

## A New Proof of the Rectifiable Slices Theorem

ROBERT L. JERRARD

**Abstract.** This paper gives a new proof of the fact that a  $k$ -dimensional normal current  $T$  in  $\mathbb{R}^m$  is integer multiplicity rectifiable if and only if for every projection  $P$  onto a  $k$ -dimensional subspace, almost every slice of  $T$  by  $P$  is 0-dimensional integer multiplicity rectifiable, in other words, a sum of Dirac masses with integer weights. This is a special case of the Rectifiable Slices Theorem, which was first proved a few years ago by B. White.

**Mathematics Subject Classification (2000):** 49Q15.

### 1. – Introduction

The main result of this paper is

**THEOREM 1** (Rectifiable Slices Theorem, [13], [2], [9]). *Suppose  $T$  is a  $k$ -dimensional normal current in  $\mathbb{R}^m$ . Then the following are equivalent:*

- (i)  *$T$  is integer multiplicity rectifiable*
- (ii) *For every projection  $P$  onto a  $k$ -dimensional subspace of  $\mathbb{R}^m$ , the slices  $\langle T, P, y \rangle$  are 0-dimensional integer multiplicity rectifiable for a.e.  $y$ .*

The definition and basic properties of slices  $\langle T, f, y \rangle$  of a current  $T$  are recalled at the end of this introduction.

Theorem 1 is a special case of a result first proved a few years ago by B. White [13]. A bit later, unaware of White's earlier work (which at that point was not yet published), I developed essentially the proof presented here in the course of joint work with H. M. Soner on spaces of functions of bounded higher variation, that is, functions whose distributional Jacobians are measures. A sketch of this proof in a simple special case appears in a paper of myself and Soner [9], where however we do not state Theorem 1 in the form given above. Our original interest was in showing that the distributional Jacobian of a function  $\mathbb{R}^m \rightarrow S^{n-1}$  is integer multiplicity rectifiable if it has locally finite mass; this follows immediately from Theorem 1 once one verifies that

the Jacobian satisfies (ii) of Theorem 1. Our paper [10] contains a complete discussion on functions of bounded higher variation.

Subsequently Ambrosio and Kirchheim [2] showed that the Rectifiable Slices Theorem holds in very general metric spaces, and they used it to deduce versions of the Closure Theorem and Boundary Rectifiability Theorem in this general setting. Their proof relies crucially on one observation from [9], see (1.1) below, but otherwise is completely different.

In this paper I give the details of the proof sketched in [9].

The basic point is the following. Suppose that  $T$  is a  $k$ -dimensional normal current in  $\mathbb{R}^m$  and  $P : \mathbb{R}^m \rightarrow \mathbb{R}^k$  is a projection, and let  $\{\langle T, P, y \rangle\}_{y \in \mathbb{R}^k}$  be the slices of  $T$  by  $P$ . Then

$$(1.1) \quad \text{the map } \mathbb{R}^k \ni y \mapsto \langle T, P, y \rangle \text{ is a function of bounded variation}$$

when the target space of 0-dimensional currents is given the appropriate (weak) metric, which is essentially the flat norm. A local version of this statement is established in Section 2.2. The proof then uses a blowup argument in which key control is provided by this BV-type estimate.

The implication (i)  $\implies$  (ii) in Theorem 1 is classical, as I recall in Theorem 3 below, and so the point is to prove the other implication. I do this by employing a blowup argument, What I in fact prove is

**THEOREM 2.** *Suppose  $T$  is a  $k$  dimensional normal current in  $\mathbb{R}^m$  and that for every projection  $P$  onto a  $k$ -dimensional subspace of  $\mathbb{R}^m$ , the slices  $\langle T, P, y \rangle$  are 0-dimensional integer multiplicity rectifiable for a.e.  $y$ .*

*Then at  $|T|$  almost every  $x_0 \in \mathbb{R}^m$ , there exists some  $k$ -dimensional subspace  $P_{x_0}$  of  $\mathbb{R}^m$ , a  $k$ -vector  $\xi_{x_0}$  that orients  $P_{x_0}$ , and an integer  $\theta_{x_0}$  such that*

$$\eta_{x_0, \lambda \#} T \rightarrow \tau(P_{x_0}, \theta_{x_0}, \xi_{x_0})$$

as  $\lambda \rightarrow 0$ , where for  $x_0 \in \mathbb{R}^m$ ,  $\lambda > 0$  we define  $\eta_{x_0, \lambda}(x) := (x - x_0)/\lambda$ .

Here  $\tau(P_{x_0}, \theta_{x_0}, \xi_{x_0})$  denotes the current defined by

$$\tau(P_{x_0}, \theta_{x_0}, \xi_{x_0})(\phi) = \int_{P_{x_0}} \langle \phi, \xi_{x_0} \rangle \theta_{x_0} d\mathcal{H}^k.$$

It is well-known that the conclusion of Theorem 2 implies (i) of Theorem 1 (see for example [6], Section 2.1.4.) Thus to prove Theorem 1 it suffices to prove Theorem 2. This is done in Section 2.

As remarked above, the proof relies on the notion of functions of bounded variation with values in certain (dual) Banach spaces. In particular I use a Poincaré-type inequality which was first proved by Ambrosio, who developed a general theory of metric space valued functions of bounded variation in [1]. In Section 3, I present for the reader's convenience a quick proof of this Poincaré inequality in the form (2.8) in which I need it for the proof of Theorem 2.

**1.1. – Background on slicing**

In the following theorem I collect all the properties of slices  $\langle T, f, y \rangle$  that are needed in this paper. In particular (1.2) is in effect the definition of slicing. These results and many other facts about slicing are established in Federer [5] 4.3.1-6.

**THEOREM 3.** *Suppose that  $T$  is a normal  $k$ -current in  $\mathbb{R}^m$ , and that  $f : \mathbb{R}^m \rightarrow \mathbb{R}^k$  is Lipschitz. Then for Lebesgue almost every  $y \in \mathbb{R}^k$  there exists a normal 0-current  $\langle T, f, y \rangle$  supported in  $f^{-1}(y)$ , such that*

$$(1.2) \quad (T \llcorner f^\#(dy))(\omega) = \int_{\mathbb{R}^k} \langle T, f, y \rangle(\omega) \mathcal{L}^k(dy)$$

for every smooth 0-form  $\omega$  with compact support. Also,  $y \mapsto |\langle T, f, y \rangle|(\mathbb{R}^m)$  is integrable, and

$$(1.3) \quad |T \llcorner f^\#(dy)|(v) = \int_{\mathbb{R}^k} |\langle T, f, y \rangle|(v) \mathcal{L}^k(dy)$$

for every bounded real-valued Borel function  $v$ . Moreover, if  $T$  is real (resp., integer) rectifiable, then  $\langle T, f, y \rangle$  is real (resp., integer) rectifiable for a.e.  $y$ . Finally, if  $\eta : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is Lipschitz and one-to-one, then

$$(1.4) \quad \eta_\# \langle T, f, y \rangle = \langle \eta_\# T, f \circ \eta^{-1}, y \rangle$$

for  $\mathcal{L}^k$  almost every  $y \in \mathbb{R}^k$ , and if  $\zeta : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a diffeomorphism then

$$(1.5) \quad \langle T, \zeta^{-1} \circ f, y \rangle = \langle T, f, \zeta(y) \rangle$$

**1.2. – Notation**

Given a set  $S \subset \mathbb{R}^m$ , we write  $\chi_S$  to denote the characteristic function of  $S$ , so that  $\chi_S(x) = 1$  if  $x \in S$  and 0 otherwise.

We write  $B_r^n(a)$  to denote the closed  $n$ -dimensional ball of radius  $r$  around a point  $a \in \mathbb{R}^n$ .  $B_r^n$  denotes a ball centered at the origin.

Given a  $k$ -form  $\phi$  on  $\mathbb{R}^M$  and a smooth function  $g : \mathbb{R}^m \rightarrow \mathbb{R}^M$  we let  $g^\# \phi$  denote the pullback of  $\phi$  by  $g$ , which is a  $k$ -form on  $\mathbb{R}^m$ .

Given a  $k$ -current  $T$  on  $\mathbb{R}^m$  and a smooth function  $g$  as above, the push-forward  $g_\# T$  is the  $k$ -current on  $\mathbb{R}^M$  defined by  $g_\# T(\phi) = T(g^\# \phi)$ .

Given a  $k$ -current  $T$  on  $\mathbb{R}^m$  and a smooth  $l$ -form  $\omega$  with  $l < k$ , we define the  $(k - l)$ -current  $T \llcorner \omega$  by  $T \llcorner \omega(\phi) = T(\omega \wedge \phi)$ .

$I(k, m)$  denotes the set of multi-indices  $\alpha$  of the form  $\alpha = (\alpha_1, \dots, \alpha_k)$  where  $1 \leq \alpha_1 < \dots < \alpha_k \leq m$ .

We write  $\Lambda_k \mathbb{R}^m$  and  $\Lambda^k \mathbb{R}^m$  respectively to denote the spaces of  $k$ -vectors and  $k$ -covectors on  $\mathbb{R}^m$ .

We write  $\{e_1, \dots, e_m\}$  to denote an orthonormal basis for  $\mathbb{R}^m$ . Then  $\Lambda_k \mathbb{R}^m$  is the space of  $k$ -vectors on  $\mathbb{R}^m$ , that is, the real vector space spanned by

$\{e_\alpha\}_{\alpha \in I(k,m)}$ , where  $e_\alpha = e_{\alpha_1} \wedge \dots \wedge e_{\alpha_k}$ . We endow  $\Lambda_k \mathbb{R}^m$  with the inner product determined by stipulating that the set  $\{e_\alpha\}_{\alpha \in I(k,m)}$  be orthonormal. We write  $|v| := (v \cdot v)^{1/2}$ . We do not use the comass norm of geometric measure theory.

We similarly write  $\Lambda^k \mathbb{R}^m$  to denote the space  $k$ -covectors on  $\mathbb{R}^m$ , that is, the inner product space generated by the orthonormal basis  $\{dx^\alpha\}_{\alpha \in I(k,m)}$ . We identify  $\Lambda^k \mathbb{R}^m$  with the dual of  $\Lambda_k \mathbb{R}^m$ , and we use brackets  $\langle \cdot, \cdot \rangle$  to indicate the pairing. We always assume that the bases for these two spaces are dual in the sense that  $\langle e_\alpha, dx^\beta \rangle = 1$  if  $\alpha = \beta$  and 0 otherwise.

A generic  $k$ -form can then be written  $\phi = \sum_{\alpha \in I_{k,m}} \phi^\alpha dx^\alpha$ .

In general I have mostly followed notational conventions from Giaquinta, Modica and Souček [6], see particularly Chapter 2.2. For the most part these are quite standard in geometric measure theory.

Further notation is introduced at the beginning of Sections 2.1 and 2.2.

ACKNOWLEDGMENTS. This paper is an outgrowth of joint work with H.M. Soner and reflects many useful discussions with him, as well as with Giovanni Alberti and Luigi Ambrosio. I am also grateful to an anonymous referee for helpful comments.

This research was partially supported by NSF grant DMS 99-70273.

## 2. – Proof of Theorem 2

In this section I give the proof of Theorem 2. We assume throughout the section that  $T$  is a normal  $k$ -current in  $\mathbb{R}^m$  satisfying the hypotheses of the theorem, so that almost every slice of  $T$  by a projection onto a  $k$ -dimensional subspace is 0-dimensional rectifiable.

### 2.1. – Finding good points

Because  $T$  is a normal  $k$ -current the Riesz Representation Theorem implies that there exists a nonnegative measure  $|T|$  and a  $|T|$ -measurable map  $\vec{T} : \text{supp } T \rightarrow \Lambda_k \mathbb{R}^m$  such that  $|\vec{T}| = 1$  almost everywhere, and for every smooth  $k$ -form  $\omega$  with finite mass,

$$T(\omega) = \int \langle \vec{T}, \omega \rangle d |T|.$$

For a discussion see [6], Section 2.2.3 Theorem 1.

First we identify a set of points of full  $|T|$  measure on which certain desirable properties hold.

We write  $P$  to denote projection onto a generic  $k$ -dimensional subspace of  $\mathbb{R}^m$ , and  $P^\perp$  for the projection onto the orthogonal complement. We define the cylinder

$$C_\lambda^P(x) := \{y \in \mathbb{R}^m : |P(y - x)| \leq \lambda, |P^\perp(y - x)| \leq \lambda\}.$$

When we write  $C_\lambda$  without a superscript, it denotes the cylinder  $B_\lambda^k \times B_\lambda^{m-k}$  corresponding to projection onto  $\mathbb{R}^k \times \{0\}$ .

Define

$$(2.1) \quad G := \{x \in \mathbb{R}^m \mid \text{conditions (2.2)-(2.5) hold at } x\}$$

where the above-mentioned conditions are:

$$(2.2) \quad \lim_{\lambda \rightarrow 0} \frac{1}{|T|(C_\lambda^P(x))} \int_{C_\lambda^P(x)} |\vec{T}(x) - \vec{T}(y)| \, d|T|(y) = 0$$

for every projection  $P$  onto a  $k$ -dimensional subspace;

$$(2.3) \quad |\vec{T}(x)| = 1;$$

$$(2.4) \quad \limsup_{r \rightarrow 0} \frac{|T|(B_r(x))}{r^k} < \infty$$

$$(2.5) \quad \limsup_{r \rightarrow 0} \frac{|\partial T|(B_r(x))}{|T|(B_r(x))} < \infty.$$

We first prove

LEMMA 1.  $|T|(\mathbb{R}^m \setminus G) = 0$ .

PROOF. First note that (2.3) holds almost everywhere by the definition of  $\vec{T}$ .

Recall that  $\vec{T}$  is the Radon-Nikodym derivative  $\frac{dT}{d|T|}$  (see [6], Section 2.2.3 Theorem 1), and so (2.2) holds  $|T|$  almost everywhere as a consequence of general results about differentiation of Radon measures, see for example Bliedtner and Loeb [3]. These results are well-known if the cylinders  $C_\lambda^P$  are replaced by balls  $B_\lambda$ . The result as stated is proved by following the standard proof but using the Morse covering lemma in place of the Besicovitch covering lemma.

From basic results about differentiation of measures (see for example [6], Section 1.1.5 Theorem 3) we know that  $\lim_{r \rightarrow 0} \frac{|\partial T|(B_r(x))}{|T|(B_r(x))}$  exists and is finite for  $|T|$  almost every  $x$ , and thus that (2.5) holds a.e..

Finally, it is quite well-known that (2.4) holds  $|T|$  almost everywhere whenever  $T$  is a normal current. We give a short proof using Theorem 3. Let  $S_0$  denote the set of points at which (2.4) fails. This set must have Hausdorff  $k$ -dimensional measure zero, as a result of standard covering arguments; see for example [6] Section 1.1.5 Theorem 6. However, it follows from (1.3) that  $|T|$  cannot charge sets of  $\mathcal{H}^k$  measure zero, and so in particular  $|T|(S_0) = 0$ .  $\square$

**2.2. – Slicing and BV estimate**

By Lemma 1, it suffices to show that the conclusion of Theorem 2 holds at any  $x_0$  at which (2.2)-(2.5) hold. We therefore fix some such  $x_0$ , which for simplicity we assume to be the origin. We will also write  $\eta_\lambda$  instead of  $\eta_{x_0,\lambda}$ .

We want to slice  $T$  in a direction that is optimal near 0. To this end we select  $\xi_0$  to be a simple  $k$ -vector such that  $|\xi_0| = 1$  and

$$(2.6) \quad \vec{T}(0) \cdot \xi_0 = \max\{\vec{T}(0) \cdot \zeta : \zeta \in \Lambda_k \mathbb{R}^m, \zeta \text{ simple}, |\zeta| \leq 1\}.$$

After a change of variables we can assume that  $\xi_0 = e_1 \wedge \dots \wedge e_k$ . We will write  $P_0$  for the projection  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_k)$ .

We will often identify  $x \in \mathbb{R}^m$  with  $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ , where  $y = P_0 x$  and  $z = P_0^\perp x$ . We will similarly identify  $dy := dy^1 \wedge \dots \wedge dy^k$  with  $dx^1 \wedge \dots \wedge dx^k$ . We will also write  $dy^{\bar{i}}$  for  $dy^1 \wedge \dots \wedge dy^{i-1} \wedge dy^{i+1} \wedge \dots \wedge dy^k$ , which we identify with  $dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^k$ . Note that  $dy^i \wedge dy^{\bar{i}} = (-1)^{i+1} dy$ .

We will study limits of  $\eta_{\lambda\#} T$  by analyzing appropriate slices of this measure. In particular, for every  $y \in \mathbb{R}_y^k$  we define a signed measure  $T^\lambda(y)$  on  $\mathbb{R}_z^{m-k}$  by

$$(2.7) \quad \int \psi(y, z) T^\lambda(y)(dz) = \langle \eta_{\lambda\#} T, P_0, y \rangle(\psi).$$

Since  $\langle \eta_{\lambda\#} T, P_0, y \rangle$  is supported in  $P_0^{-1}(y) = \{y\} \times \mathbb{R}_z^{m-k}$ , one sees that  $T^\lambda(y)$  is well-defined as a measure on  $\mathbb{R}_z^{m-k}$ .

For open sets  $W \subset \mathbb{R}^k$  and  $V \subset \mathbb{R}^{m-k}$  we define

$$\begin{aligned} & \text{Var}(T^\lambda; W, C_c^1(V)^*) \\ & := \sum_{i=1}^k \sup \left\{ \int \int \phi_{y_i}(y, z) T^\lambda(y)(dz) dy : \phi \in C_c^1(W \times V), \right. \\ & \qquad \qquad \qquad \left. \|\phi(y; \cdot)\|_{(C_c^1(V))^*} \leq 1 \quad \forall y \in W \right\}. \end{aligned}$$

This is the total variation of  $y \mapsto T^\lambda(y)$ , seen as a map from  $W$  into the dual of  $C_c^1(V)$ . The total variation controls the  $L^1$ -norm via the following Poincaré-type inequality:

$$(2.8) \quad \int_W \|T^\lambda(y) - (T^\lambda)_W\|_{C_c^1(V)^*} dy \leq C_W \text{Var}(T^\lambda; W, C_c^1(V)^*)$$

if  $W$  is connected. Here we use the notation

$$(T^\lambda)_W := \frac{1}{|W|} \int_W T^\lambda(y) dy.$$

This follows from work of Ambrosio [1]. For the convenience of the reader, we give a proof of (2.8) in Section 3, Lemma 7 for sets  $W$  that are diffeomorphic to the unit ball; this is the only case we use here.

The main point in our analysis is contained in the following

PROPOSITION 1. For any  $R > 0$ ,

$$\text{Var}(T^\lambda; B_R^k, C_c^1(B_R^{m-k})^*) = o\left(\frac{|T|(C_{R\lambda})}{\lambda^k}\right) \leq o(1) \quad \text{as } \lambda \rightarrow 0.$$

PROOF. The inequality on the right follows from (2.4), and so we only need to prove the estimate on the left.

1. First note that

$$(2.9) \quad \langle \vec{T}(0), dy^{\bar{i}} \wedge dz^{\bar{j}} \rangle = 0 \quad \text{for all } i \in \{1, \dots, k\} \text{ and all } j \in \{1, \dots, m-k\}.$$

To prove this, let  $g(t) := \langle \vec{T}(0), (-1)^{i+1}(\cos t \, dy^i + \sin t \, dz^j) \wedge dy^{\bar{i}} \rangle$ . The choice (2.6) of  $\xi_0$  implies that  $g$  attains its maximum at  $t = 0$ . Thus  $g'(0) = 0$ , which is exactly (2.9).

2. Next, consider any  $\psi \in C_c^1(C_R)$  such that  $\sup_{y \in B_R^k} \|\psi(y; \cdot)\|_{(C_c^1(B_R^{m-k}))} \leq 1$  and note that

$$d(\psi dy^{\bar{i}}) = \psi_{y_i} dy^i \wedge dy^{\bar{i}} + \psi_{z_j} dz^j \wedge dy^{\bar{i}} = (-1)^{i+1} \psi_{y_i} dy + \psi_{z_j} dz^j \wedge dy^{\bar{i}}.$$

Thus for any normal  $k$ -current  $S$ ,

$$\int_{\mathbb{R}^k_y} \langle S, P_0, y \rangle (\psi_{y_i}) dy = S(\psi_{y_i} dy) = (-1)^i (S(\psi_{z_j} dz^j \wedge dy^{\bar{i}}) - \partial S(\psi dy^{\bar{i}})).$$

Applying this identity to  $\eta_{\lambda\#}T$  and using the definition (2.7) of  $T^\lambda$  we obtain

$$(2.10) \quad \int_{\mathbb{R}^k_y} \int_{R_z^{m-k}} \psi_{y_i}(y, z) T^\lambda(y)(dz) dy = \pm \eta_{\lambda\#}T(\psi_{z_j} dz^j \wedge dy^{\bar{i}}) \mp \partial(\eta_{\lambda\#}T)(\psi dy^{\bar{i}}).$$

3. We estimate the first term on the right-hand side of (2.10). Using (2.9) and the fact that  $\eta_\lambda^\#(dz^j \wedge dy^{\bar{i}}) = \lambda^{-k} dz^j \wedge dy^{\bar{i}}$  we obtain

$$\begin{aligned} \eta_{\lambda\#}T(\psi_{z_j} dz^j \wedge dy^{\bar{i}}) &= \frac{1}{\lambda^k} \int \psi_{z_j} \left(\frac{x}{\lambda}\right) \langle dz^j \wedge dy^{\bar{i}}, \vec{T}(x) \rangle d|T|(x) \\ &= \frac{1}{\lambda^k} \int \psi_{z_j} \left(\frac{x}{\lambda}\right) \langle dz^j \wedge dy^{\bar{i}}, \vec{T}(x) - \vec{T}(0) \rangle d|T|(x). \end{aligned}$$

Our choice of  $\psi$  implies that the integrand is supported in the cylinder  $C_{\lambda R}$  and is dominated by  $|\vec{T}(x) - \vec{T}(0)|$ . Thus since (2.2) holds at  $x_0 = 0$ ,

$$(2.11) \quad |\eta_{\lambda\#}T(\psi_{z_j} dz^j \wedge dy^{\bar{i}})| \leq o\left(\frac{|T|(C_{\lambda R})}{\lambda^k}\right)$$

as  $\lambda \rightarrow 0$ .

4. To estimate the second term on the right-hand side of (2.10), we write  $\partial T(\phi) = \int \langle \phi, \partial \vec{T} \rangle d|\partial T|$  and compute

$$\begin{aligned} \partial(\eta_{\lambda\#}T)(\psi dy^{\bar{i}}) &= \eta_{\lambda\#}(\partial T)(\psi dy^{\bar{i}}) \\ &= \lambda^{-k+1} \int \psi \left( \frac{x}{\lambda} \right) \langle dy^{\bar{i}}, \partial \vec{T}(x) \rangle d|\partial T|(x). \end{aligned}$$

The factor of  $\lambda^{k-1}$  appears because  $\eta_{\lambda\#}dy^{\bar{i}} = \lambda^{k-1}dy^{\bar{i}}$ . Again the integrand is supported in  $C_{\lambda R}$ , and so it follows immediately from (2.5) that

$$(2.12) \quad |\partial(\eta_{\lambda\#}T)(\psi dy^{\bar{i}})| \leq C\lambda \frac{|T|(C_{\lambda R})}{\lambda^k} \quad \text{as } \lambda \rightarrow 0.$$

The conclusion now follows from (2.10), (2.11), and (2.12). □

### 2.3. – Finding good slices

In this subsection we prove:

LEMMA 2. *For every  $R > 0$ , there exist sets  $\mathcal{G}_{\lambda,R} \subset B_R^k$  and a continuous nondecreasing function  $h : (0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{\lambda \rightarrow 0} h(\lambda) = 0$  and*

$$(2.13) \quad \|T^\lambda(y) - (T^\lambda)_{B_R^k}\|_{C_c^1(B_R^{m-k})^*} \leq Ch(\lambda) \quad \text{for } y \in \mathcal{G}_{\lambda,R}$$

$$(2.14) \quad \|T^\lambda(y) - (T^\lambda)_{B_R^k}\|_{C_c^1(B_R^{m-k})^*} \leq h(\lambda) \frac{|T|(C_{R\lambda})}{\lambda^k} \quad \text{for } y \in \mathcal{G}_{\lambda,R}$$

$$(2.15) \quad T^\lambda(y) \text{ is a nonnegative measure in } B_R^{m-k} \quad \text{for } y \in \mathcal{G}_{\lambda,R}$$

and

$$(2.16) \quad \mathcal{L}^k(B_R^k \setminus \mathcal{G}_{\lambda,R}) \rightarrow 0$$

as  $\lambda \rightarrow 0$ .

PROOF. 1. Let  $h$  be an increasing continuous function such that  $h(0) = 0$  and

$$(2.17) \quad \text{Var}(T^\lambda; B_R^k, C_c^1(B_R^{m-k})^*) \leq \frac{|T|(C_{\lambda R})}{\lambda^k} h^2(\lambda).$$

Such a function exists as a consequence of Proposition 1. The Poincaré inequality (2.8) immediately implies that

$$\int_{B_R^k} \|T^\lambda(y) - (T^\lambda)_{B_R^k}\|_{C_c^1(B_R^{m-k})^*} dy \leq C \frac{|T|(C_{\lambda R})}{\lambda^k} h^2(\lambda),$$



and so an easy estimate shows that

$$\frac{|T|(C_{\lambda R})}{\lambda^k} h(\lambda) \mathcal{L}^k(\{y \in B_R^k : (2.14) \text{ does not hold}\}) \leq C \frac{|T|(C_{\lambda R})}{\lambda^k} h(\lambda)^2.$$

Thus (2.14) holds away from a set of measure  $Ch(\lambda)$ . We deduce (2.13) as a result of (2.14) and (2.4).

2. We define  $\mathcal{G}_{\lambda,R} \subset B_R^k$  to the set of points in  $B_R^k$  where (2.13)-(2.15) hold. In view of Step 1, if we define

$$\mathcal{S}_\lambda := \{y \in B_R^k : (2.15) \text{ does not hold}\},$$

then we need only to show that  $\mathcal{L}^k(\mathcal{S}_\lambda)$  vanishes as  $\lambda \rightarrow \infty$ .

Our assumption (ii) of Theorem 1 implies that  $T^\lambda$  is a sum of point masses with integer multiplicities for a.e.  $y$ . As a result,  $(|T^\lambda(y)| - T^\lambda(y))(B_R^{m-k}) \geq 2$  for a.e.  $y \in \mathcal{S}^\lambda$ . So

$$(2.18) \quad 2\mathcal{L}^k(\mathcal{S}_\lambda) \leq \int_{B_R^k} (|T^\lambda(y)| - T^\lambda(y))(B_R^{m-k}).$$

The definitions imply that

$$\begin{aligned} \int_{B_R^k} T^\lambda(y)(B_R^{m-k}) dy &= \int \langle \eta_{\lambda\#} T, P_0, y \rangle (\chi_{C_R}) dy \\ &= (\eta_{\lambda\#} T)_\# P_0^\#(dy) (\chi_{C_R}) = \frac{1}{\lambda^k} \int_{C_{\lambda R}} \langle dy, \vec{T}(x) \rangle d|T|, \end{aligned}$$

and similarly  $\int |T^\lambda(y)|(B_R^{m-k}) dy = \frac{1}{\lambda^k} \int_{C_{\lambda R}} |\langle dy, \vec{T}(x) \rangle| d|T|$ . Thus

$$(2.19) \quad \int_{B_R^k} (|T^\lambda(y)| - T^\lambda(y))(B_R^{m-k}) = \frac{1}{\lambda^k} \int_{C_{\lambda R}} (|\langle dy, \vec{T}(x) \rangle| - \langle dy, \vec{T}(x) \rangle) d|T|.$$

Our choice (2.6) of  $\xi_0$  implies that  $\langle dy, \vec{T}(0) \rangle > 0$ , and from this we infer that

$$|\langle dy, \vec{T}(x) \rangle| - \langle dy, \vec{T}(x) \rangle \leq 2|\langle dy, \vec{T}(0) - \vec{T}(x) \rangle| \leq 2|\vec{T}(0) - \vec{T}(x)|,$$

Combining this with (2.18) and (2.19) we find that

$$\mathcal{L}^k(\mathcal{S}_\lambda) \leq \frac{1}{\lambda^k} \int_{C_{\lambda R}} |\vec{T}(0) - \vec{T}(x)| dx,$$

and since (2.2) and (2.4) hold at  $x_0 = 0$ , this implies that  $\mathcal{L}^k(\mathcal{S}_\lambda) \rightarrow 0$ . □

**2.4. – Blowup**

We now carry out the main part of the blowup argument.

For  $z \in \mathbb{R}^{m-k}$  we write  $\eta_\lambda(z) = z/\lambda$ . Thus  $\eta_\lambda$  denotes both dilation  $\mathbb{R}^m \rightarrow \mathbb{R}^m$  and  $\mathbb{R}^{m-k} \rightarrow \mathbb{R}^{m-k}$ . I believe this will not cause any confusion.

PROPOSITION 2. *There exists a positive integer  $\theta$  such that*

$$(2.20) \quad (\eta_{\lambda\#}T)_\perp dy \rightharpoonup \theta \mathcal{H}^k \llcorner (\mathbb{R}^k \times \{0\}) \quad \text{weak-* in } (C_c^0)^*.$$

Also, for any  $R > 0$ ,

$$(2.21) \quad \lim_{\lambda \rightarrow 0} \int_{B_R^k} \|T^\lambda(y) - \theta \delta_0\|_{C_c^1(B_R^{m-k})^*} dy = 0.$$

PROOF. 1. Since (2.4) holds at  $x_0 = 0$ , the sequence  $\eta_{\lambda\#}T$  has uniformly bounded mass on compact sets, and so it suffices to verify that (2.20) holds when we integrate against smooth test functions with compact support. Any such function has compact support in the cylinder  $C_R$  for large  $R$ , so (2.20) can be deduced from (2.21) and the definition (2.7) of  $T^\lambda$ . Thus we only need to establish (2.21).

Fix  $R > 0$ . For each  $\lambda$  small enough that  $\mathcal{G}_{\lambda,R}$  is nonempty, we select some  $y_*^\lambda \in \mathcal{G}_{\lambda,R}$  and we let  $T_*^\lambda := T^\lambda(y_*^\lambda)$ . Note that

$$\int_{B_R^k} \|T^\lambda(y) - T_*^\lambda\|_{C_c^1(B_R^{m-k})^*} dy \longrightarrow 0$$

as a consequence of the Poincaré inequality (2.8) and properties of  $\mathcal{G}_{\lambda,R}$ , in particular (2.13). Because of this, it suffices to find a positive integer  $\theta$  such that

$$(2.22) \quad \|T_*^\lambda - \theta \delta_0\|_{C_c^1(B_R^{m-k})^*} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

2. We first claim that there exists some  $\lambda_0 > 0$  such that

$$(2.23) \quad \|\eta_{\sigma\#}T_*^\lambda - T_*^{\sigma\lambda}\|_{C_c^1(B_R^{m-k})^*} \leq 6h(\lambda)$$

whenever  $\lambda \leq \lambda_0$  and  $\frac{1}{2} \leq \sigma \leq 1$ . This asserts that  $T_*^{\sigma\lambda}$  is roughly a dilation of  $T_*^\lambda$ .

To see this note first that, if  $h(\lambda)$  is sufficiently small, then there exists  $\bar{y} \in B_{\sigma R}^k$  such that

$$\bar{y} \in \mathcal{G}_{\lambda,R}, \quad \text{and} \quad \frac{1}{\sigma} \bar{y} \in \mathcal{G}_{\sigma\lambda,R}.$$

In other words,  $\bar{y}$  belongs to the complement in  $B_{\sigma R}$  of  $(B_R^k \setminus \mathcal{G}_{\lambda,R}) \cup (B_{\sigma R} \setminus \sigma \mathcal{G}_{\sigma\lambda,R})$ . It follows from (2.16) that this set is nonempty when  $\lambda$  is sufficiently small, say  $\lambda \leq \lambda_0$ .

Now suppose  $\lambda < \lambda_0$ , so that we can find  $\bar{y}$  as above. Note that, because  $y_*^{\sigma\lambda}, \bar{y}/\sigma \in \mathcal{G}_{\sigma\lambda,R}$ , the triangle inequality and (2.13) imply that

$$\|T_*^{\sigma\lambda} - T^{\sigma\lambda}(\bar{y}/\sigma)\|_{C_c^1(B_R^{m-k})^*} \leq 2h(\sigma\lambda) \leq 2h(\lambda).$$

Similarly

$$\|\eta_{\sigma\#}T_*^\lambda - \eta_{\sigma\#}T^\lambda(\bar{y})\|_{C_c^1(B_R^{m-k})^*} \leq \frac{2}{\sigma}h(\lambda) \leq 4h(\lambda).$$

Thus (2.23) follows once we verify that  $\|\eta_{\sigma\#}T^\lambda(\bar{y}) - T^{\sigma\lambda}(\bar{y}/\sigma)\|_{C_c^1(B_R^{m-k})^*} = 0$ .

This holds because  $T^\lambda(\bar{y})$  and  $T^{\sigma\lambda}(\bar{y}/\sigma)$  correspond to two different dilations of the same unscaled measure  $\langle T, P_0, \lambda\bar{y} \rangle$ , and thus coincide after an appropriate dilation. We verify this by undoing the notation. First using (1.4) we get

$$\eta_{\sigma\#}T^\lambda(\bar{y}) = \eta_{\sigma\#}\langle \eta_{\lambda\#}T, P_0, \bar{y} \rangle = \langle \eta_{\sigma\#}\eta_{\lambda\#}T, P_0 \circ \eta_\sigma^{-1}, \bar{y} \rangle.$$

Now let  $\zeta_\sigma$  be the dilation  $\mathbb{R}^k \rightarrow \mathbb{R}^k$  defined by  $\zeta_\sigma(y) = y/\sigma$ . Then  $P_0 \circ \eta_\sigma^{-1} = \zeta_\sigma^{-1} \circ P_0$ , so (1.5) implies that

$$\langle \eta_{\sigma\#}\eta_{\lambda\#}T, P_0 \circ \eta_\sigma^{-1}, \bar{y} \rangle = \langle \eta_{\sigma\lambda\#}T, P_0, \zeta_\sigma(\bar{y}) \rangle = T^{\sigma\lambda}(\bar{y}/\sigma).$$

3. Now for  $j \geq 1$  let  $\lambda_j = 2^{-j}\lambda_0$ . Note that (2.23) implies that

$$(2.24) \quad \left| \int \phi(z)T_*^{\lambda_{j+1}}(dz) - \int \phi(2z)T_*^{\lambda_j}(dz) \right| \leq 6h(\lambda_j)\|\phi\|_{C^1}$$

for every  $\phi \in C_c^1(B_R^{m-k})$ .

We now show that for any  $r \in (0, R/2)$  there exists some  $K(r)$  such that

$$(2.25) \quad T_*^{\lambda_{j+1}}(B_r^{m-k}) \leq T_*^{\lambda_j}(B_r^{m-k})$$

whenever  $j \geq K(r)$ .

To see this, fix some such  $r$ , and let  $\phi$  be a smooth function such that

$$\phi \equiv 1 \text{ on } B_r^{m-k}, \quad 0 \leq \phi \leq 1, \quad \text{supp}\phi \subset B_{2r}^{m-k}.$$

Then, using (2.24) and the fact that  $T_*^\lambda$  is nonnegative in  $\text{supp}\phi \subset B_R^{m-k}$

$$T_*^{\lambda_{j+1}}(B_r^{m-k}) \leq \int \phi(z)T_*^{\lambda_{j+1}}(dz) \leq \int \phi(2z)T_*^{\lambda_j}(dz) + 6h(\lambda_j)\|\phi\|_{C^1}.$$

Since  $0 \leq \phi(2z) \leq \chi_{B_r^{m-k}}(z)$ , we deduce that

$$T_*^{\lambda_{j+1}}(B_r^{m-k}) \leq T_*^{\lambda_j}(B_r^{m-k}) + o(1)$$

as  $j \rightarrow \infty$ . Because  $T_*^\lambda(B_r^{m-k})$  is an integer for every  $\lambda$  and  $r$ , this establishes (2.25).

**4.** For any  $r$  as above, the sequence  $T_*^{\lambda_j}(B_r^{m-k})$  is nonnegative and eventually nonincreasing, and takes values in  $\mathbb{Z}$ , so there must exist some integer  $\theta(r)$  such that

$$T_*^{\lambda_j}(B_r^{m-k}) = \theta(r)$$

for all sufficiently large  $j$ .

If  $0 < s < r$ , then  $T_*^\lambda(B_s^{m-k}) \leq T_*^\lambda(B_r^{m-k})$ , which easily implies that  $r \mapsto \theta(r)$  is nondecreasing. So there is some  $r_0 > 0$  and a nonnegative integer  $\theta$  such that  $\theta(r) = \theta(s) := \theta$  whenever  $0 < r, s \leq r_0$ .

Moreover, we easily see from (2.24) that

$$\theta(r) = \lim_j T_*^{\lambda_{j+1}}(B_r^{m-k}) = \lim_j T_*^{\lambda_j}(B_{r/2}^{m-k}) = \theta(r/2)$$

whenever  $r < R$ . Thus  $\theta(r) = \theta$  for all  $r \leq R$ .

Any weak-\* limit  $\bar{T}_*$  of a subsequence  $T_*^{\lambda_{j_l}}$  must therefore satisfy

$$\bar{T}_*(B_r^{m-k}) = \theta \quad \forall r \in (0, R).$$

Since  $T_*^\lambda$  is nonnegative for every  $\lambda$ ,  $\bar{T}_*$  must have the same property. It follows that the only possible limit point is  $\bar{T}_* = \theta \delta_0$ .

**5.** We now verify that the full sequence  $T_*^{\lambda_j}$  converges. Let  $\mu_j$  be any sequence of numbers tending to zero, and for each  $j$  write  $\mu_j = \lambda_{n_j} \sigma_j$ , for some integer  $n_j$  and  $\sigma_j \in (\frac{1}{2}, 1]$ . After passing to a further subsequence, we may assume that  $\sigma_j \rightarrow \bar{\sigma}$  and that  $T_*^{\mu_j}$  converges weakly to a limit. Using (2.23) to get rid of the error term, we compute

$$\lim_j T_*^{\mu_j} = \lim_j \eta_{\sigma_j \#} T_*^{\lambda_{n_j}} - \lim_j (\eta_{\sigma_j \#} T_*^{\lambda_{n_j}} - T_*^{\sigma_j \lambda_{n_j}}) = \eta_{\bar{\sigma} \#}(\theta \delta_0) = \theta \delta_0.$$

**6.** It remains to show that  $\theta \neq 0$ . If not, then  $0 = T_*^{\lambda_j} = \frac{\lambda_j^k}{|T|(C_{\lambda_j R})} T_*^{\lambda_j}$  in  $B_R^{m-k}$  for all sufficiently large  $j$ . Using (2.14) and arguing as in Step 1, this implies that

$$\frac{\lambda_j^k}{|T|(C_{\lambda_j R})} (\eta_{\lambda_j \#} T) \lrcorner dy \rightharpoonup 0 \quad \text{weak-* in } (C_c^0)^*.$$

By integrating over  $C_R$ , rescaling via the definition of the pushforward  $\eta_\#$ , and passing to limits, this yields

$$\lim_{j \rightarrow \infty} \frac{1}{|T|(C_{\lambda_j R})} \int_{C_{\lambda_j R}} \langle \vec{T}(x), dy \rangle d|T|(x) = 0.$$

However, (2.3) implies that

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{1}{|T|(C_{\lambda R})} \int_{C_{\lambda R}} \langle \vec{T}(x), dy \rangle d|T|(x) &= \lim_{\lambda \rightarrow 0} \frac{1}{|T|(C_{\lambda R})} \int_{C_{\lambda R}} \langle \vec{T}(0), dy \rangle d|T|(x) \\ &= \langle \vec{T}(0), dy \rangle \\ &> 0 \quad \text{by our choice (2.6) of } \xi_0. \end{aligned}$$

This contradiction implies that  $\theta > 0$ , which is what we wanted to show.  $\square$

**2.5. – Mopping-up**

The following lemma completes the proof of Theorem 2.

LEMMA 3.  $\vec{T}(0)$  is simple and in fact equals  $\xi_0 = e_1 \wedge \dots \wedge e_k$ ; and

$$(2.26) \quad \eta_{x_0, \lambda\#} T \rightharpoonup \tau(\mathbb{R}^k \times \{0\}, \theta, \xi_0) \quad \text{weak-* in } (C_c^0)^*.$$

PROOF. 1. We first claim that for every  $\alpha \in I_{k,m}$ ,

$$(2.27) \quad (\eta_{\lambda\#} T) \llcorner dx^\alpha \rightharpoonup \frac{\langle \vec{T}(0), dx^\alpha \rangle}{\langle \vec{T}(0), dy \rangle} \theta \mathcal{H}^k \llcorner (\mathbb{R}^k \times \{0\}).$$

Recall that our choice (2.6) of  $\xi_0$  implies that  $\langle \vec{T}(0), dy \rangle \neq 0$ .

Fix any  $\alpha \in I_{k,m}$  and any smooth test function  $\phi$  with compact support, and observe that

$$\begin{aligned} \eta_{\lambda\#}(T \llcorner dx^\alpha)(\phi) &= \int \phi \left( \frac{x}{\lambda} \right) \left\langle \frac{dx^\alpha}{\lambda^k}, \vec{T}(x) \right\rangle d|T|(x) \\ &= \int \phi \left( \frac{x}{\lambda} \right) \left\langle \frac{dx^\alpha}{\lambda^k}, \vec{T}(0) \right\rangle d|T|(x) + o(1) \quad \text{by (2.2) and (2.4)} \\ &= \frac{\langle dy, \vec{T}(0) \rangle}{\langle dx^\alpha, \vec{T}(0) \rangle} \int \phi \left( \frac{x}{\lambda} \right) \left\langle \frac{dy}{\lambda^k}, \vec{T}(0) \right\rangle d|T|(x) + o(1) \end{aligned}$$

as  $\lambda \rightarrow 0$ . And by essentially the same argument,

$$\int \phi \left( \frac{x}{\lambda} \right) \left\langle \frac{dy}{\lambda^k}, \vec{T}(0) \right\rangle d|T|(x) = \eta_{\lambda\#}(T \llcorner dy)(\phi) + o(1)$$

as  $\lambda \rightarrow 0$ . Thus (2.27) follows from the previous estimates and Proposition 2.

2. To complete the proof we need to show that  $\vec{T}(0) = \xi_0$ , in other words, that  $\langle \vec{T}(0), dx^\alpha \rangle = 0$  for all  $\alpha$  not equal to the multivector  $(1, \dots, k)$ . This is quite standard. For the reader's convenience, we reproduce the proof from [6], Section 2.2.7, Theorem 4:

Let  $T_0(\phi) := \tau(\mathbb{R}^k \times \{0\}, \theta, \vec{T}(0))$ . The previous step implies that  $\eta_{\lambda\#}T \rightarrow T_0$  as  $\lambda \rightarrow 0$ , and so

$$\partial T_0(\phi) = T_0(d\phi) = \lim_{\lambda} \eta_{\lambda\#}T(d\phi) = \lim_{\lambda} \partial \eta_{\lambda\#}T(\phi) = 0.$$

The final equality is proved as in Step 4 of the proof of Proposition 1. Let  $\phi_0 \in C_c^\infty(\mathbb{R}^m)$  and let  $\phi := x_j \phi_0(x) dx^\beta$  for some  $j > k$  and  $\beta \in I_{k-1,m}$ . Then  $d\phi - \phi_0(x) dx^j \wedge dx^\beta$  vanishes on  $\{x \in \mathbb{R}^m : x_j = 0\} \supset \mathbb{R}^k \times \{0\}$  and so

$$0 = \partial T(\phi) = \int_{\mathbb{R}^k \times 0} \phi_0(x) \langle \vec{T}(0), dx^j \wedge dx^\beta \rangle d\mathcal{H}^k.$$

Since  $\phi$  is arbitrary this implies that  $\langle \vec{T}(0), dx^j \wedge dx^\beta \rangle = 0$  whenever  $j > k$  and  $\beta \in I_{k-1,m}$ . This implies the result. □

### 3. – Dual-space valued BV functions

In this section we prove a form of Poincaré's inequality for functions on Euclidean spaces taking values in separable dual spaces and having bounded variation in a certain sense. In particular, these functions have weak derivatives that satisfy a kind of finiteness condition.

A similar Poincaré inequality is established in a more general setting in Ambrosio [1], which develops a general theory of functions of bounded variation taking values in a separable, locally compact metric space. The approach we present here differs from that of Ambrosio.

We will write  $U$  to indicate a bounded, open subset of  $\mathbb{R}^k$ . We always assume that  $U$  is connected. We use boldface to indicate functions taking values in Banach spaces.

Throughout this section we assume that  $X$  and  $Y$  are Banach spaces such that

$$(3.1) \quad X \subset\subset Y, \quad \text{and as a result} \quad Y^* \subset\subset X^*,$$

where the notation means for example that  $X$  is compactly embedded in  $Y$  and similarly  $Y^*$  in  $X^*$ . For our application we have in mind  $Y = C_0^0(V)$  and  $X = C_0^1(V)$  with the natural norms, where  $V$  is a bounded open subset of some Euclidean space, and the subscript  $0$  indicates functions that vanish on  $\partial V$ . The compactness of the embedding then follows from the Arzela-Ascoli theorem.

With the above applications in mind, we will typically write  $\mu$  for maps taking values in  $X^*$  or  $Y^*$ , as we are thinking of  $\mu$  as a measure.

We first recall some facts about calculus of smooth functions taking values in Banach spaces.

If  $X$  is any Banach space, we say that  $\mathbf{f} \in C^1(U; X)$  if there is a continuous function  $U \mapsto X^k$ , denoted  $D\mathbf{f} := (\frac{\partial}{\partial y_1}\mathbf{f}, \dots, \frac{\partial}{\partial y_k}\mathbf{f})$  such that

$$\lim_{y' \rightarrow y} \frac{1}{|y' - y|} \|\mathbf{f}(y') - \mathbf{f}(y) - D\mathbf{f} \cdot (y' - y)\|_X = 0$$

for all  $y \in U$ . One can then check that the fundamental theorem of calculus holds, so that for example if  $U$  is convex and  $y, y' \in U$ , then

$$(3.2) \quad \int_0^1 D\mathbf{f}(sy + (1-s)y') \cdot (y - y') ds = \mathbf{f}(y) - \mathbf{f}(y'),$$

where  $D\mathbf{f} \cdot z := \sum_{i=1}^k z_i \frac{\partial}{\partial y_i}\mathbf{f}$ . Similarly, one can verify that if  $\mu \in C^1(U; X^*)$  and  $\phi \in C^1(U; X)$ , then for any  $y \in U$ ,

$$\frac{\partial}{\partial y_i} \langle \phi(y), \mu(y) \rangle = \left\langle \frac{\partial}{\partial y_i} \phi(y), \mu(y) \right\rangle + \left\langle \phi(y), \frac{\partial}{\partial y_i} \mu(y) \right\rangle.$$

If in addition  $\phi$  or  $\mu$  has compact support in  $U$ , we can integrate to find that

$$(3.3) \quad \int_U \left\langle \frac{\partial}{\partial y_i} \phi(y), \mu(y) \right\rangle dy = - \int_U \left\langle \phi(y), \frac{\partial}{\partial y_i} \mu(y) \right\rangle dy.$$

We will write  $\|D\mu\|_{(X^*)^k} := \sup \{ \langle \phi, D\mu \rangle : \|\phi\|_{X^k} \leq 1 \}$  where

$$\|\phi\|_{X^k} = \left( \sum_{i=1}^k \|\phi^i\|_X^2 \right)^{1/2}.$$

One immediately sees from this convention that, if  $\phi \in X$  and  $v \in \mathbb{R}^k$ , then

$$(3.4) \quad \langle \phi v, D\mu(y) \rangle \leq \|\phi\|_X |v| \|D\mu(y)\|_{(X^*)^k}$$

where  $|v|$  is just the Euclidean norm of  $v$  and  $\phi v = (\phi v_1, \dots, \phi v_k)$ .

For a function  $\mu : U \rightarrow X^*$  we will write  $\mu \in L^1(U; X^*)$  if  $\mu$  is Borel measurable in the sense that the inverse image of every open set (in the norm topology on  $X^*$ ) is measurable and

$$(3.5) \quad \int_U \|\mu(y)\|_{X^*} dy < \infty.$$

We write  $\mu \in L^1_w(U; X^*)$  if (3.5) holds and  $\mu$  is merely weak-\* measurable in the sense that  $y \mapsto \langle \phi, \mu(y) \rangle$  is measurable for every fixed  $\phi \in X$ . Note that  $L^1 \subset L^1_w$ .

For  $\mu \in L^1_w(U; X^*)$  we define the total variation

$$\text{Var}(\mu; U, X^*) := \sup \left\{ \int \langle \text{div}_y \phi(y), \mu(y) \rangle dy : \phi \in C^1_c(U; X^k), \|\phi(y)\|_{X^k} \leq 1 \quad \forall y \in U \right\}.$$

Here  $\text{div}_y \phi = \sum_{i=1}^k \phi^i_{y_i}$ . Clearly the total variation can be infinite for general  $\mu \in L^1_w(U; X^*)$ .

We now prove

LEMMA 4. *If  $\mu \in C^1_c(U; X^*)$  then*

$$\text{Var}(\mu; U, X^*) = \int_U \|D\mu\|_{(X^*)^k} dy.$$

PROOF. It is clear that

$$\begin{aligned} & \int_U \|D\mu\|_{(X^*)^k} dy \\ & \geq \sup \left\{ \int_U \langle \phi(y), D\mu(y) \rangle dy : \phi \in C^1_c(U; X^k), \|\phi(y)\|_{X^k} \leq 1 \forall y \in U \right\} \\ & = \sup \left\{ \int_U \langle \text{div}_y \phi(y), \mu(y) \rangle dy : \phi \in C^1_c(U; X^k), \|\phi(y)\|_{X^k} \leq 1 \forall y \in U \right\}. \end{aligned}$$

using integration by parts (3.3). To prove the opposite inequality it suffices to show that, given an open set  $W \subset\subset U$  there exists  $\phi \in C^1_c(U; X^k)$  such that  $\|\phi(y)\|_{X^k} \leq 1$  for all  $y$  and  $\langle \phi(y), D\mu(y) \rangle \geq \|D\mu(y)\|_{(X^*)^k} - \delta$  for all  $y \in W$ . In view of the continuity of  $D\mu$ , this can easily be done by defining  $\phi$  of the form

$$\phi(y) = \sum \zeta_i(y) \phi_i, \quad \phi \in X^k, \quad \zeta_i \in C^\infty_c(U),$$

for appropriate  $\phi_i \in X$  and a suitable partition of unity  $\{\zeta_i\}$ . □

We next prove some approximation lemmas.

LEMMA 5. *Suppose that  $X, Y$  are separable Banach spaces such that  $Y^* \subset\subset X^*$ . Let  $\eta^\epsilon : \mathbb{R}^k \rightarrow [0, \infty)$  be a smoothing kernel supported in  $B_\epsilon(0)$ , and for  $\mu \in L^1_w(U; Y^*)$  define*

$$\mu^\epsilon(y) = \eta^\epsilon * \mu(y) = \int \eta^\epsilon(y - y') \mu(y') dy'$$

with  $\text{dist}(y, \partial U) > \epsilon$ . Then for any  $W \subset\subset U$ ,

$$\int_W \|\mu(y) - \mu^\epsilon(y)\|_{X^*} \rightarrow 0$$

as  $\epsilon \rightarrow 0$ .



PROOF. It suffices to show that for  $\mu, W$  as above,

$$(3.6) \quad \lim_{h \in \mathbb{R}^k, h \rightarrow 0} \int_W \|\mu(y) - \mu(y - h)\|_{X^*} dy \rightarrow 0.$$

For  $M > 0$  let  $\mu_M(y) = \mu(y)\chi_{\{\|\mu(y)\|_{Y^*} \leq M\}}$ . Clearly  $\int_W \|\mu(y) - \mu_M(y)\|_{X^*} dy \rightarrow 0$  as  $M \rightarrow \infty$ , so it suffices to prove that  $\mu_M$  satisfies (3.6) for arbitrary  $M > 0$ . Changing notation slightly, we may assume that  $\mu$  in (3.6) satisfies  $\|\mu\|_{Y^*} \leq M$  for some  $M$ .

Following Sychev [12], we let  $K_M := \{\mu \in Y^* : \|\mu\|_{Y^*} \leq M\}$ , and we define

$$\rho(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i \|\phi_i\|_Y} |\langle \mu - \nu, \phi_i \rangle|,$$

where  $\{\phi_i\}_{i=1}^{\infty}$  is a dense subset of  $Y$ . Then it is easy to check that  $\rho$  is a metric, and that the metric topology induced by  $\rho$  on  $K_M$  is just the topology of weak-\* convergence. An easy case of the Banach-Alaoglu Theorem implies that  $(K_M, \rho)$  is a compact metric space. Sychev [12] Theorem 3.2, remarks that Lusin's theorem holds for functions  $\mu : U \rightarrow (K_M, \rho)$ : a function  $\mu$  is measurable (in the standard sense: the inverse image of every  $\rho$ -open set is measurable) if and only if for every  $\epsilon > 0$  there exists a compact set  $U_\epsilon$  such that  $\mathcal{L}^k(U \setminus U_\epsilon) < \epsilon$  and the restriction of  $\mu$  to  $U_\epsilon$  is continuous. In [12], Lemma 3.3, it is also shown that a function  $\mu : U \rightarrow K_M$  is weak-\* measurable if and only if it is measurable in the above sense. In particular weak-\* measurable functions enjoy the Lusin property.

From this it follows by standard arguments that if  $\mu : U \rightarrow K_M$  is weak-\* measurable then

$$(3.7) \quad \lim_{h \in \mathbb{R}^k, h \rightarrow 0} \int_W \rho(\mu(y), \mu(y - h)) dy \rightarrow 0.$$

Finally we assert that there exists a continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$

$$(3.8) \quad \|\mu - \nu\|_{X^*} \leq \omega(\rho(\mu, \nu)) \quad \text{for all } \mu, \nu \in K_M.$$

This can be deduced easily from the compactness of the embedding  $Y^* \subset X^*$ , which implies that any weak-\* convergent sequence in  $K_M$  also converges in the  $X^*$  norm. Since  $\rho(\mu, \nu) \leq C$  for  $\mu, \nu \in K_M$ , (3.8) and (3.7) imply (3.6).  $\square$

LEMMA 6. *If  $\mu \in L^1_w(U; Y^*)$  and  $\text{Var}(\mu; U, X^*) < \infty$ , then there exists a sequence of functions  $\mu_\epsilon \in C^1(U; X^*) \cap L^1_w(U; Y^*)$  such that*

$$(3.9) \quad \int_U \|\mu(y) - \mu_\epsilon(y)\|_{X^*} dy \rightarrow 0$$

as  $\epsilon \rightarrow 0$ , and

$$(3.10) \quad \lim_{\epsilon} \text{Var}(\mu_\epsilon; U, X^*) = \text{Var}(\mu; U, X^*).$$

PROOF. This exactly follows the usual argument for real-valued functions of bounded variation, as found for example in Giusti [8], Theorem 1.17. In this construction one defines approximators  $\mu_\epsilon$  of the form

$$\mu_\epsilon := \sum_i \eta^{\epsilon_i} * (\zeta_i \mu),$$

for a suitable sequence  $\{\epsilon_i\}_{i=1}^\infty$  and partition of unity  $\{\zeta_i\}_{i=1}^\infty \subset C_c^\infty(U)$ , both of which depend on the overall approximation parameter  $\epsilon$ . Let  $U_i$  denote sets such that  $\text{supp } \zeta_i \subset U_i$ ; these are chosen so that  $U_i \cap U_j = \emptyset$  if  $|i - j| \geq 2$ .

The main point in the proof is to verify that

$$(3.11) \quad \text{Var}(\mu_\epsilon; U, X^*) \leq \text{Var}(\mu; U, X^*) + O(\epsilon).$$

This is done by noting that

$$(3.12) \quad \int \langle \text{div } \phi(y), \mu_\epsilon(y) \rangle dy = \sum_{i=1}^\infty \int \langle \text{div } (\zeta_i \eta^{\epsilon_i} * \phi), \mu \rangle dy + \sum_{i=1}^\infty \int \langle \phi, \eta^{\epsilon_i} * (D\zeta_i \mu) - D\zeta_i \mu_\epsilon \rangle dy.$$

This calculation uses the fact that  $\sum D\zeta_i = 0$ ; for more details see [8]. Note that

$$\max_y \|\zeta_i(y) \eta^{\epsilon_i} * \phi(y)\|_{X^k} \leq \max_y \|\eta^{\epsilon_i} * \phi(y)\|_{X^k} \leq \max_y \|\phi(y)\|_{X^k} \leq 1$$

using the Hahn-Banach Theorem for the second inequality. Thus

$$\sum_{i=1}^\infty \left| \int \langle \text{div } (\zeta_i \eta^{\epsilon_i} * \phi), \mu \rangle dy \right| \leq \text{Var}(\mu; U_1, X^*) + \sum_{i=2}^\infty \text{Var}(\mu; U_i, X^*).$$

The sum on the right-hand side can be made arbitrarily small (say, less than  $\epsilon/2$ ) by selecting the open cover  $\{U_i\}$  carefully. Similarly, the last sum in (3.12) can be made arbitrarily small by using Lemma 5 and making a careful choice of the sequence  $\{\epsilon_i\}$ . This proves (3.11). All other conclusions are proved exactly as in the scalar case; see [8].  $\square$

We finally prove a form of Poincaré’s inequality. For this we introduce the notation

$$(\mu)_U := \frac{1}{|U|} \int_U \mu(y) dy.$$

LEMMA 7. *Suppose that  $X, Y$  are separable Banach spaces such that  $Y^* \subset\subset X^*$ . Given  $U \subset \mathbb{R}^k$  diffeomorphic to the open unit ball, there exists a constant  $C$  such that*

$$(3.13) \quad \|\mu - (\mu)_U\|_{L^1(U; X^*)} \leq C \text{Var}(\mu; U, X^*)$$

for all  $\mu \in L^1_w(U; Y^*)$ .

**PROOF. 1.** In view of Lemma 6, it suffices to prove (3.13) under the assumption that  $\mu$  is smooth. We also assume that  $U$  is the open unit ball, since more general domains can be reduced to this by a change of variables.

For any  $y \in U$ , we then have

$$\begin{aligned} \mu(y) - (\mu)_U &= \frac{1}{|U|} \int_U \mu(y) - \mu(y') dy' \\ &= \frac{1}{|U|} \int_U \int_0^1 \frac{\partial}{\partial s} \mu(sy + (1-s)y') ds dy' \\ &= \frac{1}{|U|} \int_U \int_0^1 D\mu(sy + (1-s)y') \cdot (y - y') ds dy'. \end{aligned}$$

Here we have used the convexity of  $U$  as well as (3.2). Thus, recalling (3.4),

$$\begin{aligned} \|\mu(y) - (\mu)_U\|_{X^*} &= \sup\{\langle \phi, \mu(y) - (\mu)_U \rangle : \phi \in X, \|\phi\|_X \leq 1\} \\ &\leq \frac{1}{|U|} \int_U \int_0^1 \|D\mu(sy + (1-s)y')\|_{X^*} |y - y'| ds dy' \end{aligned}$$

It follows that

$$(3.14) \quad \int_U \|\mu(y) - (\mu)_U\|_{X^*} dy \leq C(U) \int_0^1 \int_U \int_U \|D\mu(sy + (1-s)y')\|_{X^*} dy dy' ds.$$

**2.** For  $1/2 \leq s \leq 1$  and  $y' \in U$  we change variables to obtain

$$\begin{aligned} \int_U \|D\mu(sy + (1-s)y')\|_{X^*} dy &= s^{-k} \int_{\{\tilde{y}: \tilde{y}=sy+(1-s)y', y \in U\}} \|D\mu(\tilde{y})\|_{X^*} dy \\ &\leq 2^k \int_U \|D\mu(\tilde{y})\|_{X^*} dy \end{aligned}$$

We have again used the convexity of  $U$ . It follows that

$$(3.15) \quad \int_U \int_U \|D\mu(sy + (1-s)y')\|_{X^*} dy dy' \leq C(U, n) \int_U \|D\mu(\tilde{y})\|_{X^*} dy$$

whenever  $1/2 \leq s \leq 1$ . If  $0 \leq s \leq 1/2$  we obtain the same conclusion by integrating first in the  $y'$  variables and then next in the  $y$  variables. Thus (3.15) holds for all  $s \in [0, 1]$ . This fact, together with (3.14), implies that, if  $\mu$  is smooth, then

$$(3.16) \quad \begin{aligned} \int_U \|\mu_y - (\mu)_U\|_{X^*} dy &\leq C \int_U \|D\mu_{\tilde{y}}\|_{X^*} dy \\ &= C \text{Var}(\mu, U, X^*). \end{aligned}$$

## REFERENCES

- [1] L. AMBROSIO, *Metric space valued functions of bounded variation*, Ann. Scuola. Norm. Sup. Pisa Cl. Sci (4) **17** (1990), 439-478.
- [2] L. AMBROSIO – B. KIRCHHEIM, *Currents in metric spaces*, Acta Math. **185** (2000), 1-80.
- [3] J. BLIEDTNER – P. LOEB, *A reduction technique for limit theorems in analysis and probability theory*, Ark. Math. **30** (1992), 25-43.
- [4] L. C. EVANS – R. F. GARIEPY, “Measure theory and fine properties of functions”, CRC Press, Boca Raton, Florida, 1992.
- [5] H. FEDERER, “Geometric measure theory”, Springer-Verlag, Berlin-Heidelberg-New York, 1969.
- [6] M. GIAQUINTA – G. MODICA – J. SOUČEK, “Cartesian currents in the calculus of variations. I. Cartesian currents”, Springer-Verlag, 1998.
- [7] M. GIAQUINTA – G. MODICA – J. SOUČEK, “Cartesian currents in the calculus of variations. II. Variational integrals”, Springer-Verlag, 1998.
- [8] E. GIUSTI, “Minimal surfaces and functions of bounded variation”, Birkhäuser, 1984.
- [9] R. L. JERRARD – H. M. SONER, *Rectifiability of the distributional Jacobian for a class of functions*, C.R. Acad. Sci. Paris, Série I, **329** (1999), 683-688.
- [10] R. L. JERRARD – H. M. SONER, *Functions of bounded higher variation*, Indiana Univ. Math. J., **51** (2002), 645-677.
- [11] L. SIMON, “Lectures on geometric measure theory”, Australian National University, 1984.
- [12] M. SYCHEV, *A new approach to Young measure theory, relaxation and convergence in energy*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **16** (1999), 773-812.
- [13] B. WHITE, *Rectifiability of flat chains*, Ann. of Math. (2), **150** (1999), 165-184.

University of Illinois  
Urbana, IL 61801  
rjerrard@uiuc.edu