On $q$-Runge Pairs

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Abstract. We show that the converse of the approximation theorem of Andreotti and Grauert does not hold. More precisely we construct a $4$-complete open subset $D \subset \mathbb{C}^6$ (which is an analytic complement in the unit ball) such that the restriction map $H^3(\mathbb{C}^6, \mathcal{F}) \to H^3(D, \mathcal{F})$ has a dense image for every $\mathcal{F} \in Coh(\mathbb{C}^6)$ but the pair $(D, \mathbb{C}^6)$ is not a $4$-Runge pair.

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1. Introduction

Let $X$ be a $n$ dimensional $q$-complete complex manifold and $D \subset X$ a $q$-complete open subset. As is well-known (see e.g. [Col1]) the pair $(D, X)$ is called a $q$-Runge pair if for each compact subset $K \subset D$ there exists a $q$-convex exhaustion function $\varphi : X \to \mathbb{R}$ (which may depend on $K$) such that $K \subset \{\varphi < 0\} \subset D$. A main result of Andreotti and Grauert [A-G] asserts that if $(D, X)$ is a $q$-Runge pair then the following holds:

*) for every coherent sheaf $\mathcal{F} \in Coh(X)$ the restriction map $H^{q-1}(X, \mathcal{F}) \to H^{q-1}(D, \mathcal{F})$ has a dense image.

One could say that a pair $(D, X)$ of $q$-complete manifolds satisfying the assumption *) is a cohomologically $q$-Runge pair. It is then naturally to ask (see [Col]) if the converse of the approximation theorem of Andreotti and Grauert holds; in other words is it true that a cohomologically $q$-Runge pair is a $q$-Runge pair? As is well-known, the answer to this question is yes if $q = 1$ or $q = n$. For $q = n$ this condition of approximation turns out to be equivalent to a purely topological property, namely $X \setminus D$ has no compact connected components (see [Co-Sil] for a complete discussion of this case even on complex spaces with

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singularities). However, for $1 < q < n$, we shall show in this short note that the answer to the above question is negative. More precisely we prove:

**Theorem 1.1.** There exists a 3-dimensional closed analytic subset $A$ contained in the unit ball $B \subset \mathbb{C}^6$ such that the open set $D := B \setminus A$ has the following properties:

1) $D$ is 4-complete
2) for every coherent sheaf $\mathcal{F} \in \text{Coh}(\mathbb{C}^6)$ the restriction map $H^3(\mathbb{C}^6, \mathcal{F}) \to H^3(D, \mathcal{F})$ has a dense image
3) the pair $(D, \mathbb{C}^6)$ is not a 4-Runge pair

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2. – Preliminary results

We briefly recall some well-known definitions and results concerning $q$-convexity (see also [A-G], [Col]).

Let $U \subset \mathbb{C}^n$ be an open subset. A smooth function $\varphi \in C^\infty(U, \mathbb{R})$ is called $q$-convex if its Levi form $L(\varphi)$ has at least $(n - q + 1)$ strictly positive eigenvalues at any point of $U$. By means of local charts of coordinates this definition can be easily extended to complex manifolds. A complex manifold $X$ is said to be $q$-complete if there exists a smooth $q$-convex function $\varphi : X \to \mathbb{R}$ which is an exhaustion function, i.e. for every $c \in \mathbb{R}$, $\{ \varphi < c \} \subset \subset X$. If $q = 1$ then, by Grauert's solution to the Levi problem [Gra], $X$ is 1-complete iff $X$ is a Stein manifold. By Andreotti and Grauert results [A-G] a $q$-complete manifold $X$ satisfies $H^i(X, \mathcal{F}) = 0$ for $i \geq q$, $\mathcal{F} \in \text{Coh}(X)$. An open subset $D \subset X$, where $D$ and $X$ are assumed to be $q$-complete, is called $q$-Runge (or equivalently $(D, X)$ is a $q$-Runge pair) if for every compact subset $K \subset D$ there exists a $C^\infty$ smooth $q$-convex exhaustion function $\varphi : X \to \mathbb{R}$ (which may depend on $K$) such that $K \subset \{ \varphi < 0 \} \subset D$. By [A-G] a $q$-Runge pair $(D, X)$ satisfies the following approximation property: the restriction map $H^{q-1}(X, \mathcal{F}) \to H^{q-1}(D, \mathcal{F})$ has a dense image for every $\mathcal{F} \in \text{Coh}(X)$. By Morse theory it follows easily (see [Fo], [Va]) that a $q$-Runge pair $(D, X)$ satisfies the topological condition:

***) $H_{n+q-1}(X, D; \mathbb{Z})$ is torsion free and $H_{n+j}(X, D; \mathbb{Z}) = 0$ if $j \geq q$.

Unless $q = n = \dim X$ this topological condition is only necessary but not sufficient to guarantee the $q$-Runge condition. In the top degree $q = n$ the
topological condition **) is equivalent to each of the following assumptions (see e.g. [Co-Sil]):

i) \(H_{2q}(X, D; \mathbb{Z}) = 0\)

ii) \(X \setminus D\) has no compact connected components

iii) \((D, X)\) is a \(n\)-Runge pair

iv) \((D, X)\) is a cohomologically \(n\)-Runge pair

We note that the condition ‘\(H_{n+q-1}(X, D; \mathbb{Z})\) is torsion free’ will play an important role in the construction of our counter-example proving Theorem 1.1.

Another important tool needed for the proof of Theorem 1.1 are the results of W.Barth in [Ba] concerning the vanishing of the local cohomology groups \(H^i_A\) having only one singular point \(0 \in A\). In order to state the results of W.Barth we make the following notations:

Let \(A\) be a closed analytic subset of \(\mathbb{C}^n\) such that \(0 \in A\), \(A\) has pure dimension \(k \geq 2\) and \(0\) is the only one singular point of \(A\). For \(\varepsilon > 0\) we denote by \(B(\varepsilon)\) the open ball of radius \(\varepsilon\) about the origin and by \(K_\varepsilon = A \cap \partial B(\varepsilon)\). As well-known for \(\varepsilon > 0\) small enough, the cohomology groups \(H^i(K_\varepsilon, \mathbb{C})\) of the link \(K_\varepsilon\) do not depend on \(\varepsilon\). W.Barth [Ba] proves the following theorem concerning the vanishing of the cohomology groups with supports in \(A\):

Let \(2 \leq p \leq k\); then \(H^q_A(B(\varepsilon), \mathcal{O}) = 0\) for \(q \geq n - p + 2\) and the separated \(\sigma H^{n-p+1}(B(\varepsilon), \mathcal{O})\) (of \(H^{n-p+1}(B(\varepsilon), \mathcal{O})\) ) vanishes if and only if \(H^0(K_\varepsilon, \mathbb{C}) = \mathbb{C}\) and \(H^j(K_\varepsilon, \mathbb{C}) = 0\) for \(1 \leq j \leq p - 2\).

We shall apply this result in the following situation: \(n = 6\), \(p = q = 3\) and the singularity will be the vertex of the cone over the image of the Veronese embedding \(\mathbb{P}^2 \hookrightarrow \mathbb{P}^5\).

3. – The construction of the example proving Theorem 1.1

We consider the complex space \(A\), with an isolated singularity, which is obtained from \(\mathbb{C}^3\) by identifying \(z\) with \(-z\) via the quotient map \(\psi : \mathbb{C}^3 \to A\). Note that \(A\) can be embedded naturally in \(\mathbb{C}^6\). It is exactly the image of the map \(\Phi : \mathbb{C}^3 \to \mathbb{C}^6\) given by \((\alpha, \beta, \gamma) \mapsto (\alpha^2, \alpha \beta, \alpha \gamma, \beta^2, \beta \gamma, \gamma^2)\). Observe also that \(\psi : \mathbb{C}^3 \to A\) can be identified with \(\Phi : \mathbb{C}^3 \to \Phi(\mathbb{C}^3)\). This is a simple computation and so it is omitted. \(A\) has the origin of \(\mathbb{C}^6\) as the only one singular point. We are interested to compute the cohomology groups of the link of \(A\) at 0. For this we remark that the images of the balls \(B^3(0, \varepsilon) \subset \mathbb{C}^3\) by \(\psi, \varepsilon > 0\), give a fundamental system of good neighbourhoods (in the sense of Prill [Pr]) \(V_\varepsilon\) of \(0 \in A\) and the boundaries \(\partial V_\varepsilon\) of \(V_\varepsilon\) are real projective spaces of dimension 5. In particular it follows that the link \(K_\varepsilon\) of \(A\) at 0 satisfies:

\[H^0(K_\varepsilon, \mathbb{C}) = \mathbb{C}, H^1(K_\varepsilon, \mathbb{C}) = 0, H^2(K_\varepsilon, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\] (see e.g. [Gre] p. 90).

Let \(B^6(0, \varepsilon) \subset \mathbb{C}^6\) be open balls, \(\varepsilon > 0\) small enough, \(K_\varepsilon = A \cap \partial B^6(0, \varepsilon)\). We fix \(\varepsilon_0 > 0\) such that the previous mentioned results of Barth holds for
$B = B^6(0, \varepsilon_0)$ and we denote $D = B \setminus A$. Then $D$ is 4-complete because $A$ can be described in $\mathbb{C}^6$ by 4 holomorphic equations (e.g. if $(x, y, z, u, v, t)$ are coordinates in $\mathbb{C}^6$ then $A$ can be described by the equations $xu = y^2, xt = z^2, ut = v^2, xut = yzv$). We compute now the homology of $D$. Denote $A' := A \cap B$ and $K := K_{x_0}$. From the exact sequence

$$0 = H^i(A', \mathbb{Z}) \to H^i(K, \mathbb{Z}) \to H^{i+1}(A', K, \mathbb{Z}) = H^{i+1}_c(A', \mathbb{Z}) \to H^{i+1}(A', \mathbb{Z}) = 0$$

we get $H^i(K, \mathbb{Z}) = H^{i+1}_c(A', \mathbb{Z})$ for $i \geq 1$ and by Poincaré duality we have $H^i_c(A', \mathbb{Z}) = H_{12-i}(B, \mathbb{Z}, \mathbb{Z})$. It follows that $H_8(D, \mathbb{Z}) = H_9(B, \mathbb{Z}) = H^2_c(A', \mathbb{Z}) = H^2(K, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Since $H_9(\mathbb{C}^6, \mathbb{Z}) = H_8(D, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ has torsion we get, as remarked in the preliminaries that $(\mathbb{C}^6, D)$ cannot be a 4-Runge pair (the condition **) for $n = 6$ and $q = 4$. On the other hand since $H^0(\mathbb{K}, \mathbb{C}) = \mathbb{C}$ and $H^1(\mathbb{K}, \mathbb{C}) = 0$ the theorem of W.Barth (for $p = k = 3$ and $n = 6$) implies that $H^j_A(B, \mathcal{O}) = 0$ for $j \geq 5$ and $^\sigma H^3_A(B, \mathcal{O}) = 0$ where $^\sigma H^i$ denotes the separated $H^i/\{0\}$ associated to the cohomology space $H^i$. We therefore get $H^j(D, \mathcal{O}) = 0$ for $j \geq 4$ and $^\sigma H^3(D, \mathcal{O}) = 0$. So the restriction map $0 = H^3(\mathbb{C}^6, \mathcal{O}) \to H^3(D, \mathcal{O})$ has a dense image. We have to check a similar property for every $F \in Coh(\mathbb{C}^6)$. For this we note that over $D$ we have an exact sequence of coherent sheaves $0 \to K \to \mathcal{O}^p \to F \to 0$ from which we get an exact sequence of cohomology groups: $H^3(D, \mathcal{O}^p) \to H^3(D, F) \to H^3(D, K) = 0$ which implies that $^\sigma H^3(D, F) = 0$, hence the restriction map $0 = H^3(\mathbb{C}^6, F) \to H^3(D, F)$ has a dense image, as required. Theorem 1.1 is completely proved.

**Remark 3.1.** It would be interesting to know if $H^3(D, \mathcal{O}) = 0$, or more generally, if $H^3(D, F) = 0$ for every $F \in Coh(D)$. If the answer would be yes then one would get an example of a cohomologically 3-complete open subset $D \subset \mathbb{C}^6$ which is not 3-complete.

**Remark 3.2.** Let $X$ be a $q$-complete manifold and $D \subset X$ an open subset which is $q$-complete. In [So] the pair $(D, X)$ is called $q$-Runge (we shall call it weakly $q$-Runge) if the restriction map $H^{q-1}(X, \Omega^p) \to H^{q-1}(D, \Omega^p)$ has a dense image for every $p \geq 0$, where $\Omega^p$ denotes the sheaf of germs of holomorphic $p$-forms. By the previous example provided by Theorem 1.1 we see that the conditions “$q$-Runge” and “weakly $q$-Runge” are not equivalent, we have only the implication “$q$-Runge” $\implies$ “weakly $q$-Runge”. As observed in [So] a weakly $q$-Runge pair satisfies the topological condition $H_{n+j}(X, D, \mathbb{Z}) = 0$ for $j \geq q$.

**REFERENCES**


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