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On the Two-weight Problem for Singular Integral Operators

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Abstract. We give A_p type conditions which are sufficient for two-weight, strong (p, p) inequalities for Calderón-Zygmund operators, commutators, and the Littlewood-Paley square function g_λ^*. Our results extend earlier work on weak (p, p) inequalities in [13].

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1. – Introduction

1.1. – Background

The purpose of this paper is to obtain sufficient conditions for strong-type, two-weight norm inequalities for singular integral operators. We derive A_p type conditions on the weights which are better than previously known conditions. Furthermore, our approach is general enough that we can apply it to obtain results about other kinds of operators, such as Calderón-Zygmund operators, Littlewood-Paley square functions, and operators with a higher degree of singularity, such as commutators of singular integral operators with BMO functions.

To put our results in context, we first outline the analogous results for the Hardy-Littlewood maximal operator, M. Given a weight w (i.e., a non-negative, locally integrable function), and p, $1 < p < \infty$, a necessary and sufficient condition for the weighted norm inequality

$$\int_{\mathbb{R}^n} (Mf)^p \, w \, dx \leq C \int_{\mathbb{R}^n} |f|^p \, w \, dx \tag{1.1}$$

is the A_p condition: there exists a constant C such that for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q w \, dy\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'} \, dy\right)^{p-1} \leq C < \infty. \tag{1.2}$$

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This characterization is due to Muckenhoupt [24].

It is natural to consider the analogous problem for a pair of weights: find necessary and sufficient conditions on (u, v) such that

$$\int_{\mathbb{R}^n} (Mf)^p\, u\, dx \le C \int_{\mathbb{R}^n} |f|^p\, v\, dx. \tag{1.3}$$

However, simple examples show (see [18, p. 395]) that the analogous two-weight A_p condition,

$$\left(\frac{1}{|Q|}\int_Q u\, dy\right)\left(\frac{1}{|Q|}\int_Q v^{1-p'}\, dy\right)^{p-1} \le C < \infty, \tag{1.4}$$

is necessary but not sufficient for M to be bounded from $L^p(v)$ to $L^p(u)$. Sawyer [40] showed that a necessary and sufficient condition is the following: there exists a constant C such that for all cubes Q,

$$\int_Q M(v^{1-p'}\chi_Q)^p\, u\, dy \le C \int_Q v^{1-p'}\, dy < \infty.$$

(This is equivalent to saying that M is bounded on the family of "test functions" $\{v^{1-p'}\chi_Q\}$.)

A major drawback of this condition is that it involves the operator M itself, and this has motivated a search for sufficient conditions which are simpler and are, in some sense, close to the A_p condition (1.4). Partial results in this direction were first obtained by Muckenhoupt and Wheeden [26]. The first result of this kind is due to Neugebauer [27]. He proved that if (u, v) is a pair of weights such that for some $r > 1$,

$$\left(\frac{1}{|Q|}\int_Q u^r\, dy\right)^{1/r}\left(\frac{1}{|Q|}\int_Q v^{(1-p')r}\, dy\right)^{(p-1)/r} \le C < \infty, \tag{1.5}$$

then (1.3) holds. (We refer to (1.5) as a "power bump" condition.) In fact, Neugebauer proved a stronger result: he showed that (1.5) holds if and only if there exist $w \in A_p$ and positive constants c_1, c_2, such that $c_1 u \le w \le c_2 v$. (Since A_p weights satisfy the reverse Hölder inequality, in the one weight case (1.5) is equivalent to the A_p condition.)

Neugebauer's result was improved in [30], where it was shown that the power bump can be eliminated on the left-hand weight u, and that it can be replaced on the right-hand weight by a smaller "Orlicz bump." More precisely, given a Young function B, define the mean Luxemburg norm of f on a cube Q by

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f|}{\lambda}\right) dy \le 1\right\}.$$

Then if B is a doubling Young function such that

$$\int_c^\infty \left(\frac{t^{p'}}{B(t)}\right)^{p-1} \frac{dt}{t} < \infty, \tag{1.6}$$

and if (u, v) is a pair of weights such that

$$\left(\frac{1}{|Q|}\int_Q u\,dy\right) \|v^{-1/p}\|_{B,Q}^p \le C < \infty, \tag{1.7}$$

then (1.3) holds. Further, this condition is sharp, in the sense that if (1.7) implies (1.3) then (1.6) must hold. The Young functions $B(t) = t^{p'r}$, $r > 1$, satisfy (1.6), and in this case condition (1.7) is weaker than (1.5) since there is no bump on the weight u. More interesting examples are given by the functions $B(t) = t^{p'}(\log(e+t))^{p'-1+\delta}$, $\delta > 0$; if $\delta = 0$ then the result is false.

In this paper we give conditions analogous to (1.7) for the inequality

$$\int_{\mathbb{R}^n} |Sf|^p\, u\,dx \le C \int_{\mathbb{R}^n} |f|^p\, v\,dx, \tag{1.8}$$

where S is a Calderón-Zygmund operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy;$$

the Littlewood-Paley square function

$$g_\lambda^*(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} |\phi_t * f(y)|^2 \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \tag{1.9}$$

where $\phi \in \mathcal{S}$, $\int \phi\,dx = 0$, and $\lambda > 2$; or a commutator of the form

$$C_b^m f(x) = \int_{\mathbb{R}^n} [b(x)-b(y)]^m K(x,y)f(y)\,dy, \qquad b \in BMO, \quad m \in \mathbb{N}. \tag{1.10}$$

Further, our techniques extend to a variety other operators, such as weakly strongly singular integral operators, pseudo-differential operators of Hörmander type, and oscillatory integral operators of the kind introduced by Phong and Stein. We will give precise definitions below in Section 2.

In the one-weight case (i.e., when $u = v$), a sufficient condition for (1.8) to hold is $v \in A_p$. For Calderón-Zygmund operators, see Journé [21] or Duoandikoetxea [14]. For first order commutators, this result is due to Bloom [5] when T is the Hilbert transform, and follows from a result of Strömberg for Calderón-Zygmund operators. (See Janson [20].) For higher order commutators it is due to Segovia and Torrea [17], [41]. For classical Littlewood-Paley square functions with $\lambda \ge 2$ it is due to Muckenhoupt and Wheeden [25]; for (1.9) it is due to Strömberg and Torchinsky [44]. (These papers also give sufficient, A_p-type conditions for the case $1 < \lambda < 2$.)

In the two-weight case, however, $(u, v) \in A_p$ is no longer sufficient, even for the Hilbert transform: for a counter-example, see Muckenhoupt and Wheeden [26], or the one mentioned above for the Hardy-Littlewood maximal function [18, p. 395].

In the special case of the periodic Hilbert transform (i.e. the conjugate function) on the unit circle, Cotlar and Sadosky [9], [10] gave a necessary and sufficient condition (which is a generalization of the Helson-Szegö condition) for (1.8) to hold. However, their results do not extend to higher dimensions or more general operators.

Sufficient conditions for (1.8) that resemble but are stronger than the A_p condition have been given by several authors. See, for example, Fujii [16], Katz and Pereyra [22], Leckband [23], Rakotondratsimba [37], [38], and Wilson [48].

In the special case of the periodic Hilbert transform, Treil, Volberg and Zheng [46] found a stronger variant of Neugebauer's condition (1.5) similar to (1.7). Let A and B be two Young functions such that

$$\int_c^\infty \left(\frac{t^p}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty \quad \text{and} \quad \int_c^\infty \left(\frac{t^{p'}}{B(t)}\right)^{p-1} \frac{dt}{t} < \infty. \tag{1.11}$$

Also, for $z \in \mathbb{D}$, let ϕ_z be the Möbius transform in the unit circle,

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in \bar{\mathbb{D}}.$$

If (u, v) is a pair of weights such that

$$\sup_{z\in\mathbb{D}} \|u^{1/p} \circ \phi_z\|_{A,\partial\mathbb{D}} \|v^{-1/p} \circ \phi_z\|_{B,\partial\mathbb{D}} < \infty, \tag{1.12}$$

then (1.8) holds for the periodic Hilbert transform.

This theorem has two drawbacks. First, if $A(t) = t^{rp}$ and $B(t) = t^{rp'}$, $r > 1$, then (1.12) is a stronger condition than (1.5). In fact, even when $r = 1$ and $u = v$ in (1.12), this condition, the so-called "invariant A_p" condition, is stronger than the A_p condition. (See [46].) Second, as with the work of Cotlar and Sadosky, their proof relies heavily on complex analysis and so does not extend to more general operators or to higher dimensions.

1.2. – Statements of the main results

In each of our results we show that we can improve Neugebauer's condition (1.5) by replacing the power bump on the right-hand weight by an Orlicz bump. Unlike the condition for the maximal operator, (1.7), we need "bumps" on both weights. This is natural since the operators we consider are essentially self-adjoint, so any condition must also be sufficient for the dual inequality. In addition, some of the operators have a higher degree of singularity than singular integrals, so a stronger condition is natural.

Our first theorem is for Calderón-Zygmund operators.

THEOREM 1.1. *Let T be a Calderón-Zygmund operator. Given p, $1 < p < \infty$, let B be a doubling Young function such that for some constant $c > 0$,*

$$\int_c^\infty \left(\frac{t^{p'}}{B(t)}\right)^{p-1} \frac{dt}{t} < \infty. \tag{1.13}$$

If (u, v) is a pair of weights such that for some $r > 1$ and for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q u^r\, dy\right)^{1/r} \|v^{-1/p}\|_{B,Q}^p \le C < \infty, \tag{1.14}$$

then T satisfies the strong (p, p) inequality

$$\int_{\mathbb{R}^n} |Tf|^p u\, dx \le C \int_{\mathbb{R}^n} |f|^p v\, dx. \tag{1.15}$$

REMARK 1.2. If we let $B(t) = t^{rp'}$ then (1.14) is equivalent to (1.5), and so by Neugebauer's theorem [27], (1.15) follows from the one-weight results discussed above. However, there exist smaller Young functions B satisfying (1.13): for example, as we noted above, for any $\delta > 0$, $B(t) \approx t^{p'}(\log t)^{p'-1+\delta}$. Similarly, we also have

$$B(t) \approx t^{p'}\,(\log t)^{p'-1}\,(\log\log t))^{p'-1+\delta}\,.$$

REMARK 1.3. It is conceptually simple, though computationally tedious, to produce examples of weights which satisfy (1.14). For example, let $n = 1$, $p = 2$, $r = 2$, $B(t) \approx t^2 \log(t)^2$, and let $u(x) = x^{-1/2}\chi_{(1,\infty)}$. Then, since we have the inequality

$$\|v^{-1/p}\|_{B,Q} \le 1 + \frac{1}{|Q|}\int_Q B(|v(x)^{-1/p}|)\, dx,$$

(see, for instance, Rao and Ren [36, p. 69]), we can show that if $v(t) \approx \log(t)^{9/4}/t^{1/4}$, the pair (u, v) satisfies (1.14).

REMARK 1.4. We conjecture that Theorem 1.1 remains true if (1.14) is replaced by the weaker condition

$$\|u^{1/p}\|_{A,Q}\|v^{-1/p}\|_{B,Q} \le C, \tag{1.16}$$

where A and B are Young functions satisfying (1.11). If $A(t) = t^{rp}$, $r > 1$, then (1.16) is equivalent to (1.14). Clearly, this condition is similar to but weaker than (1.12).

We can prove a partial result in this direction: by applying results due to Cruz-Uribe and Fiorenza [11], we can adapt our proof of Theorem 1.1 to show that (1.16) is sufficient provided that, for example,

$$A(t) \approx t^p \exp[\log(e+t^p)^r], \qquad 0 < r < 1. \tag{1.17}$$

We will give the details after the proof of Theorem 1.1.

Further evidence for this conjecture is the fact that the analogous result holds for fractional integrals. For $0 < \alpha < n$, let I_α be the fractional integral operator. If (u, v) is a pair of weights such that for all cubes Q,

$$|Q|^{\alpha/n} \, \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{B,Q} \le C, \tag{1.18}$$

where A and B are Young functions satisfying (1.11), then

$$\int_{\mathbb{R}^n} |I_\alpha f|^p u \, dx \le C \int_{\mathbb{R}^n} |f|^p v \, dx. \tag{1.19}$$

This was proved in [29] (also see [34]). Unfortunately, the techniques in those papers are not applicable to even the Hilbert transform, and this problem seems to be more difficult.

Remark 1.5. Conditions (1.14) and (1.7) are related to the problem of weak (p, p) inequalities, $p > 1$, for Calderón-Zygmund operators:

$$u(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \le \frac{C}{t^p} \int_{\mathbb{R}^n} |f|^p v \, dx. \tag{1.20}$$

In [12] we showed that (1.20) holds whenever the pair of weights (u, v) satisfies a "dual" version of (1.7): for some $\delta > 0$,

$$\|u\|_{L(\log L)^{p-1+\delta},Q} \left(\frac{1}{|Q|} \int_Q v^{1-p'} \, dx \right)^{p-1} \le C < \infty.$$

Note that in this condition the Orlicz bump is on the left-hand weight instead of the right-hand one. In [13] we derived similar results, with power bumps instead of Orlicz bumps, for commutators and fractional integral operators. We conjecture that these results are true with the appropriate Orlicz bump replacing the power bump.

It follows from our proof that Theorem 1.1 holds for a larger class of operators. If T is an operator such that for some $0 < \delta < 1$ and for every $f \in C_c^\infty(\mathbb{R}^n)$,

$$M_\delta^\#(Tf)(x) = M^\#(|Tf|^\delta)(x)^{1/\delta} \le C_\delta \, Mf(x), \tag{1.21}$$

then (1.14) implies (1.15). The fact that we can take δ as small as desired plays an important role in the proofs. It was shown in [1] that (1.21) holds for Calderón-Zygmund operators; in [2] it was shown that that (1.21) holds for:

- weakly strongly singular integral operators, as considered by C. Fefferman [15];
- pseudo-differential operators—more precisely, pseudo-differential operators in the Hörmander class (see Hörmander [19]);
- oscillatory integral operators of the kind introduced by Phong and Stein [35].

Our second result shows that Theorem 1.1 holds for the Littlewood-Paley square function g_λ^*. The proof depends on a version of (1.21) for the square function.

LEMMA 1.6. *Fix* $\lambda > 2$ *and* $0 < \delta < 1$. *Then there exists a constant C such that for any locally integrable* f,

$$M_\delta^\#(g_\lambda^*(f))(x) \leq C\, Mf(x). \tag{1.22}$$

Given inequality (1.22), the next result follows from essentially the same proof as that of Theorem 1.1, and we omit the details.

THEOREM 1.7. *Let* g_λ^* *be the Littlewood-Paley square functions* (1.9) *with* $\lambda > 2$. *Let* p, r, *and* B *be as in the statement of Theorem* 1.1, *and suppose the pair of weights* (u, v) *satisfy condition* (1.14). *Then* g_λ^* *satisfies the strong* (p, p) *inequality*

$$\int_{\mathbb{R}^n} (g_\lambda^* f)^p\, u\, dx \leq C \int_{\mathbb{R}^n} |f|^p\, v\, dx. \tag{1.23}$$

REMARK 1.8. If we combine Lemma 1.6 with well-known estimates for the maximal operator and the sharp maximal operator (cf. Journé [21] or Duoandikoetxea [14]), then we get the following one-weight norm inequalities for g_λ^*: if $\lambda > 2$, $1 < p < \infty$, and $w \in A_p$, then $g_\lambda^* : L^p(w) \longrightarrow L^p(w)$. This is the analogue of the norm inequalities due to Muckenhoupt and Wheeden [25] and Strömberg and Torchinsky [44].

REMARK 1.9. The proofs of the weak-type inequalities in [13] also depend only on the operator satisfying (1.21). Therefore, the same proofs, combined with Lemma 1.6, yield weak (p, p) inequalities for g_λ^*, $\lambda > 2$. Details are left to the reader.

REMARK 1.10. Given the norm inequalities which g_λ^* satisfies (see Section 2 below), Lemma 1.6 should hold for $\lambda \geq 2$. In the case $1 < \lambda < 2$, we conjecture that the correct inequality is

$$M_\delta^\#(g_\lambda^*(f))(x) \leq C M_{2/\lambda} f(x) = C M(|f|^{2/\lambda})(x)^{\lambda/2}.$$

If this inequality holds then we could give another proof of the one-weight norm inequalities in [25] and [44] for the case $1 < \lambda < 2$.

Our last theorem is the corresponding result for commutators of singular integral operators with BMO functions. These operators are interesting, among other reasons, since they have a higher degree of singularity than the associated singular integral. (See, for example, [31], [32] and [33].)

THEOREM 1.11. *Given p, $1 < p < \infty$, and $m \geq 0$ an integer, let A_m and C_m be Young functions such that A_m satisfies the condition*

$$\int_c^\infty \frac{A_m(t)}{t^p}\frac{dt}{t} < \infty, \tag{1.24}$$

and

$$A_m^{-1}(t)C_m^{-1}(t) \leq B_m^{-1}(t), \tag{1.25}$$

where $B_m(t) = t\log(e+t)^m$. Let C_b^m be a commutator (1.10), *where K is the kernel of a Calderón-Zygmund operator T and $b \in BMO$. If the pair of weights (u, v) are such that for some $r > 1$ and any cube Q,*

$$\left(\frac{1}{|Q|}\int_Q u^r\,dy\right)^{1/r} \|v^{-1/p}\|_{C_m,Q}^p \leq C < \infty, \tag{1.26}$$

then the commutator C_b^m satisfies the strong (p, p) inequality

$$\int_{\mathbb{R}^n} |C_b^m f|^p\, u\,dx \leq C\int_{\mathbb{R}^n} |f|^p\, v\,dx. \tag{1.27}$$

REMARK 1.13. An example of a Young function C_m which satisfies the hypotheses of Theorem 1.1 is

$$C_m(t) = t^{p'}\log(e+t)^{(m+1)p'-1+\delta}, \qquad \delta > 0.$$

To see this, let

$$A_m(t) = t^p\log(e+t)^{-(1+\delta(p-1))},$$

where δ is the same as in the definition of C_m. Then A_m satisfies condition (1.24). The inverses of these functions are, approximately, given by

$$\begin{aligned} C_m^{-1}(t) &\approx t^{1/p'}\log(e+t)^{-\left(m+1+\frac{\delta}{p'}-\frac{1}{p'}\right)}, \\ A_m^{-1}(t) &\approx t^{1/p}\log(e+t)^{\frac{\delta}{p'}+\frac{1}{p}} \\ B_m^{-1}(t) &\approx t\log(e+t)^{-m}. \end{aligned}$$

(See [28].) It follows at once that condition (1.25) holds.

REMARK 1.13. As with Theorem 1.1, in the one weight case Theorem 1.11 reduces to the known results.

The remainder of this paper is organized as follows. In Section 2 we give precise definitions of the operators we are considering and state some preliminary results. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorem 1.11. For both proofs we draw heavily on our earlier work in [13], and we recommend that the reader consult that paper. Finally, in Section 5 we prove Lemma 1.6.

Throughout this paper all notation is standard or will be defined as needed. All cubes are assumed to have their sides parallel to the coordinate axes. Given a cube Q and $r > 0$, rQ will denote the cube with the same center as Q and whose sides are r times as long. By weights we will always mean non-negative, locally integrable functions which are positive on a set of positive measure. Given a Lebesgue measurable set E and a weight w, $|E|$ will denote the Lebesgue measure of E and $w(E) = \int_E w\,dx$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent of p. C will denote a positive constant whose value may change at each appearance.

2. – Preliminaries

In this section we precisely define the operators we are considering and state some basic results we will need in later sections.

2.1. – Calderón-Zygmund operators

A kernel K is a locally integrable function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, where $\Delta = \{(x,x) : x \in \mathbb{R}^n\}$. We say that K satisfies the standard estimates if there exist positive and finite constants γ and C such that, for all distinct $x, y \in \mathbb{R}^n$ and all z with $2|x-z| < |x-y|$,

$$\begin{aligned}
|K(x,y)| &\leq C|x-y|^{-n}, \\
|K(x,y) - K(z,y)| &\leq C\frac{|x-z|^{\gamma}}{|x-y|^{n+\gamma}}, \\
|K(y,x) - K(y,z)| &\leq C\frac{|x-z|^{\gamma}}{|x-y|^{n+\gamma}}.
\end{aligned}$$

The linear operator $T : C_0^{\infty}(\mathbb{R}^n) \to C_0^{\infty}(\mathbb{R}^n)$ is associated with the kernel K if

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

when x is not in the support of f. T is called a Calderón-Zygmund operator if K satisfies the standard estimates and if it extends to a bounded operator on $L^2(\mathbb{R}^n)$. It is well known that a Calderón-Zygmund operator extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and is weak $(1,1)$. For more information, see Christ [7], Duoandikoetxea [14], or Journé [21].

2.2. – Littlewood-Paley square functions

Let ϕ be a Schwartz function such that $\int \phi\, dx = 0$. Then for $\lambda > 1$, define the Littlewood-Paley square function g^*_λ for $f \in C^\infty_c(\mathbb{R}^n)$ by

$$(2.1) \qquad g^*_\lambda(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} |\phi_t * f(y)|^2 \left(\frac{t}{t + |x-y|} \right)^{n\lambda} \frac{dy\, dt}{t^{n+1}} \right)^{1/2}.$$

The classical Littlewood-Paley square function (cf. Stein [43]), defined in terms of the gradient of the Poisson kernel, is a variant of this operator. If $p > 2/\lambda$, then g^*_λ is bounded on L^p; in particular, when $\lambda \geq 2$ it is bounded on L^p for $1 < p < \infty$. Further, in this case g^*_λ is weak $(1,1)$. For more information, see Torchinsky [45].

2.3. – Commutators

Given a Calderón-Zygmund operator T with kernel K, and a function $b \in BMO$, let M_b denote multiplication by b and define the commutators C^m_b inductively: $C^0_b = T$, and for $m \geq 1$, $C^m_b = [M_b, C^{m-1}_b] = M_b C^{m-1}_b - C^{m-1}_b M_b$. If $f \in C^\infty_c(\mathbb{R}^n)$ then

$$C^m_b f(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]^m K(x,y) f(y)\, dy, \qquad x \notin \operatorname{supp}(f).$$

It is well–known that for all m, C^m_b is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. (See Coifman, Rochberg and Weiss [8].)

2.4. – Orlicz spaces

We will need the following facts about Orlicz spaces. (For details see Bennett and Sharpley [4] or Rao and Ren [36].) A function $B : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and increasing, and if $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. A Young function B is doubling if there exists a positive constant C such that $B(2t) \leq CB(t)$ for all $t > 0$. We are usually only concerned about the behavior of Young functions for t large. If A, B are two Young functions, we write $A(t) \approx B(t)$ if there are constants $c, c_1, c_2 > 0$ with $c_1 A(t) \leq B(t) \leq c_2 A(t)$ for $t > c$.

Given a Young function B, define the mean Luxemburg norm of f on a cube Q by

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f|}{\lambda} \right) dy \leq 1 \right\}.$$

When $B(t) = t^r$, $1 \leq r < \infty$,

$$\|f\|_{B,Q} = \left(\frac{1}{|Q|} \int_Q |f|^r\, dx \right)^{1/r},$$

so the Luxemburg norm coincides with the (normalized) L^r norm.

Given a Young function B, there exists a complementary Young function $\bar{B}$ such that

$$t \le B^{-1}(t)\bar{B}^{-1}(t) \le 2t. \tag{2.2}$$

Lemma 2.1 (Generalized Hölder's Inequality). *If A, B and C are Young functions such that $A^{-1}(t)C^{-1}(t) \le B^{-1}(t)$, then for all functions f and g and any cube Q,*

$$\|fg\|_{B,Q} \le 2\|f\|_{A,Q}\|g\|_{C,Q}. \tag{2.3}$$

In particular, given any Young function B,

$$\frac{1}{|Q|}\int_Q |fg|\,dx \le 2\|f\|_{B,Q}\|g\|_{\bar{B},Q}. \tag{2.4}$$

Inequality (2.4) is due to Weiss [47]; inequality (2.3) is due to O'Neil [28]. For a proof see [36].

2.5. – Maximal operators

Given a locally integrable function f, define the Hardy-Littlewood maximal function Mf by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes containing x. M^d is the dyadic version of this operator — i.e., the supremum is taken over all dyadic cubes containing x.

More generally, given a Young function B, define the associated Orlicz maximal operator by

$$M_B f(x) = \sup_{Q\ni x} \|f\|_{B,Q}.$$

The dyadic operator M_B^d is defined similarly. We can precisely characterize the Young functions for which M_B is a bounded operator on L^p.

Lemma 2.2. *Let B be a doubling Young function. Then*

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} < \infty \tag{2.5}$$

if and only if

$$M_B : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n).$$

For a proof, see [30].

If B satisfies (2.5), then we denote this by $B \in B_p$. Thus, in Theorem 1.11 hypothesis (1.27) is equivalent to $A_m \in B_p$. We may similarly restate the hypotheses in Theorems 1.1 and 1.7. It follows from (2.2) that the B_p condition can expressed in terms of the complementary Young function $\bar{B}$:

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} \approx \int_c^\infty \left(\frac{t^{p'}}{\bar{B}(t)}\right)^{p-1}\frac{dt}{t}.$$

Therefore, (1.13) is equivalent to $\bar{B} \in B_p$.

For $m \geq 1$, $M^m = M \circ \cdots \circ M$ denotes the mth iterate of the maximal operator. It follows essentially from a result due to Stein [42] that there exists a constant C, depending only on m and n, such that

$$M^{m+1} f(x) \leq C M_{B_m} f(x),$$

where $B_m(t) = t\log(e+t)^m$. (See [34] or Carozza and Passarelli [6] for detailed proofs.)

Define the sharp maximal function $M^\# f$ by

$$M^\# f(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy,$$

where $f_Q = \frac{1}{|Q|}\int_Q f\,dx$ is the average of f over Q. Let $M^{\#,d}$ be its dyadic analogue.

Given $\delta > 0$, define

$$M_\delta f(x) = M(|f|^\delta)(x)^{1/\delta},$$

and

$$M_\delta^\# f(x) = M^\#(|f|^\delta)(x)^{1/\delta}.$$

M_δ^d and $M_\delta^{\#,d}$ are defined similarly.

3. – Proof of Theorem 1.1

To prove Theorem 1.1 we need several lemmas.

Given $r > 1$ and a weight u, define the set function A_u^r on measurable sets $E \subset \mathbb{R}^n$ by

$$A_u^r(E) = |E|^{1/r'}\left(\int_E u^r\,dx\right)^{1/r} = |E|\left(\frac{1}{|E|}\int_E u^r\,dx\right)^{1/r}.$$

(The second equality holds provided $|E| > 0$.)

The first lemma implies that the functionals A_u^r behave in some sense as A_∞ weights.

LEMMA 3.1. *For any $r > 1$ and weight u, the set function A_u^r has the following properties:*

(1) *If $E \subset F$ then $A_u^r(E) \le (|E|/|F|)^{1/r'} A_u^r(F)$;*
(2) *$u(E) \le A_u^r(E)$;*
(3) *If $\{E_j\}$ is a sequence of disjoint sets and $\bigcup_j E_j = E$ then*

$$\sum_j A_u^r(E_j) \le A_u^r(E).$$

The proof of Lemma 3.1 is straightforward; it can be found in [13, Lemma 3.2].

LEMMA 3.2. *Given a function $g \in L^p$, $1 \le p < \infty$, and constants $a > 2^n$ and $\gamma > 0$, for each integer k form the Calderón-Zygmund decomposition $\{C_j^k\}$ of g at height γa^k. Then there exists $M > 0$ and pairwise disjoint sets $E_j^k \subset C_j^k$ such that $|C_j^k| \le M|E_j^k|$.*

The idea behind this lemma is implicit in [18]. An explicit proof can be found in [29]. (This proof has $\gamma = 1$, but the same proof works for $\gamma > 0$ with only trivial changes.)

LEMMA 3.3. *Given a Young function B, suppose f is a non-negative function such that $\|f\|_{B,Q}$ tends to zero as $|Q|$ tends to infinity. Then for each $t > 0$ there exists a disjoint collection of dyadic cubes $\{C_i^t\}$ such that for each i, $t < \|f\|_{B,C_i^t} \le 2^n t$,*

$$\{x \in \mathbb{R}^n : M_B^d f(x) > t\} = \bigcup_i C_i^t,$$

and

$$\{x \in \mathbb{R}^n : M_B f(x) > 4^n t\} \subset \bigcup_i 3C_i^t.$$

Moreover, the cubes are maximal: if Q is a dyadic cube such that $Q \subset \{M_B^d f(x) > t\}$, then $Q \subset C_i^t$ for some i.

For a proof see [30]; this is an adaptation of the classical proof given in García-Cuerva and Rubio de Francia [18, p. 137]. The collection $\{C_i^t\}$ is referred to as the Calderón-Zygmund decomposition of f with respect to B at height t.

LEMMA 3.4. *Let B be a Young function. Suppose that for some function $f \in L^q$, $1 \le q < \infty$, and for some $t > 0$ there exists a constant μ, $0 < \mu \le 1$, and a collection of dyadic cubes $\{Q_j\}$ such that for each j,*

$$|Q_j \cap \{x \in \mathbb{R}^n : M_B f(x) > t\}| \ge \mu |Q_j|.$$

Then there exists a constant $\nu > 0$, depending on n and μ, and a subcollection $\{P_k\}$ of the Calderón-Zygmund decomposition with respect to B of f at height νt, $\{C_i^{\nu t}\}$, such that for each j, $Q_j \subset 3P_k$ for some k.

If we replace M_B by M_B^d in the hypothesis then we can strengthen the conclusion by finding P_k's such that $Q_j \subset P_k$ and by letting $\mu = \nu$.

The proof of this result can be found in [13, Lemma 5.1].

PROOF OF THEOREM 1.1. We may assume without loss of generality that u is bounded. The general case follows at once: if (1.14) holds for a given pair (u, v), then it holds with u replaced by $u_N = \min(u, N)$, and if (1.15) holds with u_N on the left-hand side and the constant C independent of N, then the desired inequality follows by Fatou's lemma. We may also assume that $f \in C_c^\infty(\mathbb{R}^n)$ since the general result follows by a standard approximation argument. With these assumptions we have $\int_{\mathbb{R}^n} |Tf|^p u\, dx < \infty$.

Fix δ, $0 < \delta < 1$, and let $a = (2^n + 1)^{1/\delta}$; the reason for this choice will be clear below. For each $k \in \mathbb{Z}$ define $\Omega_k = \{x \in \mathbb{R}^n : M_\delta^d(Tf)(x) > a^k\}$. Then

$$\int_{\mathbb{R}^n} |Tf|^p u\, dx \le \int_{\mathbb{R}^n} M_\delta^d(Tf)^p u\, dx \le a^p \sum_k a^{kp} u(\Omega_k \backslash \Omega_{k+1}) \le a^p \sum_k a^{kp} u(\Omega_k).$$

Since $Tf \in L^q$, $1 < q < \infty$, for each k we can form the Calderón-Zygmund decomposition of $|Tf|^\delta$ at height $a^{k\delta}$ to get a collection of disjoint dyadic cubes $\{C_j^k\}$ such that

$$\Omega_k = \bigcup_j C_j^k,$$

and

$$a^{k\delta} \le \frac{1}{|C_j^k|} \int_{C_j^k} |Tf|^\delta\, dx \le 2^n a^{k\delta}.$$

Therefore, by Lemma 3.1, condition (2),

$$a^p \sum_k a^{kp} u(\Omega_k) = a^p \sum_{k,j} a^{kp} u(C_j^k) \le a^p \sum_{k,j} a^{kp} A_u^r(C_j^k). \tag{3.1}$$

Fix $\epsilon > 0$ so that $a^p \epsilon^{\delta/r'} = 1/2$; the reason for this choice will be clear below. For each k, divide the cubes $\{C_i^k\}$ into two sets: $C_i^k \in \mathcal{F}_k$ if

$$\left(\frac{1}{|C_i^k|} \int_{C_i^k} \left| |Tf(x)|^\delta - (|Tf|^\delta)_{C_i^k} \right| dx \right)^{1/\delta} \le \epsilon a^k,$$

and $C_i^k \in \mathcal{G}_k$ if the opposite inequality holds.

Our proof now has two steps. First, we will show that

$$a^p \sum_{k,j} a^{kp} A_u^r(C_j^k) \le 2a^{2p} \sum_{k, C_i^k \in \mathcal{G}_k} a^{kp} A_u^r(C_i^k). \tag{3.2}$$

(While doing so we will show that these sums are finite.) Second, we will show that the right-hand side of (3.2) is bounded by a multiple of $\int |f|^p v\, dx$.

To show that (3.2) holds, we first note that if we re-index by replacing k with $k+1$ we get

$$a^p \sum_{k,j} a^{kp} A_u^r(C_j^k) = a^{2p} \sum_{k,j} a^{kp} A_u^r(C_j^{k+1}).$$

By the maximality of Calderón-Zygmund cubes, for each j there exists i such that $C_j^{k+1} \subset C_i^k$. Hence, by Lemma 3.1, condition (3),

$$\begin{aligned} &= a^{2p} \sum_k \sum_{i, C_j^{k+1} \subset C_i^k} a^{kp} A_u^r(C_j^{k+1}) \\ &\leq a^{2p} \sum_{k,i} a^{kp} A_u^r(\Omega_{k+1} \cap C_i^k). \end{aligned}$$

We now claim that if $C_i^k \in \mathcal{F}_k$, then

$$A_u(\Omega_{k+1} \cap C_i^k) \leq \epsilon^{\delta/r'} A_u^r(C_i^k). \tag{3.3}$$

By Lemma 3.1, condition (1), it will suffice to show that

$$|\Omega_{k+1} \cap C_i^k| \leq \epsilon^\delta |C_i^k|.$$

By the properties of Calderón-Zygmund cubes, if $x \in \Omega_{k+1} \cap C_i^k$ then

$$M^d(|Tf|^\delta)(x) = M^d(|Tf|^\delta \chi_{C_i^k})(x).$$

Furthermore, $(|Tf|^\delta)_{C_i^k} \leq 2^n a^{k\delta}$, and by our choice of a, $a^{(k+1)\delta} - 2^n a^{k\delta} = a^{k\delta}$. Hence,

$$\begin{aligned} \Omega_{k+1} \cap C_i^k &= \left\{ x \in C_i^k : M^d(|Tf|^\delta \chi_{C_i^k})(x) > a^{(k+1)\delta} \right\} \\ &\subset \left\{ x \in C_i^k : M^d\left(\left| |Tf|^\delta - (|Tf|^\delta)_{C_i^k} \right| \chi_{C_i^k} \right)(x) > a^{(k+1)\delta} - (|Tf|^\delta)_{C_i^k} \right\} \\ &\subset \left\{ x \in \mathbb{R}^n : M^d\left(\left| |Tf|^\delta - (|Tf|^\delta)_{C_i^k} \right| \chi_{C_i^k} \right)(x) > a^{k\delta} \right\}. \end{aligned}$$

Since the dyadic maximal operator is weak $(1,1)$ with constant 1 (see Journé [21, p. 10]) and since $C_i^k \in \mathcal{F}_k$,

$$|\Omega_{k+1} \cap C_i^k| \leq a^{-k\delta} \int_{C_i^k} \left| |Tf(x)|^\delta - (|Tf|^\delta)_{C_i^k} \right| dx \leq \epsilon^\delta |C_i^k|.$$

This establishes (3.3). We have thus shown that

$$\begin{aligned}
a^p \sum_{k,j} a^{kp} A_u^r(C_j^k) &\le a^{2p} \sum_{k,i} a^{kp} A_u^r(\Omega_{k+1} \cap C_i^k) \\
&\le a^{2p} \epsilon^{\delta/r'} \sum_{k, C_i^k \in \mathcal{F}_k} a^{kp} A_u^r(C_i^k) + a^{2p} \sum_{k, C_i^k \in \mathcal{G}_k} a^{kp} A_u^r(C_i^k) \\
&\le a^{2p} \epsilon^{\delta/r'} \sum_{k,i} a^{kp} A_u^r(C_i^k) + a^{2p} \sum_{k, C_i^k \in \mathcal{G}_k} a^{kp} A_u^r(C_i^k)
\end{aligned}$$

If the first term in the last line were finite then by our choice of ϵ we could rearrange terms to get (3.2). However, since $u \in L^\infty$,

$$\begin{aligned}
\sum_{k,i} a^{kp} A_u^r(C_i^k) &\le \|u\|_\infty \sum_{k,i} a^{kp} |C_i^k| \\
&= \|u\|_\infty \sum_k a^{kp} |\Omega_k| \\
&\le C \|u\|_\infty \sum_k \int_{a^{k-1}}^{a^k} t^{p-1} |\Omega_k| \, dt \\
&\le C \|u\|_\infty \int_{\mathbb{R}^n} M_\delta(Tf)^p \, dx,
\end{aligned}$$

and the last quantity is finite since M_δ and T are bounded on L^p.

Therefore, (3.2) holds, and to complete our proof we need to show that the right-hand side of this inequality is bounded by a multiple of $\int |f|^p v \, dx$. By the definition of the sets $\mathcal{G}_k$ and by inequality (1.21), for each set $C_i^k \in \mathcal{G}_k$,

$$C_i^k \subset \{x \in \mathbb{R}^n : M_\delta^\#(Tf)(x) > \epsilon a^k\} \subset \{x \in \mathbb{R}^n : Mf(x) > \epsilon C_\delta^{-1} a^k\}. \tag{3.4}$$

By Lemma 3.4 (with $B(t) = t$ and $\mu = 1$), for each k there exists a sequence of disjoint Calderón-Zygmund cubes for f at height ρa^k, $\{P_j^k\}$, with ρ depending only on n, ϵ and C_δ, such that for each i there exists j so that $C_i^k \subset 3P_j^k$.

Therefore, by Lemma 3.1, condition (3),

$$\begin{aligned}
2a^{2p} \sum_{k, C_i^k \in \mathcal{G}_k} a^{kp} A_u^r(C_i^k) &\le 2a^{2p} \sum_k \sum_{j, C_i^k \subset 3P_j^k} a^{kp} A_u^r(C_i^k) \\
&\le 2a^{2p} \sum_{k,j} a^{kp} A_u^r(3P_j^k) \\
&\le 2a^{2p} \rho^{-p} \sum_{k,j} A_u^r(3P_j^k) \left(\frac{1}{|P_j^k|} \int_{P_j^k} |f| \, dx \right)^p \\
&= C \sum_{k,j} |P_j^k| \left(\frac{1}{|3P_j^k|} \int_{3P_j^k} u^r \, dx \right)^{1/r} \left(\frac{1}{|P_j^k|} \int_{P_j^k} |f| v^{1/p} v^{-1/p} dx \right)^p.
\end{aligned}$$

By Lemma 2.1 and (1.14),

$$\leq C\sum_{k,j}|P_j^k|\left(\frac{1}{|3P_j^k|}\int_{3P_j^k}u^r\,dx\right)^{1/r}\|v^{-1/p}\|_{B,3P_j^k}^p\|fv^{1/p}\|_{\bar{B},P_j^k}^p$$
$$\leq C\sum_{k,j}|P_j^k|\|fv^{1/p}\|_{\bar{B},P_j^k}^p.$$

By Lemma 3.2 there exists a collection of pairwise disjoint sets $\{E_j^k\}$ such that each E_j^k is a subset of P_j^k of comparable size. Hence,

$$\leq C\sum_{k,j}|E_j^k|\|fv^{1/p}\|_{\bar{B},P_j^k}^p$$
$$\leq C\sum_{k,j}\int_{E_j^k}M_{\bar{B}}(fv^{1/p})^p\,dx$$
$$\leq C\int_{\mathbb{R}^n}M_{\bar{B}}(fv^{1/p})^p\,dx.$$

By Lemma 2.2, $M_{1.7ex\bar{B}}$ is bounded on L^p since by hypothesis B satisfies (1.13), so

$$\leq C\int_{\mathbb{R}^n}|f|^p v\,dx.$$

This completes our proof. □

Remark 3.5. As we noted in the Introduction, our proof of Theorem 1.1 adapts to work with weaker hypotheses. In the proof, the only properties of the set function A_u^r that we used were the three conditions in Lemma 3.1 and the fact that $A_u^r(E)\leq\|u\|_\infty|E|$. Given a Young function A, define the set function

$$A_{u,A}(E)=|E|\|u\|_{A,E}.$$

Then, as we noted in [13], $A_{u,A}$ satisfies conditions (2) and (3) of Lemma 3.1. Further, Cruz-Uribe and Fiorenza [11] proved the following result.

Proposition 3.6. *Given a Young function A, define the function h_A by*

$$h_A(s)=\sup_{t>0}\frac{A(st)}{A(t)},\qquad s\geq 0.$$

If

$$\lim_{s\to 0}\frac{h_A(s)}{s}=0, \tag{3.5}$$

then there exists a positive function ϕ on $(0,\infty)$ with $\phi(t)\to 0$ as $t\to 0$, such that for any cube Q and $E\subset Q$,

$$A_{u,A}(E)\le \phi(|E|/|Q|)A_{u,A}(Q). \tag{3.6}$$

Condition (3.6) is an adequate replacement for condition (1) of Lemma 3.1 in the proof of Theorem 1.1. Therefore, the same proof, *mutatis mutandis*, shows that if A satisfies (3.5) and the pair (u,v) satisfies

$$\|u\|_{A,Q}\|v^{-1/p}\|_{B,Q}^{p}\le C<\infty, \tag{3.7}$$

then the strong (p,p) inequality (1.15) holds. If we replace A by $\tilde{A}(t)=A(t^p)$, the resulting condition is equivalent to (1.16). It can be shown (see [11]) that (3.5) holds if, for example,

$$A(t)=t[\exp(\log(e+t)^r)-1],\qquad 0<r<1.$$

Unfortunately, in some sense A is the smallest Young function with this property: for example, $t\log(e+t)^r$, $r>0$, does not satisfy (3.5). Therefore, this approach will not yield the conjectured result.

4. – Proof of Theorem 1.11

To prove Theorem 1.11 we need the following lemma.

LEMMA 4.1. *Given a Calderón-Zygmund operator T, a function b in BMO, constants δ_0 and δ_1, $0<\delta_0<\delta_1<1$, and $m\ge 0$, there exists a constant K, depending on the BMO norm of b, such that for every function $f\in C_c^\infty(\mathbb{R}^n)$ and any $x\in\mathbb{R}^n$,*

$$M_{\delta_0}^{\#,d}(C_b^m f)(x)\le K\sum_{i=0}^{m-1}M_{\delta_1}^{d}(C_b^i f)(x)+KM^{m+1}f(x).$$

A proof of this result can found in [31]. As given there, the non-dyadic maximal operator appears in the first term on the right-hand side, but it is immediate from the proof that it is still true with the dyadic maximal operator there.

PROOF OF THEOREM 1.11. As in the proof of Theorem 1.1 , we may again assume without loss of generality that u is bounded and $f\in C_c^\infty(\mathbb{R}^n)$. Further, when $m=0$, Theorem 1.11 reduces to Theorem 1.1, so we may assume that $m\ge 1$.

The proof initially proceeds exactly as in the proof of Theorem 1.1, with T replaced by C_b^m. We thus get, using the same notation as before, that

$$\int_{\mathbb{R}^n} |C_b^m f|^p u\,dx \le \int_{\mathbb{R}^n} M_\delta^d(C_b^m f)^p u\,dx \le a^p \sum_{k,j} a^{kp} A_u^r(C_j^k), \tag{4.1}$$

where for each k the C_j^k's are the Calderón-Zygmund cubes of $|C_b^m f|^\delta$ at height $a^{k\delta}$. We now continue with the argument that in the proof of Theorem 1.1 yielded inequality (3.2), but here using the fact that C_b^m is bounded on L^p. This shows that the right-hand side of (4.1) is bounded by

$$2a^{2p} \sum_{k, C_i^k \in \mathcal{G}_k} a^{kp} A_u^r(C_i^k),$$

where each cube $C_i^k \in \mathcal{G}_k$ has the property that

$$C_i^k \subset \{x \in \mathbb{R}^n : M_\delta^{\#,d}(C_b^m f)(x) > \epsilon a^k\}.$$

Then by Lemma 4.1, if we fix δ_1, $\delta < \delta_1 < 1$, for each i and k we have that

$$\begin{aligned} C_i^k \subset \bigcup_{j=1}^{m-1} &\{x \in \mathbb{R}^n : M_{\delta_1}^d(C_b^j f)(x) > \beta a^k\} \\ &\cup \{x \in \mathbb{R}^n : M_{\delta_1}^d(Tf)(x) > \beta a^k\} \\ &\cup \{x \in \mathbb{R}^n : M^{m+1} f(x) > \beta a^k\} \\ \equiv &\left(\bigcup_{j=1}^{m-1} H_j^k\right) \cup H_0^k \cup H_m^k, \end{aligned}$$

where $\beta = \epsilon K^{-1}(m+1)^{-1}$.

For each integer j, $0 \le j \le m$, we define $\mathcal{H}_j^k$ to be the set of all C_i^k such that $|C_i^k \cap H_j^k| \ge (m+1)^{-1}|C_i^k|$. Clearly, for every i there exists at least one j such that $C_i^k \in \mathcal{H}_j^k$. Thus

$$2a^{2p} \sum_{k, C_i^k \in \mathcal{G}_k} a^{kp} A_u^r(C_i^k) \le 2a^{2p} \sum_{j=0}^{m} \sum_{k, C_i^k \in \mathcal{H}_j^k} a^{kp} A_u^r(C_i^k).$$

To complete the proof and get inequality (1.27), we will show that each of the m terms of the outer sum on the right-hand side is dominated by $C \int |f|^p v\,dx$. There are three cases.

CASE 1: CUBES IN $\mathcal{H}_m^k$. As we noted in Section 2, there exists a constant C, depending only on m and n, such that

$$M^{m+1} f(x) \le C M_{B_m} f(x),$$

where $B_m(t) = t \log(e+t)^m$. Hence, there exists $\beta' > 0$ such that

$$\{x \in \mathbb{R}^n : M^{m+1} f(x) > \beta a^k\} \subset \{x \in \mathbb{R}^n : M_{B_m} f(x) > \beta' a^k\}.$$

Therefore, by Lemma 3.4 (with $\mu = (m+1)^{-1}$) there exists $\nu > 0$ such that for each k there is a collection of disjoint dyadic cubes $\{P_l^k\}$ with the property that for each $C_i^k \in \mathcal{H}_m^k$ there exists P_l^k with $C_i^k \subset 3P_l^k$ and $\|f\|_{B_m, P_l^k} > \nu a^k$. We can now argue exactly as we did in the proof of Theorem 1.1, beginning from inequality (3.4), to get

$$\begin{aligned}\sum_{k, C_i^k \in \mathcal{H}_m^k} a^{kp} A_u^r(C_i^k) &\le \sum_k \sum_{l, C_i^k \subset 3P_l^k} a^{kp} A_u^r(C_i^k) \\ &\le \sum_{k,l} a^{kp} A_u^r(3P_l^k) \\ &\le C \sum_{k,l} A_u^r(3P_l^k) \|f\|^p_{B_m, P_l^k};\end{aligned}$$

by our hypotheses and by Lemma 2.1,

$$\begin{aligned}&\le C \sum_{k,l} |P_l^k| \left(\frac{1}{|3P_l^k|} \int_{3P_l^k} u^r \, dx \right)^{1/r} \|f v^{1/p}\|^p_{A_m, P_l^k} \|v^{-1/p}\|^p_{C_m, 3P_l^k} \\ &\le C \sum_{k,l} |P_l^k| \|f v^{1/p}\|^p_{A_m, P_l^k};\end{aligned}$$

by Lemma 3.2 there exist pairwise disjoint sets E_l^k such that

$$\begin{aligned}&\le C \sum_{k,l} |E_l^k| \|f v^{1/p}\|^p_{A_m, P_l^k} \\ &\le C \sum_{k,l} \int_{E_l^k} M_{A_m}(f v^{1/p})^p \, dx \\ &\le C \int_{\mathbb{R}^n} M_{A_m}(f v^{1/p})^p \, dx;\end{aligned}$$

since $A_m \in B_p$, by Lemma 2.2

$$\le C \int_{\mathbb{R}^n} |f|^p \, v \, dx.$$

CASE 2: CUBES IN $\mathcal{H}_0^k$. For each k let $s_k = (\beta a^k)^{\delta_1}$. Then by Lemma 3.4 (the dyadic case), if we form the Calderón-Zygmund decomposition of $|Tf|^{\delta_1}$ at height s_k, $\{Q_l^k\}$, then for each i there exists l such that $C_i^k \subset Q_l^k$. Hence, by Lemma 3.1, condition (3),

$$\sum_{k, C_i^k \in \mathcal{H}_0^k} a^{kp} A_u^r(C_i^k) \leq \sum_{k,l} a^{kp} A_u^r(Q_l^k).$$

However, the right-hand side is precisely the quantity we had to estimate in the proof of Theorem 1.1, beginning at (3.1). Therefore, we can repeat the entire argument in that proof to get

$$\sum_{k,l} a^{kp} A_u^r(Q_l^k) \leq C \int_{\mathbb{R}^n} |f|^p v \, dx.$$

CASE 3: CUBES IN $\mathcal{H}_j^k$, $1 \leq j \leq m-1$. Fix j; we can then argue as we did in Case 2 to get a set of Calderón-Zygmund cubes for $|C_b^j|^{\delta_1}$ at height s_k, $\{Q_l^k\}$ such that

$$\sum_{k, C_i^k \in \mathcal{H}_j^k} a^{kp} A_u^r(C_i^k) \leq \sum_{k,l} a^{kp} A_u^r(Q_l^k).$$

We now apply the above argument, beginning from inequality (4.1), to the right-hand side. This yields a new collection $\{\bar{Q}_j^k\}$ of disjoint dyadic cubes and a constant $\gamma > 0$ such that if $x \in \bar{Q}_j^k$ then $M_{\delta_1}^{\#,d}(C_b^j)(x) > \gamma a^k$. Furthermore,

$$\sum_{k,l} a^{kp} A_u^r(Q_l^k) \leq \sum_{k,j} a^{kp} A_u^r(\bar{Q}_j^k).$$

We can now apply Lemma (4.1) as we did before. This yields collections of cubes such as those in Cases 1 and 2 above, or cubes as in the sets $\mathcal{H}_i^k$, $1 \leq i < j$. In particular, the degree of the highest order commutator which appears has been reduced by one. Hence, if we repeat this argument at most j times, we end up with collections of cubes as those in Cases 1 and 2, and the arguments given above yield the desired inequality. □

5. – Proof of Lemma 1.6

To prove inequality (1.22) we will use the theory of vector-valued singular integral operators as developed by Rubio de Francia, Ruiz and Torrea [39]. (The theory itself originated in [3].) The key idea is to think of $g_\lambda^*(f)$ as the norm of a vector-valued singular integral operator whose kernel satisfies a gradient-type condition. More precisely, for $\lambda > 2$ define the Hilbert space H_λ by

$$H_\lambda = L^2\left(\mathbb{R}^{n+1}_+; \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydt}{t^{n+1}}\right)$$

with norm

$$\|g\|_{H_\lambda} = \left(\int_0^\infty \int_{\mathbb{R}^n} |g(y,t)|^2 \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydt}{t^{n+1}}\right)^{1/2}.$$

Then

$$g^*_\lambda(f)(x) = \|Tf(x)\|_{H_\lambda},$$

where T is the linear operator mapping $C_0^\infty(\mathbb{R}^n)$ into the space of H_λ valued functions defined by

$$Tf(x) = \{\phi_t * f(x-y)\}_{(t,y)\in\mathbb{R}^{n+1}_+}.$$

Define the vector-valued kernel $\mathcal{K}(x) \in \mathcal{L}(\mathbb{C}, H_\lambda) \approx H_\lambda$ by $\mathcal{K}(x)(y,t) = \phi_t(x-y)$; then

$$\begin{aligned} Tf(x) &= \left\{\int_{\mathbb{R}^n} \phi_t(x-y-z) f(z)\,dz\right\}_{(t,y)\in\mathbb{R}^{n+1}_+} \\ &= \int_{\mathbb{R}^n} \{\phi_t(x-y-z)\}_{(t,y)\in R^{n+1}_+} f(z)\,dz \\ &= \int_{\mathbb{R}^n} \mathcal{K}(x-z) f(z)\,dz, \end{aligned}$$

and therefore T is a vector-valued singular integral with kernel $\mathcal{K}$. To prove inequality (1.22) we need to show that $\mathcal{K}$ satisfies the following gradient-type condition.

LEMMA 5.1. *Given $\lambda > 2$, there exist constants C, ϵ such that for any $|x| > 2|h|$,*

$$\|\mathcal{K}(x+h) - \mathcal{K}(x)\|_{H_\lambda} \le C\frac{|h|^\epsilon}{|x|^{n+\epsilon}}. \tag{5.1}$$

We give the proof of (5.1) after the proof of (1.22).

PROOF OF LEMMA 1.6. We will sketch the proof since most of the argument is well known. (For further details see, for example, [39].) First note that by homogeneity, it will suffice to prove this for $x = 0$. Further, since $0 < \delta < 1$, for $\alpha, \beta > 0$, $\left||\alpha|^\delta - |\beta|^\delta\right| \le |\alpha - \beta|^\delta$. Hence it will suffice to show that, given a cube Q centered at the origin, there exists a constant c_Q such that

$$\left(\frac{1}{|Q|}\int_Q |g^*_\lambda(f)(y) - c_Q|^\delta\,dy\right)^{1/\delta} \le C_\delta\, Mf(0). \tag{5.2}$$

Decompose f as $f_1 + f_2$, where $f_1 = f\chi_{Q^*}$ and Q^* is the cube centered at the origin whose sides are $2\sqrt{n}$ times larger, and let $c_Q = \left\|(T(f_2))_Q\right\|_{H_\lambda}$. Recall that $g_\lambda^*(f) = \|Tf\|_{H_\lambda}$. Then we can estimate the left-hand side of (5.2) as follows:

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\|Tf(y)\|_{H_\lambda} - \left\|(Tf_2)_Q\right\|_{H_\lambda}\right|^\delta dy\right)^{1/\delta} \\
&\quad \le \left(\frac{1}{|Q|}\int_Q \left\|Tf(y) - (Tf_2)_Q\right\|_{H_\lambda}^\delta dy\right)^{1/\delta} \\
&\quad \le \left(\frac{1}{|Q|}\int_Q \|Tf_1(y)\|_{H_\lambda}^\delta dy\right)^{1/\delta} + \left(\frac{1}{|Q|}\int_Q \left\|Tf_2(y) - (Tf_2)_Q\right\|_{H_\lambda}^\delta dy\right)^{1/\delta} \\
&\quad = I + II.
\end{aligned}
$$

Because $\lambda > 2$, g_λ^* is weak (1, 1); since $0 < \delta < 1$, by Kolmogorov's inequality,

$$
I \le \frac{C}{|Q|}\int_{\mathbb{R}^n} |f_1(y)|\,dy = \frac{C}{|Q^*|}\int_{Q^*} |f(y)|\,dy \le C\,Mf(0).
$$

To estimate II we use (5.1). By our choice of Q^*,

$$
\begin{aligned}
II &\le \frac{1}{|Q|}\int_Q \left\|\frac{1}{|Q|}\int_Q \int_{\mathbb{R}^n\setminus Q^*} (\mathcal{K}(y-z) - \mathcal{K}(x-z)) f(z)\,dz\,dx\right\|_{H_\lambda} dy \\
&\le \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \int_{|x-z|>2|y-x|} \|\mathcal{K}(y-z) - \mathcal{K}(x-z)\|_{H_\lambda} |f(z)|\,dz\,dx\,dy \\
&\le C\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \int_{|x-z|>2|y-x|} \frac{|y-x|^\epsilon}{|x-z|^{n+\epsilon}} |f(z)|\,dz\,dx\,dy \\
&\le CMf(0),
\end{aligned}
$$

where the last inequality follows by a standard argument. Thus (5.2) holds and our proof is complete. □

Proof of Lemma 5.1. We will show that there exists $\epsilon > 0$ such that if $|x| > 2|h|$,

$$
\|\mathcal{K}(x+h) - \mathcal{K}(x)\|_{H_\lambda}^2 \le C\frac{|h|^{2\epsilon}}{|x|^{2n+2\epsilon}}.
$$

By definition,

$$
\begin{aligned}
&\|\mathcal{K}(x+h) - \mathcal{K}(x)\|_{H_\lambda}^2 \\
&\quad = \int_0^\infty \int_{\mathbb{R}^n} |\phi_t(x+h-y) - \phi_t(x-y)|^2 \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dy\,dt}{t^{n+1}} \\
&\quad = \int_0^\infty \int_{\mathbb{R}^n} \left|\phi\left(\frac{x+h-y}{t}\right) - \phi\left(\frac{x-y}{t}\right)\right|^2 \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dy\,dt}{t^{3n+1}} \\
&\quad \le C_\phi \int_0^\infty \int_{\mathbb{R}^n} \left|\phi\left(\frac{x+h-y}{t}\right) - \phi\left(\frac{x-y}{t}\right)\right|^{2\epsilon} \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dy\,dt}{t^{3n+1}}
\end{aligned}
$$

where $\epsilon < 1$ will be fixed below. By the mean value theorem,

$$\left|\phi\left(\frac{x+h-y}{t}\right)-\phi\left(\frac{x-y}{t}\right)\right| \le \frac{|h|}{t}\int_0^1 \left|\nabla\phi\left(\frac{x+\theta h-y}{t}\right)\right| d\theta.$$

Since ϕ is a Schwartz function, $|\nabla\phi(x)| \le C\min\{1, |x|^{-(n+\gamma)/\epsilon}|x|^{-(n-\delta)/2\epsilon}\}$, where $\gamma > 0$ and $\delta > 0$ are small values which will be chosen below. Hence, if we choose $\epsilon < 1/2$ we have

$$\left|\phi\left(\frac{x+h-y}{t}\right)-\phi\left(\frac{x-y}{t}\right)\right|^{2\epsilon} \le \frac{|h|^{2\epsilon}}{t^{2\epsilon}}\int_0^1 \left|\nabla\phi\left(\frac{x+\theta h-y}{t}\right)\right|^{2\epsilon} d\theta.$$

For clarity, let

$$m(x,y,t,\theta,h) = \min\left\{1, \left|\frac{x+\theta h-y}{t}\right|^{-2(n+\gamma)}, \left|\frac{x+\theta h-y}{t}\right|^{-(n-\delta)}\right\}.$$

Then

$$\begin{aligned}
&\|K(x+h)-K(x)\|^2\\
&\quad\le C\,|h|^{2\epsilon}\int_0^\infty\int_{\mathbb{R}^n}\int_0^1 m(x,y,t,\theta,h)\left(\frac{t}{t+|y|}\right)^{n\lambda}\frac{d\theta dy dt}{t^{3n+1+2\epsilon}}\\
&\quad= C\,|h|^{2\epsilon}\left(\int_0^{|x|}+\int_{|x|}^\infty\right)\\
&\quad= C\,|h|^{2\epsilon}\,(A_1+A_2).
\end{aligned}$$

We first estimate A_2. Since $\lambda > 1$,

$$\begin{aligned}
A_2 &\le \int_{|x|}^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|y|}\right)^{n\lambda}\frac{dy\,dt}{t^{3n+1+2\epsilon}}\\
&= C\int_{|x|}^\infty\int_{\mathbb{R}^n}\left(\frac{1}{1+|y|}\right)^{n\lambda}\frac{dy\,dt}{t^{2n+1+2\epsilon}}\\
&= C_\lambda\int_{|x|}^\infty t^{-2n-\epsilon-1}\,dt\\
&= \frac{C_\lambda}{|x|^{2n+2\epsilon}},
\end{aligned}$$

which yields the desired inequality.

To estimate A_1 we split the second integral in two pieces:

$$\begin{aligned}
A_1 &= \int_0^{|x|}\int_{B_t(x)}\int_0^1 m(x,y,t,\theta,h)\left(\frac{t}{t+|y|}\right)^{n\lambda}\frac{d\theta dy dt}{t^{3n+1+2\epsilon}}\\
&\quad+\int_0^{|x|}\int_{\mathbb{R}^n\setminus B_t(x)}\int_0^1 m(x,y,t,\theta,h)\left(\frac{t}{t+|y|}\right)^{n\lambda}\frac{d\theta dy dt}{t^{3n+1+2\epsilon}}\\
&= A_3+A_4.
\end{aligned}$$

To estimate A_3, note that $y \in B_t(x)$ implies $t + |y| \geq |x|$. Therefore, since $\lambda > 2$,

$$\begin{aligned}
A_3 &\leq \int_0^{|x|} \int_{B_t(x)} \left(\frac{t}{t+|y|} \right)^{n\lambda} \frac{dydt}{t^{3n+1+2\epsilon}} \\
&\leq C \int_0^{|x|} \int_{B_t(x)} \left(\frac{t}{|x|} \right)^{n\lambda} \frac{dydt}{t^{3n+1+2\epsilon}} \\
&= \frac{C_\lambda}{|x|^{n\lambda}} \int_0^{|x|} t^{n\lambda - 2n - 2\epsilon - 1}\, dt \\
&= \frac{C_\lambda}{|x|^{2n+2\epsilon}},
\end{aligned}$$

where the last equality holds if we choose ϵ small enough that $0 < 2\epsilon < n(\lambda - 2)$.

To estimate A_4 we again split the second integral in two:

$$\begin{aligned}
A_4 = &\int_0^{|x|} \int_{\substack{y \notin B_t(x) \\ |y| > |x|/4}} \int_0^1 m(x, y, t, \theta, h) \left(\frac{t}{t+|y|} \right)^{n\lambda} \frac{d\theta dydt}{t^{3n+1+2\epsilon}} \\
&+ \int_0^{|x|} \int_{\substack{y \notin B_t(x) \\ |y| \leq |x|/4}} \int_0^1 m(x, y, t, \theta, h) \left(\frac{t}{t+|y|} \right)^{n\lambda} \frac{d\theta dydt}{t^{3n+1+2\epsilon}} \\
= &A_5 + A_6.
\end{aligned}$$

To estimate A_6, first note that for each $\theta \in [0, 1]$, $|x| > 2|h|$, and $|y| \leq |x|/4$,

$$|x + \theta h - y| \geq |x| - \theta|h| - |y| \geq |x| - |h| - |y| > \frac{|x|}{2} - |y| \geq \frac{|x|}{4}.$$

Hence, if we choose $\gamma > \epsilon$,

$$\begin{aligned}
A_6 &\leq \int_0^{|x|} \int_{\substack{y \notin B_t(x) \\ |y| \leq |x|/4}} \int_0^1 \left(\frac{t}{|x + \theta h - y|} \right)^{2(n+\gamma)} \left(\frac{t}{t+|y|} \right)^{n\lambda} \frac{d\theta dydt}{t^{3n+1+2\epsilon}} \\
&\leq C \int_0^{|x|} \int_{\mathbb{R}^n} \left(\frac{t}{|x|} \right)^{2(n+\gamma)} \left(\frac{t}{t+|y|} \right)^{n\lambda} \frac{dydt}{t^{3n+1+2\epsilon}} \\
&= C \int_0^{|x|} \int_{\mathbb{R}^n} \left(\frac{t}{|x|} \right)^{2(n+\gamma)} \left(\frac{1}{1+|y|} \right)^{n\lambda} \frac{dydt}{t^{2n+1+2\epsilon}} \\
&= \frac{C_\lambda}{|x|^{2(n+\gamma)}} \int_0^{|x|} t^{2\gamma - 2\epsilon - 1}\, dt \\
&= \frac{C_\gamma}{|x|^{2n+2\epsilon}}.
\end{aligned}$$

To estimate A_5 we use Fubini's theorem and again split the inner integral in two:

$$\begin{aligned}
A_5 &\leq \int_0^1 \int_0^{|x|} \int_{|y|>|x|/4} m(x,y,t,\theta,h) \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydtd\theta}{t^{3n+1+2\epsilon}} \\
&= \int_0^1 \int_0^{|x|} \int_{\substack{|y|>|x|/4 \\ |x+\theta h-y|<|x|/4}} m(x,y,t,\theta,h) \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydtd\theta}{t^{3n+1+2\epsilon}} \\
&\quad + \int_0^1 \int_0^{|x|} \int_{\substack{|y|>|x|/4 \\ |x+\theta h-y|>|x|/4}} m(x,y,t,\theta,h) \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydtd\theta}{t^{3n+1+2\epsilon}} \\
&= A_7 + A_8.
\end{aligned}$$

To estimate A_7, we fix δ such that $0 < \delta < n(\lambda-2) - 2\epsilon$. Then

$$\begin{aligned}
A_7 &\leq C \int_0^1 \int_0^{|x|} \int_{\substack{|y|>|x|/4 \\ |x+\theta h-y|<|x|/4}} \left(\frac{t}{|x+\theta h-y|}\right)^{n-\delta} \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydtd\theta}{t^{3n+1+2\epsilon}} \\
&\leq C \int_0^1 \int_0^{|x|} \int_{\substack{|y|>|x|/4 \\ |x+\theta h-y|<|x|/4}} \left(\frac{1}{|x+\theta h-y|}\right)^{n-\delta} \left(\frac{1}{|y|}\right)^{n\lambda} \frac{dydtd\theta}{t^{2n-\lambda n+1+2\epsilon+\delta}} \\
&\leq \frac{C_\lambda}{|x|^{n\lambda}} \int_0^1 \int_0^{|x|} \int_{|x+\theta h-y|<|x|/4} \left(\frac{1}{|x+\theta h-y|}\right)^{n-\delta} \frac{dydtd\theta}{t^{2n-\lambda n+1+2\epsilon+\delta}}.
\end{aligned}$$

If we make the change of variables $x+\theta h-y \mapsto y$, we get

$$\begin{aligned}
&= \frac{C_\lambda}{|x|^{n\lambda}} \int_0^{|x|} \int_{|y|<|x|/4} \left(\frac{1}{|y|}\right)^{n-\delta} \frac{dydt}{t^{2n-\lambda n+1+2\epsilon+\delta}} \\
&= \frac{C_\lambda}{|x|^{n\lambda-\delta}} \int_0^{|x|} t^{n\lambda-2n-1-2\epsilon-\delta}\, dt \\
&= \frac{C_\lambda}{|x|^{2n+2\epsilon}}.
\end{aligned}$$

Finally, we estimate A_8: again by our choice of δ,

$$\begin{aligned}
A_8 &\leq C \int_0^1 \int_0^{|x|} \int_{\substack{|y|>|x|/4 \\ |x+\theta h-y|>|x|/4}} \left(\frac{t}{|x+\theta h-y|}\right)^{n-\delta} \left(\frac{t}{t+|y|}\right)^{n\lambda} \frac{dydtd\theta}{t^{3n+1+2\epsilon}} \\
&\leq \frac{C}{|x|^{n-\delta}} \int_0^{|x|} \int_{|y|>|x|/4} \left(\frac{1}{|y|}\right)^{n\lambda} \frac{dydt}{t^{2n-\lambda n+1+2\epsilon+\delta}} \\
&\leq \frac{C_\lambda}{|x|^{n\lambda-\delta}} \int_0^{|x|} t^{\lambda n-2n-1-2\epsilon-\delta}\, dt \\
&= \frac{C_\lambda}{|x|^{2n+2\epsilon}}.
\end{aligned}$$

□

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