+$ (resp. $j \in J^-$) choose a vector space Y_i (resp. X_j) and define $X = \bigoplus_{j \in J^-} X_j$ and $Y = \bigoplus_{i \in J^+} Y_i$. Consider the group

$$G_{XY} = \prod_{i \in J^+} GL(Y_i) \times \prod_{j \in J^-} GL(X_j)$$

and its character $c = \prod_{i \in J^+} \det_{GL(Y_i)} \times \left(\prod_{j \in J^-} \det_{GL(X_j)}\right)^{-1}$.

We fix a matrix $r = (r_{ij})_{i \in J^+, j \in J^-}$ of integers and we consider the vector space:

$$H_r^{XY} = \bigoplus_{(i,j) \in J} \operatorname{Hom}(X_j, Y_i)^{\oplus r_{ij}}.$$

When r and the spaces X, Y will be clear from the context we will write H instead of H_r^{XY}. In the case $r_{ij} = 1$ for all i, j we obtain the space $H_{\mathbf{1}}^{XY} = H_{\mathbf{1}} = \operatorname{Hom}(X, Y)$. We fix a basis e_m^{ij} of $\mathbb{C}^{r_{ij}}$, so we have a canonical identification

$$H = \bigoplus_{(i,j) \in J} \operatorname{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}}. \tag{5}$$

We want to study c-equivariant polynomials on H. If we choose elements $\varphi_{ij} \in (\mathbb{C}^{r_{ij}})^*$ for all i, j then we can define a map $\Phi_\varphi : H \longrightarrow H_{\mathbf{1}}$ by

$$\Phi_\varphi\left(\sum_{(i,j)\in J} A_{ij} \otimes v_{ij}\right) = \sum_{(i,j)\in J} \varphi_{ij}(v_{ij})A^{ij} \tag{6}$$

where $A_{ij} \otimes v_{ij} \in \mathrm{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}}$. We will prove that the determinants of these maps generates the space of c-equivariant polynomials. We will study first the case $r_{ij} = 1$ for all i, j where we can state a more precise result. To state it we introduce the following set of matrices:

$$\mathcal{S}_n = \left\{ S = (s_{ij}) \in \mathbb{N}^{J^+ \times J^-} \ : \sum_{i,j} s_{ij} = n \right\},$$

$$\mathcal{S}^{XY} = \mathcal{S} = \left\{ S = (s_{ij}) \in \mathbb{N}^{J^+ \times J^-} \ : \sum_j s_{ij} = \dim Y_i \ \forall i \in J^+ \text{ and} \right.$$
$$\left. \sum_i s_{ij} = \dim X_j \ \forall j \in J^- \right\}.$$

As for H we will write $\mathcal{S}$ when the spaces X_j, Y_i will be clear from the context. Observe that $\mathcal{S} = \varnothing$ if $\sum_j \dim X_j \neq \sum_i \dim Y_i$ and that if $N = \sum \dim X_j = \sum \dim Y_i$ then $\mathcal{S} \subset \mathcal{S}_N$. For each $\mathrm{card}(J^+) \times \mathrm{card}(J^-)$ matrix $S = (s_{ij})$ we consider $\varphi_{ij} \in \mathbb{C}^*$ given by $\varphi_{ij}(\lambda) = s_{ij}\lambda$ and we define

$$\Phi_S = \Phi_\varphi \quad \text{and} \quad f_S(A) = f_S^{XY}(A) = \det(\Phi_S(A)).$$

We can now state the two special versions of Theorem 14.

PROPOSITION 19. *$\{f_S\}_{S\in\mathcal{S}^{XY}}$ is a basis of $\mathbb{C}[H_{\mathbf{1}}^{XY}]_c$.*

LEMMA 20. *$\mathbb{C}[H^{XY}]_c$ is generated as a vector space by the following functions:*

$$s \longmapsto \det\big(\Phi_\varphi(s)\big)$$

where $\Phi_\varphi : H^{XY} \longrightarrow H_{\mathbf{1}}^{XY}$ is as in (6) above.

2.2.1. – Proof of Proposition

Here and in the following we will use polarization: if V is a finite dimensional vector space then we can define a map

$$\wp : (V^{\otimes n})^* \longrightarrow S^n(V^*) \subset \mathbb{C}[V] \quad \text{through} \quad \wp(\varphi)(v) = \varphi(v \otimes \cdots \otimes v).$$

LEMMA 21. *$\wp$ is surjective, moreover if V is a finite dimensional representation of a reductive group Γ, and χ is a character of Γ then*

$$\wp((V^{\otimes n})^*_\chi) = \mathrm{S}^n(V^*)_\chi$$

where E_χ is the isotypic component of type χ^{-1} of a G module E.

LEMMA 22. *For $i = 1, \dots, n$ let Γ_i be a reductive group, χ_i be a character of Γ_i and E_i be a f.d. representation of Γ_i. Let $\Gamma = \prod \Gamma_i$, then $E = \otimes_i E_i$ is a representation of Γ and $\chi = \prod \chi_i$ is a character of Γ. Then*

$$E^*_\chi = (E_1)^*_{\chi_1} \otimes \cdots \otimes (E_n)^*_{\chi_n}.$$

Now we prove Proposition 19. We have to compute $\mathrm{S}^n(H^*_{\mathbf{1}})_c = (\mathrm{S}^n(H_{\mathbf{1}}))^*_c$ for all n. For all $S \in \mathcal{S}_n$ define

$$E_S = \bigotimes_{(i,j)\in J^+\times J^-} \mathrm{S}^{s_{ij}} \left(\mathrm{Hom}(X_j, Y_i)\right).$$

Observe that $\mathrm{S}^n(H_{\mathbf{1}}) = \bigoplus_{S\in\mathcal{S}_n} E_S$ as a G-module. So $\mathrm{S}^n(H_{\mathbf{1}})^*_c = \bigoplus_{S\in\mathcal{S}_n} (E_S)^*_c$. Observe now that E_S is a quotient of

$$\tilde{E}_S = \bigotimes_{(i,j)\in J^+\times J^-} (X^*_j)^{\otimes s_{ij}} \otimes Y_i^{\otimes s_{ij}}. \tag{7}$$

By the Lemmas of Subsection 2.1 we have that

$$(\tilde{E}_S)^*_c = \begin{cases} 0 & \text{if } S \notin \mathcal{S}^{XY}, \\ \mathbb{C} & \text{if } S \in \mathcal{S}^{XY}. \end{cases}$$

So in particular $(E_S)^*_c = 0$ if $S \notin \mathcal{S}^{XY}$. Hence $\dim \mathrm{S}^n(H)^*_c \leq \mathrm{card}(\mathcal{S}^{XY})$.

The functions f_S are clearly c-equivariant so the only thing that we have to prove is that they are linearly independent. To prove it we will prove a generalization of it.

If $i \in J^+$ and $j \in J^-$ let E_{ij} be the $\mathrm{card}(J^+) \times \mathrm{card}(J^-)$ matrix with a 1 in the (i, j) position and 0 elsewhere.

For each $i \in J^+$, $j \in J^-$, $m \in \mathbb{N}$ and $N \in \mathbb{N}$ we consider the following sentence $P_{i,j,m,N}$:

If $\sum_j \dim X_j = N = \sum_i \dim Y_i$ then $\{f_{S+mE_{ij}}\}_{S\in\mathcal{S}^{XY}}$ is linearly independent.

In the case $m = 0$ we call this Proposition $P_{0,N}$ since it does not depend on i, j and we observe that $\forall N\ P_{0,N}$ is equivalent to the statement of Proposition 19.

For each $N \in \mathbb{N}$ we consider also the following sentence Q_N:

If $\sum_j \dim X_j = N = \sum_i \dim Y_i$ then $P_{i,j,m,N}$ is true for all $i \in J^+, j \in J^-$ and $m \in \mathbb{N}$.

We prove Q_N by induction on N. The case $N = 1$ is trivial. Suppose now that there exist $c_S \in \mathbb{C}$, $i_0 \in J^+$, $j_0 \in J^-$ and $m \in \mathbb{N}$ such that

$$\sum_{S\in\mathcal{S}^{XY}} c_S f^{XY}_{S+mE_{i_0j_0}} = 0.$$

We shall prove $c_S = 0$ for all S in various steps.

First step: If there exists j_1 such that $\dim X_{j_1} = 1$ then $c_S = 0$ for all S. Set

$$\tilde{\mathcal{S}}_i = \{S \in \mathcal{S}^{XY} \, : \, s_{ij_1} = 1\}$$

and observe that since $\dim X_{j_1} = 1$ then $\mathcal{S}^{XY} = \coprod \tilde{\mathcal{S}}_i$. Now choose a non zero vector $x_{j_1} \in X_{j_1}$ and for all $i \in J^+$ choose a non zero vector $y_i \in Y_i$ and an hyper-plane Y_i' of Y_i such that $Y_i = \mathbb{C}y_i \oplus Y_i'$.

Now fix $i_1 \neq i_0$ such that $\dim Y_{i_1} \geq 2$ and consider $\tilde{J}^+ = J^+$ and $\tilde{J}^- = J^- - \{j_1\}$. For all $i \in \tilde{J}^+$ and for all $j \in \tilde{J}^-$ define:

$$\tilde{X}_j = X_j, \quad \tilde{Y}_i = \begin{cases} Y_i & \text{if } i \neq i_1, \\ Y_{i_1}' & \text{if } i = i_1. \end{cases}$$

For any $S \in \tilde{\mathcal{S}}_{i_1}$ we define also $t(S) \in \mathcal{S}^{\tilde{X}\tilde{Y}}$ by $t(S)_{ij} = s_{ij}$ for all $i \in \tilde{J}^+$, $j \in \tilde{J}^-$. $S \longmapsto t(S)$ is a bijection between $\tilde{\mathcal{S}}_{i_1}$ and $\mathcal{S}^{\tilde{X}\tilde{Y}}$: we call t^{-1} the inverse map. Finally we define $\Psi : H_{\mathbf{1}}^{\tilde{X}\tilde{Y}} \longrightarrow H_{\mathbf{1}}^{XY}$ by

$$\Psi(T)\Big|_{X_j} = T\Big|_{\tilde{X}_j} \text{ for all } j \in \widetilde{J}^- \text{ and } \Psi(T)(x_{j_1}) = y_{j_1}. \tag{8}$$

Observe that if $S \in \mathcal{S}^{XY}$ then $f_{S+mE_{i_0j_0}} \circ \Psi = 0$ if $S \notin \tilde{\mathcal{S}}_{i_1}$.

Now if $j_0 \neq j_1$ we have

$$\begin{aligned} 0 &= \sum_{S \in \mathcal{S}^{XY}} c_S f_{S+mE_{i_0j_0}}^{XY}(\Psi(T)) = \sum_{S \in \tilde{\mathcal{S}}_i} c_S f_{t(S)+mE_{i_0j_0}}^{\widetilde{X}\widetilde{Y}}(T) \\ &= \sum_{S \in \mathcal{S}^{\tilde{X}\tilde{Y}}} c_{t^{-1}(S)} f_{S+mE_{i_0j_0}}^{\widetilde{X}\widetilde{Y}}(T) \end{aligned}$$

and by $P_{i_0,j_0,m,N-1}$ we deduce $c_S = 0$ for all $S \in \tilde{\mathcal{S}}_{i_1}$.

If $j_0 = j_1$ we obtain similarly

$$0 = (\delta_{i_0i_1} m + 1) \sum_{S \in \mathcal{S}^{\tilde{X}\tilde{Y}}} c_{t^{-1}(S)} f_S^{\widetilde{X}\widetilde{Y}}(T)$$

and by $P_{0,N-1}$ we deduce $c_S = 0$ for all $S \in \tilde{\mathcal{S}}_{i_1}$.

In a similar way we prove $c_S = 0$ if $S \in \tilde{\mathcal{S}}_{i_1}$ and $\dim Y_{i_1} = 1$.

Second step: If there exists i_1 such that $\dim Y_{i_1} = 1$ then $c_S = 0$ for all S. This is completely analogous to the previous step.

So we can assume $\dim X_j, \dim Y_i \geq 2$ for all i, j.

Third step: If $m = 0$ then $c_S = 0$ for all S. Choose i_1, j_1 arbitrarily. Since $\dim X_{j_1}, \dim Y_{i_1} \geq 2$ we can choose a nonzero element $x_{j_1} \in X_{j_1}$ (resp. $y_{i_1} \in$

Y_{i_1}) and an hyper-plane $X'_{j_1} \subset X_{j_1}$ (resp. $Y'_{i_1} \subset Y_{i_1}$) such that $X_{j_1} = \mathbb{C}x_{j_1} \oplus X'_{j_1}$ (resp. $Y_{j_1} = \mathbb{C}y_{i_1} \oplus Y'_{i_1}$). Define:

$$\tilde{X}_j = \begin{cases} X_j & \text{if } j \neq j_1 \\ X'_{j_1} & \text{if } j = j_1 \end{cases} \quad \text{and} \quad \tilde{Y}_i = \begin{cases} Y_i & \text{if } i \neq i_1 \\ Y'_{i_1} & \text{if } i = i_1 \end{cases} \tag{9}$$

and $\Psi : H_{\mathbf{1}}^{\tilde{X}\tilde{Y}} \longrightarrow H_{\mathbf{1}}^{XY}$ by

$$\Psi(T)\Big|_{X_j} = T\Big|_{\tilde{X}_j} \quad \text{and} \quad \Psi(T)(x_{j_1}) = y_{j_1}. \tag{10}$$

Then

$$\begin{aligned} 0 &= \sum_{S \in \mathcal{S}^{XY}} c_S f_S^{XY}(\Psi(T)) = \sum_{S \in \mathcal{S}^{XY} : s_{i_1 j_1} \neq 0} s_{i_1 j_1} c_S f_S^{\tilde{X}\tilde{Y}}(T) \\ &= \sum_{S \in \mathcal{S}^{\tilde{X}\tilde{Y}}} (s_{i_1 j_1} + 1) c_{S+E_{i_1 j_1}} f^{\tilde{X}\tilde{Y}}_{S+E_{i_1 j_1}}(T) \end{aligned}$$

By induction $P_{i,j,1,N-1}$ is true for all i, j so we see that $c_S = 0$ for all $S \in \mathcal{S}$ such that $s_{i_1,j_1} \geq 1$. Now we conclude thanks to previous two steps.

Fourth step: If $m \geq 1$ then $c_S = 0$ for all S. We can construct $\tilde{X}_j, \tilde{Y}_i, \Psi$ as in the third step with $j_1 = j_0$ and $i_1 = i_0$ and we see that

$$\begin{aligned} 0 &= \sum_{S \in \mathcal{S}^{XY}} c_S f^{XY}_{S+mE_{i_0 j_0}}(\Psi(T)) = \sum_{S \in \mathcal{S}^{XY}} (s_{i_0 j_0} + m) c_S f^{\tilde{X}\tilde{Y}}_{S+mE_{i_0 j_0}}(T) \\ &= \sum_{S \in \mathcal{S}^{\tilde{X}\tilde{Y}}} (s_{i_0 j_0} + m + 1) c_{S+E_{i_0 j_0}} f^{\tilde{X}\tilde{Y}}_{S+(m+1)E_{i_0 j_0}}(T) \end{aligned}$$

and by $P_{i_0, j_0, m+1, N-1}$ we deduce $c_S = 0$ for all S.

2.3. – Proof of Lemma 20

We study first $(H^{\otimes n})^*_c$ and then we apply polarization. As in the previous section we can decompose $H^{\otimes n}$ in summands of the following form:

$$E = \bigotimes_{(i,j) \in J} (\mathrm{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}})^{\otimes s_{ij}} = \bigotimes_{(i,j) \in J} (X_j^*)^{\otimes s_{ij}} \otimes Y_i^{\otimes s_{ij}} \otimes (\mathbb{C}^{r_{ij}})^{\otimes s_{ij}} \tag{11}$$

where s_{ij} are nonnegative integers such that $\sum_{i,j} s_{ij} = n$. Notice that the order of the factors is not important for us since we will apply polarization.

We can describe easily E^*_c using the Lemmas in Subsection 2.1. In particular a necessary and sufficient condition for the existence of c-covariants is

$\sum_{i\in J^+} s_{ij} = \dim X_j$ for all $j \in J^-$ and $\sum_{j\in J^+} s_{ij} = \dim Y_i$ for all $i \in J^+$. Moreover

$$E_c^* \simeq \bigotimes_{(i,j)\in J} \left((\mathbb{C}^{r_{ij}})^*\right)^{\otimes s_{ij}}$$

To write explicit formulas we choose an order on the factors of E, for example choosing a lexicographic order in $i \in J^+$, $j \in J^-$ and $1 \le q \le s_{ij}$:

$$E = \underbrace{X_1^* \otimes Y_1 \otimes \mathbb{C}^{r_{11}}}_{q=1} \otimes \cdots \otimes \underbrace{X_1^* \otimes Y_1 \otimes \mathbb{C}^{r_{11}}}_{q=s_{11}} \otimes \underbrace{X_1^* \otimes Y_2 \otimes \mathbb{C}^{r_{12}}}_{q=1} \otimes \cdots$$

Once we have chosen such an order we can write an element of E as linear combination of elements of the form $\otimes_{(i,j,q)\in K} x^{i,j,q} \otimes y^{i,j,q} \otimes v^{i,j,q}$ with $x^{i,j,q} \in X_j^*$, $y^{i,j,q} \in Y_i$ $v^{i,j,q} \in \mathbb{C}^{r_{ij}}$ and we have set $K = \{(i,j,q) \in J \times \mathbb{N} : 1 \le q \le s_{ij}\}$. Using this convention if

$$\phi = \bigotimes_{(i,j,q)\in K} \phi^{i,j,q} \in \bigotimes_{(i,j,q)\in K} (\mathbb{C}^{r_{ij}})^* \tag{12}$$

the corresponding c equivariant linear function on E is defined on an element $s = \otimes_{(i,j,q)\in K} x^{i,j,q} \otimes y^{i,j,q} \otimes v^{i,j,q}$ by

$$\phi(s) = \prod_{i\in J^+} \langle \bigwedge_{\overrightarrow{(i,j,q)\in K}} y^{i,j,q}, o_i^* \rangle \prod_{j\in J^- \longrightarrow} \langle \bigwedge_{(i,j,q)\in K} x^{i,j,q}, o_j \rangle \prod_{(i,j,q)\in K} \phi^{i,j,q}(v^{i,j,q})$$

where $\underset{\longrightarrow}{\bigwedge}$ means that we are making the exterior product with respect to the order we have chosen above. Observe now that $\wp(E_c^*) = \wp\left(\bigotimes_{(i,j)\in J} \mathrm{S}^{s_{ij}}\left((\mathbb{C}^{r_{ij}})^*\right)\right)$. In particular since $\mathrm{S}^m(V)$ is spanned by vectors of the form $v \otimes \cdots \otimes v$, $\wp(E_c^*)$ is spanned by the functions $\wp(\phi)$ with ϕ of the following special form:

$$\phi = \bigotimes_{(i,j)\in J} (\phi^{i,j})^{\otimes s_{ij}} \tag{13}$$

The Lemma now follows from the following claim:

Claim. For each ϕ as in (13) $\wp(\phi)$ is a linear combination of the functions $A \mapsto \det(\Phi_\varphi(A))$.

We prove the claim as follows: we consider the space $H_1 = \mathrm{Hom}(X, Y)$ and we construct a G_{XY} equivariant map $\rho : H \longrightarrow H_1$ such that

1. there exists a c-covariant function f on H_1 such that $\wp(\phi) = f \circ \rho$,
2. for all $S \in \mathcal{S}^{XY}$ there exists φ as in equation (6) such that for all $A \in H$ we have $f_S(\rho(A)) = \det(\Phi_\varphi(A))$.

The claim now follows from Proposition 19.

To prove 1. fix ϕ as in (13) and define $\rho^{ij} : \mathrm{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}} \longrightarrow \mathrm{Hom}(X_j, Y_i)$ by

$$\rho^{ij}(T \otimes v) = \phi^{i,j}(v)T,$$

and define $\rho = \bigoplus_{i,j \in J} \rho^{ij} : H \longrightarrow H_{\mathbf{1}}$. Observe that ρ is G_{XY}-equivariant.

Observe now that $H_{\mathbf{1}}^{\otimes n} = \bigoplus E_S$ where $S \in \mathcal{S}_n$ and E_S is defined as in (7). In particular we choose the following summand of $H_{\mathbf{1}}^{\otimes n}$:

$$E = \bigotimes_{(i,j)\in J} \mathrm{Hom}(X_j, Y_i)^{\otimes s_{ij}}$$

where the (s_{ij}) are the same as those used to define ϕ in formula (13). Observe that $(E)_c^* = \mathbb{C}$. Choose a non zero element $\widetilde{\phi} \in (E)_c^*$ and observe that up to a nonzero scalar factor we have

$$\wp_{H_{\mathbf{1}}}(\widetilde{\phi}) \circ \rho = \wp_H(\phi). \tag{14}$$

To see this choose ϕ as in (13), and bases y_h^i of Y_i, x_k^j of X_j^* (and its dual basis z_k^j of X_j) . Choose also a bases ε_m^{ij} of $\mathbb{C}^{r_{ij}}$ such that $\phi^{i,j}(\varepsilon_m^{ij}) = \delta_{m,1}$ and set $A^{ij} = \rho^{ij}(s) = \sum_{h,k} a_{hk}^{ij} y_h^i \otimes x_k^j$ for $s \in H$. Then

$$\begin{aligned}
\wp(\phi)(t) &= \sum_{h \in \mathcal{K}_Y, k \in \mathcal{K}_X} \prod_{i \in J^+} \langle \bigwedge_{\overrightarrow{(i,j,q)} \in K} a^{ij}_{h(i,j,q)k(i,j,q)} y^i_{h(i,j,q)}, o_i^* \rangle \prod_{j \in J^-} \langle \bigwedge_{\overrightarrow{(i,j,q)} \in K} x^j_{k(i,j,q)}, o_j \rangle \\
&= \sum_{k \in \mathcal{K}_X} \prod_{i \in J^+} \langle \bigwedge_{\overrightarrow{(i,j,q)} \in K} A^{ij} z^j_{k(i,j,q)}, o_i^* \rangle \prod_{j \in J^-} \langle \bigwedge_{\overrightarrow{(i,j,q)} \in K} x^j_{k(i,j,q)}, o_j \rangle
\end{aligned}$$

where the indexes are as follows:

$$\begin{aligned}
\mathcal{K}_X &= \{k : K \longrightarrow \mathbb{N} : 1 \le k(i,j,q) \le \dim X_j\}, \\
\mathcal{K}_Y &= \{h : K \longrightarrow \mathbb{N} : 1 \le h(i,j,q) \le \dim Y_i\}.
\end{aligned}$$

The left hand side in (14) clearly gives the same expression.

Finally for any $S = (t_{ij}) \in \mathcal{S}^{XY}$ if we choose $\varphi_{ij} = t_{ij}\phi^{i,j}$ then

$$f_S(\rho(A)) = \det(\Phi_\varphi(A)).$$

REMARK 23. The basis of $\mathbb{C}[H]_c$ we have described are different from the polarization of the natural basis of E_c^*. The relation between the two basis is given by formulas of the following types

1. If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ then

$$\det \begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{pmatrix} + \det \begin{pmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{pmatrix} = \det(A+B) - \det A - \det B.$$

2. If $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$, $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$, $C=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$ and $D=\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{pmatrix}$ then

$$\det\begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21}\end{pmatrix}\det\begin{pmatrix} c_{12} & d_{12} \\ c_{22} & d_{22}\end{pmatrix}-\det\begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22}\end{pmatrix}\det\begin{pmatrix} c_{12} & d_{11} \\ c_{22} & d_{21}\end{pmatrix}$$
$$-\det\begin{pmatrix} a_{12} & b_{11} \\ a_{22} & b_{21}\end{pmatrix}\det\begin{pmatrix} c_{11} & d_{12} \\ c_{21} & d_{22}\end{pmatrix}+\det\begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22}\end{pmatrix}\det\begin{pmatrix} c_{11} & d_{11} \\ c_{21} & d_{21}\end{pmatrix}$$
$$=-\det\begin{pmatrix} A & B \\ C & D\end{pmatrix}+\det\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}+\det\begin{pmatrix} 0 & B \\ C & 0\end{pmatrix}$$

The first type of formula corresponds to the reduction of Lemma 20 to the case $r_{ij}=1$. The second type of formula correspond to the case of Proposition 19.

2.4. – Proof of Theorem 14

Write the vector space $S(Q,v)$ in the following way:

$$S=\bigoplus_{h\in H} V^*_{h_0}\otimes V_{h_1}.$$

Fix a character χ_m and m_i, m_i^+, m_i^- as in 2.0.2 and let I^+ (resp. I^0, I^-) be the set of vertices i such that $m_i>0$ (resp. $=0$, <0). We describe first the χ_m covariants of $S^{\otimes n}$. Observe that we can decompose $S^{\otimes n}$ in the following way:

$$S^{\otimes n}=\bigoplus_{\ell} E_1^{(\ell)}\otimes\cdots\otimes E_n^{(\ell)}$$

where each $E_i^{(\ell)}$ is of the form $V^*_{h_0}\otimes V_{h_1}$. So it is enough to compute the χ_m covariants of each piece $E_1^{(\ell)}\otimes\cdots\otimes E_n^{(\ell)}$. We fix such a piece $E=E_1\otimes\cdots\otimes E_n$ and we compute E^*_χ. Let I^* be a copy of I and fix an isomorphism $i\longleftrightarrow i^*$ between the two sets. For each $j=1,\dots,n$ we define the subset $\mathcal{S}_j$ of $I\coprod I^*$ as $\{h_0^*,h_1\}$ if $E_j=V^*_{h_0}\otimes V_{h_1}$.

Let now $\mathcal{S}=\coprod_{j=1}^n\mathcal{S}_j$. An element of $\mathcal{S}$ can be thought as a couple (i,j) (or (i^*,j)) where i (or i^*) is in $\mathcal{S}_j$. We consider now a special class of partitions of $\mathcal{S}$: a collection $\mathfrak{F}=\{\mathcal{C},\mathcal{M}_i^{(l)}$ for $i\in I$ and $1\le l\le m_i\}$ of disjoint subsets of $2^{\mathcal{S}}$ is called *m-special* if:

1. $\bigcup\mathfrak{F}$ is a partition of $\mathcal{S}$,
2. $\forall C\in\mathcal{C}$ card $C=2$ and $\exists i\in I$, $\mathcal{S}_{j_1},\mathcal{S}_{j_2}$ such that $i\in\mathcal{S}_{j_1}$, $i^*\in\mathcal{S}_{j_2}$ and $C=\{(i,j_1),(i^*,j_2)\}$,
3. $\forall M\in\mathcal{M}_i^{(l)}$ we have $M=\{(i,j)\}$ if $i\in I^+$ and $M=\{(i^*,j)\}$ if $i\in I^-$,
4. card $\mathcal{M}_i^{(l)}=v_i=\dim V_i$.

We can represents a special collection with a graph whose vertices are the sets $\mathcal{S}_j$ enriched according with the following rules:

1. we put an arrow from $\mathcal{S}_{j_1}$ to $\mathcal{S}_{j_2}$ if there exists $C = \{(i, j_2), (i^*, j_1)\} \in \mathcal{C}$,
2. we put an indexed circle box $\circ_i^l$ on $\mathcal{S}_j$ if there exists $M = \{(i, j)\} \in \mathcal{M}_i^{(l)}$,
3. we put an indexed square box $\square_i^l$ on $\mathcal{S}_j$ if there exists $M = \{(i^*, j)\} \in \mathcal{M}_i^{(l)}$,
4. if E_j is of type $V_{h_0}^* \otimes V_{h_1}$ then we write h at the left of the corresponding vertex.

Observe that: $i)$ a vertex can be marked with a circle and a square but that it cannot be marked with two circles or two squares, $ii)$the cardinality of the vertices marked with $\circ_i^l$ is v_i for each $i \in I^+$ and $1 \leq l \leq m_i$, $iii)$ the cardinality of verteces marked with $\square_i^l$ is v_i for each $i \in I^-$ and $1 \leq l \leq -m_i$.

To a special collection $\mathfrak{F}$ as above we attach a function $\phi_{\mathfrak{F}}$ on E. We define it by the formula

$$\phi_{\mathfrak{F}}(e_1 \otimes \cdots \otimes e_n) = \prod_{C \in \mathcal{C}} \phi_C \cdot \prod_{i \in I^+} \prod_{l=1}^{m_i} \langle o_i^*, \bigwedge \mathcal{M}_i^{(l)} \rangle \cdot \prod_{i \in I^-} \prod_{l=1}^{m_i} \langle o_i, \bigwedge \mathcal{M}_i^{(l)} \rangle$$

where o_i is a non zero element in $\bigwedge^{v_i} V_i$, o_i^* is a non zero element in $\bigwedge^{v_i} V_i^*$, $e_j = x_j^* \otimes y_j \in V_{h_0}^* \otimes V_{h_1}$ if $E_j = V_{h_0}^* \otimes V_{h_1}$, and

$$\begin{aligned}
\phi_C &= \langle x_{j_1}^*, v_{j_2} \rangle && \text{if } C = \{(i^*, j_1), (i, j_2)\}, \\
\bigwedge \mathcal{M}_i^{(l)} &= y_{j_1} \wedge \cdots \wedge y_{j_{v_i}} && \text{if } \mathcal{M}_i^{(l)} = \{\{(i, j_1)\}, \ldots, \{(i, j_{v_i})\}\} \text{ and } i \in I^+, \\
\bigwedge \mathcal{M}_i^{(l)} &= x_{j_1}^* \wedge \cdots \wedge x_{j_{v_i}}^* && \text{if } \mathcal{M}_i^{(l)} = \{\{(i, j_1)\}, \ldots, \{(i, j_{v_i})\}\} \text{ and } i \in I^-.
\end{aligned}$$

Finally we extend $\phi_{\mathfrak{F}}$ to E by linearity. By the discussion in 2.1 we deduce easily the following Lemma:

LEMMA 24. *E_χ^* is generated by the functions $\phi_{\mathfrak{F}}$.*

Theorem 14 now follows from Lemma 24 and the following claim:

CLAIM. for any special collection $\mathfrak{F}$ the function $\wp(\phi_{\mathfrak{F}})$ is a $\mathbb{C}[S]^{G_v}$-linear combination of the functions f_Δ described in 2.0.2.

We consider the connected components of the graph. There are only two possible types of paths:

1. closed paths,
2. straight paths leaving from a square boxed vertex and arriving in a circle boxed vertex.

Let now $\mathfrak{F}_0$ be the union of the connected components of the first type and $\mathfrak{F}_1$ be the union of the remaining components. Observe that

$$\wp(\phi_{\mathfrak{F}}) = \wp(\phi_{\mathfrak{F}_0}) \wp(\phi_{\mathfrak{F}_1}).$$

Observe also that $\phi_{\mathfrak{F}_0}$ is an invariant function (indeed this part of the graph corresponds to the situation studied by Lusztig in [5]). Since we are interested in generators of $\mathbb{C}[S]_{\chi,1}$as a $\mathbb{C}[S]^G$-module, we can suppose $\mathfrak{F} = \mathfrak{F}_1$.

Observe now that each connected component Γ of the graph of the second type with the initial vertex marked with a square $\square_{i_0}^{l_0}$ and the final vertex marked with a circle $\circ_{i_1}^{l_1}$ determines:

1. a path α^Γ of our quiver Q such that $\alpha_0^\Gamma = i_0$ and $\alpha_1^\Gamma = i_1$,
2. two numbers $l_0 = L_0(\Gamma)$ and $l_1 = L_0(\Gamma)$ such that $1 \leq l_0 \leq -m_{\alpha_0^\Gamma}$ and $1 \leq l_1 \leq m_{\alpha_1^\Gamma}$.

Now we prove the claim in the following way, we construct X_j, Y_i and r as in 2.2, a group homomorphism $\sigma : G_v \longrightarrow G_{XY}$ such that $\sigma^* c = \chi_m$, a G_v equivariant map $\rho : S \longrightarrow H$, and a G_{XY} c-covariant function f on H such that:

1. for all φ there exists a χ_m-good data Δ such that $\det(\Phi_\varphi(\rho(s))) = f_\Delta(s)$,
2. $\wp(\phi_{\mathfrak{F}}) = f \circ \rho$.

The claim will clearly follow from Lemma 20.

Set

$$J^- = \{(i,l) : i \in I^- \text{ and } 1 \leq l \leq -m_i\},$$
$$J^+ = \{(i,l) : i \in I^+ \text{ and } 1 \leq l \leq m_i\}.$$

For all $(i,l) \in J^-$ choose $X_{(i,l)} = V_i$ and for each $(i,l) \in J^+$ choose $Y_{(i,l)} = V_i$. For each $(i_0, l_0) \in J^-$ and for each $(i_1, l_1) \in J^+$ define:

$$r_{(i_0,l_0)(i_1,l_1)} = \operatorname{card}\{\text{connected component } \Gamma \text{ of the second type such that } \alpha_0^\Gamma = i_0,\ \alpha_1^\Gamma = i_1,\ L_0(\Gamma) = l_0 \text{ and } L_1(\Gamma) = l_1\}.$$

We use the connected component Γ of the set in the right hand side as a basis e_Γ of the vector space $\mathbb{C}^{r(i_0,l_0)(i_1,l_1)}$. This basis plays the role of the basis e_m^{ij} we used to give the identification in (5).

Now for each connected component Γ of the second type define $\rho^\Gamma : S \longrightarrow \operatorname{Hom}(X_{(\alpha_0^\Gamma, L_0(\Gamma))}, Y_{(\alpha_1^\Gamma, L_1(\Gamma))})$ by $\rho^\Gamma(s) = \alpha^\Gamma(s)$. Finally define

$$\rho : S \longrightarrow H \quad \text{by} \quad \rho = \bigoplus_\Gamma \rho^\Gamma.$$

Define also a group homomorphism $\sigma : G_v \longrightarrow G_{XY}$ by $(\sigma(g_i))_{X_{(i_0,l_0)}} = g_{i_0}$ and $(\sigma(g_i))_{Y_{(i_1,l_1)}} = g_{i_1}$, and observe that ρ is G_v equivariant.

Now we describe $\phi \in (H^{\otimes \tilde{n}})^*_c$ (in general $\tilde{n}$ is less or equal to n) such that

$$\wp(\phi_{\mathfrak{F}})(s) = \wp(\phi)(\rho(s)). \tag{15}$$

We describe ϕ by giving a summand $\tilde{E}$ of $H^{\otimes \tilde{n}}$ as in (11) and $\phi \in \tilde{E}_c^*$ as in (12). To define $\tilde{E}$ we set $s_{(i_1,l_1)(i_0,l_0)} = r_{(i_1,l_1)(i_0,l_0)}$ for all $(i_1, l_1) \in J^+$ and for all $(i_0, l_0) \in J^-$.

Observe that we can choose a bijection $q \longleftrightarrow \Gamma_q$ between $\{1,\ldots,s_{(i_1,l_1)(i_0,l_0)}\}$ and the set of connected component Γ of the second type such that $\alpha_1^\Gamma = i_1, L_1(\Gamma) = l_1$, $\alpha_0^\Gamma = i_0$ and $L_0(\Gamma) = l_0$. So we can define $\phi^{(i_1,l_1),(i_0,l_0),q}$ by

$$\phi^{(i_1,l_1),(i_0,l_0),q}(e_\Gamma) = \delta_{\alpha_1^\Gamma i_1}\delta_{\alpha_0^\Gamma i_0}\delta_{L_0(\Gamma) l_0}\delta_{L_1(\Gamma) l_1}\delta_{\Gamma,\Gamma_q}.$$

Up to a sign which depends on our choices and ordering, equation (15) is tautologically satisfied so we proved property 2. above.

To prove property 1. fix φ as in (6) and choose a χ_m-good $\Delta = \{\alpha_*^*\}$ as follows:

$$\alpha_{jh}^{ik} = \sum_{1 \leq q \leq s_{(j,h)(i,k)}} \varphi_{(j,h),(i,k)}(e_{\Gamma_q})\alpha^{\Gamma_q}.$$

The equation $\det(\Phi_\varphi(\rho(s))) = f_\Delta(s)$ follows now by definition.

3. – The action of the Weyl group

For any $v \in R$, $m \in P$ and for any $\lambda \in Z$ we have defined a variety $M_{m,\lambda}(Q, v)$. Observe that on v, m, λ there is a natural action of the Weyl group W. So it makes sense to consider the variety $M_{\sigma m,\sigma\lambda}(\sigma v)$ or the variety $\mathfrak{M}_{\sigma\zeta}(\sigma v)$ for $\zeta \in \mathfrak{Z}$.

In [11] Nakajima used analytic methods to prove, in the case of a finite Dynkin diagram, that if ζ is generic then there exists a diffeomorphism of differentiable manifolds

$$\Phi_{\sigma,\zeta} : \mathfrak{M}_\zeta(d, v) \longrightarrow \mathfrak{M}_{\sigma\zeta}(\sigma(d, v))$$

and moreover that $\Phi_{\sigma',\sigma\zeta} \circ \Phi_{\sigma,\zeta} = \Phi_{\sigma'\sigma,\zeta}$. In the same paper he also asserted that a similar construction could be obtained in the general case using reflection functors as indeed we are going to do.

In [6] Lusztig gave a purely algebraic construction of an isomorphism

$$M_{0,\lambda}(v) \simeq M_{0,s_i\lambda}(s_i v)$$

whenever $\lambda_i \neq 0$. In this section we will give a generalization of Lusztig construction and we will prove Coxeter relations.

Definition 25. If $u \in R$ and $A \subset R$ we define

$$H_u = \{(m, \lambda) \in P \oplus Z \,:\, \langle u, \lambda\rangle = \langle u, m\rangle = 0\} \quad \text{and} \quad H_A = \bigcup_{a \in A} H_a.$$

Let $K = \max\{1, a_{ij}^2 \,:\, i, j \in I\}$. If $v \in \mathbb{Z}^n$ we define

$$A_v = \{u \in R^+ \,:\, Kv - u \in R^+\} \quad \text{and} \quad H^v = H_{A_v}.$$

We define also
$$A_\infty = \bigcup_{i\in I} W\bar{\alpha}_i \quad \text{and} \quad H^\infty = H_{A_\infty}.$$
Finally we set $\mathcal{G}_v = \{(m,\lambda) \in P \times Z : (\sigma m, \sigma\lambda) \notin H^{\sigma\cdot v} \text{ for all } \sigma \in W\}$. Let
$$\mathcal{G}' = \{(v,m,\lambda) \in R \times P \times Z : (m,\lambda) \in \mathcal{G}_v\} \quad \text{and}$$
$$\mathcal{G}'' = \{(v,m,\lambda) \in R \times P \times Z : (m,\lambda) \notin H^\infty\}.$$
Observe that both $\mathcal{G}'$ and $\mathcal{G}''$ are W-stable.

If $\mathcal{G} = \mathcal{G}'$ or $\mathcal{G} = \mathcal{G}''$ then the following Theorem holds.

THEOREM 26. *For all $v \in R$, for all $\sigma \in W$ and for all (m,λ) such that $(v,m,\lambda) \in \mathcal{G}$ there exists an algebraic isomorphism:*
$$\Phi^v_{\sigma,m,\lambda}\colon M_{m,\lambda}(v) \longrightarrow M_{\sigma m,\sigma\lambda}(\sigma v).$$
Moreover these isomorphisms satisfy
$$\Phi^{\sigma v}_{\tau,\sigma m,\sigma\lambda} \circ \Phi^v_{\sigma,m,\lambda} = \Phi^v_{\tau\sigma,m,\lambda}. \tag{16}$$

3.1. – Generators

In this subsection we define the action of the generators s_i of W following [6]. We fix $i \in I$, $v \in R$, $\lambda \in Z$ and $m \in P$. We call $v' = s_i v$, $\lambda' = s_i\lambda$ and $m' = s_i m$. Through all this section we assume $v, v' \in R^+$. For the convenience of the reader we write explicit formulas in this case:
$$\begin{aligned} \lambda'_j &= \lambda_j - c_{ij}\lambda_i, & m'_j &= m_j - c_{ij}m_i \quad \text{for all } j,\\ v'_i &= -v_i + \sum_{j\neq i} a_{ij}v_j, & v'_j &= v_j \quad \text{for all } j \neq i. \end{aligned}$$
Observe that we can choose $V'_j = V_j$ for all $j \neq i$ and that in particular we have $T_i = \bigoplus_{h_1=i} V_{h_0} = T'_i$ since we suppose that our quiver has not simple loops.

DEFINITION 27 (Lusztig [6]). Fix $\lambda \in Z$ and define $Z^\lambda_i(v)$ to be the subvariety of $S_i(v) \times S_i(v')$ of pairs (s,s') such that the following conditions hold:
C1: $B_h(s) = B_h(s')$ for all h such that $h_0, h_1 \neq i$,
C2: the following sequence is exact:
$$0 \longrightarrow V'_i \xrightarrow{a_i(s')} T_i \xrightarrow{b_i(s)} V_i \longrightarrow 0,$$
C3: $a_i(s')b_i(s') = a_i(s)b_i(s) - \lambda_i \,\mathrm{Id}_{T_i}$,
C4: $s \in \Lambda_\lambda(v)$ and $s' \in \Lambda_{\lambda'}(v')$.

LEMMA 28. *Let $(s,s') \in S_i(v) \times S_i(v')$ and suppose that it satisfies conditions* C1, C2, C3 *above then:*

1. $s \in \Lambda_\lambda(v) \iff s' \in \Lambda_{\lambda'}(v')$,
2. *if $\mu_j(s) - \lambda_j \,\mathrm{Id}_{V_j} = 0$ for all $j \neq i$ then $s \in \Lambda_\lambda(d,v)$,*
3. *if $\mu_j(s') = \lambda_j \,\mathrm{Id}_{V'_j}$ for all $j \neq i$ then $s' \in \Lambda_{\lambda'}(d,v')$.*

PROOF. 2. We have to prove $b_i a_i - \lambda_i \operatorname{Id}_{V_i} = 0$ and by condition C2 it is enough to prove $b_i a_i b_i = \lambda_i b_i$. So $b_i a_i b_i = b_i(a'_i b'_i - \lambda_i) = \lambda_i b_i$ by conditions C2 and C3.

The proof of 3. is equal to the proof of 2. We prove the implication $\Rightarrow$ in 1. By 2. and 3. it is enough to prove that $b'_j a'_j = \lambda'_j$ for $j \neq i$.

$$\begin{aligned}
b'_j a'_j &= \sum_{h_1 = j} \varepsilon(h) B'_h B'_{\bar{h}} \\
&= \sum_{h_1 = j, h_0 \neq i} \varepsilon(h) B_h B_{\bar{h}} + \sum_{h_1 = j, h_0 = i} \varepsilon(h) B'_h B'_{\bar{h}} \\
&= b_j a_j + \sum_{h_1 = j, h_0 = i} \varepsilon(h) \left(B'_h B'_{\bar{h}} - B_h B_{\bar{h}} \right) \\
&= b_j a_j + \sum_{h_0 = j, h_1 = i} \left(B_{\bar{h}} \varepsilon(h) B_h - B'_{\bar{h}} \varepsilon(h) B'_h \right) \\
&= b_j a_j + \sum_{h_0 = j, h_1 = i} \left([a_i b_i]^{V_{h_0}}_{V_{h_0}} - [a'_i b'_i]^{V_{h_0}}_{V_{h_0}} \right) \\
&= \lambda_j + \sum_{h_0 = j, h_1 = i} \lambda_i = \lambda'_j .
\end{aligned}$$

The proof of the converse is completely analogous. □

LEMMA 29. *Let $\lambda \in Z$, $(s, s') \in Z_i^\lambda(v)$ and α be an element of the path algebra algebra of type (α_0, α_1) then if $\alpha_0, \alpha_1 \neq i$ there exists an element α' of the path algebra of type (α_0, α_1) such that $\alpha'(s') = \alpha(s)$.*

PROOF. By induction on the length of α we can reduce the proof of this Lemma to the following identity that is a consequence of condition C3 in Definition 2.0.2:

$$B'_h B'_k = \begin{cases} B_h B_k - \lambda_i & \text{if } k = \bar{h} \text{ and } h_0 = i, \\ B_h B_k & \text{otherwise.} \end{cases}$$ □

LEMMA 30. *Let $(s, s') \in Z_i^\lambda(v)$ and suppose $m_i \geq 0$ or $\lambda_i \neq 0$ then*

$$s \text{ is } \chi_m \text{ semistable} \iff s' \text{ is } \chi_{m'} \text{ semistable}$$

PROOF. We prove only $\Rightarrow$. Let us consider first the case $m_i \geq 0$. If s is χ_m semi-stable, then there exists $\Delta = \{\alpha^*_*\}$ m-good such that $f_\Delta(s) \neq 0$. Using the notation in 2.0.2 we have $f_\Delta = \det \Psi_\Delta$ where $\Psi_\Delta : Y \longrightarrow Z$ is a linear map. In our case we can write Z as $\mathbb{C}^{m_i} \otimes V_i \oplus \widetilde{Z}$ and we observe that no V_i summands appear in Y or $\widetilde{Z}$ since Δ is χ_m-good.

Now we construct a new data $\Delta' = \{m^+, m^-, \alpha'^*_*\}$ such that $f_{\Delta'}(s') \neq 0$ and $f_{\Delta'}$ is a χ'-covariant polynomial. Our strategy will be the following: we substitute each V_i with the space T_i in the space Z and we add m_i copies of

V'_i to Y. Let's do it more precise: first of all the new data will not be m'-good so we have to define m'^{+}_j and m'^{-}_j:

1. $m'^{+}_i = 0$ and $m'^{-}_i = m_i = m^{+}_i$,
2. $m'^{-}_j = m^{-}_j$ and $m'^{+}_j = m^{+}_j + a_{ij} m^{+}_i$ for all $j \neq i$.

Observe that $m'^{+}_j - m'^{-}_j = m'_j$ for all j so our data will furnish a $\chi_{m'}$ equivariant function. Moreover if we define

$$Z' = \mathbb{C}^{m_i} \otimes T_i \oplus \widetilde{Z} \quad \text{and} \quad Y' = \mathbb{C}^{m_i} \otimes V'_i \oplus Y$$

we observe that they have the numbers of V'_j factors prescribed by m'^{+}, m'^{-}. Now we construct the new data Δ' in such a way that with respect to the decompositions above we have:

$$[\Psi_\Delta(s)]^{Y}_{\mathbb{C}^{m_i} \otimes V_i \oplus \widetilde{Z}} = \begin{pmatrix} (\mathrm{Id} \otimes b_i) \circ \pi \\ \Phi \end{pmatrix} \quad \text{and}$$
$$[\Psi_{\Delta'}(s')]^{\mathbb{C}^{m_i} \otimes V'_i \oplus Y}_{\mathbb{C}^{m_i} \otimes T_i \oplus \widetilde{Z}} = \begin{pmatrix} \mathrm{Id} \otimes a'_i & \pi \\ 0 & \Phi \end{pmatrix}.$$

If we construct a data with this property we observe that $\Psi_\Delta(s)$ is an isomorphism if and only if $\Psi_{\Delta'}(s')$ is an isomorphism. Hence $f_\Delta(s) \neq 0$ implies $f_{\Delta'}(s') \neq 0$ and the Lemma is proved.

To construct the new data we choose first of all $\alpha'^{j_1,h_1}_{j_2,h_2}$ for $j_1 \neq i$ and $h_2 \leq m^{+}_{j_2}$ as an element constructed according to the previous Lemma (observe that j_2 is always different from i).

In this way we guarantee that the projection of $\Psi_{\Delta'}(s')$ onto $\widetilde{Z}$ is equal to $\begin{pmatrix} 0 & \Phi \end{pmatrix}$. To define the remaining part of the new data we do not give details on the indexes, but we explain how to construct it. It is clear that we can choose $\alpha'^{i,h}_{*}$ for the remaining indexes $*$ in such a way that the projection of $\Psi_{\Delta'}(s')\big|_{\mathbb{C}^{m_i} \otimes V'_i}$ on $\mathbb{C}^{m_i} \otimes T_i$ is equal to $\mathrm{Id} \otimes a'_i$. Finally we observe that a path β from V_j to V_i with $j \neq i$ need to pass through a summand of T_i so there exists a path α such that $\beta(s) = b_i \circ \alpha(s)$. Now we use the previous Lemma to change α with a α' such that $\beta(s) = b_i \circ \alpha'(s')$. More generally if β is an element of the path algebra of type (j, i) with $j \neq i$ then there exists an element of the path algebra α' such that $\beta(s) = b_i \circ \alpha'(s)$. In this way we define the elements of the path algebra connecting summands of Y and summands of $\mathbb{C}^{m_i} \otimes T_i$.

In the case $m_i < 0$ we proceed in a similar way: we choose Δ m-good and we have

$$Y = \mathbb{C}^{-m_i} \otimes V_i \oplus \widetilde{Y}, \quad Y' = \mathbb{C}^{-m_i} \otimes T_i \oplus \widetilde{Y}, \quad Z' = \mathbb{C}^{-m_i} \otimes V'_i \oplus Z.$$

As in the previous case we can find a new data Δ' such that:

$$[\Psi_\Delta(s)]^{\mathbb{C}^{-m_i} \otimes V_i \oplus \widetilde{Y}}_{Z} = \begin{pmatrix} \pi \circ (\mathrm{Id} \otimes a_i) & \Phi \end{pmatrix},$$
$$[\Psi_{\Delta'}(s')]^{\mathbb{C}^{-m_i} \otimes T_i \oplus \widetilde{Y}}_{\mathbb{C}^{-m_i} \otimes V'_i \oplus Z} = \begin{pmatrix} \mathrm{Id} \otimes b'_i & 0 \\ \pi & \Phi \end{pmatrix}.$$

Now to conclude that $\Psi_{\Delta'}(s')$ is an isomorphism if $\Psi_\Delta(s)$ is an isomorphism we need to know that b'_i is an epimorphism and this is not guarantee by $(s, s') \in Z_i^\lambda(v)$. But if $\lambda_i \neq 0$ then, since $b'_i a'_i = -\lambda_i$, we have that b'_i is surjective. $\square$

DEFINITION 31. Let p (resp. p') be the projection of $Z_i^\lambda(v)$ on $\Lambda_\lambda(v) \subset S(v)$ (resp. $\Lambda_{\lambda'}(v') \subset S(v')$). Suppose that $m_i > 0$ or $\lambda_i \neq 0$ then we define

$$Z_i^{m,\lambda} = p^{-1}\big(\Lambda_{m,\lambda}(v)\big) = p'^{-1}\big(\Lambda_{m',\lambda'}(v')\big).$$

We define also

$$G_{i,v} = \prod_{j \neq i} GL(V_j) \times GL(V_i) \times GL(V'_i).$$

Observe that there are natural projections from $G_{i,v}$ to G_v and $G_{v'}$, therefore there are natural actions of $G_{i,v}$ on $S_i(v)$ and $S_i(v')$. Observe that there is a natural action of $G_{i,v}$ on Z_i^λ and $Z_i^{m,\lambda}$ such that the projections p, p' are equivariant.

LEMMA 32. *Let $s \in \Lambda_{\lambda,m}(v)$ then*

1) *if $\lambda_i \neq 0$ then b_i is surjective and a_i is injective,*
2) *if $m_i > 0$ then b_i is surjective,*
3) *if $m_i < 0$ then a_i is injective.*

PROOF. If $\lambda_i \neq 0$ then the result is clear by $b_i a_i = \lambda_i$. Suppose now that $\lambda_i = 0$ and $m_i > 0$. Let $U_i = \operatorname{Im} b_i$ and let $V_i = U_i \oplus W_i$. Define now a one parameter subgroup $g(t)$ of G_V in the following way:

$$[g_i(t)]_{U_i \oplus W_i}^{U_i \oplus W_i} = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ and } g_j \equiv 1 \text{ for } j \neq i$$

Since $\operatorname{Im} b_i \subset U_i$ we have that there exists the limit $\lim_{t \to 0} g(t) \cdot s = s_0$. Let now $n > 0$ and f a χ^n-covariant function on S such that $f(s) \neq 0$. Then

$$f(s_0) = \lim_{t \to 0} f(g(t) \cdot s) = \lim_{t \to 0} \det{}_{GL(V_i)}^{nm_i} f(s) = \lim_{t \to 0} t^{-nm_i \dim W_i} f(s)$$

So we must have $\dim W_i = 0$. The proof of the third case is completely similar to this one. $\square$

LEMMA 33 (see also Lusztig [6]). *If $m_i > 0$ or $\lambda_i \neq 0$ then*

1) *$p : Z_i^{m,\lambda}(v) \longrightarrow \Lambda_{m,\lambda}(v)$ is a principal $GL(V'_i)$ bundle,*
2) *$p' : Z_i^{m,\lambda}(v) \longrightarrow \Lambda_{m',\lambda'}(v')$ is a principal $GL(V_i)$ bundle.*

PROOF. Lusztig's proof extends to this case without changes. Let's prove for example 1. We have to prove: i) that the action on the fiber is free, ii) that it is transitive. First of all we observe that by the previous Lemma if $s \in \Lambda_{m,\lambda}$ then $b_i(s)$ is surjective. In particular there exists $a'_i : V'_i \longrightarrow T_i$ such that sequence (17) is exact, and clearly a'_i is univocally determined up to the action of $GL(V'_i)$, moreover this action is free. So i) and ii) reduce to the following fact: if $s \in \Lambda_{m,\lambda}$ and a'_i is such that sequence (17) is exact, then there exists a unique, b'_i such that $a'_i b'_i = a_i b_i - \lambda_i$. Since a'_i is injective the unicity is clear. To prove the existence we observe that it is equivalent to $\operatorname{Im} a'_i \supset \operatorname{Im}(a_i b_i - \lambda_i)$. But the last statement is clear since we have: $\operatorname{Im} a'_i = \ker b_i$ and $b_i(a_i b_i - \lambda_i) = 0$. □

PROPOSITION 34. *If $m_i > 0$ or $\lambda_i \neq 0$ then the projections p, p' induces algebraic isomorphisms $\bar{p}$, $\bar{p}'$:*

$$\Lambda_{m,\lambda}(v)//G_v \xleftarrow[\bar{p}]{\sim} Z_i^{m,\lambda}(v)//G_{i,v} \xrightarrow[\bar{p}']{\sim} \Lambda_{m',\lambda'}(v')//G_{v'}$$

PROOF. This proposition is a straightforward consequence of the previous Lemma and the following general fact (see for example [10] Proposition 0.2): let G be an algebraic group over $\mathbb{C}$ and X, Y two irreducible algebraic variety over $\mathbb{C}$; if G acts on X and $\varphi : X \longrightarrow Y$ is such that for all $y \in Y$ the fiber X_y contains exactly one G-orbit then φ is a categorical quotient. If we apply this Lemma to the projection p, (resp. p') and to the group $GL(V'_i)$ (resp. $GL(V_i)$) we obtain the required result. □

We can use this proposition to define the action of the generators of the Weyl group.

DEFINITION 35. Let $i, \lambda, m, v, \lambda', m', v'$ be as above, and suppose $v, v' \in R^+$ and $(\lambda_i, m_i) \neq 0$. We define an isomorphism of algebraic varieties

$$\Phi^v_{s_i,\lambda,m}: M_{m,\lambda}(v) \longrightarrow M_{m',\lambda'}(v')$$

in the following way:

1. if $m_i > 0$ or $\lambda_i \neq 0$ we set $\Phi^v_{s_i,\lambda,m} = \bar{p}'\bar{p}^{-1}$,
2. if $m_i < 0$ we exchange the role of v, v' in the previous construction: more precisely we observe that $m'_i > 0$ so we can define $\Phi^{v'}_{s_i,\lambda',m'}: M_{m',\lambda'}(v') \longrightarrow M_{m,\lambda}(v)$ and we define $\Phi^v_{s_i,\lambda,m} = \left(\Phi^{v'}_{s_i,\lambda',m'}\right)^{-1}$.

REMARK 36. To see that $\Phi^v_{s_i,\lambda,m}$ is univocally defined we have to verify that if $\lambda_i \neq 0$ and $m_i < 0$ the two definitions above coincide. This fact reduces easily to the following remark: if $\lambda_i \neq 0$ then

$$(s, s') \in Z_i^{\lambda}(v) \iff (s', s) \in Z_i^{\lambda'}(v').$$

Let us prove, for example, the $\Rightarrow$ part. Since $a_i b_i = a'_i b'_i + \lambda_i = a'_i b'_i - \lambda'_i$ the only thing we have to verify is that the sequence

$$0 \longrightarrow V_i \xrightarrow{a_i} T_i \xrightarrow{b'_i} V'_i \longrightarrow 0$$

is exact. The surjectivity of b_i' and the injectivity of a_i are a consequence of $\lambda_i \neq 0$. Since $\dim T_i = \dim V_i + \dim V_i'$ we need only to prove that $b_i' a_i = 0$. Observe that $b_i' a_i = 0$ if and only if $a_i' b_i' a_i = 0$ since a_i' is injective. Finally $a_i' b_i' a_i = (a_i b_i - \lambda_i) a_i = 0$.

3.2. – Empty case

We saw how to define

$$\Phi^{v}_{s_i,m,\lambda}\colon M_{m,\lambda}(v) \longrightarrow M_{s_i(m),s_i(\lambda)}(s_i v)$$

in the case that $(\lambda_i, m_i) \neq 0$ and $v, s_i v \in R^+$. To define an action of the Weyl group we have now to guarantee that Coxeter relations hold. We will prove these relations in the next paragraph. Before doing it we observe that we have to guarantee some conditions on m, λ such that we will be able to define $\Phi^{\sigma v}_{s_i,\sigma m,\sigma\lambda}$ for any element $\sigma \in W$: this condition will be $(v, m, \lambda) \in \mathcal{G}'$ or $(v, m, \lambda) \in \mathcal{G}''$. We will make some remark also about the case $v \notin R^+$.

LEMMA 37. *Suppose that $(m, \lambda) \in \mathcal{G}_v$ and that there exists $\sigma \in W$ such that $\sigma \cdot v \notin R^+$; then $M_{m,\lambda}(v) = \varnothing$.*

PROOF. Suppose that σ is an element of minimal length such that $\sigma \cdot v \not\geq 0$ and let $l = \ell(\sigma)$. We prove the Lemma by induction on l. The case $l = 0$ is trivial.
Initial step: $l = 1$. If $s_i \cdot v \notin R^+$ then we have $0 \leq \sum a_{ij} v_j < v_i$. Hence $\dim T_i < \dim V_i$, $u = \alpha_i \in A_v$ and $(\lambda_i, m_i) \neq 0$. So $M_{m,\lambda}(v) = \varnothing$ by Lemma 32.
Inductive step: if $l \geq 2$ then $l - 1 \Rightarrow l$. Let $\sigma = \tau s_i$ with $\ell(\tau) = l - 1$ and $v' = s_i \cdot v$, $\lambda' = s_i \lambda$, $m' = s_i m$. By induction $M_{m',\lambda'}(v') = \varnothing$ and, since $l \geq 2$, $v' \in R^+$. If $(m_i, \lambda_i) \neq 0$ then we can apply Proposition 34 and we obtain $M_{m,\lambda}(v) \simeq M_{m',\lambda'}(v') = \varnothing$. If $(m_i, \lambda_i) = 0$ then $u = \alpha_i \notin A_v$, hence $v_i = 0$. Moreover $\lambda' = \lambda$ and $m' = m$ so $(m_i', \lambda_i') = 0$ and $u = \alpha_i \notin A_{v'}$. Hence $v_i' = 0$ so that $v' = v$ and $\tau v \notin R^+$ against the minimality of σ. □

LEMMA 38. *Let (I, H) be connected, $(m, \lambda) \in \mathcal{G}_v$ and suppose $\sigma \cdot v \in R^+$ for all $\sigma \in W$. If there exists $i \in I, \sigma \in W$ such that $\langle \sigma m, \omega_i \rangle = \langle \sigma\lambda, \omega_i \rangle = 0$ then $v = 0$.*

PROOF. Without loss of generality we can assume $\sigma = 1$.
First step: $v_i = 0$. This is clear since otherwise $\alpha_i \in A_v$.
Second step: $v_j = 0$ for all j such that $a_{ij} \neq 0$. Let $v' = s_i \cdot v$ and observe that $s_i \lambda = \lambda$ and $s_i m = m$. Then, as in the first step, we have $0 = v_i' = \sum_j a_{ij} v_j$ from which the claim follows.

Let now $W' = \langle \{s_j : a_{ij} = 0 \text{ and } j \neq i\} \rangle$. Since (I, H) is connected there exists $j \in I$ and $\tau \in W'$ such that $a_{ij} \neq 0$ and

$$n = \sum_{h \in I} a_{jh} \tilde{v}_h > 0.$$

where $\tilde{v} = \tau \cdot v$. Since $(\tau\lambda)_i = \lambda_i = 0 = m_i = (\tau m)_i$ we can assume $\tau = 1$. Let now $v' = s_i s_j \cdot v$, $\lambda' = s_i s_j \lambda$ and $m' = s_i s_j m$, we have:

$$\begin{aligned} v'_i &= a_{ij} n \qquad & \lambda'_i &= -a_{ij}\lambda_j \qquad & m_i &= -a_{ij} m_j \\ v'_j &= n & \lambda'_j &= (a_{ij}^2 - 1)\lambda_j & m_j &= (a_{ij}^2 - 1) m_j. \end{aligned}$$

Hence $u = a_{ij}\alpha_j + (a_{ij}^2 - 1)\alpha_i \in A_{v'}$ and $\langle u, \lambda' \rangle = \langle u, m' \rangle = 0$ against $(m, \lambda) \in \mathcal{G}_v$. $\square$

REMARK 39. The analogous Lemmas in the case of $\mathcal{G}'' = \{(m, \lambda, v) : (m, \lambda) \notin H^\infty\}$ are simpler.

3.3. – Relations

In this section we define an isomorphism of algebraic varieties

$$\Phi^v_{\sigma,m,\lambda} : M_{m,\lambda}(v) \longrightarrow M_{\sigma m,\sigma\lambda}(\sigma v).$$

in the case $(m, \lambda) \in \mathcal{G}_v$ or $(m, \lambda) \notin H^\infty$. If there exists $\sigma \in W$ such that $\sigma v \notin R^+$ or in the case $v = 0$ we have seen in the previous section that there is nothing to define or that the definition is trivial. In the remaining cases we observe that for all σ, i we have $(\langle \sigma m, \alpha_i \rangle, \langle \sigma\lambda, \alpha_i \rangle) \neq 0$ by Lemma 38. Hence we can define $\Phi^v_{\sigma,m,\lambda}$ by induction on $\ell(\sigma)$ by the formula

$$\Phi^v_{\sigma,m,\lambda} = \Phi^{s_i v}_{\sigma s_i, s_i m, s_i\lambda} \circ \Phi^v_{s_i,m,\lambda} \tag{18}$$

if $\ell(\sigma s_i) = \ell(\sigma) - 1$. Of course we have to prove that (18) is well defined by checking the Coxeter relations (2a), (2b), (2c) that in our situation take respectively the following form:

$$\Phi^{s_i v}_{s_i, s_i\lambda, s_i m} \circ \Phi^v_{s_i,\lambda,m} = \mathrm{Id} \tag{19a}$$

$$\Phi^{s_j v}_{s_i, s_j\lambda, s_j m} \circ \Phi^v_{s_j,\lambda,m} = \Phi^{s_i v}_{s_j, s_i\lambda, s_i m} \circ \Phi^v_{s_i,\lambda,m} \tag{19b}$$

$$\Phi^{s_j s_i v}_{s_i, s_j s_i\lambda, s_j s_i m} \circ \Phi^{s_i v}_{s_j, s_i\lambda, s_i m} \circ \Phi^v_{s_i,\lambda,m} = \Phi^{s_i s_j v}_{s_j, s_i s_j\lambda, s_i s_j m} \circ \Phi^{s_j v}_{s_i, s_j\lambda, s_j m} \circ \Phi^v_{s_j,\lambda,m}. \tag{19c}$$

The first equation is clear by the very definition and Remark 36. The second equation is trivial. We need to prove the third equation. We need the following two simple Lemmas of linear algebra whose proofs are trivial.

LEMMA 40. *Let V, W, X, Y, Z be finite dimensional vector spaces and α, β, γ, δ, ε, φ linear maps between them as in the diagrams below. The sequence*

$$0 \longrightarrow V \xrightarrow{\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}} W \oplus X \oplus Y \xrightarrow{\begin{pmatrix}\delta & 0 & -1\\ 0 & \varepsilon & \varphi\end{pmatrix}} Y \oplus Z \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow V \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} W \oplus X \xrightarrow{(\varphi\delta \quad \varepsilon)} Z \longrightarrow 0$$

is exact and $\gamma = \delta\alpha$.

LEMMA 41. *Let U, V, W, X, Y, Z be finite dimensional vector spaces and α, β, γ, δ, ε, φ, ψ, ρ, σ linear maps between them as in the diagrams below such that $\psi \oplus \rho : W \oplus X \longrightarrow Z$ is an epimorphism. Then the sequence*

$$0 \longrightarrow U \xrightarrow{\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}} V \oplus W \oplus X \xrightarrow{\begin{pmatrix}\delta & 0 & 1\\ \varepsilon & \varphi & 0\\ 0 & \psi & \rho\end{pmatrix}} X \oplus Y \oplus Z \xrightarrow{(\rho \quad \sigma \quad -1)} Z \longrightarrow 0$$

is exact if and only if $\gamma = -\delta\alpha$, $\psi = \sigma\phi$, $\rho\delta + \sigma\varepsilon = 0$ and the sequence

$$0 \longrightarrow U \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} V \oplus W \xrightarrow{(\varepsilon \quad \varphi)} Y \longrightarrow 0$$

is exact.

We fix now and i, j such that $a_{ij} = 1$ and we verify (19c). Let

$$\begin{array}{lll}
\lambda' = s_i\lambda & m' = s_i m & v' = s_i v\\
\lambda'' = s_j\lambda' & m'' = s_j m' & v'' = s_j v'\\
\lambda''' = s_i\lambda'' & m''' = s_i m'' & v''' = s_i v''\\
\tilde\lambda = s_j\lambda & \tilde m = s_j m & \tilde v = s_j v\\
\tilde{\tilde\lambda} = s_i\tilde\lambda & \tilde{\tilde m} = s_i\tilde m & \tilde{\tilde v} = s_i\tilde v
\end{array}$$

First of all we observe that since relation (19a) holds we can assume that:

1) $\lambda_i \neq 0$ or $m_i > 0$ and $\lambda_j \neq 0$ or $m_j > 0$,
2) $\lambda'_j \neq 0$ or $m'_j > 0$ and $\tilde\lambda_i \neq 0$ or $\tilde m_i > 0$,
3) $\lambda''_i \neq 0$ or $m''_i > 0$ and $\tilde{\tilde\lambda}_j \neq 0$ or $\tilde{\tilde m}_j > 0$.

Define

$$\begin{aligned}
Z_{iji} = \{(s''', s) \in \Lambda_{m''',\lambda'''}(v''') \times \Lambda_{m,\lambda}(v) \ :\ &\exists s'' \in S(v''), \text{ and } s' \in S(v') \text{ such}\\
&\text{that } (s''', s'') \in Z_i^{m'',\lambda''}(v''),\ (s'', s') \in Z_j^{m',\lambda'}(v') \text{ and } (s', s) \in Z_i^{m,\lambda}(v)\}\\
Z_{jij} = \{(s''', s) \in \Lambda_{m''',\lambda'''}(v''') \times \Lambda_{m,\lambda}(v) \ :\ &\exists \tilde{\tilde s} \in S(\tilde{\tilde v}), \text{ and } \tilde s \in S(\tilde v) \text{ such}\\
&\text{that } (s''', \tilde{\tilde s}) \in Z_j^{\tilde{\tilde m},\tilde{\tilde\lambda}}(\tilde{\tilde v}),\ (\tilde{\tilde s}, \tilde s) \in Z_i^{\tilde m,\tilde\lambda}(\tilde v) \text{ and } (\tilde s, s) \in Z_j^{m,\lambda}(v)\}
\end{aligned}$$

Observe that $(s''', s) \in Z_{iji} \iff p^{v'''}_{m''',\lambda'''}(s''') = \Phi_{s_i}\Phi_{s_j}\Phi_{s_i}(p^v_{m,\lambda}(s))$ and that $(s''', s) \in Z_{jij} \iff p^{v'''}_{m''',\lambda'''}(s''') = \Phi_{s_j}\Phi_{s_i}\Phi_{s_j}(p^v_{m,\lambda}(s))$. So relation (19c) is equivalent to $Z_{iji} = Z_{jij}$.

Let $R_i = \bigoplus_{h\,:\,h_1=i,h_0\neq j} V_{h_0}$, $R_j = \bigoplus_{h\,:\,h_1=i,h_0\neq j} V_{h_0}$ and observe that $T_i = R_i \oplus V_j$ and $T_j = R_j \oplus V_i$. Let k be the only element of H such that $k_0 = j$ and $k_1 = i$. Let $\varepsilon = \varepsilon(k)$. Define $A = A(s) = B_k(s)$, $B = B(s) = B_{\bar k}(s)$ and for $l = i, j$ and $\{l', l\} = \{i, j\}$ set $c_l = c_l(s) = \pi^{R_l\oplus V_{l'}}_{R_l} a_l(s)$ and $d_l = d_l(s) = b_l(s)\big|_{R_l}$.

Let $(s, s''') \in \Lambda_\lambda(v) \times \Lambda_{\lambda'''}(v''')$ and set $A^* = A(s^*)$, $B^* = B(s^*)$, $c_l^* = c_l(s^*)$ and $d_l^* = d_l(s^*)$ for $l \in \{i, j\}$ and $* \in \{\ ,'''\}$.

If we apply Lemmas 40 and 41 to our situation we obtain the following result: $(s, s''') \in Z_{iji}$ if and only if there exist vector spaces V_i', V_j', V_i'', V_j'' and linear maps A', B', c_i', d_i', c_j', d_j', A'' , B'' , c_i'' , d_i'' , c_j'' , d_j'' such that:

1. $\dim V_l^* = v_l^*$ for $l \in \{i, j\}$ and $* \in \{',''\}$,
2. for each $* \in \{',\ ''\}$ and $l \in \{i, j\}$ $A^* \in \mathrm{Hom}(V_i^*, V_j^*)$, $B^* \in \mathrm{Hom}(V_j^*, V_i^*)$, $c_l \in \mathrm{Hom}(V_l^*, R_l^*)$ and $d_l \in \mathrm{Hom}(R_l^*, V_l^*)$,
3. $V_j''' = V_j''$, $c_j''' = c_j''$, $d_j''' = d_j''$ and
$$\begin{aligned} c_i'''d_i''' &= c_id_i - \lambda_i - \lambda_j & \qquad c_i'''B''' &= c_i'B'' \\ A'''d_i''' &= A''d_i' & \varepsilon A'''B''' &= \varepsilon A''B'' - \lambda_j \end{aligned}$$
4. $V_i'' = V_i'$, $c_i'' = c_i'$, $d_i'' = d_i'$ and
$$\begin{aligned} c_j''d_j'' &= c_jd_j - \lambda_i - \lambda_j & \qquad c_j''A'' &= c_jA' \\ B''d_j'' &= B'd_j & \varepsilon A''B'' &= \varepsilon A'B' + \lambda_i + \lambda_j \end{aligned}$$
5. $V_j' = V_j$, $c_j' = c_j$, $d_j' = d_j$ and
$$\begin{aligned} c_i'd_i' &= c_id_i - \lambda_i & \qquad c_i'B' &= c_iB \\ A'd_i' &= Ad_i & \varepsilon A'B' &= \varepsilon AB - \lambda_i \end{aligned}$$
6. $\varepsilon c_i'B''A''' + c_i'd_i'c_i''' = 0$ and $\varepsilon A'B'' = d_ja_j''$,
7. the following sequences are exact
$$0 \longrightarrow V_i''' \xrightarrow{\begin{pmatrix} c_i''' \\ c_j''A''' \end{pmatrix}} R_i \oplus R_j \xrightarrow{(Ad_i \quad d_j)} V_j \longrightarrow 0$$
$$0 \longrightarrow V_j'' \xrightarrow{\begin{pmatrix} c_j'' \\ c_i'B'' \end{pmatrix}} R_j \oplus R_i \xrightarrow{(Bd_j \quad d_i)} V_i \longrightarrow 0$$
$$0 \longrightarrow V_i' \xrightarrow{\begin{pmatrix} c_i' \\ A' \end{pmatrix}} R_i \oplus V_j \xrightarrow{(d_i \quad \varepsilon B)} V_i \longrightarrow 0$$

REMARK 42. The first condition in point 6 is equivalent to $\varepsilon B''A''' = d_i'c_i'''$. Indeed this condition is certainly sufficient. To prove its necessity observe that by the injectivity of $a_i' = (c_i' \ A')^t$ it is enough to prove $\varepsilon c_i'B''A''' + c_i'd_i'c_i''' = 0$ and $\varepsilon A'B''A''' + A'd_i'c_i''' = 0$. The first equation is the first condition in point 6 and the second one is a consequence of $\varepsilon A'B'' = d_jc_i'''$, $A'd_i' = Ad_i$ and the exactness of the first sequence.

REMARK 43. The condition $(s, s''') \in Z_{jij}$ can be expressed in a similar way. In the previous conditions we have only to exchange i with j and ε with $-\varepsilon$.

We will prove now $Z_{iji} \subset Z_{jij}$. To do it we suppose that $A', \dots, d''_j$ are given as above and we construct $\tilde{A}, \tilde{B}, \tilde{c}_i, \tilde{d}_i, \tilde{c}_j, \tilde{d}_j, \tilde{\tilde{A}}, \tilde{\tilde{B}}, \tilde{\tilde{c}}_i, \tilde{\tilde{d}}_i, \tilde{\tilde{c}}_j, \tilde{\tilde{d}}_j$ such that they satisfy the analogous conditions for $(s, s''') \in Z_{jij}$.
We construct first $\tilde{A}, \tilde{B}, \tilde{c}_i, \tilde{c}_j, \tilde{d}_i, \tilde{d}_j$. Choose $\tilde{s}$ such that $(\tilde{s}, s) \in Z_j^{\chi,\lambda}$ and define $\tilde{A} = A(\tilde{s})$, $\tilde{B} = B(\tilde{s})$, $\tilde{c}_l = c_l(\tilde{s})$ and $\tilde{d}_l = d_l(\tilde{s})$ for $l \in \{i, j\}$.

I claim that there exist unique $\tilde{\tilde{A}} : V'''_i \longrightarrow \tilde{V}_j$ and $\tilde{\tilde{B}} : \tilde{V}_j \longrightarrow V'''_i$ such that:

$$\begin{cases} \tilde{c}_j \tilde{\tilde{A}} = c''_j A''' \\ \tilde{B}\tilde{\tilde{A}} = -\varepsilon d_i c'''_i \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\tilde{A}}\tilde{\tilde{B}} = \tilde{A}\tilde{B} - \varepsilon\lambda_i - \varepsilon\lambda_j \\ c'''_i \tilde{\tilde{B}} = c_i \tilde{B} \end{cases}$$

Unicity of $\tilde{\tilde{A}}$: since the map $\tilde{a}_j = (\tilde{c}_j \quad -\varepsilon\tilde{B})^t$ is injective the unicity is clear.
Existence of $\tilde{\tilde{A}}$: to prove the existence of $\tilde{\tilde{A}}$ is enough to prove:

$$\operatorname{Im}\begin{pmatrix} c''_j A''' \\ -\varepsilon d_i c'''_i \end{pmatrix} \subset \operatorname{Im}\begin{pmatrix} \tilde{c}_j \\ \tilde{B} \end{pmatrix} = \ker\left(\, d_j \quad -\varepsilon A \,\right).$$

So the thesis follows from $d_j c''_j A''' + A d_i c'''_i = 0$.

Let now $\tilde{\tilde{a}}_i = (c'''_i \quad \tilde{\tilde{A}})^t$. I claim that $\tilde{\tilde{a}}_i$ is injective and that $\operatorname{Im}\tilde{\tilde{a}}_i = \ker(d_i \quad \varepsilon\tilde{B}) = \ker\tilde{b}_i$. First of all observe that since $\tilde{m}_i > 0$ or $\lambda_i \neq 0$, $\tilde{b}_i$ is surjective. Observe also that

$$\begin{pmatrix} \tilde{c}_j & 0 \\ 0 & \mathrm{Id}_{V'''_i} \end{pmatrix} \circ \begin{pmatrix} \tilde{\tilde{A}} \\ c'''_i \end{pmatrix} = \begin{pmatrix} c''_j A''' \\ c'''_i \end{pmatrix}.$$

So $\tilde{\tilde{a}}_i$ is injective as claimed. Now since $\dim R_i + \dim \tilde{V}_j = \dim V'''_i + \dim V_i$ to prove the last part of the claim it is enough to check that $\tilde{b}_i\tilde{\tilde{a}}_i = 0$. Indeed

$$\tilde{b}_i\tilde{\tilde{a}}_i = d_i c'''_i + \varepsilon\tilde{B}\tilde{\tilde{A}} = 0.$$

Unicity of $\tilde{\tilde{B}}$: this is a consequence of $\tilde{\tilde{a}}_i$ injective.
Existence of $\tilde{\tilde{B}}$: As for the existence of $\tilde{\tilde{A}}$ this is equivalent to

$$\operatorname{Im}\begin{pmatrix} c_i\tilde{B} \\ \tilde{A}\tilde{B} - \varepsilon\lambda_i - \varepsilon\lambda_j \end{pmatrix} \subset \operatorname{Im}\begin{pmatrix} c'''_i \\ \tilde{\tilde{A}} \end{pmatrix} = \ker\left(\, d_i \quad \varepsilon\tilde{B} \,\right).$$

So the thesis follows from $\varepsilon\tilde{B}\tilde{A}\tilde{B} - \lambda_i\tilde{B} - \lambda_j\tilde{B} + d_i c_i\tilde{B} = 0$.

Finally we set

$$\begin{array}{lll} \tilde{\tilde{V}}_i = V'''_i & \tilde{\tilde{c}}_i = c'''_i & \tilde{\tilde{d}}_i = d'''_i \\ \tilde{\tilde{V}}_i = \tilde{V}_j & \tilde{\tilde{c}}_j = \tilde{c}_j & \tilde{\tilde{d}}_j = \tilde{d}_j. \end{array}$$

The verification of all the conditions is now straightforward.

The inclusion $Z_{jij} \subset Z_{iji}$ can be proved similarly and equation (16) is clear by definition. Proposition 26 is proved.

4. – Reduction to the dominant case

As a consequence of Proposition 26 we see that in the finite type case if $(m,\lambda)\in\mathcal{G}_v$ then there exists $\sigma\in W$ and $v'=\sigma\cdot v$ such that v' is dominant and $M_{\sigma m,\sigma\lambda}(v')\simeq M_{m,\lambda}(v)$. We generalize now this result to arbitrary quiver and arbitrary (m,λ).

On R we consider the following order: $v'\le v$ if and only if $v-v'\in R^+$. We say that an element v of R is dominant if $\langle v,\alpha_i\rangle\le 0$ for all $i\in I$.

We consider now the following construction: let $v'\le v$ and fix an embedding $V_i'\hookrightarrow V_i$ and a complement W_i of V_i' in V_i, then we can define a map $\widetilde{\jmath}\colon S(v')\longrightarrow S(v)$ through:

$$\widetilde{\jmath}(B_h')=\begin{pmatrix}B_h' & 0\\ 0 & 0\end{pmatrix}\tag{20}$$

where the matrices of the element on the right hand side represents the components B_h through the decomposition $V_i=V_i'\oplus W_i$.

Suppose now that $(m_i,\lambda_i)=0$ for all i such that $\langle v-v',\omega_i\rangle\neq 0$. Then it is easy to see that this map restrict to a map $\jmath_r:\Lambda_{m,\lambda}(v')\longrightarrow\Lambda_{m,\lambda}(v)$ and that induces a map $\jmath_v^{v'}=\jmath\colon M_{m,\lambda}(v')\longrightarrow M_{m,\lambda}(v)$. Clearly $\jmath$ is independent from the choice of the embedding $V_i'\hookrightarrow V_i$ and of the complement W_i.

Lemma 44. *$\jmath$ is a closed immersion*

Proof. It is enough to prove that the map $\jmath^\sharp:\mathbb{C}[\Lambda_\lambda(v)]_{\chi_m}^{G(v)}\longrightarrow\mathbb{C}[\Lambda_\lambda(v')]_{\chi_m}^{G(v')}$ is surjective. By Propositions 12 and 14 this follows by the identities:

$$\operatorname{Tr}(\alpha(\jmath(s)))=\operatorname{Tr}(\alpha(s))\quad\text{and}\quad f_\Delta(\jmath(s))=f_\Delta(s)$$

for each closed path α and for each χ_m good Δ. □

Lemma 45. *If $\langle v,\alpha_i\rangle>0$, $v'=v-\alpha_i$ and $(m_i,\lambda_i)=0$ then $\jmath$ is an isomorphism of algebraic varieties.*

Proof. It's enough to prove that $\jmath$ is surjective. Let $s\in\Lambda_{m,\lambda}(v)$ and Δ a χ_m good data for some positive n such that $f_\Delta(s)\neq 0$. Consider now the sequence (see equation (4) for the notation):

$$T_i\xrightarrow{\ b_i\ }V_i\xrightarrow{\ a_i\ }T_i.$$

Since $b_ia_i=\lambda_i\operatorname{Id}_{V_i}=0$ and $2\dim V_i-\dim T_i=\langle v,\alpha_i\rangle>0$ we have that b_i is not surjective or that a_i is not injective.

Suppose that b_i is not surjective, then up to the action of G_v we can assume that $\operatorname{Im} b_i\subset V_i'$. Then, for $t\in\mathbb{C}^*$ consider $g(t)=(g_j(t))\in G_v$ with

$$g_i(t)=\begin{pmatrix}\operatorname{Id}_{V_i'} & 0\\ 0 & t^{-1}\end{pmatrix}\quad\text{and}\quad g_j(t)=\operatorname{Id}_{V_j}\ \text{for } j\neq i.$$

Then

1. $g_i(t)B_h = B_h$ if $h_1 = i$, since $\operatorname{Im} B_h \subset \operatorname{Im} b_i \subset V_i'$,
2. $\exists \lim_{t\to 0} B_h g_i(t)^{-1} = B_h$ if $h_0 = i$,
3. $\chi_m(g(t)) = 1$ since $m_i = 0$.

So $\exists \lim_{t\to 0} g(t)s = s'$ and $f_\Delta(s') = f_\Delta(s) \neq 0$. We observe now that $s' \in \widetilde{\jmath}(\Lambda_{m,\lambda}(v'))$ and that $p^v_{m,\lambda}(s) = p^v_{m,\lambda}(s') \in \operatorname{Im} \jmath$.

If b_j is surjective and a_i is not injective the argument is similar. $\square$

PROPOSITION 46. *For all λ and for all v there exists v' and $\sigma \in W$ such that v' is dominant and*

$$M_{\sigma m,\sigma\lambda}(v') \simeq M_{m,\lambda}(v).$$

PROOF. If $v \notin R^+$ it is enough to prove that there exists a dominant v' such that $M_{m,\lambda}(v') = \varnothing$. So it is enough to prove that there exists a dominant $v \notin R^+$. By absurd suppose that v dominant implies $v \in R^+$. Observe that R^+ doesn't contain any line, so the same is true for the cone D of dominant v. So the linear functionals $\langle \alpha_i, \_ \rangle$ must be linearly independent. Then we are in the case of a Cartan matrix of finite type and we know that $D \subset -R^+$.

If $v \in R^+$ we prove the Proposition by induction on the order $\leq$ on R.

First step: $v = 0$. If $v = 0$ we can take $v' = v = 0$ and $\sigma = 1$.

Inductive step. If v is not dominant then there exists i such that $\langle v, \alpha_i \rangle > 0$.

If $(m_i, \lambda_i) \neq 0$ we observe that $s_i v = v' < v$ (that is $v' \leq v$ and $v' \neq v$) and that $M_{s_i m, s_i \lambda}(v') \simeq M_{m,\lambda}(v)$ by Proposition 34. Now we can apply the inductive hypothesis.

If $(m_i, \lambda_i) = 0$ we apply the previous Lemma and the inductive hypothesis. $\square$

REMARK 47. We observe that in the case of a Cartan matrix of finite type this implies that for all $v \in R^+$ the variety $M_{m,\lambda}(v)$ is a point if $v_i(m_i, \lambda_i) = 0$ for all i and it is empty otherwise.

5. – A representation of the Weyl group

The maps $\Phi^v_{\sigma,m,\lambda}$ induces isomorphisms between the cohomology of different quiver varieties. In this section, following Nakajima [11], we show how to use this action to construct an action of the Weyl group on the cohomology of a single quiver variety.

Let $Q = (I, H)$ be a complete subquiver of a double quiver $\widetilde{Q} = (\widetilde{I}, \widetilde{H})$: that is $I \subset \widetilde{I}$, $H \subset \widetilde{H}$ and for all $h \in \widetilde{H}$ if $h_0, h_1 \in I$ then $h \in H$. Let $\widetilde{R} = R \oplus R'$ be the root lattice of $\widetilde{Q}$ and $\widetilde{W}$ (resp. W) the Weyl groups related to $\widetilde{Q}$ (resp. Q). W is a parabolic subgroup of $\widetilde{W}$.

Choose $\tilde{v} = \sum_{i \in \widetilde{I}} v_i \alpha_i \in \widetilde{R}$ such that the following conditions are satisfied:

CON1 $\sigma \tilde{v} = \tilde{v}$ for all $\sigma \in W$,
CON2 $GCD(v_i : i \in \widetilde{I}) = 1$,
CON3 $\tilde{\mu}: S(\widetilde{Q}, \tilde{v}) \longrightarrow \mathfrak{u}_{\tilde{v}} \oplus \mathfrak{g}_{\tilde{v}}$ is surjective.

Under these hypothesis if $\lambda \in Z_{\tilde{v}}$ is generic then there exists a representation of W on the cohomology of $M_{m,\lambda}(\tilde{v})$.

We need the following definitions:

$$
\begin{aligned}
Reg(\tilde{v}) &= \{\lambda \in Z_{\tilde{v}} \ : \ \lambda \notin Z_u \text{ for all } 0 \leq u \leq \tilde{v}\},\\
\mathfrak{Reg}(\tilde{v}) &= \{\lambda \in \mathfrak{Z}_{\tilde{v}} \ : \ \lambda \notin \mathfrak{Z}_u \text{ for all } 0 \leq u \leq \tilde{v}\},\\
\Lambda(\tilde{v}) &= \{(\lambda, s) \in Z_{\tilde{v}} \times S(\widetilde{Q}, \tilde{v}) \ : \ s \in \Lambda_\lambda\},\\
M(\tilde{v}) &= \Lambda(\tilde{v})//G_{\tilde{v}} \quad \text{and} \quad p : M(\tilde{v}) \longrightarrow Z_{\tilde{v}} \text{ the projection},\\
\mathfrak{L}(\tilde{v}) &= \{(\zeta, s) \in \mathfrak{Z} \times S(\widetilde{Q}, \tilde{v}) \ : \ s \in \mathfrak{L}_\zeta\},\\
\mathfrak{M}(\tilde{v}) &= \mathfrak{L}(\tilde{v})/U_{\tilde{v}} \quad \text{and} \quad \tilde{p} : \mathfrak{M}(\tilde{v}) \longrightarrow \mathfrak{Z} \text{ the projection.}
\end{aligned}
$$

We define also $M_R = p^{-1}(Reg)$ (resp. $\mathfrak{M}_{\mathfrak{R}} = \tilde{p}^{-1}(\mathfrak{Reg})$) and we call $p_R: M_R \longrightarrow Reg$ (resp. $\tilde{p}_{\mathfrak{R}}: \mathfrak{M}_{\mathfrak{R}} \longrightarrow \mathfrak{Reg}$) the restriction of p to M_R (resp. $\tilde{p}$ to $\mathfrak{M}_{\mathfrak{R}}$).

By Proposition 6 we have the following pullback diagram

$$
\begin{array}{ccccccc}
(\lambda, s) & \in & M(\tilde{v}) & \xrightarrow{p} & Z_{\tilde{v}} & \ni & \chi\\
\downarrow & & \downarrow{\scriptstyle \iota_M} & & \downarrow{\scriptstyle \iota_Z} & & \downarrow\\
(0, \lambda, s) & \in & \mathfrak{M}(\tilde{v}) & \xrightarrow{\tilde{p}} & \mathfrak{Z}_{\tilde{v}} & \ni & (0, \lambda)
\end{array}
$$

LEMMA 48.
1) *The maps p_R and $\tilde{p}_{\mathfrak{R}}$ are locally trivial bundle.*
2) *The sheaves $R^i p_{R*}(\mathbb{Z}_{M_R})$ and $R^i \tilde{p}_{\mathfrak{R}*}(\mathbb{Z}_{\mathfrak{M}_{\mathfrak{R}}})$ are constant.*

PROOF. 1) By the surjectivity of $\tilde{\mu}$ it is enough to prove that $d\tilde{\mu}_s$ is surjective for all $s \in \mathfrak{M}_{\mathfrak{R}}$. It is easy to see that in the hyperKähler situation this is equivalent to $x \cdot s = 0 \Rightarrow x = 0$ for all $x \in \mathfrak{u}_{\tilde{v}}$. If there exists such an $x = (x_i) \in \mathfrak{u}_{\tilde{v}} \subset \oplus_{i \in \widetilde{I}} \mathfrak{u}(V_i)$ let $\zeta = (\xi, \lambda) = \tilde{\mu}(s)$, and decompose each vector space V_i in eigenspaces with respect to x_i: $V_i = \oplus_{r \in \sqrt{-1}\mathbb{R}} V_i(r)$. Now by $x \cdot s = 0$ we see $B_h(V_{h_0}(r)) \subset V_{h_1}(r)$ so $\sum_i \dim V_i(r)\lambda_i = \sum_i \dim V_i(r)\xi_i = 0$ and by $\zeta \in \mathfrak{Reg}$ we have that there exists r such that $V_i = V_i(r)$ for all i and $x = 0$.

2) By point 1) we know that these sheaves are locally constant. Observe now that $\mathfrak{Reg}$ is the complement of the the union of a finite number of linear subspaces of real codimension 3 so $\Pi_1(\mathfrak{Reg})$ is trivial and each locally constant sheaf is also constant. Finally $R^i p_{R*}(\mathbb{Z}_{M_R}) = \iota_Z^* R^i \tilde{p}_{\mathfrak{R}*}(\mathbb{Z}_{\mathfrak{M}_{\mathfrak{R}}})$. □

As a consequence of this Lemma for any $\lambda^1, \lambda^2 \in Reg$ we have a canonical isomorphism

$$\psi^i_{\lambda^1,\lambda^2} : H^i(M_{0,\lambda^1}, \mathbb{Z}) \longrightarrow H^i(M_{0,\lambda^2}, \mathbb{Z}).$$

Now observe that $\mathcal{G}_{\tilde{v}} \subset \mathfrak{Reg}$. So, for each $(0, \lambda) \in \mathcal{G}_{\tilde{v}}$ we can define a W action on $H^i(M_{0,\lambda}(\tilde{v}), \mathbb{Z})$ by

$$\sigma(c) = \psi^i_{\sigma\lambda,\lambda} \circ H^i(\Phi^{\tilde{v}}_{\sigma,0,\lambda})(c)$$

for any $\sigma \in W$. To verify that this definition is well given we have only to verify that

$$\psi^i_{\sigma\lambda^2,\lambda^1} \circ H^i(\Phi^{\tilde{v}}_{\sigma,0,\lambda^2}) \circ \psi^i_{\lambda^1,\lambda^2} = \psi^i_{\sigma\lambda^1,\lambda^1} \circ H^i(\Phi^{\tilde{v}}_{\sigma,0,\lambda^1}).$$

Since $\{\lambda : (0, \lambda) \in \mathcal{G}_v\}$ is connected and $H^i(M, \mathbb{Z})$ is discrete this is clear. So we proved the following corollary.

Corollary 49. *If $(0,\lambda) \in \mathcal{G}_v$ then there is an action of W on $H^i(M_{0\lambda}(\widetilde{Q}, v), \mathbb{Z})$.*

6. – Some remark in the case of Nakajima's quiver varieties

In the previous section we saw that enlarging the original quiver it is possible to construct representations of the Weyl group W. The simplest way to enlarge our original quiver is to add only one vertex that we call ∞. If we call $d_i = a_{i\infty}$ and we consider element v of the root lattice of the form $\tilde{v} = \alpha_\infty + \sum_{i\in I} v_i\alpha_i \in \widetilde{R}$ we construct a class of varieties depending on two discrete parameters: $v \in R$ and $d = \sum_{i\in I} d_i\omega_i \in P$. These are Nakajima's quiver varieties: $M(d, v)$. Observe that in this case the condition CON2 above is always satisfied and that the condition CON1 is equivalent to $\bar{v} = \sum_{i\in I} v_i\bar{\alpha}_i = \sum_{i\in I} d_i\omega_i = d$. In [12] par. 10 a criterion for condition CON3 is also given.

In this final section we want to show in the particular case of Nakajima's quiver varieties how to use the reduction to the dominant case to study geometric properties of quiver varieties.

In the following we fix a quiver of finite type $Q = (I, H)$, $d = \sum_{i\in I} d_i\omega_i \in P$ and $v = \sum_{i\in I} v_i\alpha_i$. We construct a new quiver $\widetilde{Q}$ as explained above and we choose $\tilde{v} = v + \alpha_\infty \in \widetilde{R} = R \oplus \mathbb{Z}\alpha_\infty$. We restrict our attention to the case $\lambda = 0$ and $m_0 = \sum_{i\in I} \omega_i - (\sum_{i\in I} v_i)\omega_\infty$. Moreover by Remark 2 we can assume without loss of generality that $v_i > 0$ for all $i \in I$. We would like to give a proof of the following conjecture in some special case:

Conjecture 50. $M_{m_0,0}(d, v)$ is connected and $M_{0,0}(d, v)$ is normal.

Remark 51. The connectedness states in the conjecture has now been proved in general by Crawley-Boevey [2]. The same author said me that he is now able to prove normality for a much bigger class of quiver varieties.

REMARK 52. If $\tilde{m} = \sum_{i\in\widetilde{I}} m_i\omega_i$ and $m_i > 0$ for all $i \in I$ it is easy to see that $\Lambda_{\tilde{m},0} = \Lambda_{m_0,0}$. Hence by the argument in Lemma 48 $M_{m_0,0}(v)$ is smooth and if it is not empty has dimension $\dim S(\widetilde{Q}, \tilde{v}) - 2\dim \mathfrak{g}_{\tilde{v}}$. On the other hand $M_{0,0}$ is a cone so it is clearly connected.

By Proposition 46 it is enough to prove the Theorem in the dominant case which in this case can be read as $\langle d-\bar{v}, \alpha_i\rangle \geq 0$ for all $i \in I$. Unfortunately I'm able to prove the conjecture only in the case $d-\bar{v}$ regular: $\langle d-\bar{v}, \alpha_i\rangle > 0$ for all i. To prove the conjecture in this case we will use the following stratification introduced by Lusztig in [6].

DEFINITION 53. For any $s \in S(\widetilde{Q}, \tilde{v})$ and $i \in I$ let

$$V_i^+ = V_i^+(s) = \sum_{\alpha \text{ path}\,:\,\alpha_1 = i \text{ and } \alpha_0 = \infty} \mathrm{Im}(\alpha(s)).$$

If $v' = \sum_{i\in I} v'_i\alpha_i \in R$ we define

$$\Lambda^{v'} = \{s \in \Lambda_0(\widetilde{Q}, \tilde{v}) \,:\, \dim V_i^+(s) = v'_i \text{ for all } i \in I\}.$$

Observe that $\Lambda^v = \Lambda_{m_0,0}(\widetilde{Q}, \tilde{v})$. To prove our result we will use the following Lemma of Lusztig.

LEMMA 54 (Lusztig: [6] Proposition 4.5 and Proposition 5.3). *If $v' \leq v$ then*

$$\dim \Lambda^{v'} = \dim S(\widetilde{Q}, \tilde{v}) - \sum_{i\in I} \dim gl(V_i) - \langle v - v', d - \bar{v}\rangle - \tfrac{1}{2}\langle v - v', v - v'\rangle$$

Our result follows trivially from the following Lemma.

LEMMA 55. 1) *If $d-\bar{v}$ is dominant then $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ is a complete intersection.* 2) *If $d-\bar{v}$ is regular then $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ is normal and irreducible and $\Lambda_{m_0,0}(\widetilde{Q}, \tilde{v})$ is connected.*

PROOF. Observe that $\Lambda_{0,0}(\widetilde{Q}, \tilde{v}) = \mu^{-1}(0)$ so each irreducible component of $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ must have dimension at least $\dim S(\widetilde{Q}, \tilde{v}) - \sum_i \dim gl(V_i) = \delta_V$.

Suppose now that $d-\bar{v}$ is dominant. By Nakajima's theorem ([12] Theorem 10.2) $M_{m_0,0}$ is not empty. Observe also that working as in Lemma 48 we obtain that $\Lambda_{m_0,0}(\widetilde{Q}, \tilde{v})$ is a smooth subset of $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ of dimension δ_V.

It is well known that $\Lambda_{m_0,0}(\widetilde{Q}, \tilde{v}) = \Lambda^v$. Hence

$$\Lambda_{0,0}(\widetilde{Q}, \tilde{v}) - \Lambda_{m_0,0}(\widetilde{Q}, \tilde{v}) = \bigcup_{v' \leq v \text{ and } v' \neq v} \Lambda^{v'}.$$

By the Lemma above we have that if $v' \leq v$ and $v' \neq v$ $\dim \Lambda^{v'} < \delta_V$. So $\Lambda_{m_0,0}(\widetilde{Q}, \tilde{v})$ must be dense in $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ and $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ is a complete intersection. Moreover if $d-\bar{v}$ is regular we have that $\dim \Lambda^{v'} < \delta_V - 1$ so the singular locus has codimension at least two and normality and irreducibility follows. Finally by our discussion it is clear that if $\Lambda_{m_0,0}(\widetilde{Q}, \tilde{v})$ is disconnected then $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ is not irreducible. $\square$

REMARK 56. In the Lemma we can substitute $\Lambda_{m_0,0}(\widetilde{Q}, \tilde{v})$ with any other subset *Regular* of regular points in $\Lambda(\widetilde{Q}, \tilde{v})$. In this way is indeed possible to improve a little bit the theorem (for example can be proved completely in the A case) but Crawley-Boevey explained to me that this strategy cannot work in general because there are cases where $d - \bar{v}$ is dominant and $\Lambda_{0,0}(\widetilde{Q}, \tilde{v})$ is not normal.

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